SOLVTIO PROBLEMATIS GEOMETRICI EX DOCTRINA SPHAERICORVM

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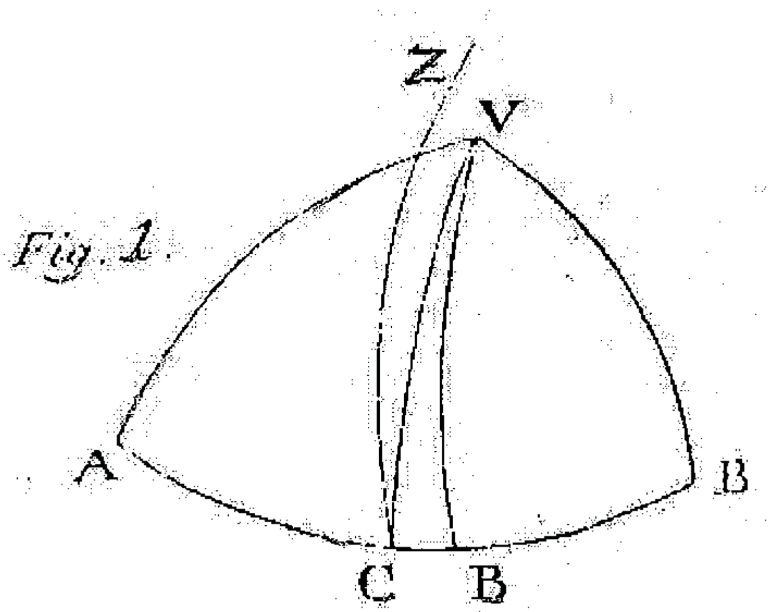
§1.

Ex eo tempore quo *Theodocii* Elementa Sphaericorum consignata habentur, vix alia quae ad doctrinam de figuris in superficie Sphaerae descriptis vlterius perficiendam pertinent, a Geometris tractata reperiuntur, ac ea quae in Elementis Trigonometriae Sphaericae tractari solent atque ad resolutionem triangulorum Sphaericorum spectant. Nullum autem est dubium, quin si symptomata linearum curuarum in superficie sphaerica descriptarum, eodem modo euoluerentur, ac curuarum in plano descriptarum affectiones explicatae fuerunt; noua Geometriae pars prodiret, quae non solum insigni varietate, sed elegantia quoque inuentorum se commendaret. Verum tamen quum huiusmodi disquisitiones curiosae magis, quam vtiles videri queant, inde sine dubio factum est, quod huic doctrinae perficiendae operam hucusque non dederint Geometrae. Quamuis autem completa Theoria curuarum in superficiae sphaerica, difficilis non minus ac parum fructuosa esse poterit; eo tamen non obstante singularia ex hac doctrina Problemata, quae praecipua quadam elegantia se commendant, non prorsus negligi debere videntur. Ex eo genere quum istud Problema sit, cuius nunc solutionem exponere constitui, eandem Geometris non prorsus ingratam fore, mihi persuasi.

§2. Ex Elementis Geometriae notum est, quod Triangula in plano descripta, quae super eadem basi et ad eandem partem huius basis collocantur, si aequalia inter se fuerint, hanc habeant proprietatem, vt eorum vertices collocentur in linea recte basi parallela; occasione igitur huius propositionis in animum induxi, vt inquirerem, de linea curua in qua collocentur vertices omnium triangulorum Sphaericorum, quae super eadem basi collocantur et quorum areae inter se sunt aequales; quum enim leui adhibita attentione perspexissem, hanc curuam nequaquam esse circulum maximum, cuius symptomata in superficie sphaerae, aliquin cum illis, quae lineae rectae in plano competunt, consentire solent, eo magis curiosa haec disquisitio mihi visa est.

§3. Priusquam vero solutionem huius Problematis adgredi licet, propositionis quasdam praeliminares Lemmatum instar, praemittere conueniet, quum illis solutio nostra superstruatur. Primum igitur nosse oportet, quod ad aequalitatem binorum triangulorum Sphaericorum id requiratur, vt summa angulorum in vno istorum triangulorum, aequalis sit summae angulorum in altero; demonstratum enim est quadruplam aream trianguli Sphaerici esse in ea ratione ad totam Sphaere superficiem, ac excessus summae omnium angulorum trianguli Sphaerici super binos rectos, est ad binos angulos rectos. Alterum Lemma quod heic tanquam propositionem praeliminarem adhibere constitui, sequenti continetur Theoremate (Vide Tab. IV, Fig. 1):



Si fuerit triangulum Sphaericum ABV, *cuius tres anguli* ABV, BAV, AVB, *litteris* B, A, V *respectiue exprimantur, lateribus illis oppositis per b, a, v designatis, erit*

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}) = \cot\frac{1}{2}\mathrm{V}\frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)}.$$

Huius propositionis variae quidem demonstrationes ab Auctoribus, qui de doctrina triangulorum Sphaericorum agunt, tradi solent; quum tamen minus commodae mihi visae sint, hic aliam meo quidem iudicio, haud inconcinnam, exponam. Ex puncto V in basin AB, demittatur normalis arcus circuli maximi VR et dicatur angulus AVR=μ atque BVR=ν. Quum igitur sit per proprietates triangulorum Sphaericorum rectangulorum:

$$\cos\mathrm{A} = \sin\mathrm{AVR}\cos\mathrm{VR} \qquad \text{et} \qquad \cos\mathrm{B} = \sin\mathrm{BVR}\cos\mathrm{VR}, \qquad \text{fiet}$$

$$\cos\mathrm{A} : \cos\mathrm{B} = \sin\mu : \sin\nu,$$

hincque

$$\cos\mathrm{A} - \cos\mathrm{B} : \cos\mathrm{A} + \cos\mathrm{B} = \sin\mu - \sin\nu : \sin\mu + \sin\nu, \qquad \text{seu}$$

$$\tan\frac{1}{2}(\mathrm{A}-\mathrm{B})\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}) = \tan\frac{1}{2}(\mu-\nu)\cot\frac{1}{2}(\mu+\nu) = \cot\frac{1}{2}\mathrm{V}\tan\frac{1}{2}(\mu-\nu).$$

Porro ob

$$\tan\mathrm{VR} = \tan\mathrm{VA}\cos\mathrm{AVR} = \tan\mathrm{VB}\cos\mathrm{BVR},$$

sit $\cos\mu\ :\cos\nu=\tan a\ :\tan b$, vnde colligitur

$$\cos\nu-\cos\mu\ :\cos\mu+\cos\nu=\tan b-\tan a\ :\tan b+\tan a,$$

seu

$$\tan\frac{1}{2}(\mu-\nu)\tan\frac{1}{2}\mathrm{V}=\frac{\sin(b-a)}{\sin(b+a)}.$$

Denique ob $\sin\mathrm{A}\ :\sin\mathrm{B}=\sin a:\sin b$, sit

$$\sin\mathrm{B}-\sin\mathrm{A}:\sin\mathrm{B}+\sin\mathrm{A}=\sin b-\sin a:\sin b+\sin a,$$

hincque

$$\cot\frac{1}{2}(\mathrm{B}-\mathrm{A})\tan\frac{1}{2}(\mathrm{A}+\mathrm{B})=\cot\frac{1}{2}(b-a)\tan\frac{1}{2}(b+a).$$

Si nunc haec aequatio ducatur in

$$\tan\frac{1}{2}(\mathrm{B}-\mathrm{A})\tan\frac{1}{2}(\mathrm{B}+\mathrm{A})=\cot\frac{1}{2}\mathrm{V}\tan\mathrm{V}\frac{1}{2}(\mu-\nu)$$

obtinebimus:

$$\tan^2\frac{1}{2}(\mathrm{B}+\mathrm{A})=\cot\frac{1}{2}\mathrm{V}\tan\frac{1}{2}(\mu-\nu)\cot\frac{1}{2}(b-a)\tan\frac{1}{2}(b+a),$$

in qua si loco $\tan\frac{1}{2}(\mu-\nu)$ substituatur $\cot\frac{1}{2}\mathrm{V}\frac{\sin(b-a)}{\sin(b+a)}$, fiet

$$\tan^2\frac{1}{2}(\mathrm{B}+\mathrm{A})=\cot^2\frac{1}{2}\mathrm{V}\frac{\cot\frac{1}{2}(b-a)}{\cot\frac{1}{2}(b+a)}\frac{\sin(b-a)}{\sin(b+a)}=\cot^2\frac{1}{2}\mathrm{V}\frac{\cos^2\frac{1}{2}(b-a)}{\cos^2\frac{1}{2}(b+a)},$$

ob $\sin(b-a)=2\sin\frac{1}{2}(b-a)\cos\frac{1}{2}(b-a)$ et $\sin(b+a)=2\sin\frac{1}{2}(b+a)\cos\frac{1}{2}(b+a)$. Extracta igitur radice quadratica sit:

$$\tan\frac{1}{2}(\mathrm{B}+\mathrm{A})=\cot\frac{1}{2}\mathrm{V}\frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)}.$$

§4. His praesuppositis solutio problematis nostri sequenti ratione adornari potest: Supponamus super data basi AB=2 CB= $2a$ descriptum esse triangulum Sphaericum AVB datae magnitudinis, tum vero ex puncto V in AB demittatur arcus circuli maximi VR normalis et dicantur CR=x; VR=y; anguli autem VAB, VBA, AVR, BVR, vti supra respectiue per litteras A, B, μ, ν exprimantur eritque per Lemma modo demonstratum:

$$\tan\frac{1}{2}(\mathrm{A}+\mu)=\frac{\cos\frac{1}{2}(\mathrm{VR}-\mathrm{AR})}{\cos\frac{1}{2}(\mathrm{VR}+\mathrm{AR})}\qquad\text{et}\qquad\tan\frac{1}{2}(\mathrm{B}+\nu)=\frac{\cos\frac{1}{2}(\mathrm{VR}-\mathrm{BR})}{\cos\frac{1}{2}(\mathrm{VR}+\mathrm{BR})},$$

ob $\tan\frac{1}{2}$ ARV=$\tan 45°$=1, hinc colligitur

$$\tan\frac{1}{2}(\mathrm{A}+\mu)=\frac{\cos\frac{1}{2}(y-a-x)}{\cos\frac{1}{2}(y+a+x)}\qquad\text{et}\qquad\tan\frac{1}{2}(\mathrm{B}+\nu)=\frac{\cos\frac{1}{2}(y-a+x)}{\cos\frac{1}{2}(y+a-x)}.$$

Hinc vero sequitur

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mu+\nu)=\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{V})$$
$$=\frac{\cos\frac{1}{2}(y-a-x)\cos\frac{1}{2}(y+a-x)+\cos\frac{1}{2}(y-a+x)\cos\frac{1}{2}(y+a+x)}{\cos\frac{1}{2}(y+a+x)\cos\frac{1}{2}(y+a-x)-\cos\frac{1}{2}(y-a-x)\cos\frac{1}{2}(y-a+x)}.$$

Est vero

$$\cos(y+x)+\cos a=2\cos\frac{1}{2}(y+x+a)\cos\frac{1}{2}(y+x-a);$$
$$\cos(y-x)+\cos a=2\cos\frac{1}{2}(y-x+a)\cos\frac{1}{2}(y-x-a);$$

tumque

$$\cos(a+y)+\cos x=2\cos\frac{1}{2}(y+a+x)\cos\frac{1}{2}(y+a-x);$$
$$\cos(y-a)+\cos x=2\cos\frac{1}{2}(y-a+x)\cos\frac{1}{2}(y-a-x);$$

quamobrem consequemur:

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{V})=\frac{\cos(y+x)+\cos(y-x)+2\cos a}{\cos(a+y)-\cos(y-a)}.$$

Atqui est

$$\cos(y+x)+\cos(y-x)=2\cos y\cos x \qquad \text{et} \qquad \cos(y-a)-\cos(a+y)=2\sin a\sin y,$$

vnde his valoribus substitutis fiet

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{V})=\frac{\cos y\cos x+\cos a}{-\sin y\sin a}.$$

Quum igitur triangulum AVB sit datae magnitudinis, summa angulorum A, B, V erit cognita, quae si statuatur $180^\circ+2\delta$, fiet

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{V})=\tan(90^\circ+\delta)=-\cot\delta,$$

vnde haec orietur aequatio:

$$\cot\delta\sin a\sin y=\cos y\cos x+\cos a,$$

qua aequatione indoles curuae in qua punctum V collocatur, erit expressa.

§5. Quum ex hac aequatione indoles huius curuae nondum satis liquido patescat, videamus quomodo propius ad scopum pertingere licebit. Per punctum C ducatur circulus maximus ZC normalis ad AB et iungatur CV, tum vero dicatur CV=z et angulus ZCV=ϕ, eritque ob

$$\cos\mathrm{CR}\cos\mathrm{VR}=\cos\mathrm{VC}, \qquad \text{et} \qquad \sin\mathrm{VR}=\sin\mathrm{VC}\sin\mathrm{VCR}=\sin\mathrm{VC}\cos\mathrm{ZCV},$$

$$\cos x\cos y=\cos z \qquad \text{et} \qquad \sin y=\sin z\cos\phi,$$

his igitur valoribus in aequatione allata substitutis, fiet

$$\cot\delta\sin a\sin z\cos\phi = \cos z + \cos a,$$

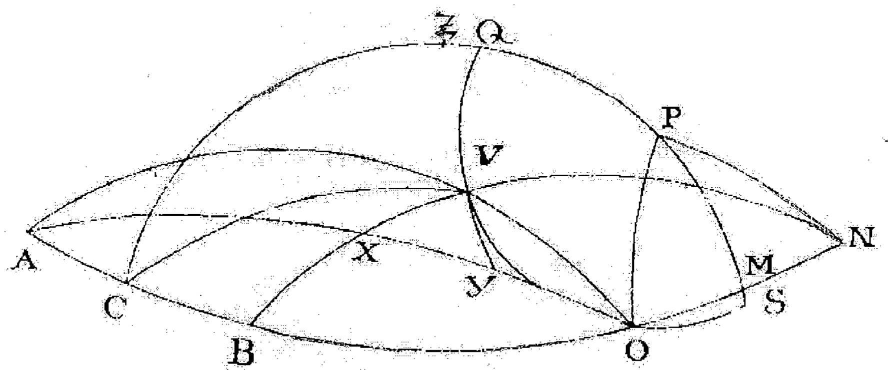
quae iam facile ad huiusmodi formam reducitur:

$$\cos\gamma = \cos z\cos\varepsilon + \sin z\sin\varepsilon\cos\phi,$$

ponendo

$$\frac{\cos\gamma}{\cos\varepsilon} = -\cos a \qquad \text{et} \qquad -\tan\varepsilon = \cot\delta\sin a,$$

ex qua aequatione manifesto liquet curuam istam quaesitam esse circulum minorem, cuius constructio hunc in modum adonatur: Concipiatur circulum maximum CZ productum iterum occurrere circulo maximo ABO in puncto M, tumque resecetur MO=CB, et ducatur PO, qui cum arcu MO facit angulum POM=$90° - \delta$; iam si polo P interuallo arcu PO describatur circulus minor OVQ, erit hic circulus locus istorum punctorum V, ita sitorum, vt si ex datis punctis A, B ad punctum quodpiam V ducantur arcus circulorum maximorum AV, BV, erit triangulum BVA datae magnitudinis, summa angulorum existente=$180° + 2\delta$.



Nam ob (Vide Tab. IV, Fig. 2)

$$\text{POM} = 90° - \delta, \qquad \text{erit} \qquad \tan\text{POM} = \cot\delta = -\frac{\tan\varepsilon}{\sin\alpha},$$

est vero

$$\tan\text{POM} = \frac{\tan\text{PM}}{\sin\text{MO}},$$

eritque igitur

$$\tan\text{PM} = -\tan\varepsilon, \qquad \text{et} \qquad \text{PM} = 180° - \varepsilon,$$

quare arcus CZP=ε, tum vero sit $\cos\text{PO} = \cos\text{OM}\cos\text{PM}$, ideoque

$$\cos\text{PO} = -\cos a\cos\varepsilon, \quad \text{ob} \quad \text{OM} = \text{CB} = a,$$

hinc obtinemus PO=γ; sponte autem liquet esse pro triangulo Sphaerico PCV;

$$\cos \mathrm{PV} = \cos \mathrm{PO} = \cos \mathrm{PC} \cos \mathrm{VC} + \sin \mathrm{PC} \sin \mathrm{VC} \cos \mathrm{PCV},$$

hoc est

$$\cos\gamma = \cos\varepsilon\cos z + \sin\varepsilon\sin z\cos\phi,$$

quae est aequatio supra inuenta. Quia vti iam obseruauimus, sit arcus

$$\mathrm{CP} = 180^\circ - \mathrm{PM} = \varepsilon, \quad \text{ideoque} \quad \tan \mathrm{CP} = \tan\varepsilon = -\cot\delta\sin a, \quad \text{erit}$$

$$\mathrm{CQ} = \mathrm{CP} - \mathrm{PQ} = \mathrm{CP} - \mathrm{PO} = \varepsilon - \gamma \quad \text{et} \quad \mathrm{CS} = \mathrm{CP} + \mathrm{PO} = \varepsilon + \gamma,$$

qua ratione bina puncta definiuntur, in quibus circulus minor QVO circulo maximo CZP occurrit.

§6. Data basi AB, cuiuscumque fuerit magnitudinis triangulum AVB, circulus minor, qui locum punctorum V exhibet, semper per punctum O circuli CBM trasibit, existente MO=CB, hinc igitur patet circulum minorem istum, in maximum non abire, nisi fuerit MV=CB=90°; pro eo autem casu quicumque demum fuerit valor trianguli AVB, semper locus punctorum V erit circulus maximus et tunc quidem triangulum AVB in segmentum sphaerae abit, binis semicirculis maximis inclusum. Si vero fuerit CB$<$ 90°, circulus minor in maximum abire non potest nisi euanescente triangulo AVB, quod per se est manifestum.

§7. Si consideretur punctum Q, vbi circulus minor OVQ intersecat circulum maximum CZM, eius situs triplici modo se habere poterit; dum enim Z supponitur esse polus circuli maximi CBM, punctum Q aut cadet inter Z et C, aut in ipsum punctum Z incidet, aut denique inter Z et M reperietur. Primus casus locus habebit, si fuerit $\cot\delta > \cot a$, secundus si sit $\cot\delta = \cot a$, et tertius denique existente $\cot\delta < \cot a$. Nam si sit $\cot\delta > \cot a$ et tan PM indicetur per $\tan\theta$, fiet ob $\frac{\tan\theta}{\sin a} = \cot\delta$, $\frac{\tan\theta}{\sin a} > \cot a$, ideoque $\tan\theta > \cos a$, vnde multiplicando vtrinque per $\cos\theta$, $\sin\theta > \cos a\cos\theta$, id est $\sin\theta > \cos\gamma$, hincque vicissim $\sin\gamma > \cos\theta$, quum igitur sit PM=θ, erit PZ=$90^\circ - \theta$, ideoque PQ>PZ, hoc est punctum Q cadet inter Z et C; simili ratione liquet posito $\cot\delta = \cot a$, fore $\sin\gamma = \cos\theta = \sin(90^\circ - \theta)$, id est PQ=PZ vnde puncta Q et Z coincident; tumque denuo posito $\cot\delta < \cot a$, fiet $\sin\gamma < \cos\theta$, ideoque PZ>PQ.

§8. Si punctum V incidat in O, abit arcus BV in BO, quamobrem ob AB+BO=180°, erit quoque arcus AO aequalis semicirculo maximo, ideoque hoc casu triangulum AVB abit in segmentum sphaerae, binis semicirculis maximis inclusum, angulo inter hos semicirculos XOB, vel XAB existente = δ; est enim

$$\frac{1}{2}(\mathrm{XAB} + \mathrm{ABO} + \mathrm{XOB}) = 90^\circ + \mathrm{XOB}, \quad \text{ob} \quad \mathrm{ABO} = 180^\circ \quad \text{et} \quad \mathrm{XAB}=\mathrm{XOB},$$

hincque fiet $90^\circ + \mathrm{XOB} = 90^\circ + \delta$ siue $\mathrm{XOB} = \delta = 90^\circ - \mathrm{POM}$, vnde perspicuum euadit arcum circuli maximi OXA, tangere circulum minorem OVQ in puncto O.

§9. Quoniam locus punctorum V est circulus minor polo P interuallo PO descriptus, merito quaeritur vtrum integer hic circulus, problemati satisfaciat, an vero ea tantum eius pars, quae super circulum maximum ABM eleuata est. Scilicet si integer hic semicirculus minor descriptus concipatur, qui occurrat circulo maximo CZM in punctis Q et S, eius pars QVO supra circulum maximum eleuatur, reliqua parte OS infra hunc circulum depressa. Tenendum vero est partem QO proprie tantum Problemati satisfacere, alteram autem partem, OS eius esse indolis, vt vbiqumque in illa sumatur punctum V′, quod cum punctis A, B iungatur arcubus circulorum maximorum AV′, BV′, triangulum AV′B quoque datae sit magnitudinis, aequale nimirum excessui, quo semissis superficiei sphaerae exsuperat aream trianguli AVB, ita vt ambo haec triangula AVB, AV′B inuicem addita aequentur semissi superficiei Sphaericae. Nam si indicentur anguli V′AB, V′BA, AV′B respectiue per litteras A′, B′, V′, tumque ponatur

$$\mathrm{A}' + \mathrm{B}' + \mathrm{A}' = 180^\circ + 2\delta,$$

ita vt sit

$$\frac{1}{2}(\mathrm{A}' + \mathrm{B}' + \mathrm{V}') = 90^\circ + \delta',$$

deinde vero statuatur ZCV′=ϕ' et arcus CV′=z, peruenietur ad hanc aequationem:

$$-\cot\delta \sin a \sin z \cos\phi = \cos z + \cos a,$$

cuius reductio ad formam

$$\cos\gamma = \cos z \cos\varepsilon + \sin z \sin\varepsilon \cos\phi',$$

requirit, vt ponatur

$$\cos\gamma = \cos a \cos\varepsilon \quad \text{et} \quad \tan\varepsilon = \cot\delta' \sin a,$$

si igitur sumatur $\delta' = 180^\circ - \delta$, pro ε et γ iidem valores reperientur ac supra §5., quare arcus OS erit locus punctorum V′, existente area trianguli ABV′ eius quantitatis, vt sit

$$\tan\frac{1}{2}(\mathrm{A}' + \mathrm{B}' + \mathrm{V}') = \tan(90^\circ + \delta').$$

Si vero fuerit

$$\mathrm{A}' + \mathrm{B}' + \mathrm{V}' = 180^\circ + 2\delta',$$

quum supra assumserimus

$$\mathrm{A}+\mathrm{B}+\mathrm{V} = 180^\circ + 2\delta,$$

addendo inuicem habebimus

$$\mathrm{A}+\mathrm{B}+\mathrm{V}+\mathrm{A}' + \mathrm{B}' + \mathrm{V}' = 360^\circ + 2\delta + 2\delta' = 2\cdot 360^\circ, \quad \text{ob} \quad \delta' = 180^\circ - \delta.$$

Hinc per §3. si superficies sphaerae indicetur per S et areae triangulorum AVB, AV′B per α, β fiet

$$4\alpha : \mathrm{S}=\mathrm{A} + \mathrm{B} + \mathrm{V} - 180^\circ : 180^\circ \quad \text{tumque}$$

$$4\beta : \mathrm{S}=\mathrm{A}' + \mathrm{B}' + \mathrm{V}' - 180^\circ : 180^\circ \quad \text{vnde}$$

$$4\alpha + 4\beta : \mathrm{S} = \mathrm{A}+ \mathrm{B}+ \mathrm{V}+ \mathrm{A}' + \mathrm{B}' + \mathrm{V}' - 360^\circ : 180^\circ,$$

id est

$$4(\alpha+\beta) : \mathrm{S} = 2 : 1, \quad \text{hinc} \quad \alpha+\beta = \frac{1}{2}\mathrm{S}.$$

In nostra figura triangulum AV′B non quidem expressimus, verum eo tamen non obstante, quae de hoc Triangulo monuimus, satis perspicua esse existimamus.

§10. Quum solutio Problematis nostri in superioribus allata Analytica sit, nunc aliam eium solutionem Geometricam eo magis adferre licebit, quod vix quidem primo intuitu iudicari possit, hoc Problema per Elementa Sphaericorum tam expedite demonstrari posse. *Problema* autem ipsum iam ita enunciatur: *Quaeritur in superficie sphaerae linea curua* QVO *eius indolis, vt vbicumque in illa sumatur punctum* V, *quod cum datis binis punctis* A, B *iungatur arcubus circulorum maximorum* AV, BV, *triangulum* AVB *semper datae sit magnitudinis, aequale nimirum segmento sphaerae* AXOBA, *binis semicirculis maximis* AXO, ABO, *incluso.*

Constructio.

Arcu AB in C bisecto describatur per C circulus maximus CZM normalis ad AB qui circulo ABO denuo in M occurrat, tum vero per O ducatur arcus circuli maximi PO normalis ad AXO, qui circulo CZM occurrat in P, iam Polo P interuallo PO describatur circulus minor QVO, dico hunc circulum minorem esse curuam istam quaesitam.

Demonstratio.

Sumatur punctum V vbicumque in hoc circulo minore, modo in arcu OVQ supra COM eleuato reperiatur, et ducantur semicirculi maximi AVO, BVN, tumque arcus circulorum maximorum PV, PN. Iam ob arcum BON=CBM=ACO, fiet CB=MO=MN, ideoque PN=PO=PV, hinc in triangulo PVN aequicruro, erit ang PVN=PNV et in Δ PVO aequicruro, ang PVO=POV. Nunc vero ob ang. PON=PNO=PNV + VNO, et VNO=VBO, fiet

$$\text{ang. PON} = 180^\circ - \text{POB=PVN} + \text{VBO},$$

ideoque

$$180^\circ - \text{VOB-POV} = 180^\circ - \text{VOB} - \text{PVO=PVN} + \text{VBO},$$

addatur vtrinque angulus BVO, eritque

$$180^\circ + \text{BVO} - \text{VOB} - \text{PVO=BVO+VBO+PVN},$$

vnde ob

$$\text{PVN=PVO}-\text{VNO=PVO}-\text{AVB, fiet:}$$
$$180^\circ + \text{BVO} - \text{VOB} - \text{PVO=BVO} + \text{VBO} + \text{PVO} - \text{AVB, vnde}$$
$$360^\circ - 2\ \text{POV=BVO} + \text{VBO} + \text{BOV},$$

quum igitur sit angulus

$$\text{VOX=}90^\circ - \text{POV, erit } 180^\circ - 2\ \text{POV=2 VOX},$$

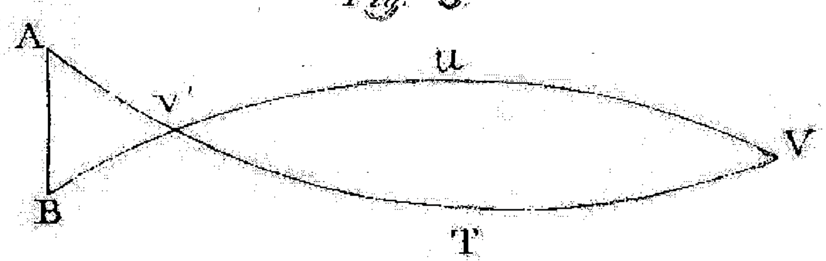
hincque

$$180° + 2VOX = BVO + VBO + BOV,$$

ex quo iam constat triangulum sphaericum BVO aequari segmento sphaerae AVOXA, hincque demto vtrinque triangulo communi VOX, fiet triangulum AVX =BOX et addito denuo vtrinque triangulo AXB fiet segmentum AXOBA=Δ AVB.

§11. Si in superficie sphaerae tria sumantur puncta A, B, V, patet semper bina quaecumque eorum, duobus arcubus circulorum maximorum iungi posse, vno scilicet arcu semicirculo minori, altero tantumdem maiore existente; hincque si per triangulum sphaericum, intelligatur figura, quae includitur tribus arcubus circulorum maximorum, patet iungendo tria puncta A, B, V omnino octoformari posse triangula sphaerica. Scilicet si bini arcus AB et 360°−AB indicentur per v, v', similiterque bini arcus AV, 360°−AV per b, b' et BV, 360−BV per a, a', octo haec triangula orientur per combinationes diuersas trium a vel a', b vel b', v vel v'. Quamuis autem in Geometria Elementari, non considerari soleant, nisi triangula quae componuntur ex arcubus circulorum maximorum, semicirculo minorum, nullum tamen est dubium, quin triangulorum sphaericorum affectiones rite explicatae ad reliqua quoque adplicari queant. Pro casu nostri Problematis vbi arcus AB supponitur cognitus, quatuor triangula formari possunt, *primum* scilicet quod in Figura indicatur AVB, *secundum* quod arcubus AB, BV et 360°−AV includitur, *tertium* quod arcubus AB, AV et 360°−BV includitur et *quartum* denique cuius latera sunt AB, 360°−AV, 360°−BV. Quod secundum et tertium attinet, facile liquet, eorum areas complementa constituere trianguli AVB ad semissim superficiei sphaerae. Pro quarto vero obseruare conuenit, arcus AV, BV antequam ad A, B peruenerit, se in V′ intersecare, vbi igitur area trianguli censeri debet aequalis differentiae inter segmentum sphaericum VUV′TV et triangulum AV′B=Δ AVB (Vide Tab. IV, Fig. 3). Deinde vero si loco basis AB adhibeatur eius complementum ad quatuor rectos, alia quatuor orientur triangula, ad quae Problema nostrum quoque adplicari potest .

Fig. 3.



§12. Ponamus circulum maximum CZM designare Meridianum cuiusdam loci et CBM horizontem, in quo C sit punctum Meridiei et M Septentrionis, iam si Polus aequatoris intelligatur esse P et distantia stellae alicuius fixae a Polo aequatoris PO, ita vt haec stella motu diurno circulum minorem QVOS describere videatur, qui horizontem in puncto O intersecant, tumque a Meridie vtrimque capiantur AC, CB singuli aequales ipsi MO, vbicumque haec stella versatur, si ex punctis horizontis A, B ad eius locum ducantur arcus circulorummaximorum, AV, BV, erit triangulum sphaericum AVB, quod inde formatur, datae magnitudinis.

Solution of a geometrical problem from the theory of the sphere

by Anders Johan Lexell (translated by J. C.-E. Sten)

§1.

From that time in which the Elements of Spherical Geometry of Theodosius [of Bithynia] had been put on record, hardly any other tracts are found from the geometers, about further perfection of the theory of figures described on spherical surfaces, and that treated in the Elements of Spherical Trigonometry is accustomed to be used and considered in the solution of spherical triangles. But there is no doubt that a new part of geometry can be brought into existence, if the properties of curved lines on the surface of a sphere are set out in the same way as the properties of curves in the plane can be described, and which comes not only in a conspicuous variety but also with elegant ways of solving problems to recommend itself. Yet truly with more careful inquiries of this kind as may be considered useful, it can thus without doubt be said that for this theory to be completed, then a hitherto unavailable aid will be given to the geometer. Although moreover in a complete theory of curves on a spherical surface, the more difficult part can be the least fruitful; yet from this theory, not without individual opposing problems, the problems that commend themselves with a particular elegance are seen in short to be the ones considered. Since I have now established the solution of a problemof this kind, I have convinced myself that the same geometers in short will not be ungrateful.

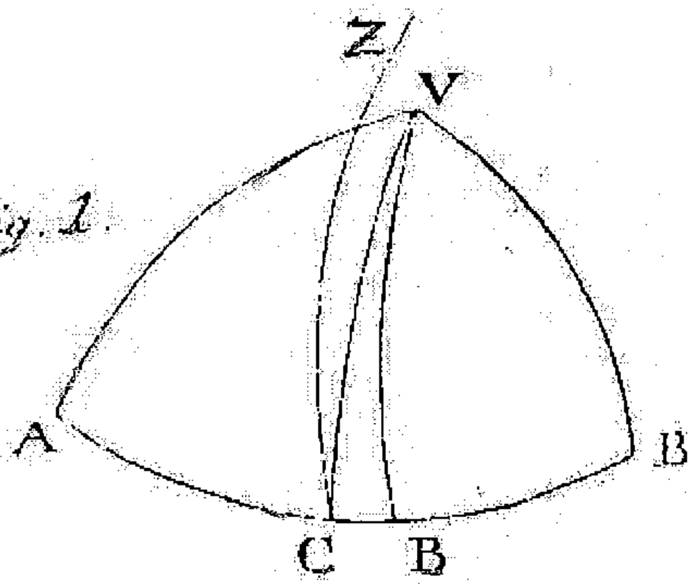
§2. It is well-known from the elements of geometry, that for two triangles in the plane, raised upon the same base and coinciding on every part of this base, to be the same, they have to have the property that their vertices coincide on the same line perpendicular to a parallel with the base. This proposition came to my mind on the occasion of my research into the curves that unite the vertices of spherical triangles that share the same baseline and have equal area. Having considered this for a while I realised that this curve cannot be a great circle, whose characteristic on a spherical surface, unlike this one, coincides with that of straight lines in the plane, the more curious this inquiry appeared to me.

§3. Prior to approaching this problem it is appropriate to put forward some preliminary propositions in the form of Lemmas, as they bridge the solution of our problem. The first thing that must be known is then, that for two spherical triangles to be equal requires that the sum of the angles of one of the triangles equals that of the other. For what is about to be proved is that the quadruple area of a spherical triangle is in the same proportion to the total area of the sphere as the excess of the sum of all the angles of the spherical triangle over two right angles to two right angles. The second Lemma, which will be adduced here as a preliminary proposition, is contained in the following Theorem (Fig. 1):

Let ABV *be a spherical triangle, whose three angles* ABV, BAV, AVB, *are denoted by the letters* B, A, V *respectively, and the opposite sides with* b, a, v. *Then*

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}) = \cot\frac{1}{2}\mathrm{V}\frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)}.$$

Authors dealing with the theory of spherical triangles usually present very different demonstrations of this proposition. However, as they seem to me less convenient, I shall here put



forward another proof, which is not the least awkward. From the point V on the base AB, an arc of a great circle is sent out normally, denoting the angle AVR$=\mu$ and BVR$=\nu$. Now, because right angled spherical triangles have this property, that:

$$\cos \mathrm{A} = \sin \mathrm{AVR} \cos \mathrm{VR} \qquad \text{and} \qquad \cos \mathrm{B} = \sin \mathrm{BVR} \cos \mathrm{VR}, \qquad \text{which leads to}$$

$$\cos \mathrm{A} : \cos \mathrm{B} = \sin \mu : \sin \nu,$$

and hence

$$\cos \mathrm{A} - \cos \mathrm{B} : \cos \mathrm{A} + \cos \mathrm{B} = \sin \mu - \sin \nu : \sin \mu + \sin \nu, \qquad \text{or}$$

$$\tan \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) \tan \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \tan \tfrac{1}{2}(\mu-\nu) \cot \tfrac{1}{2}(\mu+\nu) = \cot \tfrac{1}{2}\mathrm{V} \tan \tfrac{1}{2}(\mu-\nu).$$

Further, owing to

$$\tan \mathrm{VR} = \tan \mathrm{VA} \cos \mathrm{AVR} = \tan \mathrm{VB} \cos \mathrm{BVR},$$

it follows that $\cos \mu : \cos \nu = \tan a : \tan b$, from which it is inferred that

$$\cos \nu - \cos \mu : \cos \mu + \cos \nu = \tan b - \tan a : \tan b + \tan a,$$

or

$$\tan \tfrac{1}{2}(\mu-\nu) \tan \tfrac{1}{2}\mathrm{V} = \frac{\sin(b-a)}{\sin(b+a)}.$$

Finally, on account of $\sin \mathrm{A} : \sin \mathrm{B} = \sin a : \sin b$

$$\sin \mathrm{B} - \sin \mathrm{A} : \sin \mathrm{B} + \sin \mathrm{A} = \sin b - \sin a : \sin b + \sin a,$$

and hence

$$\cot\frac{1}{2}(\mathrm{B}-\mathrm{A})\tan\frac{1}{2}(\mathrm{A}+\mathrm{B})=\cot\frac{1}{2}(b-a)\tan\frac{1}{2}(b+a).$$

As this equation leads to

$$\tan\frac{1}{2}(\mathrm{B}-\mathrm{A})\tan\frac{1}{2}(\mathrm{B}+\mathrm{A})=\cot\frac{1}{2}\mathrm{V}\tan\mathrm{V}\frac{1}{2}(\mu-\nu),$$

we get:

$$\tan^2\frac{1}{2}(\mathrm{B}+\mathrm{A})=\cot\frac{1}{2}\mathrm{V}\tan\frac{1}{2}(\mu-\nu)\cot\frac{1}{2}(b-a)\tan\frac{1}{2}(b+a),$$

where instead of $\tan\frac{1}{2}(\mu-\nu)$ is substituted $\cot\frac{1}{2}\mathrm{V}\frac{\sin(b-a)}{\sin(b+a)}$, then

$$\tan^2\frac{1}{2}(\mathrm{B}+\mathrm{A})=\cot^2\frac{1}{2}\mathrm{V}\frac{\cot\frac{1}{2}(b-a)}{\cot\frac{1}{2}(b+a)}\frac{\sin(b-a)}{\sin(b+a)}=\cot^2\frac{1}{2}\mathrm{V}\frac{\cos^2\frac{1}{2}(b-a)}{\cos^2\frac{1}{2}(b+a)},$$

owing to $\sin(b-a)=2\sin\frac{1}{2}(b-a)\cos\frac{1}{2}(b-a)$ and $\sin(b+a)=2\sin\frac{1}{2}(b+a)\cos\frac{1}{2}(b+a)$. Finally, taking the square root gives:

$$\tan\frac{1}{2}(\mathrm{B}+\mathrm{A})=\cot\frac{1}{2}\mathrm{V}\frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)}.$$

§4. With these presuppositions the solution of our problem may be prepared as follows: Let us suppose a spherical triangle AVB of given magnitude, described upon a given base AB=2 CB=$2a$. Then, from the point V in AB, an arc of a great circle VR is sent out normally (to AB), denoting CR=x; VR=y; But while the angles VAB, VBA, AVR, BVR, which above were expressed by the letters A, B, μ, ν, it was demonstrated with the aid of the Lemma:

$$\tan\frac{1}{2}(\mathrm{A}+\mu)=\frac{\cos\frac{1}{2}(\mathrm{VR}-\mathrm{AR})}{\cos\frac{1}{2}(\mathrm{VR}+\mathrm{AR})}\qquad\text{and}\qquad\tan\frac{1}{2}(\mathrm{B}+\nu)=\frac{\cos\frac{1}{2}(\mathrm{VR}-\mathrm{BR})}{\cos\frac{1}{2}(\mathrm{VR}+\mathrm{BR})},$$

since $\tan\frac{1}{2}$ ARV=$\tan 45^\circ=1$ yields

$$\tan\frac{1}{2}(\mathrm{A}+\mu)=\frac{\cos\frac{1}{2}(y-a-x)}{\cos\frac{1}{2}(y+a+x)}\qquad\text{and}\qquad\tan\frac{1}{2}(\mathrm{B}+\nu)=\frac{\cos\frac{1}{2}(y-a+x)}{\cos\frac{1}{2}(y+a-x)}.$$

Hence it truly follows

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mu+\nu)=\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{V})$$

$$=\frac{\cos\frac{1}{2}(y-a-x)\cos\frac{1}{2}(y+a-x)+\cos\frac{1}{2}(y-a+x)\cos\frac{1}{2}(y+a+x)}{\cos\frac{1}{2}(y+a+x)\cos\frac{1}{2}(y+a-x)-\cos\frac{1}{2}(y-a-x)\cos\frac{1}{2}(y-a+x)}.$$

It is true that

$$\cos(y+x)+\cos a=2\cos\frac{1}{2}(y+x+a)\cos\frac{1}{2}(y+x-a);$$

$$\cos(y-x)+\cos a=2\cos\frac{1}{2}(y-x+a)\cos\frac{1}{2}(y-x-a);$$

and likewise that

$$\cos(a+y)+\cos x = 2\cos\frac{1}{2}(y+a+x)\cos\frac{1}{2}(y+a-x);$$

$$\cos(y-a)+\cos x = 2\cos\frac{1}{2}(y-a+x)\cos\frac{1}{2}(y-a-x);$$

wherefore it follows:

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{V}) = \frac{\cos(y+x)+\cos(y-x)+2\cos a}{\cos(a+y)-\cos(y-a)}.$$

But on the other hand

$$\cos(y+x)+\cos(y-x) = 2\cos y\cos x \qquad \text{and} \qquad \cos(y-a)-\cos(a+y) = 2\sin a\sin y,$$

whereupon substitution of these values gives

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{V}) = \frac{\cos y\cos x+\cos a}{-\sin y\sin a}.$$

Now since the magnitude of the triangle AVB is given, the sum of the angles A, B, V, is known, which, if it is set as $180^\circ+2\delta$, gives

$$\tan\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{V}) = \tan(90^\circ+\delta) = -\cot\delta,$$

giving rise to the following equation:

$$\cot\delta\sin a\sin y = \cos y\cos x+\cos a,$$

which equation innately expresses the curves at which the point V is positioned.

§5. As the curve innate in this equation does not reveal itself clearly enough, let us see how near the end it allows us to approach. Through the point C draw a great circle ZC normally on AB and join with CV, then by denoting CV=z and the angle ZCV=ϕ, owing to

$$\cos\mathrm{CR}\cos\mathrm{VR} = \cos\mathrm{VC}, \qquad \text{and} \qquad \sin\mathrm{VR} = \sin\mathrm{VC}\sin\mathrm{VCR} = \sin\mathrm{VC}\cos\mathrm{ZCV},$$

$$\cos x\cos y = \cos z \qquad \text{and} \qquad \sin y = \sin z\cos\phi,$$

substituting these values into the given equation renders

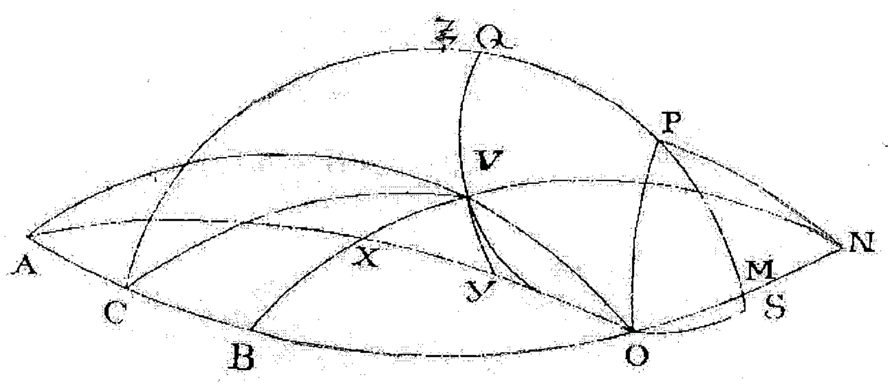
$$\cot\delta\sin a\sin z\cos\phi = \cos z+\cos a,$$

which is now easily reduced into the form:

$$\cos\gamma = \cos z\cos\varepsilon+\sin z\sin\varepsilon\cos\phi,$$

putting

$$\frac{\cos\gamma}{\cos\varepsilon} = -\cos a \qquad \text{and} \qquad -\tan\varepsilon = \cot\delta\sin a,$$



from which equation it is clearly seen, that the curve searched for is a small circle, the construction of which may be prepared as follows: Let us conceive a great circle CZ produced in such a way that it occurring the great circle ABO for the second time in the point M and cutting it so that MO=CB, then draw PO, which with the arc MO forms the angle POM=$90° - \delta$; now if the pole P describes the interval of the arc PO with the small circle OVQ, this circle is the locus of those points V in such a way situated, that if from the given points A, B be drawn to an arbitrary point V the arcs of a great circle AV, BV, respectively, the triangle BVA having a given magnitude, the sum of the angles is =$180° + 2\delta$. For owing to (see Fig. 2)

$$\mathrm{POM} = 90° - \delta, \qquad \tan \mathrm{POM} = \cot\delta = -\frac{\tan\varepsilon}{\sin\alpha}$$

it follows that

$$\tan \mathrm{POM} = \frac{\tan \mathrm{PM}}{\sin \mathrm{MO}},$$

and so

$$\tan \mathrm{PM} = -\tan\varepsilon, \qquad \text{and} \qquad \mathrm{PM} = 180° - \varepsilon,$$

because the arc CZP=ε, then it is true that $\cos\mathrm{PO} = \cos\mathrm{OM}\cos\mathrm{PM}$, and therefore

$$\cos\mathrm{PO} = -\cos a\cos\varepsilon, \quad \text{because of} \quad \mathrm{OM} = \mathrm{CB} = a,$$

whence we obtain PO=γ; on the other hand, it comes with itself that for a spherical triangle PCV;

$$\cos\mathrm{PV} = \cos\mathrm{PO} = \cos\mathrm{PC}\cos\mathrm{VC} + \sin\mathrm{PC}\sin\mathrm{VC}\cos\mathrm{PCV},$$

that is

$$\cos\gamma = \cos\varepsilon\cos z + \sin\varepsilon\sin z\cos\phi,$$

which is the equation discovered above. Because it already has been observed, if the arc be

$$\mathrm{CP} = 180° - \mathrm{PM} = \varepsilon, \quad \text{and therefore} \quad \tan\mathrm{CP} = \tan\varepsilon = -\cot\delta\sin a, \quad \text{then}$$

$$\mathrm{CQ} = \mathrm{CP} - \mathrm{PQ} = \mathrm{CP} - \mathrm{PO} = \varepsilon - \gamma \quad \text{and} \quad \mathrm{CS} = \mathrm{CP} + \mathrm{PO} = \varepsilon + \gamma,$$

which is the rule that determines the two points, in which the small circle QVO meets the great circle CZP.

§6. Given the base AB, irrespective the magnitude of the triangle AVB, the small circle, producing the locus of the points V, always crosses the circle CBM in the point O when MO=CB, whence it follows that this small circle does not go into a great circle except when MV=CB=90°; but for whatever in the end the value of the triangle AVB would be, the locus of the points V is always a great circle, and then the triangle AVB indeed changes into a spherical segment included by two great semicircles. If it was true that CB< 90°, the small circle could not become a great circle unless the triangle AVB disappears, which is evident in itself.

§7. In considering the point Q, where the small circle OVQ intersects the great circle CZM, its location can be obtained in three different ways; namely if Z is supposed to be the pole of the great circle CBM, the point Q falls either in between Z and C or in the very point Z, or at least it is found in between Z and M. The first case gives its location when $\cot\delta > \cot a$, the second if $\cot\delta = \cot a$, and finally the third when $\cot\delta < \cot a$. Namely, if $\cot\delta > \cot a$ and tan PM be indicated $\tan\theta$, owing to $\frac{\tan\theta}{\sin a} = \cot\delta$, $\frac{\tan\theta}{\sin a} > \cot a$, and therefore $\tan\theta > \cos a$, whence by multiplying from both sides with $\cos\theta$, $\sin\theta > \cos a\cos\theta$, that is $\sin\theta > \cos\gamma$, hence in turn $\sin\gamma > \cos\theta$, namely if PM=θ, then PZ=$90° - \theta$, and thus PQ>PZ, that is, the point Q falls between Z and C; by similar arguments it is clear that putting $\cot\delta = \cot a$, then $\sin\gamma = \cos\theta = \sin(90° - \theta)$, that is PQ=PZ when the points Q and Z coincide; again, setting $\cot\delta < \cot a$, then $\sin\gamma < \cos\theta$, and thus PZ>PQ.

§8. If the point V falls upon O, the arc BV becomes BO, whence due to AB + BO=180° the arc AO is indeed a great semicircle, thus in this case the triangle AVB becomes a spherical segment included by two great semicircles, the angle between these semicircles XOB, or XAB, being $=\delta$; indeed,

$$\frac{1}{2}(\mathrm{XAB} + \mathrm{ABO} + \mathrm{XOB}) = 90° + \mathrm{XOB}, \quad \text{since} \quad \mathrm{ABO} = 180° \quad \text{and} \quad \mathrm{XAB}=\mathrm{XOB},$$

whence $90° + \mathrm{XOB} = 90° + \delta$ or $\mathrm{XOB} = \delta = 90° - \mathrm{POM}$, and it becomes evident that the arc of the small circle OXA touches the small circle OVQ in the point O.

§9. As the locus of the points V lies on a small circle described by the pole P in the interval PO, it is worth investigating if the entire circle satisfies the problem, or only the part of it that rises above the great circle ABM. It is evident that if the whole of this great semicircle is conceived, which meets with the great circle CZM in the points Q and S, the part of it QVO rises above the great circle, while the remaining part OS is suppressed below this circle. If it is held as true that the part QO satisfies the problem appropriately, the other part OS has it as an innate property, that irrespective where upon it the point V′ is taken, which joins the points A, B by the arcs of great circles AV′, BV′, the triangle AV′B is indeed of a given magnitude which evidently equals the excess of half of the area of the sphere over the triangle AVB, such that both these triangles AVB, AV′B added in turn render half of the area of the sphere. For if the angles V′AB, V′BA, AV′B be denoted by the respective letters A′, B′, V′, and setting

$$\mathrm{A}' + \mathrm{B}' + \mathrm{V}' = 180° + 2\delta',$$

then

$$\frac{1}{2}(\mathrm{A}' + \mathrm{B}' + \mathrm{V}') = 90^\circ + \delta',$$

whereupon by setting ZCV$'$ = ϕ' and the arc CV$'$ = z, the following equation is obtained:

$$-\cot\delta' \sin a \sin z \cos\phi = \cos z + \cos a,$$

whose simplification into

$$\cos\gamma = \cos z \cos\varepsilon + \sin z \sin\varepsilon \cos\phi',$$

requires setting

$$\cos\gamma = -\cos a \cos\varepsilon \quad \text{and} \quad \tan\varepsilon = \cot\delta' \sin a,$$

supposing $\delta' = 180^\circ - \delta$, and for ε and γ the same values are found as before §5., whereby the arc OS is the locus of the points V'. For the area of the triangle ABV$'$ to be of this size

$$\tan\frac{1}{2}(\mathrm{A}' + \mathrm{B}' + \mathrm{V}') = \tan(90^\circ + \delta').$$

Now, if it be true that

$$\mathrm{A}' + \mathrm{B}' + \mathrm{V}' = 180^\circ + 2\delta',$$

as we assumed above, then adding with

$$\mathrm{A{+}B{+}V} = 180^\circ + 2\delta,$$

in turn gives us

$$\mathrm{A{+}B{+}V{+}A}' + \mathrm{B}' + \mathrm{V}' = 360^\circ + 2\delta + 2\delta' = 2\cdot 360^\circ, \quad \text{since} \quad \delta' = 180^\circ - \delta.$$

If the area of the sphere be denoted by S and that of the triangles AVB, AV$'$B by α, β, respectively, then on account of §3

$$4\alpha \ : \mathrm{S{=}A} + \mathrm{B} + \mathrm{V} - 180^\circ : 180^\circ \quad \text{with}$$

$$4\beta \ : \mathrm{S{=}A}' + \mathrm{B}' + \mathrm{V}' - 180^\circ : 180^\circ \quad \text{whence}$$

$$4\alpha + 4\beta \ : \mathrm{S{=}A{+}\ B{+}\ V{+}\ A}' + \mathrm{B}' + \mathrm{V}' - 360^\circ : 180^\circ,$$

that is

$$4(\alpha + \beta) \ : \mathrm{S} = 2 \ : 1, \quad \text{and thus} \quad \alpha + \beta = \frac{1}{2}\mathrm{S}.$$

In our Figure the triangle AV$'$B does not appear. However, nothing prevents us from reminding us of this triangle, in order to examine it clearly enough.

§10. Since the solution to the problem above has been produced by analysis, it may now be permitted to present in addition another geometric solution of that problem, and since it is hardly possible to judge the first solution by intuition, this problem can be readily demonstrated by the elements of Spherical Geometry. Moreover, the problem itself is now enunciated: A curved line QVO is sought on the surface of a sphere of this nature, that wherever on that line the point V is taken, for two given points A and B joined together by the arcs of the great circles AV and BV, then the triangle AVB shall always be of a given magnitude, clearly equal to the segment of the sphere AXOBA enclosed by the two great semicircles AXO and ABO.

Construction.

With the arc AB bisected at C there is described through C a great circle CZM normal to AB which again crosses the circle ABO in M, then through [the given fixed point] O there is drawn the great arc PO normal to AXO, which crosses the circle CZM at P, now with the Pole P and with the interval PO the lesser circular arc QVO is described, I say that this lesser circular arc is that curve sought.

Proof.

Take the point V anywhere on this minor circle, in a way that it is located on the arc OVQ which is above COM, and draw the great semicircles AVO, BVN, as well as the arcs of a great circle PV, PN. Now on account of the arcs BON=CBM=ACO, it follows that CB=MO=MN, and thus PN=PO=PV, hence in the isosceles triangle PVN, the angle PVN=PNV and in the isosceles Δ PVO, the angle PVO=POV. Now indeed on account of the angle PON=PNO=PNV + VNO, and VNO=VBO,

$$\text{the angle PON} = 180^\circ - \text{POB=PVN} + \text{VBO},$$

whereby

$$180^\circ - \text{VOB-POV} = 180^\circ - \text{VOB} - \text{PVO=PVN} + \text{VBO},$$

adding the angle BVO on both sides,

$$180^\circ + \text{BVO} - \text{VOB} - \text{PVO=BVO+VBO+PVN},$$

where owing to

$$\text{PVN=PVO} - \text{VNO=PVO} - \text{AVB, it follows that:}$$
$$180^\circ + \text{BVO} - \text{VOB} - \text{PVO=BVO} + \text{VBO} + \text{PVO} - \text{AVB, whence}$$
$$360^\circ - 2\ \text{POV=BVO} + \text{VBO} + \text{BOV},$$

since the angles

$$\text{VOX=}90^\circ - \text{POV, then } 180^\circ - 2\ \text{POV=2 VOX},$$

and so

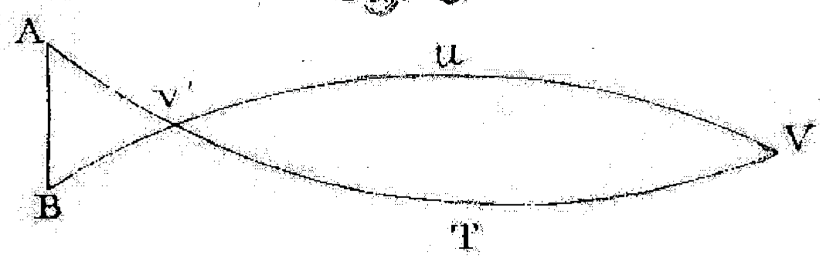
$$180^\circ + 2\text{VOX=BVO} + \text{VBO} + \text{BOV},$$

whence it can be concluded that the spherical triangle BVO equals the spherical segment AVOXA, and if from both members be subtracted VOX, the triangle AVX=BOX, and if both members be added the triangle AXB the segment becomes AXOBA=Δ AVB.

§11. If there be taken on a spherical surface three points A, B, V, it will always be possible to connect two of them by means of two arcs of great circles, namely one of them on the smaller semicircle and the other on the larger semicircle. Hence, if by a spherical triangle it is understood a figure composed of the three arcs of great circles, by joining the three points A, B, V, a total of eight spherical triangles can be formed. If for example the two arcs AB and

360°−AB are denoted by v, v', and in the same fashion AV, 360°−AV are denoted by b, b' as well as BV, 360°−BV by a, a', in all eight triangles are formed by different combinations of three of a or a', b or b', v or v'. Even if this is not usually treated in elementary geometry, if not the triangle consisting of the smaller semicircles of the great circles, there still is no doubt that the properties of spherical triangles duly set out could be applied to the remaining triangles as well. In our case where the arc AB is supposed to be known, four triangles may be formed, the *first* of course the one that in the Figure is denoted AVB, the *second* one included by the arcs AB, BV and 360°−AV, the *third* one included by the arcs AB, AV and 360°−BV and finally the *fourth*, whose sides are AB, 360°−AV, 360°−BV. Considering the second and third, it is quite obvious that their areas are the complements of the triangle AVB over half of the spherical surface. Concerning the fourth, it is appropriate to remark that the arcs AV, BV before reaching up to A, B, intersect at V′, namely where the area of the triangle must be conceived as the difference between the spherical segment VUV′TV and the triangle AV′B=Δ AVB Fig. 3. Finally, if the base AB is applied to its complement of four right angles, this gives rise to four new triangles, to which our problem can again be applied.

Fig. 3.



§12. Let us put the great circle CZM to designate the Meridian of a certain place and let CBM be the horizontal, where C is for the Meridian (South) and M is for North, then if P designates the pole of the equator and the distance from some fixed star to the pole PO, such that this star is seen to describe a diurnal movement along the small circle QVOS, which crosses the horizontal at the point O, and then from the Meridian there are taken on both sides the particular arcs AC and CB equal to MO itself, wherever this star is moving, if from the horizontal points A and B to this place the great arcs AV and BV are drawn, then the spherical triangle AVB, which thence is formed, is of the given magnitude.