CHAPTER IX

CONCERNING THE SUBDIVISION OF LINES OF THE THIRD ORDER INTO KINDS

219. It is considered that the nature and number of branches extending to infinity rightly constitute the essential distinction between curved lines, and on this basis an account most conveniently is chosen of the subdivision of lines of any order into their diverse kinds. Hence indeed also the same division of lines of the second order into their kinds arises, as the nature above will have given support to this idea. For if the proposed general equation for lines of the second order shall be

$$\alpha yy + \beta yx + \gamma xx + \delta y + \varepsilon x + \zeta = 0,$$

the greatest member of this will be seen to be chiefly $\alpha yy + \beta yx + \gamma xx$, whether or not it may have simple real factors. Because if indeed it may be without real factors, the first kind arises, called the *ellipse*, but if the factors shall be real, it is required to be seen, whether they shall be unequal or equal; in the former case the *hyperbola* is produced, truly in the latter the *parabola.*

220. Therefore in the case, in which the factors of the greatest member are real and unequal, the curve will have two right asymptotes ; towards investigating the nature of which there shall be

$$\alpha yy + \beta yx + \gamma xx = (ay - bx)(cy - dx),$$

thus so that there shall be

$$(ay - bx)\ (cy - dx) + \delta y + \varepsilon x + \zeta = 0.$$

The first factor will be considered $ay - bx$, which gives $\frac{y}{x} = \frac{b}{a}$ at infinity, and thus the equation becomes [at infinity]

$$ay - bx + \frac{\delta b + \varepsilon a}{bc - ad} + \frac{\zeta}{cy - dx} = 0,$$

from which the equation

$$ay - bx + \frac{\delta b + \varepsilon a}{bc - ad} = 0$$

defines the position of one of the right asymptotes ; in a similar manner this equation

$$cy - dx + \frac{\delta d + \varepsilon c}{ad - bc} = 0$$

will show the other asymptote.

221. Towards scrutinizing the nature of any asymptote we will transfer the equation to another axis by putting

$$y = \frac{au + bt}{\sqrt{(aa + bb)}} \text{ and } x = \frac{at - bu}{\sqrt{(aa + bb)}},$$

and there shall be $\sqrt{(aa + bb)} = g$, and the equation will become

$$u((ac + bd)u + (bc - ad)t) + \frac{(\delta a - \varepsilon b)u + (\delta b + \varepsilon a)t}{g} + \zeta = 0$$

and thus

$$g(bc - ad)tu + g(ac + bd)uu + (\delta b + \varepsilon a)t + (\delta a - \varepsilon b)u + \zeta g = 0.$$

Hence, an putting [from the coefficients of the terms involving *t*]

$$u = -\frac{\delta b + \varepsilon a}{g(bc - ad)}$$

into the remaining members [*i.e.* terms], there will be

$$(g(bc - ad)u + \delta b + \varepsilon a)t + \frac{(ac + bd)(\delta b + \varepsilon a)^2}{g(bc - ad)} - \frac{(\delta a - \varepsilon b)(\delta b + \varepsilon a)}{g(bc - ad)} + \zeta g = 0$$

or

$$g(bc - ad)u + \delta b + \varepsilon a + \frac{g(\delta d + \varepsilon c)(\delta b + \varepsilon a)}{(bc - ad)^2 t} + \frac{\zeta g}{t} = 0;$$

therefore there will be a hyperbolic asymptote of the kind $u = \frac{A}{t}$. In a similar manner the other asymptote may be defined arising from the factor $cy - dx$, from which the curve

will have two equal branches extending to infinity, each expressed by the equation
$u = \frac{A}{t}$.

222. Now both factors shall be equal or

$$\alpha yy + \beta xy + \gamma xx = (ay - bx)^2;$$

and, with the same transfer made to another axis, from which there becomes

$$y = \frac{au + bt}{g} \text{ et } x = \frac{at - bu}{g},$$

there will be

$$gguu + \frac{(\delta a - \varepsilon b)u}{g} + \frac{(\delta b + \varepsilon a)t}{g} + \zeta = 0;$$

and with the factor t infinite , the equation will become

$$uu + \frac{(\delta b + \varepsilon a)t}{g^3} = 0;$$

which equation will show two parabolic branches of the kind $uu = At$, evidently the curve will be itself a parabola, and with that its own asymptotes. But if it should become $\delta b + \varepsilon a = 0$, then the equation will become

$$gguu + \frac{\delta gu}{a} + \zeta = 0;$$

for two right lines parallel to each other, which is the case, in which an equation of the second order is resolvable in total into two simple factors.

Therefore we have thus found the kinds of lines of the second order, even if they had not yet been elicited.

223. Therefore in the same manner we may approach lines of the third order, the general equation of which is

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0.$$

The greatest member $\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3$ therefore, because it is of odd dimensions, either will have one simple real factor or all three factors will be real and simple. The following cases therefore are to be shown :

I.
If a single real simple factor may stand out.

II.
If all three factors shall be real and unequal to each other.

III.
If two factors were equal.

IV.
If all three factors were equal.

Because truly in any case it is sufficient to adapt the calculation to a single factor, this factor $ay - bx$ itself shall be present alone or with other equal or unequal factors ; and the position of the axis with respect of this will remain unchanged, as we have done up to the present ; with which done this equation may arise, which we may use in place of the above, since it is equally general

$$\alpha ttu + \beta tuu + \gamma u^3 + \delta tt + \varepsilon tu + \zeta uu + \eta t + \theta u + \iota = 0 \ ,$$

where the greatest member $\alpha ttu + \beta tuu + \gamma u^3$ has a certain factor u .

CASE 1

224. The largest member therefore may have a single real factor u , which comes about, if $\beta\beta$ shall be less than $4\alpha\gamma$; and by putting t infinite there will be $\alpha u + \delta = 0$, which is the equation for a right line asymptote. This equation may provide the value $u = c$, and there will be

$$\alpha tt(u - c) + t(\beta cc + \varepsilon c + \eta) + \gamma c^3 + \zeta cc + \theta c + \iota = 0 \,,$$

which is the equation for the nature of the asymptote.
[Thus, a judicious change has already been made from the initial rectilinear coordinates (*x, y*) to a new set of coordinates (*t, u*) by a rotation about some point, to facilitate the finding of asymptotes ; now a line parallel to the new abscissa axes is found, having the value $u = c$ on making t infinite; curved line asymptotes can now be found relative to this line by putting $u - c$ in the greatest terms and $u = c$ in the other terms, thus giving the nature of asymptotic curves that may exist, relative to the fixed right asymptotic line, as Euler has shown in the previous chapter. Further development depends on the relative magnitudes of the terms in the equation, in the present case leading either to an asymptote varying inversely with distance, or with distance squared along the abscissa axis.]
Hence, just as $\beta cc + \varepsilon c + \eta$ either shall not be $= 0$ or shall be $= 0$, the nature of the asymptotes produced will be twofold : surely

$$\text{either } u-c=\frac{A}{t} \text{ or } u-c=\frac{A}{tt};$$

from which the first two kinds of lines of the third order may be formed, which thus themselves will become :

1.

The *first* kind has a single right line asymptote of the kind $u=\frac{A}{t}$.

2.

The *second* kind has a single right line asymptote of the kind $u=\frac{A}{tt}$.

CASE 2

225. The three factors of the greatest member shall be simple, real, and unequal to each other ; which arises, if in the [quadratic] equation [in *t*],

$$\alpha ttu+\beta tuu+\gamma u^3+\delta tt+\varepsilon tu+\zeta uu+\eta t+\theta u+\iota=0$$

$\beta\beta$ were greater than $4\alpha\gamma$. In this case therefore the same are to be understood concerning each factor, which are set out in the manner as concerned a single factor. Clearly each one will supply two hyperbolic branches either of the kind $u=\frac{A}{t}$ or of the kind $u=\frac{A}{tt}$, from which in this case four different kinds of lines of the third order will be contained, for three right asymptotes given inclined in some manner to each other in turn, which kinds are :

3.

The *third* kind has three asymptotes of the kind $u=\frac{A}{t}$.

4.

The *fourth* kind has two asymptotes of the kind $u=\frac{A}{t}$, and one of the kind $u=\frac{A}{tt}$.

The *fifth* kind has one asymptote of the kind $u=\frac{A}{t}$ and two of the kind $u=\frac{A}{tt}$.

The *sixth* kind has three asymptotes of the kind $u = \frac{A}{tt}$.

226. But we may consider, whether all these kinds shall be possible ; towards which end we may take this equation extended the most

$$y(\alpha y - \beta x)(\gamma y - \delta x) + \varepsilon xy + \zeta yy + \eta x + \theta y + \iota = 0 ,$$

the greatest member of which has three real factors ; for although the term *xx* has been omitted, yet the equation appears no less wide. Moreover from the preceding it is understood that the factor *y* provides an asymptote of the form $u = \frac{A}{t}$, provided η is not $= 0$. Whereby we may see, what kind of asymptote the factor $\alpha y - \beta x$ may produce. For this we may put $y = \alpha u + \beta t$ and $x = \alpha t - \beta u$; and therefore for brevity there shall be $\alpha^2 + \beta^2 = 1$, which it is always possible to assume ; and the equation will be transformed into this form :

$$\left.\begin{array}{l} \beta(\beta\gamma - \alpha\delta)ttu + (2\alpha\beta\gamma - (\alpha\alpha - \beta\beta)\ \delta)tuu + \alpha(\alpha\gamma + \beta\delta)u^3 \\ +\beta\left(\alpha\varepsilon + \beta\zeta\right)tt + (2\alpha\beta\zeta + \left(\alpha\alpha - \beta\beta\right)\varepsilon)tu \quad + \alpha\left(\alpha\zeta - \beta\varepsilon\right)uu \\ \qquad + \ (\alpha\eta + \beta\theta)t \qquad\qquad\qquad\qquad + (\alpha\theta - \beta\eta)u \qquad + \iota \end{array}\right\} = 0.$$

Here the factor $\alpha y - \beta x$ will be changed into *u;* from which with *t* made infinite in the first place it becomes

$$u = \frac{\alpha\varepsilon + \beta\zeta}{\alpha\delta - \beta\gamma} = c,$$

which value if it may be substituted in place of *u* into the second member with *t* retained, shows from this factor *u* or $\alpha y - \beta x$ an asymptote to arise of the form $u = \frac{A}{t}$, unless there were

$$\frac{\alpha\eta + \beta\theta}{\beta} + \frac{(\alpha\varepsilon + \beta\zeta)(\gamma\varepsilon + \delta\zeta)}{(\alpha\delta - \beta\gamma)^2} = 0 .$$

In a similar manner the factor $\gamma y - \delta x$ will provide an asymptote of the form $u = \frac{A}{t}$, unless there were

$$\frac{\gamma\eta + \delta\theta}{\delta} + \frac{(\alpha\varepsilon + \beta\zeta)(\gamma\varepsilon + \delta\zeta)}{(\alpha\delta - \beta\gamma)^2} = 0 .$$

227. Hence it is certainly appears to be possible, that neither η nor each formula just found may vanish, from which the third kind certainly will be possible always. Because to retain the fourth kind, there may be put $\eta = 0$, from which a single asymptote of the form $u = \frac{A}{tt}$ may be produced ; but then both the remaining expressions merge into one and thus the two remaining asymptotes will be of the form $u = \frac{A}{t}$, unless there were

$$\theta + \frac{(\alpha\varepsilon + \beta\zeta)(\gamma\varepsilon + \delta\zeta)}{(\alpha\delta - \beta\gamma)^2} = 0\,;$$

from which also the fourth kind is possible. But, if besides $\eta = 0$, one of the two remaining expressions may become $= 0$, and likewise the other will vanish; on which account it cannot happen that two asymptotes become of the form $u = \frac{A}{tt}$, without the third likewise adopting the same form ; from which the fifth kind is impossible. But the sixth kind on this account will itself be possible, because it may arise, if $\eta = 0$ and

$$\theta = \frac{-(\alpha\varepsilon + \beta\zeta)(\gamma\varepsilon + \delta\zeta)}{(\alpha\delta - \beta\gamma)^2}.$$

Therefore these two cases will provided only five kinds of lines of the third order, because these, as we have put five in place, must be dismissed instead, and

5.

The *fifth* kind has three asymptotes of the kind, $u = \frac{A}{tt}$.

CASE 3

228. The greatest member may have two equal factors u ; which comes about, if in the equation of the preceding case the first term αttu may vanish. Therefore the general equation belonging to this case will be of this kind :

$$\alpha tuu - \beta u^3 + \gamma tt + \delta tu + \varepsilon uu + \zeta t + \eta u + \theta = 0\,,$$

therefore has the two factors u of the greatest term equal, and $\alpha t - \beta u$ of the third unequal. This third factor will produce an asymptote either of the form
$u = \frac{A}{t}$ or of the form $u = \frac{A}{tt}$, according as this expression

$$(\alpha\delta+2\beta\gamma)(\alpha\alpha\varepsilon+\alpha\beta\delta+\beta\beta\gamma)-\alpha^3(\alpha\eta+\beta\zeta) \text{ were either not } =0, \text{ or were } =0.$$

[This is obtained by putting $t=\frac{\beta}{\alpha}u+a+\frac{b}{u}$, and by making the coefficients both of u and u^2 equal to zero. (J.B. Labey in the French translation.)]

229. So that these two equal factors may be reached, in the first case they will occur if γ were not $=0$; then indeed on making $t=\infty$ there becomes $\alpha uu+\gamma t=0$, which is the equation for a parabolic asymptote of the kind $uu=At$. Hence these two new kinds of the third order will arise, surely :

6.

The *sixth* kind has a single asymptote of the kind $u=\frac{A}{t}$ and a single asymptote of the kind $uu=At$.

7.

The *seventh* kind has a single asymptote of the kind $u=\frac{A}{tt}$ and a single parabolic asymptote of the kind $uu=At$.

230. If now there shall be $\gamma=0$; and the third factor $\alpha t-\beta u$ will give an asymptote of the form $u=\frac{A}{tt}$, if there were

$$\delta(\alpha\varepsilon+\beta\delta)=\alpha(\alpha\eta+\beta\zeta),$$

but if these two equations cannot be accommodated, the asymptote will be of the form $u=\frac{A}{t}$.

Therefore we will have this equation

$$\left.\begin{array}{ll} +\ \alpha tuu & -\beta u^3 \\ +\ \delta tu & +\varepsilon uu \\ +\ \zeta t & +\eta u \\ +\theta & \end{array}\right\}=0.$$

This on making $t=\infty$ becomes $\alpha uu+\delta u+\zeta=0$.

In the first place $\delta\delta$ shall be less than $4\alpha\zeta$, and hence no asymptote may arise ; whereby from this case two kinds originate :

8.

The *eighth* kind has a single asymptote of the kind $u = \frac{A}{t}$.

9.

The *ninth* kind has a single asymptote of the kind $u = \frac{A}{tt}$.

231. Both roots of the equation $\alpha uu + \delta u + \zeta = 0$ shall be real and unequal, namely $\delta\delta$ shall be greater than $4\alpha\zeta$; and hence two right asymptotes will be produced parallel to each other, and each of the form $u = \frac{A}{t}$, which case will support two new kinds :

10.

The *tenth* kind has a single asymptote of the kind $u = \frac{A}{t}$ and two asymptotes parallel to each other of the kind $u = \frac{A}{t}$.

11.

The *eleventh* kind has a single asymptote of the kind $u = \frac{A}{tt}$ and two parallel to each other of the kind $u = \frac{A}{t}$.

232. Both roots of the equation $\alpha uu + \delta u + \zeta = 0$ shall be equal to each other, or $\alpha\alpha = 4\alpha\zeta$ or $\alpha uu + \delta u + \zeta = \alpha(u - c)^2$, and the equation becomes

$$\alpha(u - c)^2 = \beta c^3 - \varepsilon cc - \eta c - \theta ,$$

from which a single right asymptote of the kind $uu = \frac{A}{t}$ arises. Hence therefore the two new species are produced :

12.

The *twelfth* kind has a single asymptote of the kind $u = \frac{A}{t}$ and one of the kind $uu = \frac{A}{t}$.

13.

The *thirteenth* kind has a single asymptote of the kind $u = \frac{A}{tt}$ and one of the kind $u = \frac{A}{t}$.

CASUS 4

233. But if the greatest term shall have all three factors equal

[*i.e*. in the equation $\alpha ttu+\beta tuu+\gamma u^3+\delta tt+\varepsilon tu+\zeta uu+\eta t+\theta u+\iota=0$ in §225; where the first three terms $\alpha ttu+\beta tuu+\gamma u^3$ are replaced by αu^3, and the other constants are 'slid along' to accommodate the other terms; a trick used by Euler a number of times in this chapter],

an equation of this form will be had

$$\alpha u^3+\beta tt+\gamma tu+\delta uu+\varepsilon t+\zeta u+\eta=0.$$

Here at first the term βtt is to be examined, which if it shall not be missing, the curve will have a parabolic asymptote of the kind $u^3=Att$ and thus a single kind arises :

14.

The *fourteenth* kind has a single asymptote of the kind $u^3=Att$.

234. Now the term βtt shall be missing, and there becomes

$$\alpha u^3+\beta tu+\delta uu+\varepsilon t+\zeta u+\eta=0\,;\;;$$

from which on making t infinite the equation becomes $\alpha u^3+\gamma tu+\varepsilon t=0$, unless γ and $\varepsilon=0$. Therefore γ shall not be $=0$, and two equations may be present in this equation $\alpha uu+\gamma t=0$ and $\gamma u+\varepsilon=0$; the former is for a parabolic asymptote of the kind $uu=At$; truly the latter, if one may put $\frac{-\varepsilon}{\gamma}=c$, will give this equation

$$\gamma t(u-c)+\alpha c^3+\delta cc+\zeta c+\eta=0,$$

and therefore there will be a hyperbolic asymptote of this kind $u=\frac{A}{t}$ from which

15.

The *fifteenth* kind has a single asymptote of the kind $uu=At$ and a right one of the kind $u=\frac{A}{t}$, and the parallel axis of the parabola is the other right asymptote.

235. Also there may be $\gamma = 0$, so that this equation shall become

$$\alpha u^3 + \delta uu + \varepsilon t + \zeta u + \eta = 0,$$

where ε cannot vanish, unless the curve likewise may cease to be a curve. But with t made infinite, by necessity u must become infinite, unless the equation becomes $\alpha u^3 + \varepsilon t = 0$, which presents this final form :

16.

The *sixteenth* kind has a single asymptote of the kind $u^3 = At$.

236. Therefore we have reduced all these lines of the third order to sixteen kinds, in which therefore all these seventy two kinds will be contained into which Newton divided lines of the third order. Because truly this is no wonder, as only the manner of distinguishing comes between our division and that of Newton; for here we have chosen to examine only the nature of the branches extending to infinity, since Newton examined also the standing of curves in a finite space and from the variation of this the diverse kinds he might put in place. But although this account of the division may appear to be arbitrary, yet Newton following his own reasoning was able finally to produce many more kinds, since indeed using my method I have been able to elicit neither more nor less kinds.

237. Therefore so that the nature and complexity of each kind may be understood better, I will show the general equation for any kind of curve, and that in the simplest form, which can be done without harming their general nature. And for each indeed I will review the Newtonian kind relating to that.

FIRST KIND

$$y(xx - 2mxy + nnyy) + ayy + bx + cy + d = 0$$

with *mm* present less than *nn* and if b were not $= 0$.

This pertains to the Newtonian kinds 33, 34, 35, 36, 37, 38.

SECOND KIND

$$y(xx - 2mxy + nnyy) + ayy + cy + d = 0$$

with *mm* less than *nn*.
This pertains to the Newtonian kinds 39, 40, 41, 42, 43, 44, 45.

THIRD KIND

$$y(x-my)(x-ny)+ayy+bx+cy+d=0,$$

Where neither $b=0$ nor $mb+c+\frac{aa}{(m-n)^2}=0$ nor $nb+c+\frac{aa}{(m-n)^2}=0$ nor $m=n$.

This pertains to the Newtonian kinds 1, 2, 3, 4, 5, 6, 7, 8, 9; likewise 24, 25, 26, 27, if $a=0$.

FOURTH KIND

$$y\left(x-my\right)\left(x-ny\right)+ayy+cy+d=0,$$

where neither $c+\frac{aa}{(m-n)^2}=0$, nor $m=n$.

This pertains to the Newtonian kinds 10, 11, 12, 13, 14, 15, 16, 17,18,19, 20, 21; likewise, if $a=0$, to these 28, 29, 30, 31.

FIFTH KIND

$$y(x-my)(x-ny)+ayy-\frac{aay}{(m-n)^2}+d=0,$$

without $m=n$.

This pertains to the Newtonian kinds 22, 23 and 32.

SIXTH KIND

$$yy(x-my)+axx+bx+cy+d=0,$$

if neither $a=0$ nor $2m^3aa-mb-c=0$.

This pertains to the Newtonian kinds 46, 47, 48, 49, 50, 51, 52.

SEVENTH KIND

$$yy(x-my)+axx+bx+m(2mmaa-b)y+d=0$$

without $a=0$.

This pertains to the Newtonian kinds 53, 54, 55, 56.

EIGHTH KIND

$$yy(x-my)+bbx+cy+d=0,$$

with neither $c=-mbb$ nor $b=0$.

This pertains to the Newtonian kinds 61 and 62.

NINTH KIND

$$yy(x-my)+bbx-mbby+d=0$$

without $b=0$.

This pertains to the Newtonian kind 63.

TENTH KIND

$$yy(x-my)-bbx+cy+d=0,$$

without $c=mbb$ nor $b=0$.

This pertains to the Newtonian kinds 57, 58, 59.

ELEVENTH KIND

$$yy(x-my)-bbx+mbby+d=0,$$

without $b=0$.

This pertains to the Newtonian kind 60.

TWELFTH KIND

$$yy(x-my)+cy+d=0,$$

without $c=0$.

This pertains to the Newtonian kind 64.

THIRTEENTH KIND

$$yy(x-my)+d=0.$$

This pertains to the Newtonian kind 65.

FOURTEENTH KIND

$$y^3+axx+bxy+cy+d=0,$$

without $a=0$.

This pertains to the Newtonian kinds 67, 68, 69, 70, 71.

FIFETEENTH KIND

$$y^3+bxy+cx+d=0,$$

without $b=0$.

This pertains to the Newtonian kind 66.

SIXTEENTH KIND

$$y^3+ay+bx=0,$$

without $b=0$.

This pertains to the Newtonian kind 72.

238. But generally these kinds are so widely apparent, so that each may be contained well enough under a known variety, if indeed we may consider according to the form, which the curves have in finite space. And because of this reason the number of Newtonian kinds has been multiplied, so that these curves, which may disagree in a finite space, may be separated from each other. Therefore it will be convenient to call these *genera,* which we have called *species* or kinds until now, and to call the varieties, which are taken under one or another name, to be referred to as *species*. But this will be required to be preserved especially, if for which it was required to subdivide the lines of the fourth or higher orders in the same manner ; where indeed a much greater variety thus will be accommodated in whatever species thus found.

CAPUT IX

DE LINEARUM TERTII ORDINIS SUBDIVISIONE IN SPECIES

219. Natura atque numerus ramorum in infinitum extensorum merito essentiale discrimen in lineis curvis constituere censetur atque ex hoc fonte commodissime desumitur ratio subdivisionis linearum cuiusque ordinis in suas species diversas. Hinc enim quoque oritur eadem linearum secundi ordinis divisio in suas species, quam ipsa rei natura supra suppeditaverat. Sit enim proposita aequatio generalis pro lineis secundi ordinis

$$\alpha yy+\beta yx+\gamma xx+\delta y+\varepsilon x+\zeta=0,$$

cuius supremum membrum $\alpha yy+\beta yx+\gamma xx$ potissimum spectetur, utrum habeat factores simplices reales an secus. Quodsi enim careat factoribus, nascitur prima species, *ellipsis* dicta, sin autem factores sint reales, videndum est, utrum sint inaequales an aequales; illo casu oritur *hyperbola*, hoc vero *parabola.*

220. Casu ergo, quo membri supremi factores sunt reales et inaequales, curva duas habebit asymptotas rectas; ad quarum naturam investigandam sit

$$\alpha yy+\beta yx+\gamma xx=(ay-bx)(cy-dx),$$

ita ut sit

$$(ay-bx)\,(cy-dx)+\delta y+\varepsilon x+\zeta=0.$$

Consideretur primum factor $ay-bx$, qui in infinito dat $\frac{y}{x}=\frac{b}{a}$, fiet itaque

$$ay-bx+\frac{\delta b+\varepsilon a}{bc-ad}+\frac{\zeta}{cy-dx}=0,$$

unde aequatio

$$ay-bx+\frac{\delta b+\varepsilon a}{bc-ad}=0,$$

definit positionem unius asymptotae rectae; similique modo aequatio haec

$$cy-dx+\frac{\delta d+\varepsilon c}{ad-bc}=0$$

ostendet asymptotam alteram.

221. Ad naturam cuiusque asymptotae scrutandam aequationem ad alium axem transferamus ponendo

$$y=\frac{au+bt}{\sqrt{(aa+bb)}} \text{ et } x=\frac{at-bu}{\sqrt{(aa+bb)}},$$

sitque $\sqrt{(aa+bb)}=g$, erit

$$u((ac+bd)u+(bc-ad)t)+\frac{(\delta a-\varepsilon b)u+(\delta b+\varepsilon a)t}{g}+\zeta=0$$

ideoque

$$g(bc-ad)tu+g(ac+bd)uu+(\delta b+\varepsilon a)t+(\delta a-\varepsilon b)u+\zeta g=0.$$

Hinc, posito in reliquis membris

$$u=-\frac{\delta b+\varepsilon a}{g(bc-ad)},$$

erit

$$(g(bc-ad)u+\delta b+\varepsilon a)t+\frac{(ac+bd)(\delta b+\varepsilon a)^2}{g(bc-ad)}-\frac{(\delta a-\varepsilon b)(\delta b+\varepsilon a)}{g(bc-ad)}+\zeta g=0$$

seu

$$g(bc-ad)u+\delta b+\varepsilon a+\frac{g(\delta d+\varepsilon c)(\delta b+\varepsilon a)}{(bc-ad)^2 t}+\frac{\zeta g}{t}=0;$$

erit ergo asymptota hyperbolica generis $u=\frac{A}{t}$. Simili vero modo asymptota altera ex factore $cy-dx$ oriunda definietur, unde curva habebit duo ramorum in infinitum extensorum paria, utrumque aequatione $u=\frac{A}{t}$ expressum.

222. Sint iam ambo factores aequales seu

$$\alpha yy+\beta xy+\gamma xx=(ay-bx)^2;$$

atque, facta eadem ad alium axem translatione, qua fit

$$y=\frac{au+bt}{g} \text{ et } x=\frac{at-bu}{g},$$

erit

$$gguu+\frac{(\delta a-\varepsilon b)u}{g}+\frac{(\delta b+\varepsilon a)t}{g}+\zeta=0\,;$$

et facto t infinito erit

$$uu+\frac{(\delta b+\varepsilon a)t}{g^3}=0\,;$$

quae aequatio ostendit duos ramos parabolicos speciei $uu=At$, quippe curva ipsa erit parabola, ipsaque sua asymptota. Sin autem esset $\delta b+\varepsilon a=0$, tum aequatio foret

$$gguu+\frac{\delta gu}{a}+\zeta=0\,;$$

pro duabus rectis inter se parallelis, qui est casus, quo aequatio secundi ordinis tota in duos factores simplices est resolubilis.

Sic igitur species linearum secundi ordinis invenissemus, etiam si nondum erutae fuissent.

223. Eodem igitur modo aggrediamur lineas tertii ordinis, quarum aequatio generalis est

$$\alpha y^3+\beta yyx+\gamma yxx+\delta x^3+\varepsilon yy+\zeta yx+\eta xx+\theta y+\iota x+\chi=0\,.$$

Supremum igitur membrum $\alpha y^3+\beta yyx+\gamma yxx+\delta x^3$, quia est imparium dimensionum, vel unum habet factorem simplicem realem vel omnes tres factores simplices erunt reales. Sequentes igitur casus sunt evolvendi:

I.
Si unicus extet factor simplex realis.

II.
Si omnes tres sint reales et inter se inaequales.

III.
Si duo factores fuerint aequales.

IV.
Si omnes tres factores fuerint aequales.

Quoniam vero in quovis casu ad unicum factorem calculum accommodasse sufficit, sit iste factor, sive solus adsit sive cum allis sui aequalibus inaequalibusve, $ay - bx$; atque ad hunc positio axis ita immutetur, ut hactenus fecimus; quo facto oriatur haec aequatio, qua vice superioris utamur, cum aeque late pateat

$$\alpha ttu + \beta tuu + \gamma u^3 + \delta tt + \varepsilon tu + \zeta uu + \eta t + \theta u + \iota = 0 ,$$

ubi membrum supremum $\alpha ttu + \beta tuu + \gamma u^3$ unum certe habet factorem u.

CASUS 1

224. Habeat ergo membrum supremum unicum factorem realem u, quod evenit, si $\beta\beta$ sit minor quam $4\alpha\gamma$; atque posito t infinito erit $\alpha u + \delta = 0$, quae est aequatio pro asymptota recta. Praebeat haec aequatio valorem $u = c$, eritque

$$\alpha tt(u-c) + t(\beta cc + \varepsilon c + \eta) + \gamma c^3 + \zeta cc + \theta c + \iota = 0 ,$$

quae est aequatio pro natura asymptotae. Hinc, prout $\beta cc + \varepsilon c + \eta$ vel non fuerit $= 0$ vel sit $= 0$, duplex asymptotae indoles prodit; nempe

$$\text{vel } u - c = \frac{A}{t} \text{ vel } u - c = \frac{A}{tt};$$

unde duae primae linearum tertii ordinis species formantur, quae ita se habebunt:

1.

Prima species unicam habet asymptotam rectam speciei $u = \frac{A}{t}$.

2.

Secunda species unicam habet asymptotam rectam speciei $u = \frac{A}{tt}$.

CASUS 2

225. Sint membri supremi tres factores simplices reales et inter se inaequales; quod evenit, si in aequatione

$$\alpha ttu + \beta tuu + \gamma u^3 + \delta tt + \varepsilon tu + \zeta uu + \eta t + \theta u + \iota = 0$$

fuerit $\beta\beta$ maior quam $4\alpha\gamma$. Hoc igitur casu de unoquoque factore eadem sunt tenenda, quae modo de unico factore sunt exposita. Unusquisque scilicet suppeditat binos ramos hyperbolicos vel speciei $u=\frac{A}{t}$ vel speciei $u=\frac{A}{tt}$, unde in hoc casu quatuor diversae species linearum tertii ordinis continentur, tribus asymptotis rectis ad se invicem utcunque inclinatis praeditae, quae species sunt:

3.

Tertia species tres habet asymptotas speciei $u=\frac{A}{t}$.

4.

Quarta species duas habet asymptotas speciei $u=\frac{A}{t}$ et unam speciei $u=\frac{A}{tt}$.

Quinta species unam habet asymptotam speciei $u=\frac{A}{t}$ et duas speciei $u=\frac{A}{tt}$

Sexta species tres habet asymptotas speciei $u=\frac{A}{tt}$.

226. Videamus autem, an hae omnes species sint possibiles; quem in finem sumamus hanc aequationem latissime patentem

$$y(\alpha y-\beta x)(\gamma y-\delta x)+\varepsilon xy+\zeta yy+\eta x+\theta y+\iota=0\,,$$

cuius supremum membrum tres habet factores reales; quanquam enim terminus *xx* est omissus, tamen aequatio non minus late patet. Ex praecedentibus autem intelligitur factorem *y* praebere asymptotam formae $u=\frac{A}{t}$, si non fuerit $\eta=0$. Quare videamus, cuiusmodi asymptotam praebeat factor $\alpha y-\beta x$. Ad hoc ponamus $y=\alpha u+\beta t$ et $x=\alpha t-\beta u$*;* sitque brevitatis ergo $\alpha^2+\beta^2=1$, quod semper assumere licet; atque aequatio transformabitur in hanc formam:

$$\left.\begin{array}{l}\beta(\beta\gamma-\alpha\delta)ttu+(2\alpha\beta\gamma-(\alpha\alpha-\beta\beta)\ \delta)tuu+\alpha(\alpha\gamma+\beta\delta)u^3\\ +\beta\left(\alpha\varepsilon+\beta\zeta\right)tt+(2\alpha\beta\zeta+\left(\alpha\alpha-\beta\beta\right)\varepsilon)tu\quad+\alpha\left(\alpha\zeta-\beta\varepsilon\right)uu\\ \qquad +\ (\alpha\eta+\beta\theta)t\qquad\qquad\qquad +(\alpha\theta-\beta\eta)u\qquad +\iota\end{array}\right\}=0.$$

Hic factor $\alpha y-\beta x$ transiit in *u;* ex quo posito *t* infinito primum fit

$$u = \frac{\alpha\varepsilon + \beta\zeta}{\alpha\delta - \beta\gamma} = c,$$

qui valor si loco u in secundo membro continente t substituatur, ostendet ex hoc factore u seu $\alpha y - \beta x$ asymptotam oriri formae $u = \frac{A}{t}$, nisi fuerit

$$\frac{\alpha\eta + \beta\theta}{\beta} + \frac{(\alpha\varepsilon + \beta\zeta)(\gamma\varepsilon + \delta\zeta)}{(\alpha\delta - \beta\gamma)^2} = 0.$$

Simili modo factor $\gamma y - \delta x$ asymptotam praebebit formae $u = \frac{A}{t}$, nisi fuerit

$$\frac{\gamma\eta + \delta\theta}{\delta} + \frac{(\alpha\varepsilon + \beta\zeta)(\gamma\varepsilon + \delta\zeta)}{(\alpha\delta - \beta\gamma)^2} = 0 .$$

227. Hinc patet fieri utique posse, ut neque η neque utraque formula modo inventa evanescat, ex quo species tertia utique erit possibilis. Quod ad speciem quartam attinet, ponatur $\eta = 0$, quo una asymptota formae $u = \frac{A}{tt}$ prodeat; tum autem ambae reliquae expressiones in unam coalescunt ideoque binae reliquae asymptotae erunt formae $u = \frac{A}{t}$, nisi fuerit

$$\theta + \frac{(\alpha\varepsilon + \beta\zeta)(\gamma\varepsilon + \delta\zeta)}{(\alpha\delta - \beta\gamma)^2} = 0;$$

unde et species quarta est possibilis. At, si praeter $\eta = 0$ una ex binis reliquis expressionibus reddatur $= 0$, simul altera evanescit; quamobrem fieri non potest, ut duae asymptotae fiant formae $u = \frac{A}{tt}$, quin simul tertia eandem formam induat; ex quo species quinta est impossibilis. Sexta autem ob hoc ipsum erit possibilis, quia oritur, si $\eta = 0$:

$$\theta = \frac{-(\alpha\varepsilon + \beta\zeta)(\gamma\varepsilon + \delta\zeta)}{(\alpha\delta - \beta\gamma)^2}.$$

Hi ergo duo casus quinque tantum praebuerunt species linearum tertii ordinis, quod ea, quam quintam posuimus, praetermitti debet, et

5.

Quinta species tres habet asymptotas speciei u , $u = \frac{A}{tt}$.

CASUS 3

228. Habeat membrum supremum duos factores u aequales; quod evenit, si in aequatione casus praecedentis primus terminus αttu evanescat. Aequatio ergo generalis ad hunc casum pertinens erit huiusmodi

$$\alpha tuu - \beta u^3 + \gamma tt + \delta tu + \varepsilon uu + \zeta t + \eta u + \theta = 0,$$

habet ergo membrum supremum duos factores u aequales, ac tertium $\alpha t - \beta u$ reliquis inaequalem. Iste tertius factor producet asymptotam vel formae $u = \frac{A}{t}$ vel formae $u = \frac{A}{tt}$, prout fuerit haec expressio

$$(\alpha\delta + 2\beta\gamma)(\alpha\alpha\varepsilon + \alpha\beta\delta + \beta\beta\gamma) - \alpha^3(\alpha\eta + \beta\zeta) \text{ vel} = 0 \text{ vel} = 0.$$

229. Quod ad duos factores aequales attinet, primum casus occurrit, si γ non fuerit $= 0$; tum enim facto $t = \infty$ fiet $\alpha uu + \gamma t = 0$, quae est aequatio pro asymptota parabolica speciei $uu = At$. Hinc istae duae nascentur species novae linearum tertii ordinis, nempe:

6.

Sexta species habet unam asymptotam speciei $u = \frac{A}{t}$ unam asymptotam speciei $uu = At$.

7.

Septima species habet unam asymptotam speciei $u = \frac{A}{tt}$ et unam parabolicam speciei $uu = At$.

230. Sit iam $\gamma = 0$; atque factor tertius $\alpha t - \beta u$ dabit asymptotam formae $u = \frac{A}{tt}$, si fuerit

$$\delta(\alpha\varepsilon + \beta\delta) = \alpha(\alpha\eta + \beta\zeta),$$

sin autem haec aequalitas non habeat locum, asymptota erit formae $u = \frac{A}{t}$.

Habebimus ergo hanc aequationem

$$\left.\begin{array}{ll} +\ \alpha tuu - \beta u^3 & \\ +\ \delta tu & +\varepsilon uu \\ +\ \zeta t & +\eta u \\ +\theta & \end{array}\right\} = 0.$$

Hic facto $t = \infty$ fiet $\alpha uu + \delta u + \zeta = 0$.

Sit primum $\delta\delta$ minor quam $4\alpha\zeta$, atque hinc nulla orietur asymptota; quare ex hoc casu duae oriuntur species:

8.

Octava species habet unicam asymptotam speciei $u = \frac{A}{t}$.

9.

Nona species habet unicam asymptotam speciei $u = \frac{A}{tt}$.

231. Sint aequationis $\alpha uu + \delta u + \zeta = 0$ ambae radices reales et inaequales, nempe $\delta\delta$ maior quam $4\alpha\zeta$; atque hinc duae prodibunt asymptotae rectae inter se parallelae, utraque formae $u = \frac{A}{t}$, qui casus denuo duas suppeditat species:

10.

Decima species habet unam asymptotam speciei $u = \frac{A}{t}$ et duas inter se parallelas speciei $u = \frac{A}{t}$.

11.

Undecima species habet uaam asymptotaen speciei $u = \frac{A}{tt}$ et duas inter se parallelas speciei $u = \frac{A}{t}$.

232. Sint aequationis $\alpha uu + \delta u + \zeta = 0$ ambae radices inter se aequales seu $\alpha\alpha = 4\alpha\zeta$ seu $\alpha uu + \delta u + \zeta = \alpha(u-c)^2$, fietque

$$\alpha(u-c)^2 = \beta c^3 - \varepsilon cc - \eta c - \theta ,$$

unde oritur asymptota recta una speciei $uu = \frac{A}{t}$. Hinc ergo duae nascuntur species novae:

12.

Duodecima species habet unam asymptotam speciei $u = \frac{A}{t}$ et unam speciei $uu = \frac{A}{t}$.

13.

Decimatertia species habet unam asymptotam speciei $u = \frac{A}{tt}$ et unam $u = \frac{A}{t}$.

CASUS 4

233. Quodsi membri supremi omnes tres factores fuerint aequales, aequatio habebit huiusmodi formam

$$\alpha u^3 + \beta tt + \gamma tu + \delta uu + \varepsilon t + \zeta u + \eta = 0.$$

Hic primum spectandus est terminus βtt, qui si non desit, curva habebit asymptotam parabolicam speciei $u^3 = Att$ sicque una oritur species:

14.

Decimaquarta species habet unicam asymptotam parabolicam speciei $u^3 = Att$.

234. Desit iam terminus βtt, eritque

$$\alpha u^3 + \beta tu + \delta uu + \varepsilon t + \zeta u + \eta = 0 ; \;;$$

unde posito *t* infinito fiet $\alpha u^3 + \gamma tu + \varepsilon t = 0$, nisi sint γ et $\varepsilon = 0$. Non igitur sit $\gamma = 0$, atque in hac aequatione duae continentur aequationes $\alpha uu + \gamma t = 0$ et $\gamma u + \varepsilon = 0$; prior est pro asymptota parabolica speciei $uu = At$; posterior vero, si ponatur $\frac{-\varepsilon}{\gamma} = c$, dabit aequationem hanc

$$\gamma t(u - c) + \alpha c^3 + \delta cc + \zeta c + \eta = 0,$$

eritque ergo pro asymptota hyperbolica speciei $u = \frac{A}{t}$ unde

15.

Decimaquinta species unam habet asymptotam parabolicam speciei $uu = At$ et unam rectam speciei $u = \frac{A}{t}$, atque axis parabolae parallelus est alteri asymptotae rectae.

235. Sit etiam $\gamma = 0$, ut sit haec aequatio

$$\alpha u^3 + \delta uu + \varepsilon t + \zeta u + \eta = 0,$$

ubi ε evanescere non potest, nisi simul linea cesset esse curva. Facto autem t infinito, necessario u debet esse infinita, unde fit $\alpha u^3 + \varepsilon t = 0$, quae praebet speciem ultimam:

16.

Decimasexta species unam habet asymptotam parabolicam speciei $u^3 = At$.

236. Omnes ergo lineas tertii ordinis reduximus ad *sedecim species*, in quibus propterea omnes illae *species septuaginta duae*, in quas NEWTONUS lineas tertii ordinis divisit, continentur. Quod vero inter hanc nostram divisionem ac NEWTONIANAM tantum intercedat discrimen, mirum non est; hic enim tantum ex ramorum in infinitum excurrentium indole specierum diversitatem desumsimus, cum NEWTONUS quoque ad statum curvarum in spatio finito spectasset atque ex huius varietate diversas species constituisset. Quanquam autem haec divisionis ratio arbitraria videtur, tamen NEWTONUS suam tandem rationem sequens multo plures species producere potuisset, cum equidem mea methodo utens neque plures neque pauciores species eruere queam.

237. Quo igitur natura et complexus cuiusque speciei melius perspiciatur, aequationem generalem pro qualibet specie exhibebo, idque in simplicissima forma, quae salva universitate locum habere potest. Pro unaquaque vero simul species NEWTONIANAS eo pertinentes recensebo.

SPECIES PRIMA

$$y(xx - 2mxy + nnyy) + ayy + bx + cy + d = 0$$

existente mm minore quam nn et nisi fuerit $b = 0$.

Huc pertinent NEWTONI species 33, 34, 35, 36, 37, 38.

SPECIES SECUNDA

$$y(xx - 2mxy + nnyy) + ayy + cy + d = 0$$

existente *mm* minore quam *nn*.
Huc pertinent NEWTONI species 39, 40, 41, 42, 43, 44, 45.

SPECIES TERTIA

$$y(x-my)(x-ny)+ayy+bx+cy+d=0,$$

Ubi nec $b=0$ nec $mb+c+\frac{aa}{(m-n)^2}=0$ nec $nb+c+\frac{aa}{(m-n)^2}=0$ neque $m=n$.

Huc pertinent NEWTONI species 1, 2, 3, 4, 5, 6, 7, 8, 9; item 24, 25, 26, 27, si $a=0$.

SPECIES QUARTA

$$y\left(x-my\right)\left(x-ny\right)+ayy+cy+d=0,$$

ubi nec $c+\frac{aa}{(m-n)^2}=0$ nec $m=n$.

Huc pertinent NEWTONI species 10, 11, 12, 13, 14, 15, 16, 17,18,19, 20, 21;
item, si $a=0$, hae 28, 29, 30, 31.

SPECIES QUINTA

$$y(x-my)(x-ny)+ayy-\frac{aay}{(m-n)^2}+d=0$$

non existente $m=n$.

Huc pertinent NEWTONI species 22, 23 et 32.

SPECIES SEXTA

$$yy(x-my)+axx+bx+cy+d=0,$$

si neque $a=0$ neque $2m^3aa-mb-c=0$.

Huc pertinent NEWTONI species 46, 47, 48, 49, 50, 51, 52.

SPECIES SEPTIMA

$$yy(x-my)+axx+bx+m(2mmaa-b)y+d=0$$

non existente $a=0$.

Hue pertinent NEWTONI species 53, 54, 55, 56.

SPECIES OCTAVA

$$yy(x-my)+bbx+cy+d=0$$

non existente $c=-mbb$ nec $b=0$.

Huc pertinent NEWTONI species 61 et 62.

SPECIES NONA

$$yy(x-my)+bbx-mbby+d=0$$

non existente $b=0$.

Hue pertinet NEWTONI species 63.

SPECIES DECIMA

$$yy(x-my)-bbx+cy+d=0$$

non existente $c=mbb$ nec $b=0$.

Huc pertinent NEWTONI species 57, 58, 59.

SPECIES UNDECIMA

$$yy(x-my)-bbx+mbby+d=0$$

non existente $b=0$.

Hue pertinet NEWTONI species 60.

SPECIES DUODECIMA

$$yy\left(x-my\right)+cy+d=0$$

non existente $c=0$.

Huc pertinet NEWTONI species 64.

SPECIES TERTIA-DECIMA

$$yy(x-my)+d=0.$$

Huc pertinet NEWTONI species 65.

SPECIES QUARTA-DECIMA

$$y^3+axx+bxy+cy+d=0$$

non existente $a=0$.

Huc pertinent NEWTONI species 67, 68, 69, 70, 71.

SPECIES QUINTA-DECIMA

$$y^3+bxy+cx+d=0$$

non existents $b=0$.

Huc pertinet species 66.

SPECIES SEXTA-DECIMA

$$y^3+ay+bx=0$$

non existente $b=0$.

Huc pertinet NEWTONI species 72.

238. Species autem hae plerumque tam late patent, ut sub unaquaque varietates satis notabiles contineantur, siquidem ad formam, quam curvae habent in spatio finito, respiciamus, Hancque ob causam NEWTONUS numerum specierum multiplicavit, ut eas curvas, quae in spatio finite notabiliter discrepant, a se invicem secerneret. Expediet ergo has, quas *species* nominavimus, *genera* appellare atque varietates, quae sub unoquoque deprehenduntur, ad *species* referre. Imprimis autem hoc erit tenendum, si quis lineas quarti altiorisve ordinis simili modo subdividere voluerit; ibi enim multo maior varietaa in quavis specie sic inventa locum habebit.