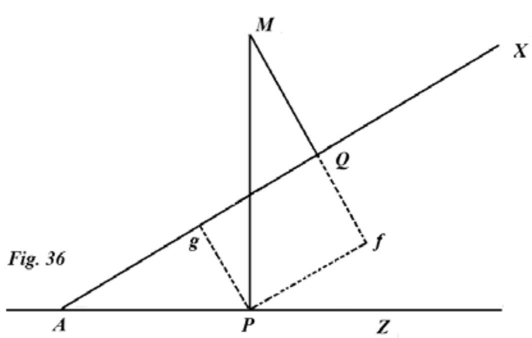
CHAPTER VIII

CONCERNING ASYMPTOTIC LINES

198. We have seen several kinds of asymptotes given in the preceding chapter ; for besides the right line we have found several curved line asymptotes expressed by this equation $u^{\mu} = Ct^{\nu}$. And a right line will make available other curved asymptotes, which will converge more with the curved lines, than with right lines. Moreover whenever a right line is found to be an asymptote of a certain curve, so it will be possible to assign another curve for the right line, which will be an asymptote of the proposed curve also. But curvilinear asymptotes of this kind will express the nature of the curve of which it is the asymptote much more accurately ; indeed likewise it shows the number of branches converging with the straight line and the region, whether above or below, and whether approaching to the right line from the right or the left.

199. Therefore these varieties of infinite asymptotes may be arranged most conveniently in order, if we may follow the source itself, from which we have adapted these. Clearly some asymptotes give rise to individual factors of the greatest member unequal to each other, others give rise to two equal factors, others to three, four, and thus so on. And thus the proposed equation of any order *n* between the coordinates *x* and *y,* which shall be $P + Q + R + S + \text{etc.} = 0$, where *P* shall be the greatest member containing all the terms of *n* dimensions, *Q* shall be the second order member containing the terms of $n-1$ dimensions, and in a like manner *R* the third, *S* the fourth and thus so on.

200. Now $ay - bx$ shall be a simple factor of *P*, for which another similar shall not be present, and putting $P = (ay - bx)M$, *M* will be a homogeneous function of $n-1$ dimensions not divisible by $ay - bx$. Without doubt (Fig. 35) *AZ* shall be the axis, in which the abscissa shall be $AP = x$ and the applied line $PM = y$. So that the factor $ay - bx$ may be expressed more briefly, the other right line *AX* is taken for the axis cutting the first line at the start of the abscissas *A* and making the angle *XAZ*, of which the



$$\text{tangent } = \frac{b}{a} \text{ and the sine } = \frac{b}{\sqrt{(aa+bb)}} \text{ and also the cosine } = \frac{a}{\sqrt{(aa+bb)}}.$$

In this axis putting the abscissa and the applied line $QM = u$; there will be, with *Pg, Pf* drawn parallel to the new coordinates *u* and *t*,

$$Pg = Qf = \frac{bx}{\sqrt{(aa+bb)}}, \quad Ag = \frac{ax}{\sqrt{(aa+bb)}},$$

$$Mf = \frac{ay}{\sqrt{(aa+bb)}}, \quad Pf = Qg = \frac{by}{\sqrt{(aa+bb)}},$$

and thus

$$t = Ag + Qg = \frac{ax+by}{\sqrt{(aa+bb)}} \text{ et } u = Mf - Qf = \frac{ay-bx}{\sqrt{(aa+bb)}}.$$

Therefore the applied line *u* now will be a factor of the greatest member *P*.

201. From these in turn there will be

$$y = \frac{au+bt}{\sqrt{(aa+bb)}} \text{ and } x = \frac{at-bu}{\sqrt{(aa+bb)}};$$

which values if they may be substituted into the equation $P+Q+R+\text{ etc.} = 0$, will produce an equation for the same curve between *t* and *u* related to the axis *AX*. But in order that we may avoid a multitude of coefficients, $\alpha, \beta, \gamma, \delta$ etc. shall take the place of all the coefficients ; and with the substitution made, the individual letters will adopt the following values :

$$M = \alpha t^{n-1} + \alpha t^{n-2}u + \alpha t^{n-3}uu + \text{etc.},$$
$$Q = \beta t^{n-1} + \beta t^{n-2}u + \beta t^{n-3}uu + \text{etc.},$$
$$R = \gamma t^{n-2} + \gamma t^{n-3}u + \gamma t^{n-4}uu + \text{etc.},$$
$$S = \delta \mathrm{t}^{n-3} + \delta t^{n-4}u + \delta t^{n-5}uu + \text{etc.},$$
$$T = \varepsilon t^{n-4} + \varepsilon t^{n-5}u + \varepsilon t^{n-6}uu + \text{etc.}$$
etc.

But because for finding the asymptote it is necessary to make the abscissa *t* infinite, in any term all the members besides the first vanish. Whereby, if some first term of each member shall be present, the following will vanish ; but if the first may be absent, the second may be taken ; if the first and second may be absent, it will require to begin from the third.

202. Because u does not divide the function M, the first term of this cannot be absent ; therefore becoming $\alpha t^{n-1}u+\beta t^{n-1}=0$, from which a finite value emerges for u, which shall be $=c$; that is, the right line parallel to the axis AX separated from that by the interval c will be the asymptote. Now towards finding a curvilinear asymptote approaching more towards the same curve putting $u=c$ everywhere, except in the first term, and this equation will be found :

$$\alpha t^{n-1}u+\beta t^{n-1}+t^{n-2}(\alpha cc+\beta c+\gamma)+t^{n-3}(\alpha c^3+\beta cc+yc+\delta)+\text{ etc.}=0;$$

or on account of $\alpha u+\beta=u-c$ there will be [essentially redefining the constants $\alpha,\beta,\gamma,\delta$ etc.]

$$(u-c)t^{n-1}+t^{n-2}(\alpha cc+\beta c+\gamma)+t^{n-3}(\alpha c^3+\beta cc+\gamma c+\delta)+\text{etc.}=0.$$

Now unless the second term may be absent, all the following terms can be ignored, and there becomes

$$\left(u-c\right)+\frac{A}{t}=0\ ;$$

[Thus, this asymptotic curve is hyperbolic, with its '*y*-coordinate ' or $\left(u-c\right)$ proportional to the negative of $\frac{A}{t}$, where t is its '*x*-coordinate ', etc. , as Euler goes on to show.]
if the second shall be absent, the third may be taken and there will be

$$\left(u-c\right)+\frac{A}{tt}=0.$$

Truly if the third may be missing, there becomes

$$\left(u-c\right)+\frac{A}{t^3}=0$$

and so on thus. If all except the final may be missing, there will be

$$(u-c)+\frac{A}{t^{n-1}}=0.$$

Moreover in a straight forwards manner if all the terms may be missing, the whole equation becomes divisible by $u-c$ and thus the right line $u-c=0$ will become a part of the curve.

203. If $u-c$ is put equal to z, or if the abscissas themselves may be taken on the right line asymptote, all the curvilinear asymptotes, which a factor of the greatest member may supply, will be found in this general equation $z=\frac{C}{t^k}$, with *k* denoting some whole number with the exponent less than *n*. Therefore just as we may see therefore that these curvilinear asymptotes may be prepared, if the abscissa *t* may be made infinite. Therefore (Fig. 36) *XY* shall be the right asymptotes for the axis taken and *A* the start of the abscissas, with the right line *CD* drawn four regions will arise, which we may designate by the letters *P*, *Q*, *R* and *S*. Now in the first place there shall be $z=\frac{C}{t}$ and, because with *t* taken negative *z* also becomes negative, the curve will have two branches *EX* and *FY* in the opposite regions *P* and *S* converging to the axis *XY*. The same will come about if *k* were some odd number. But if there were $k=2$ or $z=\frac{C}{tt}$, because, if *t* were taken either positive or negative, *z* remains always positive, and the curve will be made from the two branches (Fig. 37) *EX* and *FY* in the regions *P* and *Q* converging to the right line *XY*; which likewise will be produced, if *k* were any even number, yet with this provision, that the convergence will be quicker there, where the exponent *k* shall be greater .

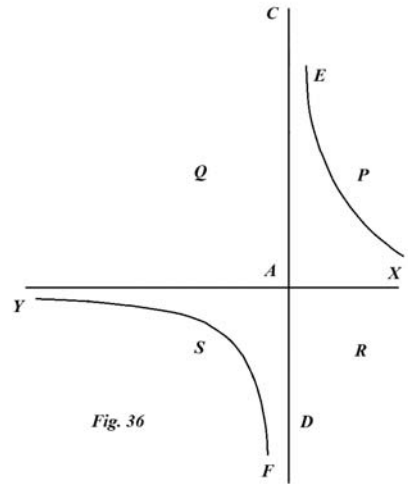


Fig. 36

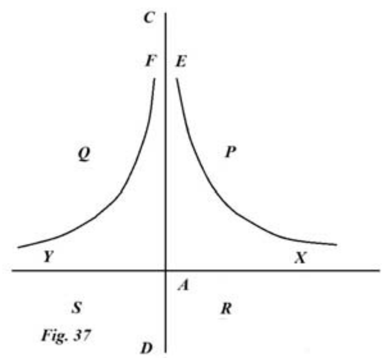


Fig. 37

204. The greatest term *P* may have two equal factors $ay-bx$;and with the same done as before by a translation to another axis, there becomes

$$\begin{aligned}
P &= \qquad\qquad +\alpha t^{n-2}uu+\alpha t^{n-3}\,u^3+\text{ etc. },\\
Q &= \beta t^{n-1}+\beta t^{n-2}u+\beta t^{n-3}uu+\beta t^{n-4}u^3\ +\text{ etc. },\\
R &= \gamma t^{n-2}+\gamma t^{n-3}u\ +\gamma t^{n-4}uu+\gamma t^{n-5}u^3\ \ +\text{ etc. },\\
S &= \delta t^{n-3}+\delta t^{n-4}u\ +\delta t^{n-5}uu+\delta t^{n-6}u^3\ \ +\text{etc.}\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

Hence, provided the first term of the member *Q* were present or not, two equations arise

I.

$$\alpha t^{n-2}uu+\beta t^{n-1}=\ 0 \quad\text{or}\quad \alpha uu+\beta t=0,$$

II.

$$\alpha t^{n-2}uu + \beta t^{n-2}u + \gamma t^{n-2} = 0 \quad \text{or} \quad \alpha uu + \beta u + \gamma = 0 .$$

But if therefore the first equation $\alpha uu + \beta t = 0$ has a place, the asymptote becomes a parabola (Fig. 38), since the two curves will merge together with both branches at infinity. Therefore the curve will have branches in the two regions *P* and *R* and thence congruent with the parabola *EAF*.

205. But if the other equation may result $\alpha uu + \beta u + \gamma = 0$, then it can be seen, that either it may have two real roots or not any. For in the latter case at once no branches are indicated extending to infinity. Therefore both the roots shall be real and unequal, the one $u = c$, and the other $u = d$, and the curve will have two right line asymptotes parallel to each other. Truly as before we will investigate what the nature of each shall be, as before; evidently, since there shall be

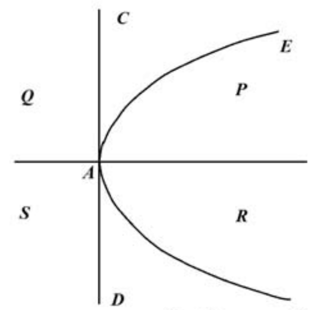


Fig. 38

$$\alpha uu + \beta u + \gamma = (u-c)(u-d),$$

everywhere there is put $u = c$, except in the factor $u - c$, and there will be produced

$$(c-d)t^{n-2}(u-c) + t^{n-3}(\alpha c^3 + \beta cc + \gamma c + \delta) + t^{n-4}(\alpha c^4 + \beta c^3 + \gamma cc + \delta c + \varepsilon) + \text{ etc.} = 0$$

or $\alpha uu + \beta u + \gamma = 0$

Therefore if the second term does not vanish, all the following vanish on putting $t = \infty$ and the asymptote will be

$$(u-c) + \frac{A}{t} = 0;$$

if the second term vanishes, there becomes

$$(u-c) + \frac{A}{tt} = 0$$

and thus so forth. If all the terms except the final constant were $= 0$, there will be

$$(u-c) + \frac{A}{t^{n-2}} = 0$$

all the figures of which curves, if $t = \infty$, we have now described.

206. But, if both the roots of the equation $\alpha uu + \beta u + \gamma = 0$ were equal, or $\alpha uu + \beta u + \gamma = (u-c)^2$, because $u = c$, if this value may be substituted in the rest of the terms, this equation will be produced

$$t^{n-2}(u-c)^2 + t^{n-3}(\alpha c^3 + \beta cc + \gamma c + \delta) + t^{n-4}(\alpha c^4 + \beta c^3 + \gamma cc + \delta c + \varepsilon) + \text{ etc.} = 0;$$

from which, therefore, with the first term excepted, either the second shall not be absent, nor the third with the first lacking, nor the fourth with the second and third lacking, the following equations will arise for the asymptotes :

$$(u-c)^2 + \frac{A}{t} = 0,$$
$$(u-c)^2 + \frac{A}{tt} = 0,$$
$$(u-c)^2 + \frac{A}{t^3} = 0,$$
as far as to
$$(u-c)^2 + \frac{A}{t^{n-2}} = 0,$$

if all the terms besides the final constant shall be absent. Truly if also the final term may vanish, there becomes $(u-c)^2 = 0$ and thus a right line itself becomes a portion of the curve, and the curve thus infolded.

207. Thus although all the cases will be seen to be enumerated, which two equal factors provide, yet the final equation can be adapted at this point to other forms, from which different asymptotes follow. This arises, if the factor of the power t^{n-3} may be taken divided by $u-c$; for then, as in the first term, $u-c$ may remain and the following term above which is closest may be added, and in this case equations of this kind emerge :

$$(u-c)^2 + \frac{A(u-c)}{t} + \frac{B}{tt} = 0,$$
$$(u-c)^2 + \frac{A(u-c)}{t} + \frac{B}{t^3} = 0,$$
as far as to
$$(u-c)^2 + \frac{A(u-c)}{t} + \frac{B}{t^{n-2}} = 0.$$

But if moreover the second term were missing completely or were divisible by $(u-c)^2$, then it will be seen that the third term, which if it may be taken divisible by $u-c$, $u-c$ is left in that and besides the following nearest term may be added on. In this case equations of this kind will arise :

$$\left(u-c\right)^2+\frac{A\left(u-c\right)}{tt}+\frac{B}{t^3}=0,$$

$$\left(u-c\right)^2+\frac{A\left(u-c\right)}{tt}+\frac{B}{t^4}=0,$$

as far as to

$$\left(u-c\right)^2+\frac{A\left(u-c\right)}{tt}+\frac{B}{t^{n-2}}=0.$$

But if also the third term may be absent and the fourth may be found divisible by $u-c$, or also with this missing, the fifth and thus so on, an equation of this kind may emerge for the asymptotic curve,

$$\left(u-c\right)^2+\frac{A\left(u-c\right)}{t^p}+\frac{B}{t^q}=0,$$

where the exponent p always will be less than q and q less than $n-1$.

208. We may put $u-c=z$, and all these equations will be present in this form :

$$z^2+\frac{Az}{t^p}+\frac{B}{t^q}=0.$$

Towards explaining that, three cases are required to be considered, provided q were either greater than $2p$, or q were equal to $2p$, or q were less than $2p$.

In the first case, in which q is greater than $2p$, two equations will be contained in that,

$$z-\frac{A}{t^p}=0 \text{ and } Az-\frac{B}{t^{q-p}}=0;$$

for each is satisfied on making $t=\infty$;

[Essentially from the sum and product of the roots of the quadratic equation : for the roots are $z=\frac{-A}{2t^p}\pm\sqrt{\frac{A^2}{4t^{2p}}-\frac{B}{t^q}}$.] For on putting $z=\frac{A}{t^p}$ the above equation will become

$$\frac{AA}{t^{2p}}-\frac{AA}{t^{2p}}+\frac{B}{t^q} \text{ or } AA-AA+\frac{B}{t^{q-2p}}=0,$$

which is true on account of *q* being greater than 2*p*, moreover *p* will be less than $\frac{n-2}{2}$.

But if $z=\frac{B}{At^{q-p}}$, there becomes

$$\frac{BB}{AAt^{2q-2p}}-\frac{B}{t^q}+\frac{B}{t^q} \text{ or } \frac{BB}{AAt^{q-2p}}-B+B=0\,,$$

which is true on account of the first term vanishing by making $t=\infty$. Therefore in this case above the same right asymptotes will have two curvilinear asymptotes and thus four branches extending to infinity.

The second case, in which $q=2p$, provided the equation

$$zz-\frac{Az}{t^p}+\frac{B}{t^{2p}}=0\,,$$

which either is imaginary, if *AA* shall be less than 4*B*, in which case no asymptote is apparent, or it provides two similar asymptotes $z=\frac{C}{t^p}$, if *AA* shall be greater than 4*B* .

In the third case, if *q* shall be less than *2p*, the middle term of the equation will vanish always on putting $t=\infty$; and therefore there will be $zz+\frac{B}{t^q}=0$, the equation for a single asymptote. Indeed now that we have put in place certain forms of the preceding asymptotes, whereby we may examine these asymptotes contained in this form $zz=\frac{C}{t^k}$.

209. Therefore if in the right line asymptote itself there is taken $u=c$ and the applied line $u-\mathrm{c}$ may be put $=z$, all these curvilinear asymptotes will be contained in this equation

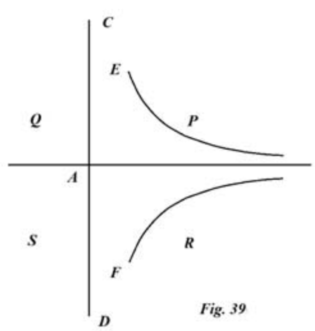


Fig. 39

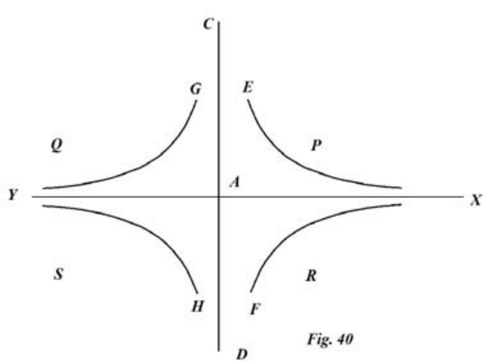


Fig. 40

$zz=\frac{C}{t^k}$, with *k* denoting a whole number less than $n-1$. Moreover the branches of these curves extending to infinity or on making $t=\infty$ thus will be had themselves. If $k=1$ or

$zz = \frac{C}{t}$, because t cannot become negative, the curve (Fig. 39) will have two branches EX and FX in the regions P and R extending to infinity ; which likewise will come about, if k were some odd number. But, if k shall be some even number such as 2, or $zz = \frac{C}{tt}$, at first it is required to be seen, whether C shall be a negative or positive quantity. In the former case the equation cannot be real and thus the equation hence can have no branches extending to infinity. In the latter case the curve (Fig. 40) will have four branches extending to infinity and since the asymptotes XY are concurrent, evidently EX, FX, GY and BY spread out in all four regions P, R, Q and S.

210. We may put the greatest member of the equation P to have three equal factors, and the equation reduced to the coordinates t and u, so that u shall be this triple factor P, will be

$$
\begin{aligned}
P &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \alpha t^{n-3}u^3 + \alpha t^{n-4}u^4 + \text{ etc. },\\
Q &= \beta t^{n-1} + \beta t^{n-2}u + \beta t^{n-3}uu + \beta t^{n-4}u^3 + \beta t^{n-5}u^4 + \text{ etc. },\\
R &= \gamma t^{n-2} + \gamma t^{n-3}u + \gamma t^{n-4}uu + \gamma t^{n-5}u^3 + \gamma t^{n-6}u^4 + \text{ etc. },\\
S &= \delta t^{n-3} + \delta t^{n-4}u + \delta t^{n-5}uu + \delta t^{n-6}u^3 + \delta t^{n-7}u^4 + \text{ etc.}\\
&\text{etc.}
\end{aligned}
$$

Hence the following equations arise for the different arrangements of the members Q and R :

I.

$$\alpha t^{n-3}u^3 + \beta t^{n-1} = 0,$$

II.

$$\alpha t^{n-3}u^3 + \beta t^{n-2}u + \gamma t^{n-2} = 0,$$

III.

$$\alpha t^{n-3}u^3 + \beta t^{n-3}uu + \gamma t^{n-2} = 0,$$

IV.

$$\alpha t^{n-3}u^3 + \beta t^{n-3}uu + \gamma t^{n-3}u + \delta t^{n-3} = 0.$$

211. The first equation will change into $\alpha u^3 + \beta tt = 0$ and thus this asymptote is a line of the third order (Fig. 41), the figure of which shall be such, if the abscissas t are taken from a point axe A on the axis SY. Clearly it will have two branches E and F extending to infinity in the regions P et Q.

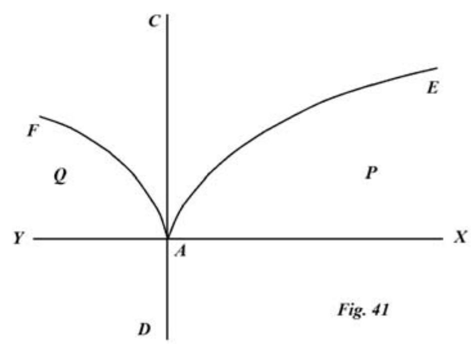


Fig. 41

The second equation thus itself will be $\alpha u^3 + \beta tu + \gamma t = 0$. From which on putting

$t = \infty$, u will have a twofold value, either finite or infinite, and thus will be resolved into these two equations $\beta u + \gamma = 0$ and $\alpha uu + \beta t = 0$; the latter is the equation of a parabola, as we have seen before, and therefore the curve will have two branches extending to infinity approaching to a parabola. Truly the first equation provides $u - c = 0$, which is for a right line asymptote, the nature of which will be seen, if except in $\beta u + \gamma = u - c$, everywhere c is written in place of u; and therefore there will be :

$$t^{n-2}(u-c) + t^{n-3}(\alpha c^3 + \beta cc + \gamma c + \delta) + t^{n-4}(\alpha c^4 + \beta c^3 + \gamma cc + \delta c + \varepsilon) + \text{ etc.} = 0;$$

from which, as above, it may be concluded that

$$\text{either } (u-c) + \frac{A}{t} = 0; \quad \text{or } (u-c) + \frac{A}{tt} = 0, \text{ etc.}$$

Truly the final equation, which can arise, is $(u-c) + \frac{A}{t^{n-2}} = 0$. Therefore in this case the curve has a two-fold asymptote, jointly one a right line of a determined nature, the other truly parabolic.

212. The third equation $\alpha u^3 + \beta uu + \gamma t = 0$ cannot be terminated on putting $t = \infty$, unless u shall be $= \infty$; and thus the term βuu shall vanish before αu^3 and this equation of the third order will be produced $\alpha u^3 + \gamma t = 0$ for the asymptote, of which this is the figure (Fig. 42), as it may have branches AE and AF in the two opposite regions P and S extending to infinity.

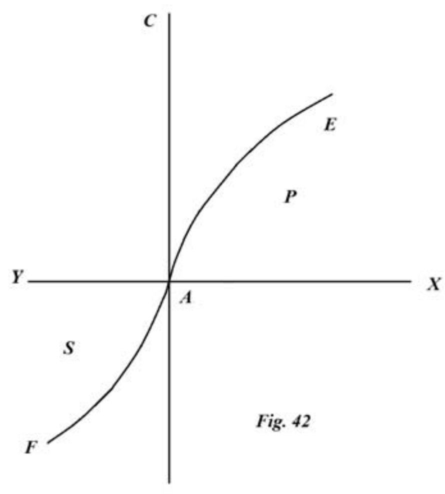


Fig. 42

But the fourth equation $\alpha u^3 + \beta uu + \gamma u + \delta = 0$ will show either one or three right line asymptotes parallel to each other, unless two or all shall be equal to each other, according to the nature of which requiring to be found the first root of the equation shall be $u = c$, not having another similar to itself, and there will be

$$\alpha u^3 + \beta uu + \gamma u + \delta = (u-c)\ (fuu + gu + h).$$

Putting everywhere $u = c$, except in the factor $u - c$, and an equation of this kind will be produced

$$t^{n-3}(u-c) + At^{n-4} + Bt^{n-5} + Ct^{n-6} + \text{etc.} = 0;$$

from which an asymptote will arise of the form $u-c=\frac{K}{t^k}$ with k being some number less than $n-2$.

213. If two roots of the equation $\alpha u^3+\beta uu+\gamma u+\delta=0$ were equal, thus so that the expression shall be $=(u-c)^2(fu+g)$; and on putting $u=c$, unless in some member there were a factor $u-c$, it will arrive at an equation of this kind :

$$(u-c)^2+\frac{A(u-c)}{t^p}+\frac{B}{t^q}=0;$$

where q will be less than $n-2$ and p less than q, which case we have set out before. Therefore the case remains, in which the equation $\alpha u^3+\beta uu+\gamma u+\delta=0$ has three equal roots, for example, $(u-c)^3$, and an equation of this kind will be obtained:

$$(u-c)^3t^{n-3}+Pt^{n-4}+Qt^{n-5}+Rt^{n-6}+St^{n-7}+\text{ etc.}=0.$$

But if P were not divisible by $u-c$, there may be put $u=c$ and there becomes

$$(u-c)^3+\frac{A}{t}=0.$$

But if P may have a single divisor $u-c$, and $u=c$ is put everywhere, except in this factor, and an equation of this form will arise

$$(u-c)^3+\frac{A(u-c)}{t}+\frac{B}{t^q}=0,$$

with q being a number less than $n-2$; truly $\frac{B}{t^q}$ is the second term following, which does not vanish on putting $u=c$. But if P thus were divisible by $(u-c)^2$, Q truly may not have the factor $u-c$, and an equation of this form will arise

$$(u-c)^3+\frac{A(u-c)^2}{t}+\frac{B}{tt}=0.$$

But if moreover the second term thus were divisible by $(u-c)^3$, then it is necessary to proceed in order, until a term may be reached not divisible by $(u-c)^3$, which shall be divisible by $u-c$, it is required to progress further until a term not divisible by

$u-c$ may be reached. In the first case the equation will be ended, in the latter it may progress further, until it may reach a term not divisible by per $u-c$. Therefore an equation will be obtained, always thus contained in this general form :

$$(u-c)^3+\frac{A(u-c)^2}{t^p}+\frac{B(u-c)}{t^q}+\frac{C}{t^r}=0 \ ,$$

where *r* will be less than $n-2$, *q* less than *r* and *p* less than *q*.

214. In this equation there will be contained either three equations of the form $u-c=\frac{K}{t^k}$, or one of this kind and one of the kind $(u-c)^2=\frac{K}{t^k}$, or finally a single one of the form $(u-c)^3=\frac{K}{t^k}$: so that the latter will come about, if *3p* were greater than *r* and *3q* were greater than *2r*. Then truly it can also arise, so that the two equations become imaginary, which therefore will show no asymptote. I have now explained the rest of the forms of these asymptotes except the final equation present $(u-c)^3=\frac{K}{t^k}$. Moreover this equation will present, if k shall be an odd number, the form designated by figure thirty six with two branches *EX* and *FY* extending to infinity from the opposite regions *P* and *S*. But if *k* shall be an even number, the shape will arise for the figure shown in figure thirty seven, in which the two branches *EX* and *FY* are on the same side of the right asymptote *XY* , or extending out from the regions *P* and *Q*.

215. Because it is seen easily from these, in what way the form of the asymptotes ought to be investigated, if four or more simple factors were equal in the greatest term of the equation, I will not proceed further here ; indeed I will finish this chapter with an application of the rules given to a simple example.

EXAMPLE

Let the proposed curves line be expressed by this equation :

$$y^3xx(y-x)-xy(yy+xx)+1=0,$$

the greatest term of which $y^3xx(y-x)$ *has a single solitary factor* $y-x$, *two equal factors* xx *and in addition the three equal* y^3.

We will examine first the simple factor $y-x$; from which, on putting $y=x$, there becomes $y-x-\frac{2}{x}=0$; and on account of $x=\infty$ it will be $y - x = 0$, which is the equation (Fig. 43) for the rectilinear asymptote *BAC* with the axis *XY* making a semi right angle *BAY* with the start of the abscissas. Just as the equation may be transferred to this line for the axis, which is done by putting $y=\frac{u+t}{\sqrt{2}}$ and $x=\frac{t-u}{\sqrt{2}}$; with which accomplished this equation will arise :

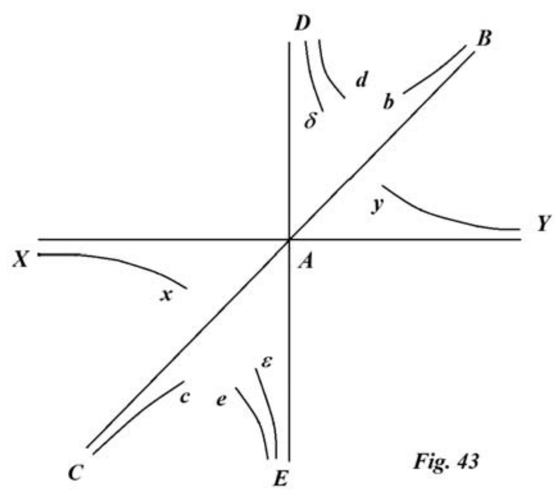


Fig. 43

$\frac{(u+t)(tt-uu)^2u}{4}-\frac{(tt-uu)(tt+uu)}{2}+1=0$; from which , on multiplying by 4, it becomes

$$\begin{aligned}0=\;&t^5u+t^4uu-2t^3u^3-2ttu^4+tu^5+u^6\;;\\&-2t^4 \qquad\qquad\qquad\qquad +\;2u^4\\&+4\end{aligned}$$

from this equation, by making $t=\infty$, it is found that $u=0$; and thus the remaining terms vanish besides these two terms $t^5u,-2t^4$; from which the curvilinear asymptote will be $u=\frac{2}{t}$. Therefore on account of this factor the curve sought will have the two branches $bB,\ Cc$ extending to infinity.

216. Now the twin equal factors may be taken xx and there will be

$$xx = \frac{xy(yy+xx)-1}{y^3(y-x)}.$$

Therefore with the right axis AD taken normal to the former XY making $y = t$ and $x = u$, from which this equation results :

$$\begin{aligned} 0 = \quad & \qquad t^4uu - t^3u^3 \\ & -t^3u \qquad -tu^3 \\ & +1 \end{aligned}$$

which with t made infinite will change into $t^4u^2 - t^3u + 1 = 0$, from which the two equations arise

$$u = \frac{1}{t} \text{ and } u = \frac{1}{t^3}.$$

Whereby this factor provides branches extending to infinity, in the first place clearly the two dD, eE arising from the $u = \frac{1}{t}$, and the two δD and εE arising from the equation $u = \frac{1}{t^3}$ placed on the same side.

217. The three equal factors y^3 are referred to the XY axis itself and become $x = t$ and $y = u$, from which this equation arises

$$0 = -t^3u^3 + ttu^4 - t^3u - tu^3 + 1,$$

which on putting t infinite gives $t^3u^3 + t^3u = 0$ or $u(uu+1) = 0$; from which, on account of $uu + 1 = 0$ being an impossible equation, the single right line asymptote $u = 0$ will be obtained, meeting with the axis XY itself, the nature of which will be expressed from this equation $t^3u = 1$ or $u = \frac{1}{t^3}$; and therefore this triple factor bears only two branches extending to infinity yY and xX. Therefore the curve sought will have eight branches extending to infinity in full, which in some manner may be connected together in the finite space among themselves, concerning which this is not the place to be explained.

218. Therefore from this and the preceding chapter the variety of branches extending to infinity is understood more clearly. For in the first place these branches of curves either converge to certain right lines as asymptotes, as it becomes in the hyperbola, or the curve does not have a right asymptote, as with the parabola. In the first case the branches of the curves are called *hyperbolic,* in the latter *parabolic.* Innumerable kinds are given of each class ; for the kinds of hyperbolic branches are expressed from these equations between the variables t and u , of which that one t is put infinite :

$$u=\frac{A}{t},\quad u=\frac{A}{tt},\quad u=\frac{A}{t^3},\quad u=\frac{A}{t^4},\quad \text{etc.},$$
$$uu=\frac{A}{t},\quad uu=\frac{A}{tt},\quad uu=\frac{A}{t^3},\quad uu=\frac{A}{t^4},\quad \text{etc.}$$
$$u^3=\frac{A}{t},\quad u^3=\frac{A}{tt},\quad u^3=\frac{A}{t^3},\quad u^4=\frac{A}{t^4},\quad \text{etc.}$$
etc.

Truly the kinds of parabolic branches are shown by the following equations :

$$uu=At\quad u^3=At\quad u^4=At\quad u^5=At\quad \text{etc.}$$
$$u^3=Att\quad u^4=Att\quad u^5=Att\quad u^6=Att\quad \text{etc.}$$
$$u^4=At^3\quad u^5=At^3\quad u^6=At^3\quad u^7=At^3\quad \text{etc.}$$
etc.

But whatever the equation of these set out at least two will show branches extending to infinity, if not each of the exponents of t and u were an even number ; but if each exponent were an even number, then either no branch to infinity will be present, or four will be present; clearly the first arises, if the equation shall be impossible, the other truly if it shall be possible.

CAPUT VIII

DE LINEIS ASYMPTOTIS

198. In capite praecedente vidimus plures dari asymptotarum species; praeter lineam rectam enim invenimus plures lineas curvas asymptotas hac aequatione $u^{\mu} = Ct^{\nu}$ expressas. Atque ipsa linea recta suppeditavit alias asymptotas curvilineas, cum quibus linea curva magis convergat, quam cum linea recta. Quoties autem linea recta reperitur esse asymptota cuiuspiam curvae, toties linea curva eandem rectam pro asymptota habens assignari poterit, quae etiam sit asymptota curvae propositae. Huiusmodi autem asymptota curvilinea multo accuratius exprimit indolem curvae, cuius est asymptota; ostendit enim simul ramorum numerum cum recta convergentium atque plagam, utrum supra an infra, an antrorsum retrorsumve ad rectam appropinquent.

199. Haec igitur infinita asymptotarum varietas commodissime in ordinem digeretur, si ipsum fontem, unde eas sumus adepti, sequamur. Alias scilicet asymptotas praebent singuli membri supremi factores inter se inaequales, alias bini factores aequales, alias terni aequales, alias quaterni et ita porro. Sit itaque proposita aequatio cuiusque ordinis n inter coordinatas x et y, quae sit $P + Q + R + S + \text{etc.} = 0$, ubi P sit membrum supremum continens omnes terminos n dimensionum, Q sit membrum secundum continens terminos $n-1$ dimensionum, similique modo R tertium, S quartum et ita porro.

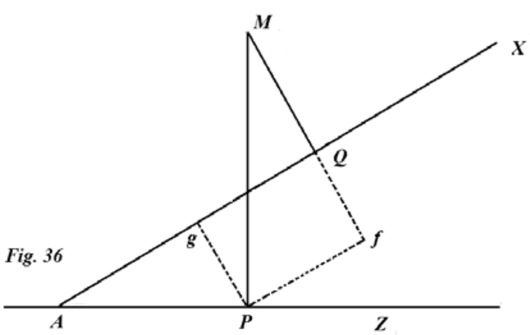


Fig. 36

200. Sit iam $ay - bx$ factor simplex ipsius P, cui alius similis non adsit, ac ponatur $P = (ay - bx)M$, eritque M functio homogenea $n-1$ dimensionum non divisibilis per $ay - bx$. Sit nimirum (Fig. 35) AZ axis, in quo sit abscissa $AP = x$ et applicata $PM = y$. Quo factor $ay - bx$ succinctius exprimatur, sumatur alia recta AX pro axe secans priorem in ipso abscissarum initio A et faciens angulum XAZ, cuius

$$\text{tangens } = \frac{b}{a} \text{ ideoque sinus } = \frac{b}{\sqrt{(aa+bb)}} \text{ et cosinus } = \frac{a}{\sqrt{(aa+bb)}}.$$

In hoc axe ponatur abscissa $AQ = t$ et applicata $QM = u$; erit, ductis Pg, Pf novis coordinatis u et t parallelis,

$$Pg = Qf = \frac{bx}{\sqrt{(aa+bb)}}, \quad Ag = \frac{ax}{\sqrt{(aa+bb)}},$$

$$Mf = \frac{ay}{\sqrt{(aa+bb)}}, \quad Pf = Qg = \frac{by}{\sqrt{(aa+bb)}},$$

ideoque

$$t = Ag + Qg = \frac{ax+by}{\sqrt{(aa+bb)}} \quad \text{et} \quad u = Mf - Qf = \frac{ay-bx}{\sqrt{(aa+bb)}}.$$

Erit ergo nunc applicata *u* factor supremi membri *P*.

201. Ex his erit vicissim

$$y = \frac{au+bt}{\sqrt{(aa+bb)}} \quad \text{et} \quad x = \frac{at-bu}{\sqrt{(aa+bb)}};$$

qui valores si in aequatione $P+Q+R+$ etc. $=0$ substituantur, prodibit aequatio pro curva eadem ad axem *AX* relata inter *t* et *u* . Ut autem coefficientium multitudinem evitemus, sustineant $\alpha, \beta, \gamma, \delta$ etc. loca omnium coefficientium; ac facta substitutione singulae litterae sequentes valores induent:

$$M = \alpha t^{n-1} + \alpha t^{n-2}u + \alpha t^{n-3}uu + \text{etc.},$$
$$Q = \beta t^{n-1} + \beta t^{n-2}u + \beta t^{n-3}uu + \text{etc.},$$
$$R = \gamma t^{n-2} + \gamma t^{n-3}u + \gamma t^{n-4}uu + \text{etc.},$$
$$S = \delta t^{n-3} + \delta t^{n-4}u + \delta t^{n-5}uu + \text{etc.},$$
$$T = \varepsilon t^{n-4} + \varepsilon t^{n-5}u + \varepsilon t^{n-6}uu + \text{etc.}$$
$$\text{etc.}$$

Quia autem ad asymptotam inveniendam abscissam *t* infinitam statui oportet, in quovis membro omnes termini prae primo evanescent. Quare, si cuiusvis terminus primus adsit, sequentes negligi possunt; sin primus desit, capiatur secundus; sin primus et secundus desint, a tertio erit incipiendum.

202. Quia *u* non dividit functionem *M*, eius primus terminus deesse non potest; fiet ergo $\alpha t^{n-1}u + \beta t^{n-1} = 0$, unde pro *u* oritur valor finitus, qui sit $=c$; hoc est, recta axi *AX* parallela ab eoque intervallo *c* distans erit asymptota. Iam ad asymptotam curvilineam magis ad ipsam curvam accedentem inveniendam ponatur ubique, praeterquam in primo termino, $u = c$, ac reperietur haec aequatio

$$\alpha t^{n-1}u+\beta t^{n-1}+t^{n-2}(\alpha cc+\beta c+\gamma)+t^{n-3}(\alpha c^3+\beta cc+yc+\delta)+\text{ etc.}=0;$$

vel ob $\alpha u+\beta=u-c$ erit

$$((u-c)t^{n-1}+t^{n-2}(\alpha cc+\beta c+\gamma)+t^{n-3}(\alpha c^3+\beta cc+\gamma c+\delta)+\text{etc.}=0.$$

Nisi iam terminus secundus desit, omnes sequentes negligi possunt, fietque

$$(u-c)+\frac{A}{t}=0 \;;$$

si secundus desit, tertius sumatur eritque

$$(u-c)+\frac{A}{tt}=0.$$

Tertio vero etiam deficiente fiet

$$(u-c)+\frac{A}{t^3}=0$$

et ita porro. Si omnes praeter ultimum constantem deficiant, erit

$$(u-c)+\frac{A}{t^{n-1}}=0.$$

Prorsus autem omnes si deessent, tota aequatio divisibilis foret per $u-c$ ideoque ipsa recta $u-c=0$ foret curvae portio.

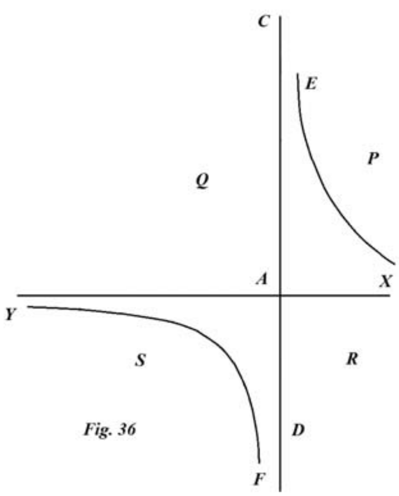


Fig. 36

203. Si ponatur $u-c=z$ seu si abscissae in ipsa asymptota recta capiantur, omnes asymptotae curvilineae, quas unicus supremi membri factor suppeditat, in hac aequatione generali comprehenduntur $z=\frac{C}{t^k}$, denotante k numerum quemvis integrum exponente n minorem. Quemadmodum ergo hae asymptotae curvilineae sint comparatae, si abscissa t ponatur infinita, videamus. Sit ergo (Fig. 36) XY asymptota recta pro axe sumta et A initium abscissarum, ducta recta CD orientur quatuor regiones, quas litteris P, Q, R et S designemus. Sit nunc primum $z=\frac{C}{t}$ et, quia sumto t negativo fit z quoque negativa,

curva duos habebit ramos *EX* et *FY* in regionibus oppositis *P* et *S* ad rectam *XY* convergentes. Idem eveniet, si *k* fuerit numerus quicunque impar. At si fuerit $k = 2$ seu $z = \frac{C}{tt}$, quia, sive *t* statuatur affirmativa sive negativa, *z* perpetuo affirmativa manet, curva constabit (Fig. 37) duobus ramis *EX* et *FY* in regionibus *P* et *Q* ad rectam *XY* convergentibus; quod idem contingit, si *k* fuerit numerus par quicunque, hoc tantum discrimine, quod convergentia eo fiat promtior, quo maior sit exponens *k* .

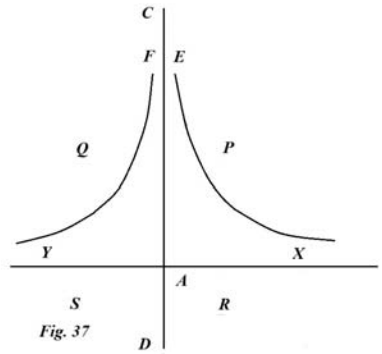


Fig. 37

204. Habeat supremum membrum *P* binos factores $ay - bx$ inter se aequales; atque facta eadem qua ante ad alium axem translatione, fiet

$$\begin{aligned}
P &= \qquad\qquad + \alpha t^{n-2}uu + \alpha t^{n-3}u^3 + \text{ etc. },\\
Q &= \beta t^{n-1} + \beta t^{n-2}u + \beta t^{n-3}uu + \beta t^{n-4}u^3 + \text{ etc. },\\
R &= \gamma t^{n-2} + \gamma t^{n-3}u + \gamma t^{n-4}uu + \gamma t^{n-5}u^3 + \text{ etc. },\\
S &= \delta t^{n-3} + \delta t^{n-4}u + \delta t^{n-5}uu + \delta t^{n-6}u^3 + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

Hinc, prout primus membri *Q* terminus affuerit, sive minus, duae oriuntur aequationes

I.

$$\alpha t^{n-2}uu + \beta t^{n-1} = 0 \quad \text{seu} \quad \alpha uu + \beta t = 0,$$

II.

$$\alpha t^{n-2}uu + \beta t^{n-2}u + \gamma t^{n-2} = 0 \quad \text{seu} \quad \alpha uu + \beta u + \gamma = 0.$$

Quodsi ergo prima aequatio $\alpha uu + \beta t = 0$ locum habet, asymptota fit parabola (Fig. 38), cum cuius ambobus ramis duo curvae rami in infinito confundentur. Curva ergo *Q* in binis regionibus *P* et *R* ramos habebit cum parabola *EAF* denique congruentes.

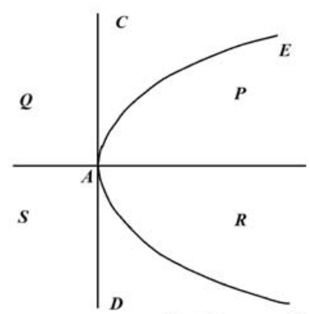


Fig. 38

205. Sin autem altera aequatio $\alpha uu + \beta u + \gamma = 0$ resultet, tum videndum est, an habeat duas radices reales an secus. Posteriori casu enim hac aequatione nulli prorsus rami in infinitum excurrentes denotantur. Sint ergo ambae radices reales et inaequales, altera $u = c$, altera $u = d$, atque curva duas habebit asymptotas rectas inter se parallelas. Cuiusnam vero utraque sit indolis, ut ante, investigabimus; scilicet, cum sit

$$\alpha uu + \beta u + \gamma = (u - c)(u - d),$$

ponatur ubique $u = c$, praeterquam in factore $u - c$, ac prodibit

$$(c-d)t^{n-2}(u-c) + t^{n-3}(\alpha c^3 + \beta cc + \gamma c + \delta) + t^{n-4}(\alpha c^4 + \beta c^3 + \gamma cc + \delta c + \varepsilon) + \text{ etc.} = 0$$

seu $\alpha uu + \beta u + \gamma = 0$

Nisi ergo secundus terminus evanescat, sequentes omnes posito $t = \infty$ evanescent eritque asymptota

$$(u-c) + \frac{A}{t} = 0;$$

si terminus secundus evanescat, fiet

$$(u-c) + \frac{A}{tt} = 0$$

atque ita porro. Si omnes termini praeter ultimum constantem fuerint $= 0$, erit

$$(u-c) + \frac{A}{t^{n-2}} = 0$$

quarum curvarum figuras, si $t = \infty$, iam supra omnes descripsimus.

206. At, si ambae radices aequationis $\alpha uu + \beta u + \gamma = 0$ fuerint aequales, seu $\alpha uu + \beta u + \gamma = (u-c)^2$, quia $u = c$, si hic valor in reliquis terminis substituatur, prodibit ista aequatio

$$t^{n-2}(u-c)^2 + t^{n-3}(\alpha c^3 + \beta cc + \gamma c + \delta) + t^{n-4}(\alpha c^4 + \beta c^3 + \gamma cc + \delta c + \varepsilon) + \text{ etc.} = 0;$$

unde, prout, excepto primo, vel non desit secundus, vel non desit tertius deficiente primo, vel non quartus deficientibus secundo et tertio, sequentes oriuntur aequationes pro asymptotis:

$$(u-c)^2+\frac{A}{t}=0,$$
$$(u-c)^2+\frac{A}{tt}=0,$$
$$(u-c)^2+\frac{A}{t^3}=0,$$
usque ad
$$(u-c)^2+\frac{A}{t^{n-2}}=0,$$

si omnes termini praeter ultimum constantem desint. Verum si etiam ultimus evanesceret, foret $(u-c)^2=0$ ideoque linea recta ipsa foret curvae portio curvaque adeo complexa.

207. Quanquam sic omnes casus, quos duo factores aequales praebeant, enumerati videntur, tamen ultima aequatio alias adhuc induere potest formas, unde diversae asymptotae sequuntur. Evenit hoc, si factor potestatis t^{n-3} per $u-c$ divisibilis deprehendatur; tum enim, uti in primo termino, relinquatur $u-c$ ac adiiciatur insuper terminus sequens, qui proxime adest, hocque casu eiusmodi emergent aequationes:

$$(u-c)^2+\frac{A(u-c)}{t}+\frac{B}{tt}=0,$$
$$(u-c)^2+\frac{A(u-c)}{t}+\frac{B}{t^3}=0,$$
usque ad
$$(u-c)^2+\frac{A(u-c)}{t}+\frac{B}{t^{n-2}}=0.$$

Sin autem secundus terminus penitus desit vel per $(u-c)^2$ divisibilis fuerit, tum spectetur terminus tertius, qui si per $u-c$ divisibilis deprehendatur, in eo $u-c$ relinquatur atque praeterea sequens proximus terminus adiungatur. Hocque casu eiusmodi orientur aequationes:

$$(u-c)^2+\frac{A(u-c)}{tt}+\frac{B}{t^3}=0,$$
$$(u-c)^2+\frac{A(u-c)}{tt}+\frac{B}{t^4}=0,$$
usque ad
$$(u-c)^2+\frac{A(u-c)}{tt}+\frac{B}{t^{n-2}}=0.$$

Quodsi etiam tertius terminus desit et quartus per $u-c$ divisibilis reperiatur vel etiam hoc deficiente quintus et ita porro, nascetur huiusmodi aequatio pro curva asymptota,

$$(u-c)^2+\frac{A(u-c)}{t^p}+\frac{B}{t^q}=0,$$

ubi exponens *p* semper minor erit quam *q* et *q* minor quam $n-1$.

208. Ponamus $u-c=z$, atque hae aequationes omnes in hac forma

$$z^2+\frac{Az}{t^p}+\frac{B}{t^q}=0$$

continentur. Ad quam evolvendam tres casus sunt spectandi, prout fuerit *q* maior quam *2p* vel *q* aequalis *2p* vel *q* minor quam *2p*.

Casu primo, quo *q* superat *2p,* duae aequationes in illa continentur,

$$z-\frac{A}{t^p}=0 \text{ et } Az-\frac{B}{t^{q-p}}=0;$$

utraque enim facto $t=\infty$ satisfacit. Nam posito $z=\frac{A}{t^p}$ aequatio superior abit in

$$\frac{AA}{t^{2p}}-\frac{AA}{t^{2p}}+\frac{B}{t^q} \text{ seu } AA-AA+\frac{B}{t^{q-2p}}=0,$$

quod ob *q* maiorem quam 2*p* verum est, erit autem *p* minor quam $\frac{n-2}{2}$.

At si $z=\frac{B}{At^{q-p}}$, fiet

$$\frac{BB}{AAt^{2q-2p}}-\frac{B}{t^q}+\frac{B}{t^q} \text{ seu } \frac{BB}{AAt^{q-2p}}-B+B=0,$$

quod verum est ob terminum primum evanescentem facto $t=\infty$. Hoc ergo casu super eadem asymptota recta duae habentur asymptotae curvilineae ideoque quatuor rami in infinitum excurrentes.

Secundus casus, quo $q=2p$, praebet aequationem

$$zz-\frac{Az}{t^p}+\frac{B}{t^{2p}}=0,$$

quae vel est imaginaria, si AA minor quam $4B$, quo casu nulla asymptota extat, vel duas praebet asymptotas similes $z=\frac{C}{t^p}$, si AA maior quam $4B$.

In tertio casu, si q minor quam $2p$, aequationis medius terminus semper evanescit posito $t=\infty$; eritque ergo $zz+\frac{B}{t^q}=0$ aequatio pro una asymptota. Formas quidem praecedentium asymptotarum iam exposuimus, quare istas asymptotas hac forma $zz=\frac{C}{t^k}$ contentas examinemus.

209. Si igitur axis in ipsa asymptota recta $u=c$ sumatur et applicata $u-\mathrm{c}$ ponatur $=z$, omnes illae asymptotae curvilineae continebuntur in hac aequatione $zz=\frac{C}{t^k}$, denotante k

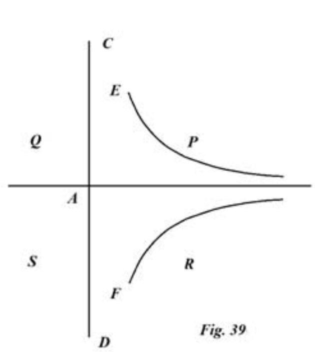


Fig. 39

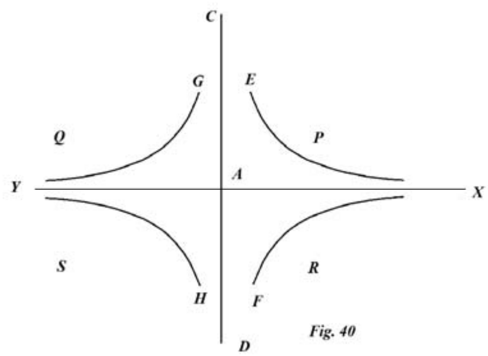


Fig. 40

numerum integrum minorem quam $n-1$. Harum autem curvarum rami in infinitum excurrentes seu facto $t=\infty$ ita se habebunt. Si $k=1$ seu $zz=\frac{C}{t}$, quia t negativum fieri nequit, curva (Fig. 39) duos habebit ramos EX et FX in regionibus P et R in infinitum excurrentes ; quod idem eveniet, si fuerit k numerus quicunque impar. At, si sit k numerus par ut 2 seu $zz=\frac{C}{tt}$, primum dispiciendum est, utrum C sit quantitas negativa an affirmativa. Priori casu aequatio realis esse nequit ideoque curva hinc nullum habebit ramum in infinitum extensum. Posteriori casu curva (Fig. 40) quatuor habebit ramos in infinitum excurrentes et cum asymptota XY concurrentes, scilicet EX, FX, GY et BY in omnibus quatuor regionibus P, R, Q et S dispersos.

210. Ponamus supremum membrum aequationis *P* habere tres factores aequales, atque aequatione ad coordinatas *t* et *u* reducta, ut sit *u* iste factor triplex ipsius *P*, erit

$$\begin{aligned} P &= +\alpha t^{n-3}u^3 + \alpha t^{n-4} u^4 + \text{ etc.}, \\ Q &= \beta t^{n-1} + \beta t^{n-2}u + \beta t^{n-3}uu + \beta t^{n-4}u^3 + \beta t^{n-5}u^4 + \text{ etc.}, \\ R &= \gamma t^{n-2} + \gamma t^{n-3}u + \gamma t^{n-4}uu + \gamma t^{n-5}u^3 + \gamma t^{n-6}u^4 + \text{ etc.}, \\ S &= \delta t^{n-3} + \delta t^{n-4}u + \delta t^{n-5}uu + \delta t^{n-6}u^3 + \delta t^{n-7}u^4 + \text{ etc.} \\ &\text{etc.} \end{aligned}$$

Hinc pro diversis constitutionibus membrorum *Q* et *R* sequentes oriuntur aequationes:

I.
$$\alpha t^{n-3}u^3 + \beta t^{n-1} = 0,$$
II.
$$\alpha t^{n-3}u^3 + \beta t^{n-2}u + \gamma t^{n-2} = 0,$$
III.
$$\alpha t^{n-3}u^3 + \beta t^{n-3}uu + \gamma t^{n-2} = 0,$$
IV.
$$\alpha t^{n-3}u^3 + \beta t^{n-3}uu + \gamma t^{n-3}u + \delta t^{n-3} = 0.$$

211. Prima aequatio abit in $\alpha u^3 + \beta tt = 0$ ideoque haec asymptota est linea tertii ordinis (Fig. 41), cuius talis erit figura, si abscissae *t* super axe *SY* a puncto *A*. sumantur. Duos scilicet habebit ramos *E* et *F* in regionibus *P* et *Q* in infinitum excurrentes.

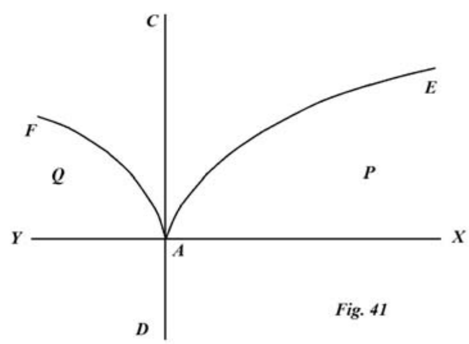


Fig. 41

Secunda aequatio ita se habet $\alpha u^3 + \beta tu + \gamma t = 0$. Ex qua *u* posito $t = \infty$ duplicem valorem habere potest, vel finitum vel infinitum, ideoque in has duas aequationes resolvitur $\beta u + \gamma = 0$ et $\alpha uu + \beta t = 0$; posterior est pro parabola, uti ante vidimus, ac propterea curva habebit duos ramos in infinitum extensos ad parabolam appropinquantes. Prior vero aequatio praebeat $u - c = 0$, quae est pro linea recta asymptota, cuius indoles perspicietur, si, praeterquam in $\beta u + \gamma = u - c$, ubique loco *u* scribatur *c;* eritque ergo

$$t^{n-2}(u-c) + t^{n-3}(\alpha c^3 + \beta cc + \gamma c + \delta) + t^{n-4}(\alpha c^4 + \beta c^3 + \gamma cc + \delta c + \varepsilon) + \text{ etc.} = 0;$$

unde, uti supra, sequitur fore

$$\text{vel } (u-c)+\frac{A}{t}=0 \quad \text{vel } (u-c)+\frac{A}{tt}=0 \text{ etc.}$$

Ultima vero aequatio, quae oriri potest, est $(u-c)+\frac{A}{t^{n-2}}=0$. Hoc ergo casu curva duplicem habebit asymptotam, alteram rectam indolis hic declaratae, alteram vero parabolam coniunctim.

212. Tertia aequatio $\alpha u^3+\beta uu+\gamma t=0$ posito $t=\infty$ subsistere nequit, nisi sit $u=\infty$; ideoque terminus βuu prae αu^3 evanescit proditque ista aequatio tertii ordinis $\alpha u^3+\gamma t=0$ pro asymptota, cuius haec est figura (Fig. 42), ut in regionibus oppositis *P* et *S* duos habeat ramos *AE* et *AF* in infinitum excurrentes.

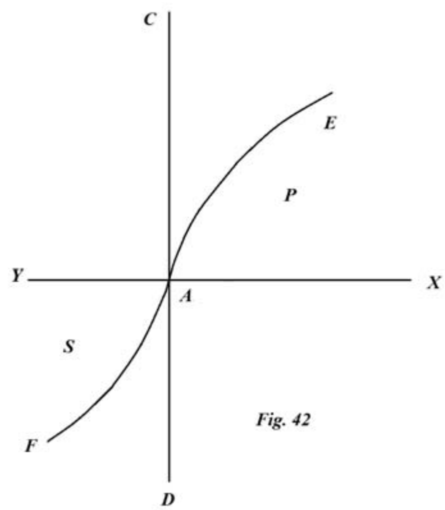


Fig. 42

Quarta aequatio autem $\alpha u^3+\beta uu+\gamma u+\delta=0$ vel unam vel tres asymptotas rectas inter se parallelas exhibet, nisi duae vel omnes inter se sint aequales, ad quarum indolem indagandam sit primum $u=c$ radix aequationis una aliam sui similem non habens sitque

$$\alpha u^3+\beta uu+\gamma u+\delta=(u-c)\ (fuu+gu+h).$$

Ponatur ubique $u=c$, praeterquam in hoc factore $u-c$, ac prodibit huiusmodi aequatio

$$t^{n-3}(u-c)+At^{n-4}+Bt^{n-5}+Ct^{n-6}+\text{etc.}=0;$$

unde asymptota orietur formae $u-c=\frac{K}{t^k}$ existente *k* numero minore quam $n-2$.

213. Si aequationis $\alpha u^3+\beta uu+\gamma u+\delta=0$ duae radices fuerint aequales, ita ut ea expressio sit $=(u-c)^2(fu+g)$; atque statuendo $u=c$, nisi in quopiam membro fuerit factor $u-c$, ad huiusmodi aequationem pervenietur

$$(u-c)^2+\frac{A(u-c)}{t^p}+\frac{B}{t^q}=0;$$

ubi erit *q* minor quam $n-2$ et *p* minor quam *q*, quem casum ante evolvimus.

Superest ergo casus, quo aequatio $\alpha u^3+\beta uu+\gamma u+\delta=0$ tres habet radices aequales, puta $(u-c)^3$, atque huiusmodi obtinebitur aequatio

$$(u-c)^3 t^{n-3}+Pt^{n-4}+Qt^{n-5}+Rt^{n-6}+St^{n-7}+\text{ etc.}=0\,.$$

Quodsi P non fuerit divisibile per $u-c$, ponatur $u=c$ fietque

$$\left(u-c\right)^3+\frac{A}{t}=0\,.$$

Sin autem P divisorem habeat $u-c$ semel, ponatur ubique, praeterquam in hoc factore, $u=c$ atque orietur aequatio huius formae

$$\left(u-c\right)^3+\frac{A\left(u-c\right)}{t}+\frac{B}{t^q}=0\,,$$

existente q numero minore quam $n-2$; est vero $\frac{B}{t^q}$ terminus secundum proxime sequens, qui non evanescit facto $u=c$. Sin P adeo per $(u-c)^2$ fuerit divisibilis, Q vero non habeat factorem $u-c$, orietur aequatio huius formae

$$\left(u-c\right)^3+\frac{A\left(u-c\right)^2}{t}+\frac{B}{tt}=0\,.$$

Quodsi autem secundus adeo per $(u-c)^3$ fuerit divisibilis, tum ordine procedendum est, donec ad terminum perveniatur non divisibilem per $(u-c)^3$, qui si fuerit divisibilis per $u-c$, ulterius est progrediendum, donec ad terminum non divisibilem per $u-c$ perveniatur. Sin autem ille terminus per $(u-c)^2$ divisibilis fuerit, procedatur ulterius, donec perveniatur ad terminum vel non divisibilem per $u-c$ vel divisibilem. Priori casu aequatio terminetur, posteriori ulterius pergatur, donec ad terminum non divisibilem per $u-c$ perveniatur. Sic itaque obtinebitur semper aequatio in hac forma generali contenta

$$\left(u-c\right)^3+\frac{A\left(u-c\right)^2}{t^p}+\frac{B\left(u-c\right)}{t^q}+\frac{C}{t^r}=0\,,$$

ubi erit r minor quam $n-2$, q minor quam r et p minor quam q.

214. In hac aequatione vel tres continentur aequationes formae $u-c=\frac{K}{t^k}$, vel una huiusmodi et una $(u-c)^2=\frac{K}{t^k}$, vel unica tantum formae $(u-c)^3=\frac{K}{t^k}$: quod postremum evenit, si fuerit et *3p* maior quam *r* et *3q* maior quam *2r.* Tum vero etiam fieri potest, ut duae aequationes fiant imaginariae, quae ergo nullam asymptotam indicabunt. Ceterum formas harum asymptotarum iam explicavimus praeter ultimam aequatione $(u-c)^3=\frac{K}{t^k}$ contentam. Praebet autem ista aequatio, si *k* sit numerus impar, formam figura trigesima sexta designatam cum duobus ramis *EX* et *FY* in regionibus oppositis *P* et *S* in infinitum excurrentibus. Sin autem *k* sit numerus par, orietur forma figura trigesima septima repraesentata, in qua sunt duo rami *EX* et *FY* ad eandem asymptotae rectae *XY* partem seu in regionibus *P* et *Q* excurrentes.

215. Quoniam ex his facile perspicitur, quemadmodum asymptotarum forma, si quatuor pluresve factores simplices in membro aequationis supremo fuerint aequales, investigari debeat, ulterius hic non progredior; verum hoc caput applicatione regularum datarum ad unum exemplum finiam.

EXEMPLUM

Sit igitur proposita linea curva hac aequatione expressa

$$y^3xx(y-x)-xy(yy+xx)+1=0,$$

cuius supremum membrum $y^3xx(y-x)$ *unum factorem habet solitarium* $y-x$, *duos aequales xx et insuper tres aequales* y^3.

Consideremus primum factorem simplicem $y-x$; ex quo, posito $y=x$, fiet $y-x-\frac{2}{x}=0$; et ob $x=\infty$ erit *y* - *x* = 0, quae est aequatio (Fig. 43) pro asymptotarectilinea *BAC* cum axe *XY* in initio abscissarum faciens angulum semirectum *BAY.* Ad hanc lineam transferatur tanquam ad axem aequatio, quod fiet ponendo $y=\frac{u+t}{\sqrt{2}}$ et $x=\frac{t-u}{\sqrt{2}}$; quo facto orietur haec aequatio

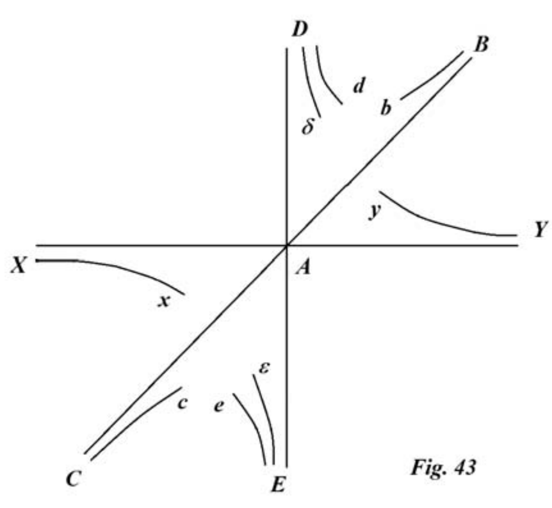


Fig. 43

$$\frac{(u+t)(tt-uu)^2u}{4}-\frac{(tt-uu)(tt+uu)}{2}+1=0\,;$$

unde, per 4 multiplicando, fiet

$$\begin{aligned}0=\;& t^5u+t^4uu-2t^3u^3-2ttu^4+tu^5+u^6\;;\\ & -2t^4 \qquad\qquad\qquad\qquad +\;2u^4\\ & +4\end{aligned}$$

ex hac aequatione, facto $t=\infty$, invenitur $u=0$; ideo que reliqui termini praeter hos duos $t^5u, -2t^4$ evanescunt; unde pro asymptota curvilinea erit $u=\frac{2}{t}$. Ob hunc ergo factorem curva quaesita duos habebit ramos $bB,\ Cc$ in infinitum excurrentes.

216. Sumantur nunc factores aequales gemini *xx* eritque

$$xx=\frac{xy(yy+xx)-1}{y^3(y-x)}.$$

Axe ergo sumto recta *AD* ad priorem *XY* normali fiet $y=t$ et $x=u$, pro quo ista aequatio resultat

$$\begin{aligned}0=\;& t^4uu-t^3u^3\\ & -t^3u \qquad -tu^3\\ & +1\end{aligned}$$

quae facto *t* infinito abit in $t^4u^2-t^3u+1=0$, unde duae nascuntur aequationes

$$u=\frac{1}{t}\text{ et }u=\frac{1}{t^3}.$$

Quare hic factor quatuor praebet ramos in infinitum excurrentes, primo nempe duos *dD, eE* ex aequatione $u=\frac{1}{t}$, et duos ad easdem partes sitos δD et εE ex aequatione $u=\frac{1}{t^3}$.

217. Tres factores aequales y^3 referuntur ad ipsum axem *XY* fietque $x = t$ et $y = u$, unde nascitur aequatio haec

$$0 = -t^3u^3 + ttu^4 - t^3u - tu^3 + 1,$$

quae posito *t* infinito dat $t^3u^3 + t^3u = 0$ seu $u(uu+1) = 0$; unde, ob $uu+1=0$ aequationem impossibilem, unica obtinetur asymptota recta $u = 0$, conveniens cum ipso axe *XY*, cuius indoles exprimetur hac aequatione $t^3u = 1$ seu $u = \frac{1}{t^3}$; ac propterea iste factor triplex duos tantum praebet ramos *yY* et *xX* in infinitum excurrentes. Omnino ergo curva quaesita octo ramos in infinitum extensos habebit, qui quomodo in spatio finito inter se coniungantur, huius non est loci explicare.

218. Ex hoc ergo et praecedente capite ramorum in infinitum extensorum varietas luculenter perspicitur. Primum enim hi rami curvarum vel ad lineam quampiam rectam tanquam asymptotam convergunt, uti fit in hyperbola, vel asymptotam rectam non habent, uti parabola. Priori casu rami curvarum vocantur *hyperbolici,* posteriori *parabolici.* Utriusque classis innumerabiles dantur species; ramorum enim hyperbolicorum species his exprimuntur aequationibus inter coordinatas *t* et *u* , quarum illa *t* statuitur infinita:

$$u = \frac{A}{t}, \quad u = \frac{A}{tt}, \quad u = \frac{A}{t^3}, \quad u = \frac{A}{t^4}, \quad \text{etc.,}$$
$$uu = \frac{A}{t}, \quad uu = \frac{A}{tt}, \quad uu = \frac{A}{t^3}, \quad uu = \frac{A}{t^4}, \quad \text{etc.}$$
$$u^3 = \frac{A}{t}, \quad u^3 = \frac{A}{tt}, \quad u^3 = \frac{A}{t^3}, \quad u^4 = \frac{A}{t^4}, \quad \text{etc.}$$
etc.

Ramorum vero parabolicorum species indicantur sequentibus aequationibus:

$$uu = At \quad u^3 = At \quad u^4 = At \quad u^5 = At \quad \text{etc.}$$
$$u^3 = Att \quad u^4 = Att \quad u^5 = Att \quad u^6 = Att \quad \text{etc.}$$
$$u^4 = At^3 \quad u^5 = At^3 \quad u^6 = At^3 \quad u^7 = At^3 \quad \text{etc.}$$
etc.

Quaelibet autem aequatio harum expositarum ad minimum duos exhibit ramos in infinitum excurrentes, si exponentium ipsarum *t* et *u* non uterque fuerit numerus par; sin autem uterque exponens fuerit numerus par, tum vel nullum ramum infinitum praebet vel quatuor; illud scilicet evenit, si aequatio sit impossibilis, hoc vero, si sit realis.