CHAPTER IV

CONCERNING THE PARTICULAR PROPERTIES OF THE LINES OF EACH ORDER

66. Among the particular properties of the lines of each order, of these the first place is held by the meeting with a right line or with the number of intersections, which a right line is able to make with the lines of each order. For since a line of the first order or a right line shall be cut by another right line only at a single point, moreover curved lines may be able to be cut at several points by a right line, therefore deservedly it is customary to ask, in how many points a curved line of each order shall be able to be cut by some drawn right line; for from that question, the nature of the curved lines relating to the various orders may be understood better. But it will be found that lines of the second order will not be able to be cut by a straight line in more than two points ; moreover lines of the third order cannot be cut by a right line in more then three points, and so on thus.

67. Now above we have made mention of the manner, by which it can be determined, into what number of points the axis of each curve may be cut by the curve itself. Indeed from the given equation between the abscissa x and the applied line y, because, where a point of the curve falls on the axis, there the applied line y becomes $=0$, $y=0$ may be put into the equation, and the resulting equation, which will contain x only, will show the values of x and hence the points of the axis, where the curve will cut that itself. Thus in the equation for the circle, which we have found above, $yy=2ax-xx$ if we may put $y=0$, there becomes $0=2ax-xx$, from which the two resulting values of x, $x=0$ and $x=2a$, which indicate that the axis RS is cut by the circle at the starting point itself of the abscissas A, then truly at the point B, by the line present $AB=2a$. And in a similar manner with other curved lines by putting $y=0$ into the equation, the roots of x will indicate the intersections of the curve with the axis.

68. Because in the general equation for some curved line some right line in turn supports the axis, if in the general equation the applied line $y=0$, the remaining equation will indicate in how many points the curved line may be cut by some right line. Moreover it will produce an equation including the abscissa x alone as the unknown, the individual roots of which will show the intersections of the curve with the axis. Therefore the number of intersections will depend on the maximum power of x in the equation and hence cannot be greater than the maximum exponent of the power of x. Indeed there will be just as many intersections, as the maximum power of x will contain units, if all the roots of the equation were real, but if some roots of the equation were imaginary, the number of intersections shall be with a smaller number.

69. Therefore since for any order of lines we have shown the most general equations, from these set out in this manner we will be able to find, in how many points lines of each order may be able to be cut by some right line. Therefore we may take the equation

$0 = \alpha + \beta x + \gamma y$ for lines of the first order or for a general right line, from which on putting $y = 0$ it becomes $0 = \alpha + \beta x$, which equation cannot have more than a single root, from which it is apparent the right line can be cut by another right line at a single point. But if there shall be $\beta = 0$, the equation of the impossible $0 = \alpha$ indicates in this case an axis to be cut at no point by the right line, for both these lines will be parallel to each other, as appears from the equation $0 = \alpha + \gamma y$, which arises if $\beta = 0$.

70. If we may put $y = 0$ into the general equation for lines of the second order

$$0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy,$$

this equation will be produced :

$$0 = \alpha + \beta x + \delta xx,$$

which equation either has two real roots or no real roots, and also a single one, if $\delta = 0$. Hence a line of the second order will be cut by a right line in two points, or in one point, or nowhere. Which cases all are understood thus in one, so that we may say a line of the second order cannot be cut by a right line in more than two points.

71. If we may put $y = 0$ in the general equation of the third order, an equation of this kind will be produced :

$$0 = \alpha + \beta x + \gamma xx + \delta x^3,$$

which since it cannot have more than three roots, it is seen that lines of the third order cannot be cut by a right line at more than three points. Truly it can happen, that a line of the third order may be cut by a right line in fewer points, evidently either in two points, if $\delta = 0$ and both the roots of the equation $0 = \alpha + \beta x + \gamma xx$ were real, or in one point, if the two roots of the above equation were imaginary or if there shall be both $\delta = 0$ and $\gamma = 0$, or also nowhere, if $\delta = 0$ and both the roots of the remaining equation were imaginary, which likewise happens, if β, γ and δ vanish, but α were a quantity not equal to zero.

72. In a similar manner lines it may be deduced lines of the fourth order cannot be cut by a right line in more than four points ; and this property thus may be extended to all orders of lines, so that lines of order *n* may not be cut by a right line in more than *n* points. Truly hence neither does it follow that a whole line of order *n* be cut by some right line into *n* points, but certainly it can happen, that the number of points of intersection may be smaller, indeed from that towards zero, as we have noted with lines of the second and third orders. Yet the strength of this proposition has been established therefore, that a

greater number of intersections never can be greater, than the exponent of the order, to which the curved line is referred.

73. Therefore from the number of intersections, that some right line makes with a given curved line, the order to which the curved line may be related, cannot be defined. For if the number of intersections shall be $= n$, it does not follow that the curve belongs to order *n* of lines, but will be able to be referred equally to some higher order : there is no reason also why it cannot happen, that the curve shall not indeed be algebraic but transcending. But by being excluded with risk it can be affirmed t always hat a curved lines , which may be cut by a right line in *n* points, can be related to a curved line of lesser order. Thus, if a proposed curved line may be cut in four points, it is certain that neither can that be referred to a line of the second or third order ; but hence it cannot be judged, whether it may be contained in the fourth order or some higher order, or whether it shall be transcending.

74. The general equations, which we have shown for lines of each order, contain several arbitrary constant quantities, from which if determined values are to be attributed, the curved lines will be determined completely and thus may be described to a given axis, so that all the remaining curved lines may be excluded, which certainly may be contained in the same general equation. Thus, whatever right line may be contained alone in the equation of the first order $0 = \alpha + \beta x + \gamma y$, yet its position with respect to the axis can be varied in an infinite number of ways for the infinitely different values of the constant quantities α, β, γ. But with these constant quantities being granted defined values to the highest degree, the position of the right line may be determined, so that besides this no other equation may be satisfied.

75. Therefore this equation $0 = \alpha + \beta x + \gamma y$ may be able to be seen to admit three determined values, on account of the three arbitrary constants $\alpha, \beta,$ and γ. Truly from the nature of the equation it is understood now to be determined, if only the ratio between these constants may be defined, evidently the ratio of two to one ; from which that equation allows only two determinations. For if β and γ by α thus may be determined, so that there shall be $\beta = -\alpha$ and $\gamma = 2\alpha$, the equation $0 = \alpha - \alpha x + 2\alpha y$, because α leaves on division, now in short will be determined. By a like account the general equation for lines of the second order, which contain six arbitrary constants, allow only five determinations, the general equation of the third order nine, and generally for lines of order n, $\frac{(n+1)(n+2)}{2} - 1$ determinations may be apparent.

76. But these arbitrary constants thus can be defined always, so that the curved line may pass through a given point, and in this way may arise be a single determination. For let the general equation be proposed for some order of lines, which thus must be defined, so that the curved line (see Fig. 17) may pass through the point *B*. With the axis taken as it pleases with the start of the abscissas on that at *A*, a perpendicular *Bb* may be sent from the point *B* to the axis, and it is clear, if the curve may pass through the point *B*, then on putting the interval *Ab* for x the perpendicular *Bb* provides the value of the applied line *y*. Whereby in the general equation proposed in place of *x* there may be substituted *Ab* and *Bb* in place of *y*, and thus an equation will be generated, from which one of the constant quantities $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. will be able to be defined ; with which done all the curves, which may be contained in the general equation determined in this way, will pass through the give point *B*.

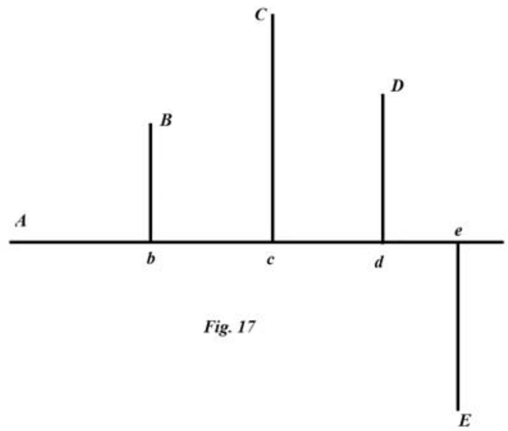

Fig. 17

77. If the above curved line must pass through the point C, by sending the perpendicular *Cc* from that to the axis and in the equation on putting $x = Ac$ and $y = Cc$ a new equation will emerge, from which equally one of the constants may be defined from $\alpha, \beta, \gamma, \delta$ etc. It is understood in the same manner, if the three points *B,C, D* may be prescribed, through which the curved line must pass, thence three constants are defined ; and moreover from four points *B, C, D, E* four constant letters are taken to be defined. But if as many points, through which the line may pass, therefore may be proposed, as the general equation will admit, then the curved line will be determined completely, and thus the only one, which indeed may pass through all the proposed points.

78. Therefore since the general equation for lines of the first order or for right lines admits only two determinations, with two points proposed, through which a line of the first order or a right line may pass, the right line may be determined completely ; nor will more than one right line be able to be drawn through two given points, which indeed is understood from [Euclid's] elements. But if only a single point may be proposed, then, on account of the equation not yet being determined, at this stage an infinite number of right lines can be drawn through the point.

79. The general equation for lines of the second order admit five determinations ; from which, if five points may be proposed, through which the given curved line must be drawn, a line of the second order may be determined completely. On this account a single line of the second order can be drawn through five given points ; but if only four or fewer points may be proposed, because from these the equation is not yet completely

determined, innumerable lines, which shall all be of the second order, will be able to be drawn through these points. But if moreover of these five points, three of these may lie in a direction, because a line of the second order cannot be cut by a right line in three points, no continued curved line will be found, but a complex line will be produced, evidently of two right lines, which, as we have now warned, may be contained in the general equation of the second order.

80. Because again the general equation for lines of the third order grant nine determinations, through the nine points taken freely a line of the third order will always be able to be drawn and uniquely. But if the number of points were smaller, then through these innumerable lines of the third order will be able to be drawn. In a similar manner a unique line of the fourth order can be drawn through fourteen points, through twenty points a unique line of the fifth order can be drawn, and thus henceforth. And thus in general the lines of order n will be determined by as many points, as this formula contains units :

$$\frac{(n+1)(n+2)}{2}-1 = \frac{n(n+3)}{2};$$

thus so that, if the number of given points were smaller, through that point innumerable lines of order n may be drawn.

81. Therefore unless more points than $=\frac{n(n+3)}{2}$ may be proposed, always one or an infinite number of lines of order n can be drawn through these points : clearly a single line, if the number of points given were $=\frac{n(n+3)}{2}$, and an infinite number, if it shall be less. But on no occasion, however these points may be arranged, will the solution emerge to be impossible ; for the determination of the coefficients $\alpha, \beta, \gamma, \delta$ etc. at no time will require the resolution of quadratic equations or equations of higher powers, but may be resolved wholly by simple equations. From which neither at any time will imaginary values be found for the quantities $\alpha, \beta, \gamma, \delta$ etc. nor multiform values, and for that reason real lines passing through the proposed points will be produced always ; and a unique line, if indeed just as many points may be proposed, as the general equation permits determinations.
[See E147 for an apparent exception.]

82. Because the axis can be assumed as it pleases, this determination of the coefficients becomes easier, if the axis is drawn through a single point of the given points and the start of the abscissas may be put in place at that point A itself; for thus on putting $x=0$ there will become $y=0$, from which in the proposed general equation :

$$0=\alpha+\beta x+\gamma y+\delta xx+\varepsilon xy+\zeta yy+\eta x^3+ \text{ etc.}$$

at once there will be $\alpha = 0$. Then the axis too will be able to pass through another of the given points, with which agreed upon the number of quantities, by which the position of the given points is defined, will be reduced. And then in place of orthogonal applied lines of this kind oblique angled lines can be chosen, so that the applied lines drawn from the start of the abscissas equally may pass through a given point. For the recognition and construction of a curve is deduced equally easily, whether orthogonal or oblique angled applied lines may be put in place.

83. If a line of the second order may be sought (see Fig. 18), which may pass through the five given points *A*, *B*, *C*, *D* and *E*, the axis may be drawn through the two points *A*, *B* and the start of the abscissas may be taken from the first point *A*. Then this point *A* may be joined with the third point *C* and the angle *CAB* may be taken for the obliquity of the applied lines. Whereby the applied lines *Dd* and *Ee* are drawn from the remaining points *D* and *E* to the axis, parallel to that line *AC*. Putting $AB = a$, $AC = b$, $Ad = c$, $Dd = d$, $Ae = e$ and $eE = f$; and with the general equation of lines of the second order taken :

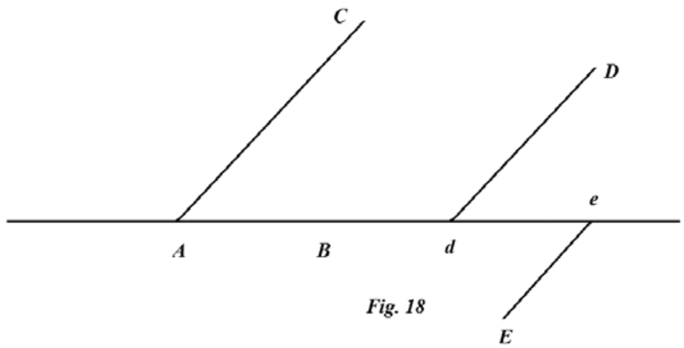

Fig. 18

$$0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy ,$$

it is clear,

on putting	there becomes
$x = 0$	$y = 0$,
$x = 0$	$y = b$,
$x = a$	$y = 0$,
$x = c$	$y = d$,
$x = e$	$y = f$.

Hence the five following equations will arise :

I. $0 = \alpha,$

II. $0 = \alpha + \gamma b + \zeta bb,$

III. $0 = \alpha + \beta a + \delta aa,$

IV. $0 = \alpha + \beta c + \gamma d + \delta cc + \varepsilon cd + \zeta dd,$

V. $0 = \alpha + \beta e + \gamma f + \delta ee + \varepsilon ef + \zeta ff .$

Therefore there will be $\alpha = 0,\ \gamma = -\zeta b,\ \beta = -\delta a$; which give the following values substituted in the remaining equations :

$$0 = -\delta ac - \zeta bd + \delta cc + \varepsilon cd + \zeta dd,$$
$$0 = -\delta ae - \zeta bf + \delta ee + \varepsilon ef + \zeta ff;$$

the upper may be multiplied by *ef* and the lower by *cd* and the one may be take from the other, so that ε may be eliminated, and there will emerge :

$$\begin{aligned} 0 = -\delta acef - \zeta bdef + \delta ccef + \zeta ddef \\ +\delta acde + \zeta bcdf - \delta cdee - \zeta cdff \end{aligned}$$

or

$$\frac{\delta}{\zeta} = \frac{bdef - bcdf - ddef + cdff}{acde - acef - cdee + ccef}$$

from which there becomes

$$\delta = df\left(be - bc - de + cf\right),$$
$$\zeta = ce\left(ad - af - de + cf\right)$$

and hence all the coefficients will be determined.

84. But with all the coefficients of the general equation $0 = \alpha + \beta x + \gamma y + \delta xx + \text{etc.}$ determined in this manner, with the above axis assumed and with the putting in place of oblique angled applied lines, a curved line may be described through endless points by the equation found and this curved line will pass through all the points proposed. If the general equation may allow more determinations than there were points proposed, then with the remaining points taken as it pleases a curved line will be described completely, with the aid of the described equation. Moreover the *x* abscissas may be granted successively more values both positive as well as negative, as $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$ etc. and $-1, -2, -3, -4$ etc., and for the individual points, the agreeing applied values *y* are found from the equation and thus many satisfying points in the vicinity become known, through which the curve will pass, from which the course of the curve will be seen readily in a straight forwards manner.

CAPUT IV

DE LINEARUM CUIUSQUE ORDINIS PRAECIPUIS PROPRIETATIBUS

66. Inter praecipuas proprietates linearum cuiusque ordinis primum locum tenet earum concursus cum linea recta seu intersectionum multitudo, quas linea recta cum lineis cuiusque ordinis facere potest. Cum enim linea primi ordinis seu recta ab alia linea recta nonnisi in unico puncto secari possit, lineae curvae autem in pluribus punctis a linea recta secari queant, merito ergo quaeri solet, in quot punctis linea curva cuiusque ordinis secari possit a linea recta utcunque ducta; ex ipsa enim hac quaestione natura linearum curvarum ad varios ordines pertinentium melius cognoscetur. Reperietur autem linea secundi ordinis a recta in pluribus quam duobus punctis secari non posse; linea autem tertii ordinis a recta in pluribus quam tribus punctis secari nequit et ita porro.

67. Supra iam mentionem fecimus modi, quo determinari potest, in quot punctis axis cuiusque curvae ab ipsa curva secetur. Data enim aequatione inter abscissam x et applicatam y, quia, ubi curvae punctum in axem incidit, ibi applicata y fit $=0$, ponatur in aequatione $y=0$, atque aequatio resultans, quae tantum x continebit, monstrabit valores ipsius x hincque axis puncta, ubi curva ipsum secabit. Ita in aequatione pro circulo, quam supra invenimus, $yy=2ax-xx$ si ponamus $y=0$, fit $0=2ax-xx$, unde duo valores ipsius x resultant, $x=0$ et $x=2a$, qui indicant axem RS primo in ipso abscissarum initio A, tum vero in puncto B, existente $AB=2a$, a circulo intersecari. Similique modo in aliis lineis curvis posito in aequatione $y=0$, radices ipsius x indicabunt intersectiones curvae cum axe.

68. Quoniam in aequatione generali pro quavis curva linea recta quaecunque vicem axis sustinet, si in aequatione generali ponatur applicata $y=0$, aequatio remanens indicabit, in quot punctis linea curva a recta quacunque traiiciatur. Prodibit autem aequatio abscissam solam x tanquam incognitam complectens, cuius singulae radices ostendent intersectiones curvae cum axe. Pendebit ergo intersectionum numerus a maxima ipsius x in aequatione potestate hincque maior esse non poterit quam exponens summae ipsius x potestatis. Tot vero erunt intersectiones, quot exponens maximae potestatis ipsius x continet unitates, si omnes radices aequationis fuerint reales, sin autem aliquot radices fuerint imaginariae, intersectionum numerus tanto erit minor.

69. Cum igitur pro quovis linearum ordine aequationes generalissimas exhibuerimus, ex iis modo exposito invenire poterimus, in quot punctis lineae cuiusque ordinis a recta quacunque secari queant. Sumamus ergo aequationem pro lineis primi ordinis seu pro linea recta generalem $0=\alpha+\beta x+\gamma y$, ex qua posito $y=0$ fit $0=\alpha+\beta x$, quae aequatio plus una radice habere nequit, unde patet lineam rectam ab alia recta in unico puncto secari. Sin autem sit $\beta=0$, aequatio $0=\alpha$ impossibilis indicat hoc casu axem a linea recta nusquam secari, erunt enim ambae hae lineae rectae inter se parallelae, uti patet ex aequatione $0=\alpha+\gamma y$, quae oritur, si $\beta=0$.

70. Si in aequatione generali pro lineis secundi ordinis

$$0=\alpha+\beta x+\gamma y+\delta xx+\varepsilon xy+\zeta yy$$

ponamus $y=0$, prodibit haec aequatio

$$0=\alpha+\beta x+\delta xx,$$

quae aequatio vel duas habet radices reales vel nullam vel etiam unicam, si $\delta=0$. Hinc linea secundi ordinis a linea recta vel in duobus punctis secabitur vel in unico vel nusquam. Qui casus omnes sic in unum comprehendi possunt, ut dicamus lineam secundi ordinis a linea recta plusquam in duobus punctis secari non posse.

71. Si in aequatione generali pro lineis tertii ordinis ponamus $y=0$, prodibit huiusmodi aequatio

$$0=\alpha+\beta x+\gamma xx+\delta x^3,$$

quae cum plures tribus radicibus habere nequeat, perspicuum est lineas tertii ordinis a linea recta in pluribus quam tribus punctis secari non posse. Fieri vero potest, ut linea tertii ordinis a linea recta in paucioribus punctis secetur, nempe vel in duobus, si $\delta=0$ et aequationis $0=\alpha+\beta x+\gamma xx$ ambae radices fuerint reales, vel in unico, si superioris aequationis duae radices fuerint imaginariae aut si sit et $\delta=0$ et $\gamma=0$, vel etiam nusquam, si $\delta=0$ et reliquae aequationis ambae radices fuerint imaginariae, quod idem evenit, si β, γ et δ evanescant, at α fuerit quantitas non aequalis nihilo.

72. Simili modo colligetur lineas quarti ordinis a recta in pluribus quam quatuor punctis secari non posse; haecque proprietas ad omnes linearum ordines ita extendetur, ut lineae ordinis *n* a linea recta in pluribus quam *n* punctis secari nequeant. Neque vero hinc sequitur omnem lineam ordinis *n* a quavis linea recta in *n* punctis secari, sed utique fieri potest, ut numerus intersectionum sit minor, imo subinde prorsus nullus, uti de lineis secundi et tertii ordinis annotavimus. In hoc ergo tantum propositionis vis est posita,

quod intersectionum numerus maior nunquam esse possit, quam exponens ordinis, ad quem linea curva refertur.

73. Ex numero igitur intersectionum, quas linea recta quaecunque cum data linea curva facit, ordo, ad quem linea curva pertineat, definiri non poterit. Si enim intersectionum numerus sit $=n$, non sequitur curvam ad ordinem linearum n pertinere, sed ad quemvis ordinem superiorem aeque referri poterit: quin etiam fieri potest, ut curva ne quidem sit algebraica sed transcendens. Excludendo autem semper tuto affirmari potest lineam curvam, quae a recta in n punctis secetur, ad nullum linearum ordinem inferiorem pertinere posse. Sic, si proposita linea curva a recta in quatuor punctis secetur, certum est eam neque ad ordinem secundum neque tertium referri; utrum autem in ordine quarto aut superiori quopiam contineatur, an sit transcendens, hinc diiudicari non potest.

74. Aequationes generales, quas pro lineis cuiusque ordinis exhibuimus, plures continent quantitates constantes arbitrarias, quibus si valores determinati tribuantur, lineae curvae penitus determinabuntur atque ad datum axem ita describentur, ut reliquae lineae curvae omnes, quae quidem in eadem aequatione generali continebantur, excludantur. Ita, quamvis in aequatione primi ordinis $0=\alpha+\beta x+\gamma y$ sola linea recta contineatur, tamen eius positio respectu axis infinitis modis variari potest pro diversis infinitis valoribus quantitatum constantium $\alpha,\beta,\ \gamma$. Quamprimum autem his quantitatibus constantibus definiti valores tribuuntur, positio lineae rectae determinatur, ut praeter hanc nulla alia aequationi satisfacere queat.

75. Haec igitur aequatio $0=\alpha+\beta x+\gamma y$ tres determinationes admittere videri posset, ob tres constantes arbitrarias $\alpha,\beta,$ et γ. Verum ex natura aequationum intelligitur aequationem iam determinari, si tantum ratio inter has constantes definiatur, scilicet ratio binarum ad unam; ex quo ista aequatio duas tantum admittet determinationes. Si enim β et γ per α ita determinentur, ut sit $\beta=-\alpha$ et $\gamma=2\alpha$, aequatio $0=\alpha-\alpha x+2\alpha y$, quia α per divisionem exit, iam prorsus erit determinata. Similem ob rationem aequatio generalis pro lineis secundi ordinis, quae sex continet constantes arbitrarias, quinque tantum admittit determinationes, aequatio generalis pro lineis tertii ordinis novem et generaliter aequatio generalis pro lineis ordinis n patietur $\frac{(n+1)(n+2)}{2}-1$ determinationes.

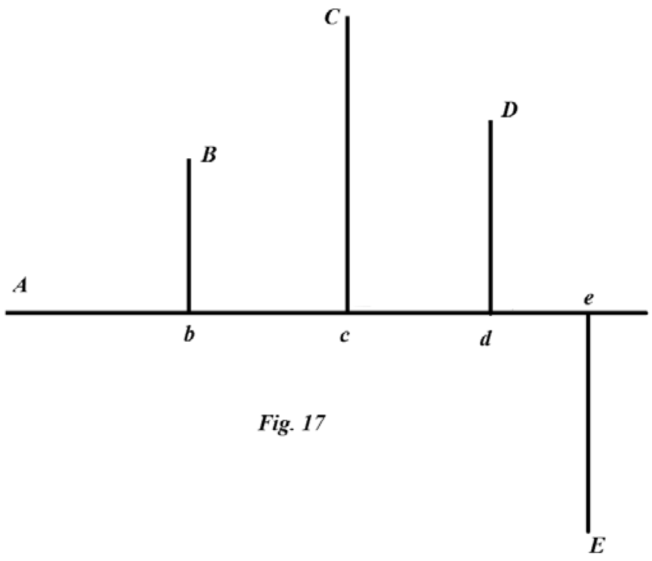

Fig. 17

76. Semper autem istae constantes arbitrariae ita definiri possunt, ut linea curva per datum punctum transeat, hocque modo una determinatio orietur. Sit enim proposita aequatio generalis pro quovis ordine linearum, quae ita definiri debeat, ut linea curva (Fig. 17) per datum punctum *B* transeat. Sumto pro lubitu axe in eoque abscissarum initio *A*, demittatur ex puncto *B* in axem perpendicularis *Bb*, atque manifestum est, si curva transeat per punctum *B*, tum posito intervallo *Ab* pro *x* perpendicularem *Bb* praebere valorem applicatae *y*. Quare in aequatione generali proposita loco *x* substituatur *Ab* et *Bb* loco *y*, sicque orietur aequatio, ex qua una quantitatum constantium $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. definiri poterit; quo facto omnes curvae, quae in aequatione generali hoc modo determinata continentur, per punctum datum *B* transibunt.

77. Si linea curva insuper per punctum C transire debeat, inde ad axem perpendiculo *Cc* demisso et in aequatione posito $x = Ac$ et $y = Cc$ nova orietur aequatio, ex qua pariter una ex quantitatibus constantibus $\alpha, \beta, \gamma, \delta$ etc. definietur. Eodem modo intelligitur, si tria puncta *B,C, D* praescribantur, per quae linea curva transire debeat, inde tres constantes definiri; ex quatuor autem punctis *B, C, D, E* quatuor litteras constantes determinationem accipere. Quodsi ergo tot puncta, per quae linea curva transeat, proponantur, quot determinationes aequatio generalis admittit, tum linea curva penitus erit determinata, ideo que unica, quae quidem per omnia puncta proposita transeat.

78. Cum igitur aequatio generalis pro lineis primi ordinis seu pro linea recta duas tantum determinationes admittat, propositis duobus punctis, per quae linea primi ordinis seu recta transeat, linea recta penitus determinatur; neque per duo puncta data plures quam una linea recta duci poterunt, quod quidem ex elementis intelligitur. Sin autem unum tantum proponeretur punctum, tum, ob aequationem nondum determinatam, adhuc infinitae lineae rectae per idem punctum duci possunt.

79. Aequatio generalis pro lineis secundi ordinis quinque admittit determinationes; unde, si quinque proponantur puncta, per quae linea curva transire debeat, linea secundi ordinis penitus determinatur. Hanc ob rem per quinque data puncta unica linea secundi ordinis duci potest; sin autem quatuor tantum vel pauciora puncta proponantur, quia iis

aequatio nondum penitus determinatur, innumerabiles lineae, quae omnes sint ordinis secundi, per ea duci poterunt. Quodsi autem quinque illorum punctorum tria in directum iaceant, quia linea secundi ordinis a recta in tribus punctis secari nequit, nulla linea curva continua reperietur, sed prodibit linea complexa, duae nempe lineae rectae, quae, uti iam monuimus, in aequatione generali secundi ordinis continentur.

80. Quia porro aequatio generalis pro lineis tertii ordinis novem determinationes admittit, per novem puncta pro libitu assumta linea tertii ordinis semper duci poterit atque unica. Sin autem numerus punctorum novenario fuerit minor, tum per ea innumerabiles lineae tertii ordinis duci poterunt. Simili modo per quatuordecim puncta data unica linea quarti ordinis, per viginti puncta unica linea quinti ordinis duci poterit et ita porro. Atque in genere lineae ordinis n determinabuntur per tot puncta, quot haec formula

$$\frac{(n+1)(n+2)}{2}-1 = \frac{n(n+3)}{2}$$

continet unitates; ita ut, si numerus punctorum datorum fuerit minor, per ea puncta innumerabiles lineae ordinis n duci queant.

81. Nisi ergo plura puncta quam $=\frac{n(n+3)}{2}$ proponantur, semper una vel infinitae lineae ordinis n per ea duci poterunt: unica scilicet, si numerus punctorum datorum fuerit $=\frac{n(n+3)}{2}$, et infinitae, si sit minor. Nunquam autem, utcunque haec puncta fuerint disposita, solutio evadet impossibilis; determinatio enim coefficientium $\alpha, \beta, \gamma, \delta$ etc. nunquam resolutionem aequationis quadraticae vel altioris potestatis requirit, sed tota per aequationes simplices absolvetur. Ex quo neque unquam valores imaginarii pro quantitatibus $\alpha, \beta, \gamma, \delta$ etc. reperientur neque valores multiformes; hancque ob causam semper linea realis per proposita puncta transiens prodibit; atque unica, siquidem tot puncta proponantur, quot determinationes aequatio generalis admittit.

82. Quoniam axis pro lubitu assumi potest, ista coefficientium determinatio facilior fiet, si axis per unum punctorum datorum ducatur atque initium abscissarum in ipso hoc puncto A statuatur; sic enim posito $x=0$ fieri debebit $y=0$, unde in aequatione generali proposita

$$0=\alpha+\beta x+\gamma y+\delta xx+\varepsilon xy+\zeta yy+\eta x^3+ \text{ etc.}$$

statim fit $\alpha=0$. Deinde axis quoque per aliud punctum datorum transire poterit, quo pacto numerus quantitatum, quibus positio punctorum datorum definitur, minuetur. Denique loco applicatarum orthogonalium eiusmodi obliquangulae eligi possunt, ut applicata in initio abscissarum ducta pariter per punctum datum transeat. Curvae enim cognitio et constructio ex aequatione aeque facile deducitur, sive applicatae orthogonales sive obliquangulae statuantur.

83. Si quaeratur (Fig. 18) linea secundi ordinis, quae per quinque data puncta *A*, *B*, *C*, *D* et *E* transeat, ducatur axis per duo puncta *A*, *B* sumaturque initium abscissarum in altero puncto *A*. Tum iungatur hoc punctum *A* cum tertio *C* sumaturque angulus *CAB* pro obliquitate applicatarum. Quare ex reliquis punctis *D* et *E* ad axem ducantur applicatae *Dd* et *Ee* illi *AC* parallelae, Ponatur $AB = a,\ AC = b,\ Ad = c,$ $Dd = d,\ Ae = e$ et $eE = f$;

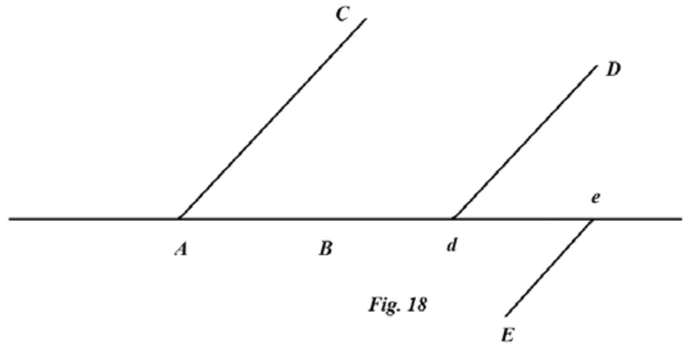

Fig. 18

atque sumta aequatione generali linearum secundi ordinis

$$0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy,$$

manifestum est,

posito	fore
$x = 0$	$y = 0$,
$x = 0$	$y = b$,
$x = a$	$y = 0$,
$x = c$	$y = d$,
$x = e$	$y = f$.

Hinc orientur sequentes quinque aequationes

I. $0 = \alpha,$
II. $0 = \alpha + \gamma b + \zeta bb,$
III. $0 = \alpha + \beta a + \delta aa,$
IV. $0 = \alpha + \beta c + \gamma d + \delta cc + \varepsilon cd + \zeta dd,$
V. $0 = \alpha + \beta e + \gamma f + \delta ee + \varepsilon ef + \zeta ff.$

Erit ergo $\alpha = 0,\ \gamma = -\zeta b,\ \beta = -\delta a$; qui valores in reliquis substituti dant

$$0 = -\delta ac - \zeta bd + \delta cc + \varepsilon cd + \zeta dd,$$
$$0 = -\delta ae - \zeta bf + \delta ee + \varepsilon ef + \zeta ff;$$

multiplicentur superior per *ef* et inferior per *cd* et altera ab altera subtrahatur, ut eliminetur ε , ac proveniet

$$
\begin{aligned}
0 = & -\delta acef - \zeta bdef + \delta ccef + \zeta ddef \\
& + \delta acde + \zeta bcdf - \delta cdee - \zeta cdff
\end{aligned}
$$

seu

$$\frac{\delta}{\zeta} = \frac{bdef - bcdf - ddef + cdff}{acde - acef - cdee + ccef}$$

unde fit

$$
\begin{aligned}
\delta &= df\left(be - bc - de + cf\right), \\
\zeta &= ce\left(ad - af - de + cf\right)
\end{aligned}
$$

hincque omnes coefficientes determinabuntur.

84. Determinatis autem hoc modo omnibus coefficientibus aequationis generalis $0 = \alpha + \beta x + \gamma y + \delta xx + \text{etc.}$, super axe assumto et sub constituta applicatarum obliquitate, linea curva describetur per puncta infinita per aequationem invenienda haecque linea curva transibit per omnia puncta proposita. Si aequatio generalis plures admittat determinationes, quam fuerint puncta proposita, tum reliquis pro lubitu assumtis linea curva per singula puncta data describetur ope aequationis omnino determinatae. Tribuuntur autem abscissae *x* successive plures valores tam affirmativi quam negativi, ut $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$ etc. et $-1, -2, -3, -4$ etc., ac pro singulis ex aequatione investigantur valores applicatae *y* convenientes sicque plurima innotescunt puncta satis vicina, per quae curva transibit, ex quibus proinde tractus curvae facile perspicietur.