CHAPTER III

CONCERNING THE DIVISION OF ALGEBRAIC CURVED LINES INTO ORDERS

47. Since the variety of curved lines and functions shall be equally infinite, it will not be possible to acquire an understanding of these in any way, unless the infinite multitude may be separated into certain classes, and in this manner one may be aided and directed in the careful examination of these. Now we have divided certain curved lines into *algebraic* and *transcending* classes, truly each class, on account of the infinite variation of the curves, has a need for further subdivision. But here we look only at *algebraic* curves, just as we may consider which most conveniently may be agreed to be distributed into classes. Therefore at first characteristics are to be defined, by which the variation of the classes may be determined, thus so that these curves which shall be endowed with the same character shall be in the same class, which otherwise are referred to different classes.

48. These various classes therefore requiring to be distinguished from each other cannot be found other than from the functions or equations, from which the nature of the curved lines may be represented ; since at this time, another way may not be apparent towards arriving at the nature of a curved line, because then if there is no other character which indeed may be given, all curves included under algebraic may be placed together. Indeed functions and equations between two coordinates are able to be distributed into different kinds in many ways, as we have made out in the above book. And in the first place the multiforms of functions present themselves, which may be seen to be more suitable than others for the distribution of curved lines into various classes ; so that these curved lines, which are generated by single form functions, may be referred to the first kind, those from two forms, to the second, which from three forms, to the third and so on thus.

[Thus, according to Euler at this point, the maximum number of times a vertical line cuts the curve, at some places on the abscissa axis of orthogonal coordinates, may be taken as a measure of its multiformity.]

49. But although this division may seem natural, yet, if it may be considered more carefully, the nature of each of the curved lines may be understood to be minimally of the innate character. For multiform functions depend mainly on the position of the axis, which is arbitrary, thus so that, if for one axis the applied line were a uniform [*i.e.* of one form] function of the abscissas, the same with another axis assumed may be able to be a multiform function ; therefore in this manner the same curved line may occur in different kinds, which is against the principles established. For thus the curved line expressed by this equation $a^3y = aaxx - x^4$ may relate to the first kind, because the applied line y is a function of one form of x; truly with the coordinates interchanged, or with the axis taken

normal to the first, the same curve may be expressed by the equation $y^4 - aayy + a^3 x = 0$ and thus belongs to the fourth order. Therefore because of this reason the multiformity of functions according to the character, by which curved lines may be distributed into classes, is unable to be allowed in the establishment of classes.

50. Equally the nature of simple pairs of equations expressing curved lines, the numbers in the ratio of the terms, may constitute a distinguishing characteristic. For if these curves may be referred to the first kind, of which the equation may depend on two terms, such as $y^m = \alpha x^n$, to the second, of which the equation may contain three terms, as $\alpha y^m + \beta y^p x^q + \gamma x^n = 0$, and thus henceforth, it is evident that the same curved line occurs in more kinds. For example by §36 with the curved line with the equation $yy - ax = 0$ likewise contained must be referred to the first kind and to the fourth, because by changing the axis, it is expressed by this equation also :

$$16uu - 24tu + 9tt - 55au + 10at = 0 .$$

Truly also, by assuming the axis and the starting point of the abscissas in another way, likewise the curve must relate to curves of the second, third, and fifth kind; from which this manner of division generally cannot be used.

51. This inconvenience will be avoided, if orders may be used by which the relation between the coordinates of the equations is expressed, constituting the classes of curves. Since for a given curved line, an equation of the same order may remain always, however both the axis and the start of the abscissas as well as the inclination of the coordinate axes may be varied, and the same curved line will not be carried off to different classes. Therefore with the character in the number of dimensions furnished in the equation, whether orthogonal or oblique angled ; neither by changing the axis, the origin of abscissas, nor the angle of inclination of the coordinates, will a variation of the classes be put in place. And the same curve will be inumerated in the same class, whether an equation between some special or general coordinates or also the most general may be seen. On which account the character of distinct curved lines is sought most conveniently from the order of the equations.

52. Therefore because these different kinds of equations, which are set up from the number of dimensions, we have called *orders*, also the diverse kinds of curved lines, which hence arise, we will call by the same name *order*. Therefore since the general equation of the first order shall be

$$0 = \alpha + \beta x + \gamma y ,$$

all curved lines, which with *x* and *y* taken for the coordinates, either orthogonal or oblique angled, proceed from this equation, we will refer to the first order. But above we have seen that only a right line is contained in this equation, and on this account the first order includes only the right line, which certainly among all lines is the most simple.

Therefore since the name of a curve may not agree with this first order, we will not call these the orders of *curved lines*, but we will call them more widely by the simple name of *lines*. Therefore the first order of lines contains no curved line, but comprises right lines only.

53. Moreover it will be in the same manner, whether the coordinates may be set up to be either rectangular or oblique angled ; for if the applied lines shall make an angle φ with the axis, the sine of which shall be μ and the cosine ν, the equation may be reduced to orthogonal coordinates on putting

$$y = \frac{u}{\mu} \text{ and } x = \frac{\nu u}{\mu} + t,$$

from that an equation arises between the orthogonal coordinates t and u :

$$0 = \alpha + \beta t + \left(\frac{\beta \nu}{\mu} + \frac{\gamma}{\mu} \right) u \ ,$$

which since it may not appear less widely than before, for each is general, it is evident that the designation of the equation is not restricted, even if the angle which the applied line makes with the axis may be made a right angle. This likewise may come about in general order of the following equations, which will not be less apparent, even if orthogonal coordinates may be put in place. Therefore since the general equation of each order loses none of its force by the determination of the inclination of the applied line to the axis, we will not restrict its significance, if we may put orthogonal coordinates in place. For just as a curved line will be contained in a general equation of each order with oblique angles taken, the same curved line will be contained in the same equation, if rectangular coordinates may be put in place.

54. Again all the curved lines of the second order will be contained in this general equation of the second order :

$$0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy.$$

Clearly all curved lines which this equation includes within it, with the letters x and y denoting orthogonal coordinates, we may consider as the second order of lines. These curved lines therefore are the most simple, because no curved lines will be contained in the first order, and on this account by some are accustomed to be called curved lines of the first order. Indeed the curved lines contained in this equation have commonly become known as *conic sections*, because all the same curves are generated from the section of a cone. The different kinds of these lines are the *circle*, the *ellipse*, the *parabola* and the *hyperbola*, which we will deduce from the general equation below.

55. Again all curved lines may be referred to the third order, which the following general equation of the third order makes available :

$$0 = \alpha + \beta x + yy + \delta xx + \varepsilon xy + \zeta yy + \eta x^3 + \vartheta xxy + \iota xyy + \chi y^3,$$

with x and y taken for orthogonal coordinates, because the condition of the applied lines does not lead to a fuller designation for this equation, as we have noted now. Because in this equation many more constant letters will be found than in the preceding, it is allowed to define these by choice, also a much greater number of different kinds will be present in this order, the enumeration of which was shown by Newton.

56. All the curved lines relate to the fourth order of lines, which this general equation of the fourth order shows :

$$\begin{aligned}0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy + \eta x^3 + \vartheta xxy + \iota xyy + \chi y^3 \\ + \lambda x^4 + \mu x^3 y + \nu xxyy + \xi xy^3 + o y^4,\end{aligned}$$

with the orthogonal coordinates x and y taken, because the obliquity of the applied lines does not lead to an equation of greater generality. Therefore in this equation fifteen arbitrary constants occur to be defined by choice, from which a much greater variety of kinds occur in this order, than in the preceding. These lines of the fourth order are accustomed also to be called curved lines of the third order, because the order of lines of the second order may be taken for the first order of curved lines ; and in a similar manner lines of the third order agree with curved lines of the second order.

57. Now from these it is understood, to which order the curved lines for the fifth, sixth, seventh and following orders may be related. Moreover the general equation containing within it all the lines of the fifth order, the terms

$$x^5,\ x^4y,\ x^3yy,\ xxy^3,\ xy^4,\ y^5\ ,$$

which are added to the above general equation of the fourth order, will constitute twenty one terms, and the general equation containing all the lines of the sixth order will have twenty eight terms, and thus henceforth, following the triangular numbers. Evidently the general equation for the lines of order n will contain $\frac{(n+1)(n+2)}{1\cdot 2}$ terms and there will be just as many constant letters present in that, which can be defined by choice.

58. Indeed nor will any diverse determination of the constant letters produce diverse curved lines. For we have seen in the preceding chapter for the same curved line, with the axis and origin of coordinates changed, infinitely many diverse equations can be shown ; from which diversity of the equations relating to the same order, a difference of the curves indicated does not follow from these equations. On account of which in the

enumeration of the kinds and forms relating to the same order, which are deduced from the general equation, great caution must be exercised, lest the same curved line may refer to two or more kinds.

59. Since therefore from the order of the equation, which is given between the coordinates, the order of the curved line may be known, it will be agreed upon by some proposed algebraic equation between the coordinates x and y put in place, to which at once the order for that curved line shall be referred by that indicated equation. Clearly the first equation, if it shall be irrational, is to be freed from irrationality and then, if fractions might remain, it must have these removed, with which done the number of the greatest dimension, which the variables x and y constitute in that equation, will indicate the order, to which the curved line is related. Thus the curved line, which this equation $yy - ax = 0$ gives, will be of the second order ; but the curved line in this equation $yy = x\sqrt{(aa - xx)}$, (which freed from irrationality shall become of the fourth order), will be contained by an equation of the fourth order. and the curved line, which this equation proposes, $y = \frac{a^3 - axx}{aa + xx}$, will be of the third order, because the equation freed from fractions will become $aay + xxy = a^3 - axx$, in the term xxy of which, the dimensions are three.

60. But several different curved lines can be retained in one and the same equation, provided the applied lines are put in place either normal to the axis or constituted under some obliquity. Thus this equation $yy - ax = 0$, if orthogonal coordinates may be put in place, provides a circle, but if oblique angled coordinates may be put in place, then the curve will be an ellipse. Yet all these diverse curves pertain to the same order, because in a reduction of the oblique angled coordinates to rectangular coordinates does not change the order of the equation. Therefore although the general equation for the curved line of each order on account of the angle, by which the applied lines make with the axis, is returned appearing neither more nor less extended, yet in the proposed equation for a specific curved line may not retained in that be determined, unless the angle may be determined, which the coordinates make amongst themselves.

61. So that a curved line may be properly referenced to that order which the equation indicates, it is necessary that the equation cannot be resolved into rational factors. For if the equation may have two or more factors, then the two or more may involve equations, each of which will generate some particular curved line, which taken together will exhaust the strength of the proposed equation. Therefore equations of this kind, resolvable into factors, include within themselves not one but several continuous curves, in which some peculiar equation may be expressed, and which otherwise are not connected to each other, except that the equations of these may be multiplied together. Since which shall be tied together depending on our choice, curved lines of this kind can be agreed not to constitute a continuous curved line. Therefore such equations, which we have called complex above, produce curved lines which are not continuous, but yet which are composed from continuous curved lines, which properly we will call complex.

62. Thus this equation $yy = ay + xy - ax$, which is seen to be of the second order, if it may be reduced to zero, so that it shall be $yy - ay - xy + ax = 0$, will be constructed from these factors $(y-x)\ (y-a) = 0$; therefore it includes these two equations $y - x = 0$ and $y - a = 0$, each of which is for a right line, clearly that one makes a half right angle with the start of the abscissas, and truly the other one is parallel to the axis at a distance $= a$. Therefore these two right lines considered together are contained in the proposed equation $yy = ay + xy - ax$. In a similar manner, this equation is complex

$$y^4 - xy^3 - aaxx - ay^3 + axxy + aaxy = 0$$

neither therefore shows a continuous curved line of the fourth order ; since indeed the factors shall be

$$(y-x)(y-a)(yy-ax),$$

it will contain three discrete lines, two clearly right, and the curve contained by the equation $yy - ax = 0$.

63. Therefore any complex lines can be formed as it pleases, which may include two or more lines which are either right or curves described according to choice. For if the nature of each of the single lines may be expressed by an equation according to the same axis and likewise the same related start of the abscissas, these individual equations , after they were reduced to zero, were multiplied by each other, and a complex equation will be produced, in which all the lines taken may be contained at the same time. Thus, if the proposed circle were described, with centre C (see Fig. 16) and radius $CA = a$; and besides the right line LN passing through the centre, an equation for some axis can be shown, which jointly includes the circle and the right line, as if both might constitute a single line.

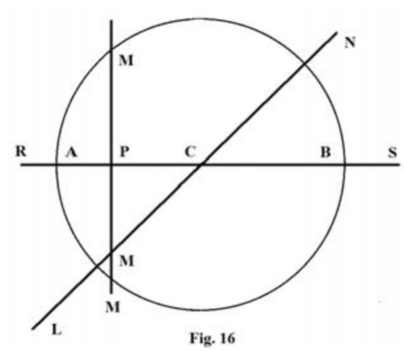

Fig. 16

64. The diameter AB may be taken for the axis, which with the right line LN make a half right angle, and with the start of the abscissas at A and with the abscissa $AP = x$ and the applied line $PM = y$, and for the right line PM there will be $PM = CP = a - x$ and, because the point M of the right line falls in the region of negative applied lines, there will be $y = -a + x$, or $y - x + a = 0$. But for the circle, since there shall be $PM^2 = AP \cdot PB$, on account of $BP = 2a - x$ it becomes $yy = 2ax - xx$ or $yy + xx - 2ax = 0$. Now these two equations may be multiplied together and a complex equation of the third order will be produced

$$y^3 - yyx + yxx - x^3 + ayy - 2axy + 3axx - 2aax = 0 ,$$

which will include both the circle as well as the right line at the same time. Clearly three applied lines will be found to correspond to the abscissa $AP = x$, two for the circle and one for the right line ; if without doubt we may put $x = \frac{1}{2}a$, it becomes

$$y^3 + \tfrac{1}{2}ayy - \tfrac{3}{4}aay - \tfrac{3}{8}a^3 = 0 ,$$

from which in the first place there becomes $y + \frac{1}{2}a = 0$, then on dividing through by this root, $yy - \frac{3}{4}aa = 0$ will be put in place, from which the three values of y will be:

$$\text{I. } y = -\tfrac{1}{2}a, \text{ II. } y = \tfrac{1}{2}a\sqrt{3}, \text{ III. } y = -\tfrac{1}{2}a\sqrt{3} .$$

Therefore the circle with the right line *LN*, as it were, will constitute a continuous curve, and thus may be represented by an equation.

65. With this distinction observed between non complex curves and complex curves, it is evident that lines of the second order are to be either continuous curves or be complexes composed from two right lines; for if the general equation has factors, these will be of the first order and thus will denote right lines. Moreover lines of the third order will be either non complex curves, or complexes composed from a single right line together with a complex line of the second order, or from three right lines. Again lines of the fourth order will be composed either from continuous or non complex curves, or from a single right line and a complex line of the third order, or from two lines of the second complex order, or from a line of the second order together with two right lines, or finally from four right lines. Similarly the account of complex lines of the fifth order and of higher orders is prepared, and will be enumerated in the same way. From which it is apparent that in lines of whatever order all the lines of inferior order are to be taken at the same time, and that is to say, that any of the orders of the lines contains at the same time all the lines of lesser orders, that is, that it may contain a complex line from the lesser orders with one or more right lines, one or more lines of the second order, of the third , and of the following orders, thus finally, so that if the number of individual orders to which the simpler orders belong, may be added into a single sum, a number will be produced, by which the order of the complex line is indicated.

CAPUT III

DE LINEARUM CURVARUM ALGEBRAICARUM IN ORDINES DIVISIONE

47. Cum linearum curvarum pariter ac functionum varietas sit infinita, earum cognitio nullo modo acquiri poterit, nisi infinita multitudo in certas classes digeratur hocque modo mens in earum scrutatione dirigatur atque adiuvetur. Divisimus iam quidem lineas curvas in *algebraicas* et *transcendentes,* verum utraque classis ob infinitam curvarum varietatem ulteriori subdivisione opus habet. Hic autem tantum curvas *algebraicas* spectamus, quas quemadmodum commodissime in classes distribui conveniat, dispiciamus. Characteres igitur primum definiendi sunt, quibus classium varietates determinentur, ita ut, quae curvae eodem charactere sint praeditae, eae ad eandem, quae contra, ad diversas classes referantur.

48. Characteres ergo isti varias classes distinguentes aliunde, nisi ex functionibus seu aequationibus, quibus linearum curvarum natura continetur, peti nequeunt; cum, quia alia via ad curvarum cognitionem perveniendi adhuc non patet, tum, quia nulla alia, quae quidem datur, omnes curvas algebraicas sub se complectitur. Functiones vero et aequationes inter binas coordinatas pluribus modis in diversa genera distribui possunt, uti fecimus in libro superiori. Ac primo quidem functionum multiformitas se offert, quae ad linearum curvarum in varias classes distributionem prae allis apta videtur ; unde huiusmodi divisio oriretur, ut eae lineae curvae, quae ex functionibus uniformibus oriuntur, ad genus primum, quae ex biformibus, ad secundum, quae ex triformibus, ad tertium referantur et ita porro.

49. Quamvis autem haec divisio videatur naturalis, tamen, si diligentius perpendatur, naturae linearum curvarum earumque indoli minime conformis deprehendetur. Multiformitas enim functionum ab axis positione, quae est arbitraria, potissimum pendet, ita ut, si pro uno axe applicata fuerit functio uniformis abscissae, eadem alia assumto axe functio multiformis esse queat; hoc ergo modo eadem linea curva in diversis generibus occurreret, quod est contra institutum. Sic enim linea curva hac aequatione $a^3y = aaxx - x^4$ expressa pertineret ad genus primum, quia applicata y est functio uniformis ipsius x; permutatis vero coordinatis seu axe sumto ad priorem normali eadem curva exprimitur aequatione $y^4 - aayy + a^3x = 0$ sicque ad genus quartum pertineret. Hanc igitur ob causam multiformitas functionum ad characterem, quo lineae curvae in classes distribuantur, constituendum admitti nequit.

50. Aeque parum simplicitas aequationum naturam linearum curvarum exprimentium, ratione numeri terminorum, characterem distinctionis constituere poterit. Si enim eae curvae ad genus primum referantur, quarum aequatio constet duobus terminis, ut

$y^m = \alpha x^n$, ad secundum, quarum aequatio contineat tres terminos, ut $\alpha y^m + \beta y^p x^q + y x^n = 0$, et ita porro, manifestum est eandem lineam curvam in pluribus generibus occurrere. Per exemplum enim § 36 subiunctum linea curva aequatione $yy - ax = 0$ contenta simul ad genus primum et quartum referri deberet, quia mutato axe etiam hac aequatione

$$16uu - 24tu + 9tt - 55au + 10at = 0$$

exprimitur. Deberet vero etiam, aliter assumto axe et abscissarum initio, simul ad genus secundum, tertium et quintum pertinere; ex quo ista divisio adhiberi omnino non potest.

51. Haec incommoda evitabuntur, si aequationum, quibus relatio inter coordinatas exprimitur, ordines ad curvarum classes constituendas adhibeantur. Cum enim pro eadem linea curva, utcunque tam axis et principium abscissarum quam inclinatio coordinatarum varietur, aequatio eiusdem semper ordinis maneat, eadem linea curva non ad diversas classes referetur. Charactere ergo in numero dimensionum, quas coordinatae, sive orthogonales sive obliquangulae, in aequatione complent, constitute, neque axis neque principii abscissarum mutatio neque inclinationis coordinatarum variatio classium constitutionem perturbabit. Atque eadem curva, sive aequatio inter coordinatas specialis quaeque sive generalis sive etiam generalissima spectetur, ad eandem semper classem annumerabitur. Quam ob rem character distinctionis linearum curvarum convenientissime ab ordine aequationum petitur.

52. Quoniam igitur haec diversa aequationum genera, quae ex dimensionum numero constituuntur, ordines vocavimus, diversa quoque linearum curvarum genera, quae hinc oriuntur, ordinum nomine appellabimus. Cum ergo aequatio primi ordinis generalis sit

$$0 = \alpha + \beta x + yy ,$$

omnes lineas curvas, quae sumtis x et y pro coordinatis, sive orthogonalibus sive obliquangulis, ex hac aequatione proficiscuntur, ad ordinem primum referemus, Supra autem vidimus in hac aequatione tantum lineam rectam contineri, et hanc ob rem primus ordo solam lineam rectam in se complectitur, quae utique inter omnes lineas est simplicissima. Cum igitur nomen curvae huic primo ordini non conveniat, hos ordines non *linearum curvarum,* sed vocabulo latiori simpliciter *linearum* vocabimus. Ordo ergo linearum primus nullam lineam curvam continet, sed a sola linea recta exhauritur.

53. Perinde autem est, sive coordinatae statuantur rectangulae sive obliquangulae; quodsi enim applicatae cum axe faciant angulum φ, cuius sinus sit μ et cosinus v, aequatio ad coordinatas orthogonales reducetur ponendo

$$y = \frac{u}{\mu} \text{ et } x = \frac{vu}{\mu} + t \quad (44),$$

unde ista inter coordinatas orthogonales *t* et *u* aequatio nascitur

$$0 = \alpha + \beta t + \left(\frac{\beta v}{\mu} + \frac{\gamma}{\mu} \right) u\ ,$$

quae cum non minus late pateat quam prior, utraque enim est generalis, manifestum est significationem aequationis non restringi, etiamsi angulus, quem applicatae cum axe faciant, rectus statuatur. Hoc idem eveniet in aequationibus sequentium ordinum generalibus, quae non minus late patebunt, etsi coordinatae orthogonales statuantur. Cum igitur aequatio generalis cuiusque ordinis per determinationem inclinationis applicatarum ad axem nihil de vi sua perdat, eius significatum non restringemus, si coordinatas orthogonales statuamus. Quaecunque enim linea curva in aequatione generali cuiusque ordinis continetur sumtis coordinatis obliquangulis, eadem linea curva in eadem aequatione continebitur, si coordinatae rectangulae statuantur.

54. Lineae porro secundi ordinis omnes continebuntur in hac aequatione generali ordinis secundi

$$0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy.$$

Omnes scilicet lineas curvas, quas haec aequatio, denotantibus litteris *x* et *y* coordinatas orthogonales, in se complectitur, ad ordinem linearum secundum numeramus. Sunt igitur hae lineae curvae simplicissimae, quia in ordine primo nulla linea curva continetur, et hanc ob rem a quibusdam lineae curvae primi ordinis vocari solent. Lineae vero istae curvae in hac aequatione contentae sub nomine *sectionum conicarum* vulgo innotuerunt, quia eaedem omnes ex sectione coni nascuntur. Diversae harum linearum species sunt *circulus, ellipsis, parabola* et *hyperbola,* quas infra ex aequatione generali deducemus.

55. Ad tertium porro linearum ordinem referuntur omnes lineae curvae, quas sequens aequatio tertii ordinis generalis suppeditat

$$0 = \alpha + \beta x + yy + \delta xx + \varepsilon xy + \zeta yy + \eta x^3 + \vartheta xxy + \iota xyy + \chi y^3,$$

sumtis *x* et *y* pro coordinatis orthogonalibus, quia conditio obliquitatis applicatarum ampliorem significatum huic aequationi non inducit, ut iam notavimus. Quia in hac aequatione multo plures quam in praecedente habentur litterae constantes, quas pro arbitrio definire licet, etiam multo maior specierum diversarum numerus in hoc ordine continetur, quarum enumerationem exhibuit Newtonus.

56. Ad quartum linearum ordinem pertinent omnes lineae curvae, quas haec aequatio generalis quarti ordinis exhibet

$$0=\alpha+\beta x+\gamma y+\delta xx+\varepsilon xy+\zeta yy+\eta x^3+\vartheta xxy+\iota xyy+\chi y^3$$
$$+\ \lambda x^4+\mu x^3y+\nu xxyy+\xi xy^3+oy^4,$$

sumtis x et y pro coordinatis orthogonalibus, quia obliquitas applicatarum aequationi maiorem generalitatem non inducit. Occurrunt ergo in hac aequatione quindecim quantitates constantes pro arbitrio definiendae, unde multo maior specierum diversarum varietas in hoc ordine occurrit, quam in praecedente. Lineae istae quarti ordinis vocari etiam solent lineae curvae tertii ordinis, quia linearum ordo secundus pro linearum curvarum ordine primo reputatur; similique modo lineae tertii ordinis conveniunt cum lineis curvis secundi ordinis.

57. Ex his iam intelligitur, quaenam lineae curvae ad ordinem quintum, sextum, septimum et sequentes pertineant. Aequatio autem generalis omnes lineas quinti ordinis in se complectens, quia ad aequationem generalem quarti ordinis insuper accedunt termini $x^5,\ x^4y,\ x^3yy,\ xxy^3,\ xy^4,\ y^5$, constabit omnino terminis viginti et uno, et aequatio generalis omnes lineas sexti ordinis continens habebit viginti et octo terminos, et ita porro secundum numeros trigonales. Scilicet aequatio generalis pro lineis ordinis n continebit $\frac{(n+1)(n+2)}{1\cdot 2}$ terminos totidemque in ea inerunt litterae constantes, quas pro arbitrio definire licet.

58. Neque vero quaelibet litterarum constantium diversa determinatio diversas lineas curvas producit. Vidimus enim in praecedente capite pro eadem linea curva, mutatis axe et abscissarum initio, infinitas exhiberi posse aequationes diversas ; unde ex diversitate aequationum ad eundem ordinem pertinentium non sequitur curvarum iis aequationibus indicatarum diversitas. Quamobrem in enumeratione generum ac specierum ad eundem ordinem pertinentium, quae ex aequatione generali deducitur, admodum cautum esse oportet, ne eadem linea curva ad duas pluresve species referatur.

59. Cum igitur ex ordine aequationis, quae inter coordinatas datur, lineae curvae ordo cognoscatur, proposita quacunque aequatione algebraica inter coordinatas x et y statim constabit, ad quemnam ordinem linea curva illa aequatione indicata sit referenda. Primum scilicet aequatio, si sit irrationalis, ab irrationalitate liberari tumque, si fractiones superfuerint, ab his purgari debebit, quo facto maximus dimensionum numerus, quem variabiles x et y in ea constituunt, ordinem, ad quem linea curva pertinet, indicabit. Sic linea curva, quam haec aequatio $yy-ax=0$ dat, erit ordinis secundi; linea curva autem in hac aequatione $yy=x\sqrt{(aa-xx)}$, (quae ab irrationalitate liberata fit ordinis quarti), contenta erit ordinis quarti. Et linea curva, quam haec aequatio praebet $y=\frac{a^3-axx}{aa+xx}$, erit ordinis tertii, quia aequatio a fractionibus liberata fit $aay+xxy=a^3-axx$, in cuius termino xxy tres sunt dimensiones.

60. In una eademque autem aequatione plures lineae curvae diversae contineri possunt, prout applicatae ad axem vel normales vel sub data obliquitate constitutae ponuntur. Sic haec aequatio $yy - ax = 0$, si coordinatae ponantur orthogonales, praebet circulum, sin autem coordinatae obliquangulae statuantur, tum curva erit ellipsis. Omnes tamen istae curvae diversae ad eundem ordinem pertinent, quia reductione coordinatarum obliquangularum ad rectangulas ordo curvae non mutatur. Quanquam ergo aequatio generalis pro lineis curvis cuiusque ordinis ob angulum, quo applicatae axi insistunt, neque latius neque minus late patens redditur, tamen proposita aequatione speciali linea curva in ea contenta non determinatur, nisi angulus, quem coordinatae inter se constituunt, determinetur.

61. Quo linea curva ad eum ordinem, quem aequatio indicat, proprie referatur, necesse est, ut aequatio in factores rationales resolvi nequeat. Si enim aequatio duos pluresve habeat factores, tum duas pluresve involvet aequationes, quarum quaelibet peculiarem lineam curvam generabit, quae iunctim sumtae aequationis propositae vim exhaurient. Huiusmodi ergo aequationes in factores resolubiles non unam sed plures curvas continuas in se complectuntur, quarum quaevis peculiari aequatione exprimi queat et quae aliter inter se non sunt connexae, nisi quod earum aequationes in se mutuo sint multiplicatae. Qui cum sit nexus ab arbitrio nostro pendens, eiusmodi lineae curvae non unam continuam lineam constituere censeri possunt. Tales ergo aequationes, quas supra complexas vocavimus, producent lineas curvas non continuas, attamen ex continuis compositas, quas propterea complexas vocabimus.

62. Sic haec aequatio $yy = ay + xy - ax$, quae ad lineam secundi ordinis esse videtur, si ad nihilum reducatur, ut sit $yy - ay - xy + ax = 0$, constabit ex his factoribus $(y-x)\ (y-a) = 0$; complectitur ergo has duas aequationes $y - x = 0$ et $y - a = 0$, quarum utraque est pro linea recta, ilia scilicet cum axe in initio abscissarum angulum semirectum constituit, haec vero axi ad distantiam $= a$ est parallela. Duae ergo istae lineae rectae simul consideratae in aequatione proposita $yy = ay + xy - ax$ continentur. Simili modo haec aequatio est complexa

$$y^4 - xy^3 - aaxx - ay^3 + axxy + aaxy = 0$$

neque propterea lineam continuam quarti ordinis exhibet; cum enim factores sint

$$(y-x)(y-a)(yy-ax),$$

tres continebit lineas discretas, duas scilicet rectas et unam curvam in aequatione $yy - ax = 0$ contentam.

63. Possunt ergo pro lubitu lineae complexae quaecunque formari, quae complectantur duas pluresve lineas sive rectas sive curvas ad arbitrium descriptas. Quodsi enim unius

cuiusque lineae natura exprimatur per aequationem ad eundem axem idemque abscissarum initium relatam haeque aequationes singulae, postquam ad cyphram fuerint reductae, in se multiplicentur, prodibit aequatio complexa, in qua omnes lineae assumtae simul continentur. Ita, si propositus fuerit (Fig. 16) circulus centro C et radio $CA = a$ descriptus ac praeterea linea recta LN per centrum C transiens, aequatio pro quovis axe exhiberi poterit, quae circulum et lineam rectam, quasi ambo unam lineam constituerent, coniunctim complectatur.

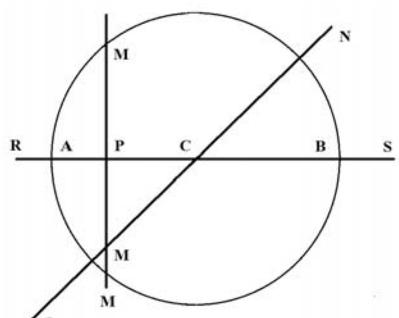

Fig. 16

64. Sumatur diameter AB, quae cum recta LN angulum semirectum constituat, pro axe ac sumto initio abscissarum in A vocatisque abscissa $AP = x$ et applicata $PM = y$ erit pro linea recta $PM = CP = a - x$ et, quia punctum rectae M in regionem applicatarum negativarum cadit, erit $y = -a + x$, seu $y - x + a = 0$. Pro circulo autem cum sit $PM^2 = AP \cdot PB$, ob $BP = 2a - x$ erit $yy = 2ax - xx$ seu $yy + xx - 2ax = 0$. Multiplicentur iam hae duae aequationes in se invicem ac prodibit aequatio tertii ordinis complexa

$$y^3 - yyx + yxx - x^3 + ayy - 2axy + 3axx - 2aax = 0,$$

quae tam circulum quam lineam rectam simul in se complectetur. Abscissae scilicet $AP = x$ respondere invenientur tres applicatae, binae circuli et una rectae; sit nimirum $x = \frac{1}{2}a$, fiet

$$y^3 + \tfrac{1}{2}ayy - \tfrac{3}{4}aay - \tfrac{3}{8}a^3 = 0,$$

unde fit primo $y + \frac{1}{2}a = 0$, tum divisione per hanc radicem instituta erit $yy - \frac{3}{4}aa = 0$, unde tres valores ipsius y erunt:

$$\text{I. } y = -\tfrac{1}{2}a, \text{ II. } y = \tfrac{1}{2}a\sqrt{3}, \text{ III. } y = -\tfrac{1}{2}a\sqrt{3}.$$

Quasi ergo circulus cum recta LN unum continuum constituerit, ita in aequatione repraesentatur.

65. Notato hoc discrimine inter curvas incomplexas et complexas, perspicuum est lineas secundi ordinis vel esse curvas continuas vel ex duabus lineis rectis complexas; si enim aequatio generalis habet factores, hi erunt primi ordinis ideoque lineas rectas denotabunt. Lineae autem tertii ordinis erunt vel incomplexae vel ex una recta et una linea secundi ordinis complexae vel ex tribus lineis rectis complexae. Porro lineae quarti ordinis erunt vel continuae seu incomplexae vel ex una linea recta et una linea tertii

ordinis complexae vel ex duabus lineis secundi ordinis complexae vel ex linea secundi ordinis una et duabus rectis, vel denique ex quatuor lineis rectis complexae erunt. Similiter ratio linearum complexarum ordinis quinti altiorumque ordinum est comparata parique modo enumerari poterit. Ex quo patet in quovis linearum ordine simul omnes lineas ordinum inferiorum comprehendi, neque vero simpliciter, sed quaelibet ordinum inferiorum complexa cum linea vel lineis rectis vel cum lineis secundi, tertii sequentiumve ordinum, ita tamen, ut, si numeri singulorum ordinum, ad quos lineae simplices pertinent,
in unam summam addantur, prodeat numerus, quo ordo lineae complexae indicatur.