CHAPTER XXII

THE SOLUTION OF SOME PROBLEMS RELATING TO THE CIRCLE

529. With the above radius of the circle put $=1$ we have seen the semi circumference π or the arc of 180 degrees

$$=3,14159265358979323846264338,$$

of which number the decimal or common logarithm is

$$0,49714987269413385435126828 8 ;$$

which if it may be multiplied by 2,30258 etc., will produce the hyperbolic logarithm of the same number, which will be

$$=1,1447298858494001741434237.$$

Therefore since the length of the arc of 180 shall be known, thence the length can be assigned of any given arc in degree. The arc of *n* degrees shall proposed, the length of which shall be $=z$*;* there will be $180:n=\pi:z$ and thus $z=\frac{\pi n}{180}$; hence the logarithm of *z* is found, if this logarithm

$$1,7581226324091722154525264 13.$$

is taken from the logarithm of the number *n*. But if the proposed arc may be given in first minutes, so that it shall be *n',* then from the logarithm of n this logarithm will have to be taken :

$$3,5362738827928158479612932 11.$$

If moreover the proposed arc may be given in second minutes [*i.e.* the modern definition of seconds of arc], so that it shall be *n'',* then the logarithm of the length of the arc will be found, if this logarithm may be taken from the logarithm of the number n :

$$5,3144251331764594804700600 09,$$

or if to the logarithm of the number n there is added

4,68557486682354051952993999 0,

and the characteristic of the sum is taken from10.

530. Therefore from these in turn the radius and any of its parts, such as the sines, tangents, and secants, are to be converted into arcs and these arcs can be expressed following the customary manner into degrees, and first and second minutes. Let *z* be a line of this kind expressed by the radius 1 and its decimal parts, its logarithm may be taken and its characteristic may be increased by, just as they are accustomed to be shown in tables of the logarithms of sines, tangents, and of secants ; with which done either by subtraction from this logarithm

4,6855748668235405195299399 90

or by adding to the same

5,3144251331764594804700600 09 ;

in each case the logarithm will be produced, of which the corresponding number will be expressed in second minutes of arcs. Indeed in the latter case the characteristic must be diminished by ten. But if the arc equal to the radius itself may be sought, this is found easily by the golden rule without logarithms, since there shall be π to 180° as 1 to the arc equal to the radius; hence moreover this arc found is expressed in degrees

57°, 2957795130823208767 98,

likewise truly the arc expressed in first minutes will be

3437' , 7467707849392526078 8 ;

truly in second minutes likewise the arc will be

$= 206264'',\ 8062470963551564728$.

But in the accustomed manner this expressed arc will contain

57°, 17', 44'', 48''', 22'''', 29''''', 22''''''.

It is shown in the above section that the sine of its arc is found from the series

$= 0,84147098480789$

and the

$$\text{cosine} = 0,54030230586814$$

of which numbers the first divided by the second will give the tangent of the angle

57°, 17', 44", 48''', 22'''', 29''''', 22'''''' etc.

531. Therefore from these presented, in which circular arcs are able to be compared with sines and tangents, we will be able to resolve general questions regarding the nature of circles. And indeed in the first place it is apparent that any arc is to be greater than its sine, unless it shall be vanishing ; but the ratio of the cosines is prepared otherwise, because the cosine of the vanishing angle is 1 and thus greater than the arc, truly the cosine of the right angle is 0 and thus less than the arc; from which it is apparent an arc to be given between the limits 0° and 90°, which shall be equal to its cosine, which we will investigate in the following problem.

PROBLEM I

To find the arc of the circle, which shall be equal to its cosine.

SOLUTION

Let s be this arc sought, and there will be $s = \cos.s$; from which equation the value of s will be able to be found quickly more conveniently by the rule called as *false position*. But now for this it is necessary to know a nearby value of s, or which it is permitted to bring to the light following a conjecture, but unless this may be apparent, three or more values may be substituted in place of s and the cosine may be returned equally to the same unit. We may put $s = 30°$, which arc we may recall to the parts of the radius by the given rule above

$$\begin{array}{rl} l.30 = & 1,4771213 \\ \text{take} & \underline{1,7581226} \\ l.\text{arc}.30° = & 9,7189987, \end{array}$$

but there is

$$l.\cos.30° = 9,9375306,$$

from which it is apparent the cosine of 30° to be much greater than this, and thus the arc sought must be greater than 30°. Therefore we may devise

$$s = 40°$$

and

$$\begin{array}{rl} l.40 = & 1,6020600 \\ \text{take} & \underline{1,7581226} \\ l.\text{arc}.40° = & 9,8439374, \end{array}$$

but there is

$$l.\cos.40° = 9,8842540\,,$$

hence it is understood that the arc sought is a little greater than 40°, and on this account we may try $s = 45°$, there becomes

$l.45 =$	1,6532125
take	1,7581226
$l.$arc.45° =	9,8950899,

but there is

$$l.\cos.45° = 9,8494850\,,$$

therefore the angle sought is contained 40° and 45° and thus hence it will be defined approximately. For on putting $s = 40°$,

the error is	+ 403166,
but on putting	$s = 45°$,
the error is	$= -456049$,
and the difference	$= -859215$.

Therefore the difference thus becomes as 859215 to 403166 of the hypothesized 5° to the excess of the angle sought above 40°, from which the arc sought shall be greater than 42°, indeed the limits have been too distant, which we may be able to define more exactly. Therefore we may take the closer limits

$l.s =$	$s = 42°$	$s = 43°$
take	1,6232493	1,6334685,
$l.s =$	1,7581226	1,7581226,
	9,8651267	9,8753459,
	and there becomes	
$l.\cos.s =$	9,8710735	9,8641275
	+ 59468	-112184
	112184	

171652: 59468 = 1°: 20', 47".

Therefore we have obtained the closer limits 42°, 20' and 42°, 21', between which the true value of s may be contained. First we will return these angles to minutes

	$s = 2540'$	$s = 2541'$
$l.s =$	3,4048337	3,4050047
take	3,5362739	3,5362739
$l.s =$	9,8685598	9,8687308
$l.\cos.s =$	9,8687851	9,8686700
	+ 2253	-608
	608	

$2861 : 2253 = 1' : 47'', 15'''.$

Hence we conclude the arc sought, which shall be equal its own cosine, to be = 42°, 20', 47", 15'", the cosine of which , or the length itself, will be $= 0,73908502$. Q. E. I.

532. The sector of a circle (Fig.112) *ACB* may be cut by a chord *AB* into two parts, the segment *AEB* and the triangle *ACB*, of which the former will be smaller than the latter, if the angle *ACB* were small, but greater, if the angle *ACB* shall be much greater. Therefore the case will be given, in which the sector *ACB* shall be cut by the chord *AB* into two equal parts, from which arises :

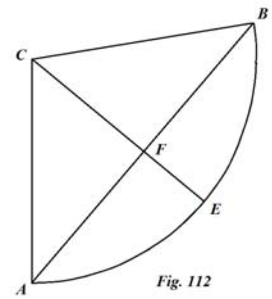

Fig. 112

PROBLEM II

To find the sector of the circle ACB, which may be cut by the chord AB into two equal parts, thus so that the area of the triangle ACB shall be equal to the area of the segment AEB.

SOLUTION

On putting the radius $AC = 1$ the arc sought will be $AEB = 2s$, so that its half shall be $AE = BE = s$; therefore with the radius *CE* drawn there will be $AF = \sin.s$ and $CF = \cos.s$. From which the triangle shall be $ACB = \sin.s \cos.s = \frac{1}{2} \cdot \sin.2s$ and the sector itself *ACB* is *s*, which since it must be equal to the two-fold triangle, there will be $s = \sin.2s$; and thus the arc must be sought, which shall be equal to the sine of the arc doubled. Indeed in the first place it is apparent that the angle *ACB* to be greater than a right angle and thus *s* to surpass 45°, from which we can make the following hypothesis :

	$s = 50°$	$s = 55°$	$s = 54°$
$l.s =$	1,6989700	1,7403627	1,7323938
take	1,7581226	1,7581226	1,7581226
	9,9408474	9,9822401	9,9742712

$l.\sin.2s =$	9,9933515	9,9729858	9,9782063
	+ 525041	- 92543	+ 39351
	92543		

$$617584 : 525041 = 5° : 4°,15'.$$

Therefore there will be just about $s = 54°,\ 15'$, from which according to the above hypothesis we may add $s = 54°$, and from the errors it may be concluded $s = 54°,17',54''$, which value will not disagree from the true value by a whole minute; therefore we may put in place the following positions disagreeing by a minute only

$s = 54°,\ 17'$	$s = 54°,\ 18'$	$s = 54°,\ 19'$
or	or	or
$s = 3257'$	$s = 3258'$	$s = 3259'$
and	and	and
$2s = 108°,\ 34'$	$2s = 108°,\ 36'$	$2s = 108°,\ 38'$
$\text{compl.} = 71°,\ 26'$	$\text{compl.} = 71°,\ 24'$	$\text{compl.} = 71°,\ 22'$
$l.s = 3,5128178$	3,5129511	3,5130844
take 3,5362739	3,5362739	3,5362739
$l.s = 9,9765439$	9,9766772	9,9768105
$l.\sin.2s = 9,9767872$	9,9767022	9,9766171
+ 2433	+ 250	− 1934
	1934	
	2184	

therefore there becomes $2184 : 250 = 1' : 6'',52'''$.
Hence there will be $s = 54°,\ 18',\ 6'',\ 52'''$. If we wish to determine this angle more accurately, it will be necessary to use larger tables ; from which we may make the following hypothetical 10'' differences

$s = 54°,\ 18',\ 0''$	$s = 54°,\ 18',\ 10''$
or	or
$s = 195480''$	$s = 195490''$
$2s = 108°,\ 36',\ 0''$	$2s = 108°,\ 36',\ 20''$
	$\text{comp.} = 71°,\ 23',\ 40''$
$\text{compl.} = 71°,\ 24',\ 0''$	$5{,}2911245466$
$l.\ s = 5{,}2911023304$	$\underline{5{,}3144251332}$
take $\underline{5{,}3144251332}$	$9{,}9766994134$
$9{,}9766771972$	$\underline{9{,}9766880552}$
$l.\sin.2s = 9{,}9767022291$	$-\quad 113582$
$+\quad 250319$	
$\underline{113582}$	

363901: 250319 = 10'': 6'',52''',43'''',33'''''

Therefore there will be $s = 54°,\ 18',\ 6'',\ 52''',\ 43'''',\ 33'''''$
and thus the angle $ACB = 108°,\ 36',\ 13'',\ 45''',\ 27'''',\ 6'''''$
and its complement $= 71°,\ 23',\ 46'',\ 14''',\ 32'''',\ 54'''''$
of which the logarithm of the sine or

$$l.\sin.2s = 9{,}9766924791$$

and the sine itself $= 0{,}9477470.$

Then there will be

$$\sin.s = AF = BF = 0{,}8121029$$

and thus its duplicate or

$$\text{chord } AB = 1{,}6242058\ .$$

Truly besides there will be

$$\cos.CF = 0{,}5835143.$$

And thus truly the sector sought will be able to be constructed. Q.E.I.

533. In a similar manner the sine can be determined, by which the quadrant of the circle may be cut into two equal parts.

PROBLEM III

In the quadrant of the circle (Fig.113) *ACB to apply the sine DE, which bisects the quadrant into two equal parts.*

SOLUTION

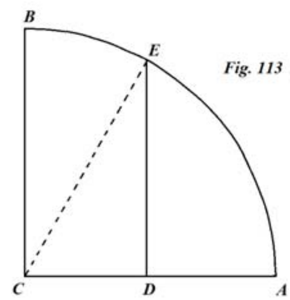

Fig. 113

Let the arc $AE = s$; BE will be $= \frac{\pi}{2} - s$ on account of $AEB = \frac{\pi}{2}$ and the area of the quadrant $= \frac{\pi}{4}$. Now the area of the sector ACE is $= \frac{1}{2}s$,
from which the triangle

$$CDE = \tfrac{1}{2}\cdot \sin.s \cdot \cos.s$$

taken away leaves the space

$$ADE = \tfrac{1}{2}s - \tfrac{1}{2}\cdot \sin.s \cdot \cos.s,$$

twice which must give the quadrant ; from which there will be

$$\tfrac{1}{4}\pi = s - \tfrac{1}{2}\cdot \sin.2s, \text{ therefore } s - \tfrac{1}{4}\pi = \tfrac{1}{2}\cdot \sin.2s$$

Putting the arc

$$s - \tfrac{1}{4}\pi = s - 45° = u,$$

there will be $2s = 90° + 2u$ and thus there is required to be

$$u = \tfrac{1}{2}\cdot \cos.\, 2u \text{ et } 2u = \cos.2u\, .$$

Therefore since the arc may be found, which will be equal to its own cosine, from the problem we found first, there will be,

$$2u = 42°, 20', 47'', 15''' \text{ et } u = 21°, 10', 23'', 37'''.$$

On account of which there will be

$$\text{the arc } AE = s = 66°,\ 10',\ 23'',\ 37''' \text{ and the arc } BE = 23°,\ 49',\ 36'',\ 23'''.$$

Hence the part of the radius will be

$$CD = 0,4039718 \text{ and } AD = 0,5960281$$

and thus the sine $DE = 0,9147711$.

Therefore in this manner, by which the quadrant of the circle is bisected, the whole circle will be cut into 8 equal parts. Q. E. F.

534. Just as a right line drawn through the centre cuts a circle in two parts, thus from some point on the periphery right lines will be drawn, thus from some point on the periphery right lines will be drawn, which will cut the circle into three or more equal parts. We will inquire into and resolve quadrisection.

PROBLEM IV

For the proposed semicircle AEDB (Fig.114) *from the point A the chord AD is erected, which will cut the area of the semicircle into two equal parts.*

SOLUTION

The arc sought shall be $AD = s$, and with the radius *CD* drawn the area of the sector $ACD = \frac{1}{2}s$, from which the triangle

$ACD = \frac{1}{2}AC \cdot DE = \frac{1}{2} \cdot \sin.s$, is taken

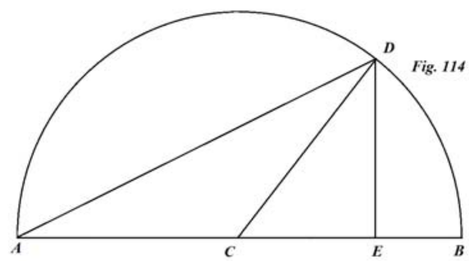

Fig. 114

and the segment will remain

$$AD = \tfrac{1}{2}s - \tfrac{1}{2} \cdot \sin.s\,,$$

which must be equal to half of the semicircle *ADB;* but the area of the semicircle is $= \frac{1}{2}\pi$, from which there will be

$$s - \sin.s = \frac{1}{2}\pi = 90° \text{ and thus } s - 90° = \sin.s\,.$$

Putting $s - 90° = u$; there will be $\sin.s = \cos.u$ and on that account $u = \cos.u$.
By the first problem therefore there will be

$$u = 42°,\ 20',\ 47'',\ 14'''$$

and hence

$$s = \text{angle } ACD = 132°,\ 20',\ 47'',\ 14''' \text{ and the angle } BCD = 47°,\ 39',\ 12'',\ 46'''.$$

Truly the chord itself AD will be $= 1,8295422$. Q. E. F.

535. Therefore a segment is cut thus in the circle, the area of which shall be the quarter part of the whole circle; but a segment equal to half the area of the circle, is itself a semicircle, a diameter of which is the chord. In a similar manner the segment can be found, which shall be a third of the whole circle, which we will investigate in the following problem.

PROBLEM V

From a point A of the periphery (Fig.115) *to erect two chords AB, AC, by which the area of the circle may be divided into three equal parts.*

SOLUTION

With the radius of the circle put $=1$ and the half periphery $=\pi$, let the arc *AB* or *AC* = *s ;* and the area of the segment *AEB* or *AFC*

$$=\tfrac{1}{2}s-\tfrac{1}{2}\cdot\sin.s\,;$$

but the area of the circle is $=\pi$; from which, since the area of the segment *AEB* shall be a third of the circle, the equation becomes

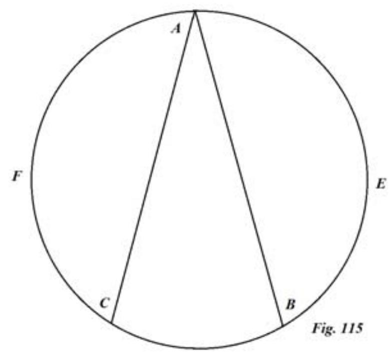

Fig. 115

$$\tfrac{1}{2}s-\tfrac{1}{2}\cdot\sin.s=\frac{\pi}{3}=\ 60^\circ \text{ or } \ s-\sin.s=120^\circ$$

and thus

$$s-120^\circ=\sin.s$$

Let $s-120^\circ=u$, there will be $u=\sin.(u+120^\circ)=\sin.(60^\circ-u)$. Therefore the arc *u* must be found, which shall be equal to the sine of the angle $60^\circ-u$. Therefore *u* will be less than 60°; towards finding which arc we may put the following in place :

$u=20^\circ$	$u=30^\circ$	$u=40^\circ$
$60^\circ-u=40^\circ$	$60^\circ-u=30^\circ$	$60^\circ-u=20^\circ$
$l.u=1,3010300$	1,4771213	1,6020600
take $\underline{1,7581226}$	$\underline{1,7581226}$	$\underline{1,7581226}$
$l.u=9,5429074$	9,7189987	9,8439374
$l.\sin.(60^\circ-u)=9,8080675$	$\underline{9,6989700}$	$\underline{9,5340517}$
	−200287	−3098857
+2651601		

Therefore it is apparent that the angle *u* is a little less than 30° and by the calculation accounted for below it must be greater than 29°; therefore there shall be :

$$u = 29°$$
$$60° - u = 31°$$
$$l.u = 1{,}4623980$$
$$\text{take } \underline{1{,}7581226}$$
$$l.u = 9{,}7042754$$
$$l.\sin.(60° - u) = \underline{9{,}7118393}$$
$$+\ 75639$$
$$-200287$$
$$275926 : 75639 \quad = 1°: 16', 26''.$$

Therefore the angle becomes u = 29°, 16', 26", according to which we may construct a hypothesis differing by one minute only to find a more accurate angle :

$u = 29°, 16'$	$u = 29°,\ 17'$
or	or
$u = 1756'$	$u = 1757'$
$60° - u = 30°,\ 44'$	$60° - u = 30°,\ 43'$
$l.\ u = 3{,}2445245$	$3{,}2447718$
take $\underline{3{,}5362739}$	$\underline{3{,}5362739}$
$l.u = 9{,}7082506$	$9{,}7084979$
$l.\sin.(60° - u) = \underline{9{,}7084575}$	$\underline{9{,}7082450}$
$+\quad 2069$	$-\quad 2529$
$\underline{2529}$	
4598	$1' : 27\ '''\ \ 0''''.$

Therefore truly there will be

$$u = 29°,\ 16',\ 27'',\ 0'''$$

and hence the arc $s = AEB = 149°, 16', 27'', 0''' = AFC$; from which the arc $BC = 61°, 27', 6'', 0'''$, and in truth the chords $AB = AC = 19285340$. Q. E. F.

536. To these problems, in which a certain arc is sought equal to a given sine or cosine, we may add the following, in which indeed the same matter is proposed, but yet it meets with greater difficulty.

PROBLEM VI

In the semicircle (Fig.116) *AEB the arc AE is to be cut off, thus so that the arc AE taken with its sine ED shall be equal to the sum of the right lines* $AD + DE$.

SOLUTION

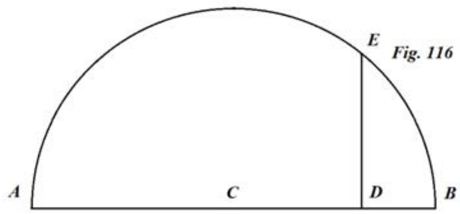

Fig. 116

Because it is apparent at once that this arc is to be greater than a quadrant, we shall seek its complement *BE* and we will call the arc $BE = s$,thus so that the arc $AE = 180° - s$, and on account of $AC = 1,\ CD = \cos.s,\ DE = \sin.s$, there will be $180° - s = 1 + \cos.s + \sin.s$

But there is

$$\sin.s = 2\sin.\tfrac{1}{2}s\cdot\cos.\tfrac{1}{2}s \quad\text{and}\quad 1+\cos.s = 2\cos.\tfrac{1}{2}s\cdot\cos.\tfrac{1}{2}s\,;$$

from which the equation arises

$$180° - s = 2\cos.\tfrac{1}{2}s\left(\sin.\tfrac{1}{2}s + \cos.\tfrac{1}{2}s\right).$$

But there is

$$\cos.\left(45° - \tfrac{1}{2}s\right) = \frac{1}{\sqrt{2}}\cdot\cos.\tfrac{1}{2}s + \frac{1}{\sqrt{2}}\cdot\sin.\tfrac{1}{2}s,$$

therefore

$$\sin.\tfrac{1}{2}s + \cos.\tfrac{1}{2}s = \sqrt{2}\cos.\left(45° - \tfrac{1}{2}s\right),$$

from which there will be

$$180° - s = 2\sqrt{2}\cdot\cos.\tfrac{1}{2}s\cdot\cos.\left(45° - \tfrac{1}{2}s\right).$$

With this reduction made, we may put the following in place :

$\tfrac{1}{2}s = 20°$	$\tfrac{1}{2}s = 21°$
$45° - \tfrac{1}{2}s = 25°$	$45° - \tfrac{1}{2}s = 24°$
$180° - s = 140°$	$180° - s = 138°$
$l.(180° - s) = 2{,}1461280$	2,1398791
take 1,7581226	1,7581226
	0,3817565

$l.(180° - s) = 0{,}3880054$	
$l.\cos.\frac{1}{2}z = 9{,}9729858$	9,9701517
$l.\cos.(45° - \frac{1}{2}s) = 9{,}9572757$	9,9607302
$l.2\sqrt{2} = 0{,}4515450$	0,4515450
0,3818065	0,3824269
Error + 61989	− 6704
6704	
68693	: 61989 = 1° : 54'

Hence $\frac{1}{2}s$ will be enclosed within the limits 20°, 54' and 20° 55' and thus the following hypotheses are made :

$\frac{1}{2}s = 20°,\ 54'$	$\frac{1}{2}s = 20°,\ 55'$
$45° - \frac{1}{2}s = 24°,\ 6'$	$45° - \frac{1}{2}s = 24°,\ 5'$
$s = 41°,\ 48'$	$s = 41°,\ 50'$
$180° - s = 138°,\ 12'$	$180° - s = 138°,\ 10'$
or	or
$180° - s = 8292'$	$180° - s = 8290'$
$l.(180° - s) = 3{,}9186593$	3,9185545
take 3,5362739	3,5362739
0,3823854	0,3822806
$l.\cos.\frac{1}{2}s = 9{,}9704419$	9,9703937
$l.\cos.(45° - \frac{1}{2}s) = 9{,}9603919$	9,9604484
$l.2\sqrt{2} = 0{,}4515450$	0,4515450
0,3823788	0,3823871
Error + 66	− 1065
1065	
1131: 66 = 1' : 3", 30"'.	

On this account the equation will be $\frac{1}{2}s = 20°,\ 54',\ 3'',\ 30'''$, thence

$$s = 41°,\ 48',\ 7'',\ 0''' = BE$$

and thus the arc sought

$$AE = 138°,\ 11',\ 53'',\ 0'''.$$

Truly the lines will be

$$DE = 0,6665578 \text{ et } AD = 1,7454535. \text{ Q. E. F.}$$

537. Now we will compare an arc with its tangents ; and, since in the first quadrant the tangents shall be less than the arcs, we may seek the arc, which shall be equal to half of its tangent, so that the following may be solved :

PROBLEM VII

To cut the sector ACD (Fig.117), *which shall be half the area of the triangle ACE, formed by the radius AC, tangent AE and secant CE .*

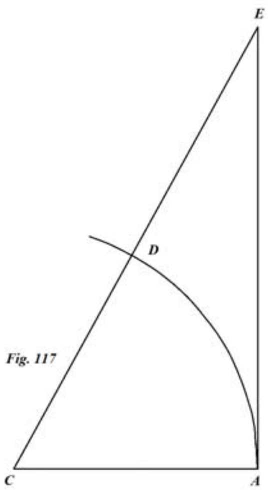

Fig. 117

SOLUTION

Putting the arc $AD = s$, the area of the sector will be $ACD = \frac{1}{2}s$, truly the area of the triangle will be $ACE = \frac{1}{2}\cdot\text{tang.}s$; from which there must be

$$\tfrac{1}{2}\cdot\text{tang.}s = s \text{ or } 2s = \text{tang.}s.$$

Therefore we may make this hypothesis

$s = 60°$	$s = 70°$	$s = 66°$	$s = 67°$
$l.2s =$ 2,0791812	2,1461280	2,1205739	2,1271048
1,7581226	1,7581226	1,7581226	1,7581226
$l.2s =$ 0,3210586	0,3880054	0,3624513	0,3689822
$l.\text{tang.}s =$ 0,2385606	0,4389341	0,3514169	0,3721481
+ 824980	− 509287	+ 110344	− 31659

Hence the closer limits of *s* may be found 66°, 46' and 66°, 47" whereby there becomes

$s = 66°,\ 46''$	$s = 66°,\ 47$
or	or
$s = 4006'$	$s = 4007'$
$2s = 8012'$	$2s = 8014'$
$l.2s = 3,9037409$	$3,9038493$
$\underline{3,5362739}$	$\underline{3,5362739}$
$l.2s = 0,3674670$	$0,3675754$
$l.\text{tang.}s = \underline{0.3672499}$	$\underline{0,3675985}$
Error $+ \quad 2171$	$- \quad 231$
$\underline{231}$	
2402	

$$2402\ :\ 2171 = 1' : 54'',\ 14''',$$

from which the arc will be $s = AD = 66°,\ 46',\ 54'',\ 14'''$
and hence the tangent $AE = 2,3311220$. Q. E. F.

538. Now the following may be proposed

PROBLEM VIII

For the proposed quadrant of the circle ACB (Fig.118) *to find the arc AE, which shall be equal to its chord AE produced as far as F.*

SOLUTION

Let the arc be $AE = s$, its chord will be $AE = 2 \cdot \sin.\frac{1}{2}s$, the versed sine $AD = 1 - \cos.s = 2 \cdot \sin.\frac{1}{2}s \cdot \sin.\frac{1}{2}s$; from which the similar triangles *ADE, ACF* will give

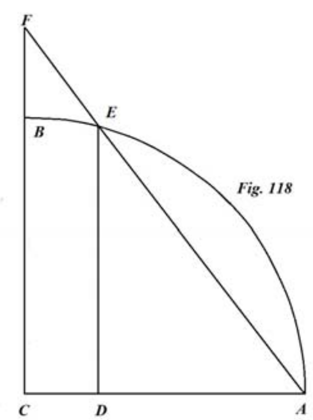

Fig. 118

$$2 \cdot \sin.\tfrac{1}{2}s \cdot \sin.\ \tfrac{1}{2}s :\ 2 \cdot \sin.\tfrac{1}{2}s = 1 : s$$

and therefore there will be $s \cdot \sin.\frac{1}{2}s = 1$. There the following may be put in place:

$s = 70°$	$s = 80°$	$s = 84°$	$s = 85°$
$l.\,s = 1{,}8450980$	1,9030900	1,9242793	1,9294189
Take away 1,7581226	1,7581226	1,7581226	1,7581226
0,0869754	0,1449674	0,1661567	0,1712963
$l.\sin\frac{1}{2}s = 9{,}7585913$	9,8080675	9,8255109	9,8296833
9,8455667	9,9530349	9,9916676	0,0009796
Error + 0,1544332	0,0469650	+ 83223	− 9796

From which *s* will be contained within the limits 84°, 53' and 84°, 54'.
Therefore there shall be

$s = 84°,\ 53'$	$s = 84°,\ 54'$
or	or
$s = 5093'$	$s = 5094'$
$\frac{1}{2}s = 42°,\ 26\frac{1}{2}'$	$\frac{1}{2}s = 42°,\ 27'$
$l.\,s = 3{,}7069737$	3,7070589
subtract 3,5362739	3,5362739
0,1706998	0,1707850
$l.\sin.\frac{1}{2}s = 9{,}8292003$	9,8292694
0,9999001	0,0000544
Error + 998	− 544

And hence there arises:

$$\text{the arc } s = AE = 84°,\ 53',\ 38'',\ 51'''$$

and

$$\text{the arc } BE = 5°,\ 6',\ 21'',\ 9'''.\ \text{Q. E. I.}$$

539. Although in the first quadrant all the arcs are smaller than their tangents, yet in the following quadrants arcs of this kind will be given, which shall be equal to their tangents, which we will investigate in the following problem by the method drawn from series.

PROBLEM IX

To find all the arcs, which shall be equal to their tangents.

SOLUTION

The first arc with this aforementioned is infinitely small. Then in the second quadrant, because here the tangents are negative, no arcs of this kind are given ; truly in the third quadrant one will be given a little less than 270°; again arcs of this kind will be given in the fifth, seventh etc. The fourth part of the periphery may be put $=q$, and the arc sought may be contained in this formula $(2n+1)q-s$,thus so that there shall be

$$(2n+1)q-s=\cot.s=\frac{1}{\text{tang}.s}$$

Let the tangent be $\text{tang}.s=x$; there will be

$$s=\frac{1}{x}+x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\text{etc.}$$

and thus

$$(2n+1)q=\frac{1}{x}+x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\text{etc.}$$

But it is apparent, on account of the arc *s* becoming smaller with that, as the number *n* becomes greater , *x* becomes a much smaller quantity and thus approximately :

$$x=\frac{1}{(2n+1)q} \text{ or } \frac{1}{x}=(2n+1)q\,;$$

but a closer value is found :

$$\frac{1}{x}=(2n+1)q-s=(2n+1)q-\frac{1}{(2n+1)q}-\frac{1}{3(2n+1)^3q^3}$$
$$-\frac{13}{15(2n+1)^5q^5}-\frac{146}{105(2n+1)^7q^7}-\frac{2343}{945(2n+1)^9q^9}-\text{etc.}$$

Therefore since there shall be

$$q=\frac{\pi}{2}=1,5707963267948,$$

the arc sought will be

$$=(2n+1)1,57079632679-\frac{1}{(2n+1)}0,63661977$$
$$-\frac{0,17200818}{(2n+1)^3}-\frac{0,09062598}{(2n+1)^5}-\frac{0,05892837}{(2n+1)^7}-\frac{0,04258548}{(2n+1)^9}-\text{etc.}$$

Or if these terms, which may be expressed in fractions of the radius, may be reduced to the measure of the arc, the arc sought in the kind considered

$$= (2n+1)\ 90° - \frac{131313''}{2n+1} - \frac{35479''}{(2n+1)^3} - \frac{18693''}{(2n+1)^5} - \frac{12155''}{(2n+1)^7} - \frac{8784''}{(2n+1)^9}.$$

Therefore the arcs sought satisfying the questions will be in order :

I.	1.	90° - 90°
II.	3.	90°−12°,32',48"
III.	5.	90° −7°, 22', 32",
IV.	7.	90° −5°, 14', 22",
V.	9.	90° −4°, 3', 59",
VI.	11.	90° −3°, 19', 24",
VII.	13.	90° −2°, 48', 37",
VIII.	15.	90° −2°, 26', 5",
IX.	17.	90° −2°, 8', 51",
X.	19.	90° −1°, 55', 16" .

540. I do not propose more questions of this kind, since the method for resolving these may be seen clearly from these examples. Moreover these problems have been thought out chiefly according to this end, so that the nature of the circle, the quadrature of which up to the present has been attempted by all methods with the usual disappointment, may be examined in more depth. If indeed it should come about, that in the solution of some problem either an arc should be produced commensurable with the whole circumference, or whether a construction might emerge of its sine or tangent through the radius, then certainly a certain kind of quadrature of the circle would be had. Evidently in the solution of problem VI. the sine DE (Fig. 116), which produced $= 0,6665578$, should it be found to be $= 0,6666666 = \frac{2}{3}$, certainly would become an elegant property of the circle, obviously the arc AE would allow the equal right line to be constructed

$$AD + DE = 1 + \frac{3}{3} + \sqrt{\frac{5}{9}}.$$

Truly even now no account is apparent, which vindicates that the quadrature of the circle is impossible, and, if such may be given, no other way besides this, that we have uncovered in this chapter, is seen to be more fitting for that investigation.

END OF BOOK II.

CAPUT XXII

SOLUTIO NONNULLORUM PROBLEMATUM AD CIRCULUM PERTINENTIUM

529. Posito radio circuli $=1$ supra vidimus fore semicircumferentiam π seu arcum 180 graduum

$$=3{,}14159265358979323846264338,$$

cuius numeri logarithmus decimalis seu vulgaris est

$$0{,}497149872694133854351268288\ ;$$

qui si multiplicetur per 2,30258 etc., prodibit eiusdem numeri logarithmus hyperbolicus, qui erit

$$=1{,}14472988584940017414342\!37\,.$$

Cum igitur longitudo arcus 180 graduum sit cognita, inde cuiusvis arcus in gradibus dati longitudo poterit assignari. Propositus sit arcus *n* graduum, cuius longitudo, quae quaeritur, sit $=z$; erit $180:n=\pi:z$ ideoque $z=\frac{\pi n}{180}$; hinc logarithmus ipsius *z* reperitur, si a logarithmo numeri *n* subtrahatur iste logarithmus

$$1{,}758122632409172215452526413.$$

Quodsi autem arcus propositus detur in minutis primis, ut sit *n'*, tum a logarithmo ipsius *n* subtrahi debebit iste logarithmus

$$3{,}536273882792815847961293211.$$

Sin autem arcus propositus detur in minutis secundis, ut sit *n''*, tum longitudinis istius arcus logarithmus reperietur, si a logarithmo numeri *n* subtrahatur iste logarithmus

$$5{,}314425133176459480470060009$$

vel si ad logarithmum numeri *n* addatur

$$4{,}6855748668235405195299399 90$$

et a characteristica summae 10 subtrahantur.

530. Ex his ergo vicissim radius et eius partes quaecunque, cuiusmodi sunt sinus, tangentes et secantes, in arcus converti hicque arcus more solito secundum gradus, minuta et secunda exprimi possunt. Sit *z* huiusmodi linea per radium 1 eiusque partes decimales expressa, sumatur eius logarithmus eiusque characteristica denario augeatur, quemadmodum in tabulis logarithmi sinuum, tangentium et secantium repraesentari solent; quo facto vel subtrahatur ab isto logarithmo

$$4{,}68557486682354051952993990$$

vel ad eundem logarithmum addatur

$$5{,}31442513317645948047006009\ ;$$

utroque casu prodibit logarithmus, cuius numerus respondens praebebit arcum in minutis secundis expressum. Posteriori quidem casu characteristica denario minui debet. Quodsi autem quaeratur arcus ipsi radio aequalis, hic sine logarithmis facilius per regulam auream invenitur, cum sit π ad 180° ut 1 ad arcum radio aequalem; hinc autem reperitur iste arcus in gradibus expressus

$$57°,\ 2957795130823208767 98,$$

idem vero arcus in minutis primis expressus erit

$$3437',\ 74677078493925260788\ ;$$

in minutis vero secundis erit idem arcus

$$= 206264'',\ 8062470963551564728\,.$$

Consueto autem more hic arcus expressus continebit

$$57°,\ 17',\ 44'',\ 48''',\ 22'''',\ 29''''',\ 22''''''.$$

Huius arcus per series in sectione superiori exhibitas reperitur

$$\text{sinus} = 0{,}84147098480789$$

et

$$\text{cosinus} = 0,54030230586814$$

quorum numerorum ille per hunc divisus dabit tangentem anguli

57°, 17', 44", 48"', 22"", 29""', *22*""" etc.

531. His igitur praemissis, quibus arcus circulares cum sinibus et tangentibus comparari possunt, plurimas quaestiones ad naturam circuli spectantes resolvere poterimus. Ac primo quidem patet omnem arcum sinu suo esse maiorem, nisi sit evanescens; aliter autem ratio cosinuum est comparata, quoniam anguli evanescentis cosinus est 1 ideoque arcu maior, anguli vero recti cosinus est 0 ideoque arcu est minor; ex quo patet intra limites 0° et 90° dari arcum, qui sit suo cosinui aequalis, quem sequente problemate investigemus.

PROBLEMA I

Invenire arcum circuli, qui sit suo cosinui aequalis.

SOLUTIO

Sit *s* iste arcus quaesitus, eritque $s = \cos.s$ *;* ex qua aequatione valor ipsius *s* commodius quam per regulam *falsi* dictam vix inveniri poterit. Ad hoc autem iam propemodum valorem ipsius *s* nosse oportet, quod vel levi coniectura assequi licet, nisi autem hoc pateat, tres pluresve valores loco *s* substituantur et cosinus pariter ad eandem unitatem revocetur. Ponamus $s = 30°$, quam arcum ad partes radii revocemus regula supra data

$$\begin{array}{rl} l.30 = & 1,4771213 \\ \text{subtrahe} & \underline{1,7581226} \\ l.\text{arc}.30° = & 9,7189987, \end{array}$$

at est

$$l.\cos.30° = 9,9375306,$$

unde patet cosinum 30° multo esse maiorem arcu ideoque arcum quaesitum maiorem esse 30°. Fingamus ergo

$$s = 40°$$

eritque

$$\begin{array}{rl} l.40 = & 1,6020600 \\ \text{subtrahe} & \underline{1,7581226} \\ l.\text{arc}.40° = & 9,8439374, \end{array}$$

at est

$$l.\cos.40° = 9,8842540,$$

hinc intelligitur arcum quaesitum aliquanto maiorem esse quam 40°, hancque ob rem fingamus $s = 45°$, erit

$$\begin{array}{rl} l.45 = & 1,6532125 \\ \text{subtrahe} & \underline{1,7581226} \\ l.\text{arc}.45° = & 9,8950899, \end{array}$$

at est

$$l.\cos.45° = 9,8494850,$$

continetur ergo angulus quaesitus inter 40° et 45° atque adeo hinc proxime definiri poterit. Nam posito $s = 40°$

$$\begin{array}{ll} \text{est error} & +403166 \\ \text{posito autem} & s = 45° \\ \text{est error} & \underline{= -456049} \\ \text{et differentia} & = -859215. \end{array}$$

Fiat ergo ut 859215 ad 403166 ita differentia hypothesium 5° ad excessum arcus quaesiti supra 40°, unde arcus quaesitus maior fit quam 42°, limites enim illi nimis sunt remoti, quam ut exactius definire queamus. Sumamus ergo limites propiores

	$s = 42°$	$s = 43°$
$l.s =$	1,6232493	1,6334685
subtrahe	1,7581226	1,7581226
$l.s =$	9,8651267	9,8753459
	et est	et est
$l.\cos.s =$	9,8710735	9,8641275
	+ 59468	-112184
	112184	

171652: 59468 = 1°: 20', 47".

Arctissimos ergo obtinuimus limites 42°, 20' et 42°, 21', intra quos verus ipsius s valor contineatur. Hos angulos ad minuta prima revocemus

	$s = 2540'$	$s = 2541'$
$l.s =$	3,4048337	3,4050047
subtrahe	3,5362739	3,5362739
$l.s =$	9,8685598	9,8687308
$l.\cos.s =$	9,8687851	9,8686700
	+ 2253	-608

608
2861: 2253 = 1': 47", 15'".

Hinc concludimus arcum quaesitum, qui suo cosinui sit aequalis, fore = 42°, 20', 47", 15'", huiusque cosinus, seu ipsa longitudo, erit $=0,73908502$. Q. E. I.

532. Sector circuli (Fig.112) *ACB* a chorda *AB* in duas partes secatur, segmentum *AEB* et triangulum *ACB*, quorum illud hoc minus est, si angulus *ACB* fuerit exiguus, maius autem, si angulus *ACB* sit admodum obtusus. Dabitur ergo casus, quo sector *ACB* per chordam *AB* in duas partes aequales secatur, unde nascitur

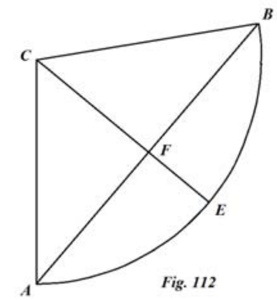

Fig. 112

PROBLEMA II

Invenire sectorem circuli ACB, qui a chorda AB in duas partes aequales secetur, ita ut triangulum ACB aequale sit segmento AEB.

SOLUTIO

Posito radio $AC=1$ sit arcus quaesitus $AEB=2s$, ut sit eius semissis $AE=BE=s$; ducto ergo radio *CE* erit $AF=\sin.s$ et $CF=\cos.s.$ Unde fit triangulum $ACB=\sin.s\cos.s=\frac{1}{2}\cdot\sin.2s$ et ipse sector *ACB* est *s*, qui cum aequari debeat duplo triangulo, erit $s=\sin.2s$; ideoque arcus quaeri debet, qui aequalis sit sinui arcus dupli. Primum quidem patet angulum *ACB* recto esse maiorem ideoque *s* superare 45°, unde sequentes faciamus hypotheses

	$s=50°$	$s=55°$	$s=54°$
$l.s=$	1,6989700	1,7403627	1,7323938
subtrahe	1,7581226	1,7581226	1,7581226
	9,9408474	9,9822401	9,9742712
$l.\sin.2s=$	9,9933515	9,9729858	9,9782063
	+ 525041	- 92543	+ 39351
	92543		

617584: 525041 =5°: 4°, 15'.

Erit ergo propemodum $s=54°,\ 15'$, unde ad superiores hypotheses addamus $s=54°$, et ex erroribus concludetur $s=54°,17',54''$, qui valor a vero minuto integro non discrepat; faciamus ergo sequentes positiones minuto tantum discrepantes

$s = 54°,\ 17'$	$s = 54°,\ 18'$	$s = 54°,\ 19'$
seu	seu	seu
$s = 3257'$	$s = 3258'$	$s = 3259'$
et	et	et
$2s = 108°,\ 34'$	$2s = 108°,\ 36'$	$2s = 108°,\ 38'$
$\text{compl.} = 71°,\ 26'$	$\text{compl.} = 71°,\ 24'$	$\text{compl.} = 71°,\ 22'$
$l.s = 3{,}5128178$	$3{,}5129511$	$3{,}5130844$
subtrahe $\underline{3{,}5362739}$	$\underline{3{,}5362739}$	$\underline{3{,}5362739}$
$l.s = 9{,}9765439$	$9{,}9766772$	$9{,}9768105$
$l.\sin.2s = 9{,}9767872$	$\underline{9{,}9767022}$	$\underline{9{,}9766171}$
$+\ 2433$	$+\ 250$	$-\ 1934$
	$\underline{1934}$	
	2184	

fiat ergo $2184 : 250 = 1' : 6'',52'''$.

Hinc erit $s = 54°,\ 18',\ 6'',\ 52'''$. Si hunc angulum accuratius determinare velimus, maioribus tabulis uti oportet; unde faciamus sequentes hypotheses

10" differentes

$s = 54°,\ 18',\ 0''$	$s = 54°,\ 18',\ 10''$
seu	seu
$s = 195480''$	$s = 195490''$
$2s = 108°,\ 36',\ 0''$	$2s = 108°,\ 36',\ 20''$
$\text{compl.} = 71°,\ 24',\ 0''$	$\text{comp.} = 71°,\ 23',\ 40''$
$l.\ s = 5{,}2911023304$	$5{,}2911245466$
subtrahe $\underline{5{,}3144251332}$	$\underline{5{,}3144251332}$
$9{,}9766771972$	$9{,}9766994134$
$l.\sin.2s = 9{,}9767022291$	$\underline{9{,}9766880552}$
$+\ 250319$	$-\ 113582$
$\underline{113582}$	

363901: 250319 = 10": 6",52''',43'''',33'''''

Erit ergo $s = 54°,\ 18',\ 6'',\ 52''',\ 43'''',\ 33'''''$

ideoque angulus $ACB = 108°,\ 36',\ 13'',\ 45''',\ 27'''',\ 6'''''$

eiusque complementum $= 71°,\ 23',\ 46'',\ 14''',\ 32'''',\ 54'''''$

cuius sinus logarithmus seu

$$l.\sin.2s = 9{,}9766924791$$

$$\text{et ipse sin.} = 0{,}9477470.$$

Deinde erit

$$\sin.s = AF = BF = 0{,}8121029$$

ideoque eius duplum seu

$$\text{chorda } AB = 1,6242058 .$$

Praeterea vero erit

$$\cos.CF = 0,5835143.$$

Sicque vero proxime sector quaesitus construi poterit. Q.E.I.

533. Simili modo determinari potest sinus, quo circuli quadrans in duas partes aequales secatur.

PROBLEMA III

In quadrante circuli (Fig.113) *ACB applicare sinum DE, qui arcam quadrantis in duas partes aequales bisecet.*

SOLUTIO

Sit arcus $AE = s;$ erit $BE = \frac{\pi}{2} - s$ ob $AEB = \frac{\pi}{2}$ et

area quadrantis $= \frac{\pi}{4}$. Iam area sectoris ACE est $= \frac{1}{2}s$,

a qua triangulum

$$CDE = \tfrac{1}{2} \cdot \sin.s \cdot \cos.s$$

subtractum relinquet spatium

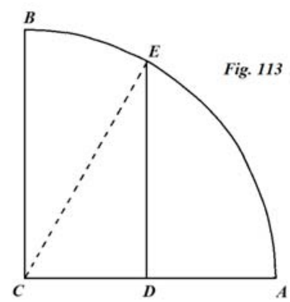

Fig. 113

$$ADE = \tfrac{1}{2}s - \tfrac{1}{2} \cdot \sin.s \cdot \cos.s,$$

cuius duplum dare debet quadrantem; ex quo erit

$\frac{1}{4}\pi = s - \frac{1}{2} \cdot \sin.2s,$ ergo $s - \frac{1}{4}\pi = \frac{1}{2} \cdot \sin.2s$

Ponatur arcus

$$s - \tfrac{1}{4}\pi = s - 45° = u,$$

erit $2s = 90° + 2u$ ideoque esse oportet

$$u = \tfrac{1}{2} \cdot \cos.\ 2u \text{ et } 2u = \cos.2u .$$

Cum ergo arcus requiratur, qui suo cosinui aequetur, eumque problemate primo invenerimus, erit

$2u$ = 42°, 20', 47'', 15''' et u = 21°, 10', 23'', 37'''.

Quocirca erit

arcus $AE = s = 66°,\ 10',\ 23'',\ 37'''$ et arcus $BE = 23°,\ 49',\ 36'',\ 23'''$.

Hinc erit radii pars

$$CD = 0,4039718 \text{ et } AD = 0,5960281$$

$$\text{atque sinus } DE = 0,9147711.$$

Hoc ergo modo, quo circuli quadrans bisecatur, totus circulus secabitur in 8 partes aequales. Q. E. F.

534. Quemadmodum circulum omnis recta per centrum ducta bifariam secat, ita ex quovis peripheriae puncto rectae educi poterunt, quae circulum in tres pluresve partes aequales secent. Inquiramus in quadrisectionem ac resolvamus.

PROBLEMA IV

Proposito semicirculo AEDB (Fig.114) *ex puncto A educere chordam AD, quae aream semicirculi in duas partes aequales secet.*

SOLUTIO

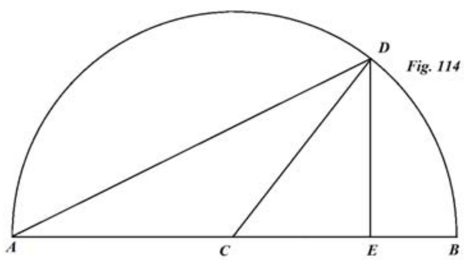
Fig. 114

Sit arcus quaesitus $AD = s$, ductoque radio CD erit area sectoris $ACD = \frac{1}{2}s$, a qua auferatur triangulum

$$ACD = \tfrac{1}{2}AC \cdot DE = \tfrac{1}{2} \cdot \sin.s,$$

remanebitque segmentum

$$AD = \tfrac{1}{2}s - \tfrac{1}{2} \cdot \sin.s,$$

quod aequale esse debet semissi semicirculi *ADB;* at area semicirculi est $= \frac{1}{2}\pi$, unde erit

$$s - \sin.s = \frac{1}{2}\pi = 90° \text{ ideoque } s - 90° = \sin.s.$$

Ponatur $s - 90° = u$; erit $\sin.s = \cos.u$ et hanc ob rem $u = \cos.u$.
Per problema ergo primum erit

$$u = 42°,\ 20',\ 47'',\ 14'''$$

hincque

$$s = \text{angulo } ACD = 132°,\ 20',\ 47'',\ 14''' \text{ et angulus } BCD = 47°,\ 39',\ 12'',\ 46'''.$$

Ipsa vero corda AD erit $= 1,8295422$. Q. E. F.

535. Sic igitur in circulo segmentum abscinditur, cuius area sit totius circuli pars quarta; segmentum autem semissi circuli aequale, est ipse semicirculus eiusque corda diameter. Simili modo segmentum inveniri potest, quod sit triens totius circuli, quod sequenti problemate investigemus.

PROBLEMATA V

Ex puncto peripheriae A (Fig.115) *educere duas cordas AB, AC, quibus area circuli in tres partes aequales dividatur.*

SOLUTIO

Posito circuli radio $=1$ et hemiperipheria $=\pi$, sit arcus *AB* vel $AC = s$; eritque area segmenti *AEB* vel *AFC*

$$=\tfrac{1}{2}s-\tfrac{1}{2}\cdot\sin.s\,;$$

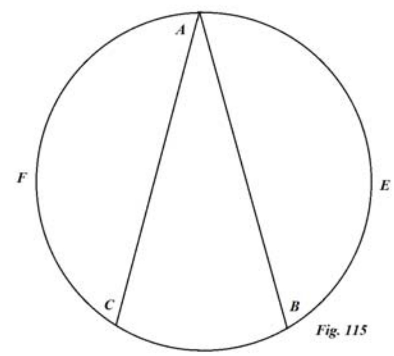

Fig. 115

at area circuli est $=\pi$; unde, cum segmenti *AEB* area debeat esse triens circuli, fiet

$$\tfrac{1}{2}s-\tfrac{1}{2}\cdot\sin.s=\frac{\pi}{3}=\ 60°\ \text{seu}\ \ s-\sin.s=120°$$

ideoque

$$s-120°=\sin.s$$

Sit $s-120°=u$, erit $u=\sin.(u+120°)=\sin.(60°-u)$. Arcus ergo u quaeri debet, qui sit aequalis sinui anguli $60°-u$. Erit ergo u minor quam 60°; ad quem arcum inveniendum faciamus sequentes positiones

$u=20°$	$u=30°$	$u=40°$
$60°-u=40°$	$60°-u=30°$	$60°-u=20°$
$l.u=1,3010300$	1,4771213	1,6020600
subtrahe 1,7581226	1,7581226	1,7581226
$l.u=9,5429074$	9,7189987	9,8439374
$l.\sin.(60°-u)=9,8080675$	9,6989700	9,5340517
+2651601	−200287	−3098857

Patet ergo angulum u aliquanto esse minorem quam 30° et calculo subducto maior esse debet quam 29°; sit ergo

$$\begin{aligned} u &= 29^\circ \\ 60^\circ - u &= 31^\circ \\ l.u &= 1{,}4623980 \\ \text{Subtrahe}\quad & \underline{1{,}7581226} \\ l.u &= 9{,}7042754 \\ l.\sin.(60^\circ - u) &= \underline{9{,}7118393} \\ & +75639 \\ & -200287 \\ \hline 275926 : 75639 & = 1^\circ{:}\ 16',\ 26''. \end{aligned}$$

Foret ergo angulus u = 29°, 16', 26", ad quem accuratius inveniendum faciamus has hypotheses uno tantum minuto differentes

$u = 29^\circ, 16'$	$u = 29^\circ,\ 17'$
seu	seu
$u = 1756'$	$u = 1757'$
$60^\circ - u = 30^\circ,\ 44'$	$60^\circ - u = 30^\circ,\ 43'$
$l.\ u = 3{,}2445245$	$3{,}2447718$
subtrahe $\underline{3{,}5362739}$	$\underline{3{,}5362739}$
$l.u = 9{,}7082506$	$9{,}7084979$
$l.\sin.(60^\circ - u) = \underline{9{,}7084575}$	$9{,}7082450$
$+ \quad 2069$	$- \quad 2529$
$\underline{2529}$	

$$4598{:}2069 : \ 1' : 27''' \ 0''''.$$

Erit ergo vere

$$u = 29^\circ,\ 16',\ 27'',\ 0'''$$

hincque

$$\text{arcus}\ \ s = AEB = 149^\circ, 16', 27'', 0''' = AFC\ ;$$

unde resultat

$$\text{arcus}\ \ BC = 61^\circ, 27', 6'', 0''',$$

ipsa vero

chorda $AB = AC = 19285340$. Q. E. F.

536. His problematis, quibus arcus quispiam quaeritur dato sinui vel cosinui aequalis, adiungamus sequens, quo quidem idem negotium proponitur, attamen maior difficultas occurrit.

PROBLEMA VI

In semicirculo (Fig.116) *AEB arcum AE abscindere, ita ut ducto eius sinu ED arcus AE sit aequalis summae rectarum* $AD + DE$.

SOLUTIO

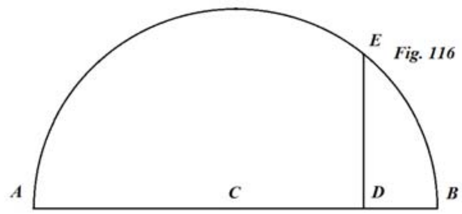

Fig. 116

Quoniam statim patet hunc arcum quadrante esse maiorem, quaeramus eius complementum *BE* et vocemus arcum $BE = s$, ita ut sit arcus $AE = 180° - s$, atque ob $AC = 1,\ CD = \cos.s,\ DE = \sin.s$, erit

$180° - s = 1 + \cos.s + \sin.s$

At est

$$\sin.s = 2\sin.\tfrac{1}{2}s \cdot \cos.\tfrac{1}{2}s \quad \text{et} \quad 1 + \cos.s = 2\cos.\tfrac{1}{2}s \cdot \cos.\tfrac{1}{2}s\,;$$

unde fit

$$180° - s = 2\cos.\tfrac{1}{2}s\left(\sin.\tfrac{1}{2}s + \cos.\tfrac{1}{2}s\right).$$

At est

$$\cos.\left(45° - \tfrac{1}{2}s\right) = \frac{1}{\sqrt{2}} \cdot \cos.\tfrac{1}{2}s + \frac{1}{\sqrt{2}} \cdot \sin.\tfrac{1}{2}s,$$

ergo

$$\sin.\tfrac{1}{2}s + \cos.\tfrac{1}{2}s = \sqrt{2}\cos.\left(45° - \tfrac{1}{2}s\right),$$

unde erit

$$180° - s = 2\sqrt{2} \cdot \cos.\tfrac{1}{2}s \cdot \cos.\left(45° - \tfrac{1}{2}s\right).$$

Hac facta reductione faciamus sequentes positiones :

$\tfrac{1}{2}s = 20°$	$\tfrac{1}{2}s = 21°$
$45° - \tfrac{1}{2}s = 25°$	$45° - \tfrac{1}{2}s = 24°$
$180° - s = 140°$	$180° - s = 138°$
$l.(180° - s) = 2{,}1461280$	$2{,}1398791$
subtrahe $\underline{1{,}7581226}$	$\underline{1{,}7581226}$
$l.(180° - s) = 0{,}3880054$	$0{,}3817565$
$l.\cos.\tfrac{1}{2}z = 9{,}9729858$	$9{,}9701517$
$l.\cos.(45° - \tfrac{1}{2}s) = 9{,}9572757$	$9{,}9607302$
$l.2\sqrt{2} = 0{,}4515450$	$0{,}4515450$
$0{,}3818065$	$0{,}3824269$

Error	+ 61989 6704	− 6704
	68693 : 61989 = 1° : 54'	

Hinc continetur $\frac{1}{2}s$ intra limites 20°, 54' et 20° 55' ideoque sequentes hypotheses fiant

$\frac{1}{2}s = 20°,\ 54'$	$\frac{1}{2}s = 20°,\ 55'$
$45° - \frac{1}{2}s = 24°,\ 6'$	$45° - \frac{1}{2}s = 24°,\ 5'$
$s = 41°,\ 48'$	$s = 41°,\ 50'$
$180° - s = 138°,\ 12'$	$180° - s = 138°,\ 10'$
seu	seu
$180° - s = 8292'$	$180° - s = 8290'$
$l.(180° - s) = 3,9186593$	$3,9185545$
subtrahe $3,5362739$	$3,5362739$
$0,3823854$	$0,3822806$
$l.\cos.\frac{1}{2}s = 9,9704419$	$9,9703937$
$l.\cos.\left(45° - \frac{1}{2}s\right) = 9,9603919$	$9,9604484$
$l.2\sqrt{2} = 0,4515450$	$0,4515450$
$0,3823788$	$0,3823871$
Error + 66	− 1065
1065	
1131 : 66 = 1' : 3", 30'".	

Hanc ob rem erit $\frac{1}{2}s = 20°,\ 54',\ 3'',\ 30'''$, inde

$$s = 41°,\ 48',\ 7'',\ 0''' = BE$$

ideoque arcus quaesitus

$$AE = 138°,\ 11',\ 53'',\ 0'''.$$

Erit vero linea

$$DE = 0,6665578 \text{ et } AD = 1,7454535. \text{ Q. E. F.}$$

537. Comparemus nunc arcus cum suis tangentibus; et, cum in primo quadrante tangentes sint arcubus minores, quaeramus arcum, qui suae tangentis semissi sit aequalis, quo solvetur

PROBLEMA VII

Abscindere sectorem ACD (Fig.117), *qui sit semissis trianguli ACE a radio AC, tangente AE et secante CE comprehensi.*

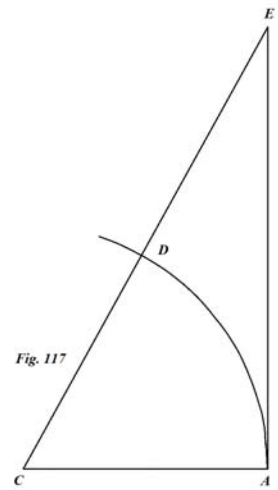

Fig. 117

SOLUTIO

Posito arcu $AD = s$ erit sector $ACD = \frac{1}{2}s$, triangulum vero $ACE = \frac{1}{2}\cdot \text{tang.}s$; unde debet esse $\frac{1}{2}\cdot\text{tang.}s = s$ seu $2s = \text{tang.}s$. Faciamus ergo has hypotheses

$s = 60°$	$s = 70°$	$s = 66°$	$s = 67°$
$l.2s = 2{,}0791812$	$2{,}1461280$	$2{,}1205739$	$2{,}1271048$
$\underline{1{,}7581226}$	$\underline{1{,}7581226}$	$\underline{1{,}7581226}$	$\underline{1{,}7581226}$
$l.2s = 0{,}3210586$	$0{,}3880054$	$0{,}3624513$	$0{,}3689822$
$l.\text{tang}.s = 0{,}2385606$	$0{,}4389341$	$0{,}3514169$	$0{,}3721481$
$+\ 824980$	$-\ 509287$	$+\ 110344$	$-\ 31659$

Hinc ipsius *s* reperiuntur limites arctiores 66°, 46' et 66°, 47" quare fiat

$s = 66°,\ 46''$	$s = 66°,\ 47$
seu	seu
$s = 4006'$	$s = 4007'$
$2s = 8012'$	$2s = 8014'$
$l.2s = 3{,}9037409$	$3{,}9038493$
$\underline{3{,}5362739}$	$\underline{3{,}5362739}$
$l.2s = 0{,}3674670$	$0{,}3675754$
$l.\text{tang}.s = \underline{0.3672499}$	$\underline{0{,}3675985}$
Error $+\ 2171$	$-\ 231$
$\underline{231}$	
2402	

$$2402 : 2171 = 1' : 54'',\ 14''',$$

unde erit

$$\text{arcus } s = AD = 66°,\ 46',\ 54'',\ 14'''$$

hincque

tangens $AE = 2,3311220$. Q. E. F.

538. Proponatur nunc sequens

PROBLEMA VIII

Proposito circuli quadrante ACB (Fig.118) *invenire arcum AE, qui aequalis sit chordae suae AE ad occursum F usque productae.*

SOLUTIO

Sit arcus $AE = s$, erit eius chorda $AE = 2\cdot\sin.\frac{1}{2}s$, sinus versus $AD = 1-\cos.s = 2\cdot\sin.\frac{1}{2}s\cdot\sin.\frac{1}{2}s$; unde triangula similia *ADE, ACF* dabunt

$$2\cdot\sin.\tfrac{1}{2}s\cdot\sin.\tfrac{1}{2}s : 2\cdot\sin.\tfrac{1}{2}s = 1 : s$$

eritque ergo $s\cdot\sin.\frac{1}{2}s = 1$. Fiant ergo sequentes positiones

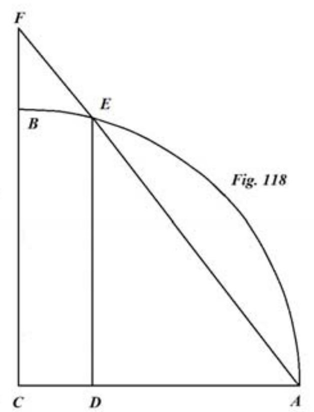

Fig. 118

$s = 70°$	$s = 80°$	$s = 84°$	$s = 85°$
$l.\ s = 1,8450980$	1,9030900	1,9242793	1,9294189
subtrahe 1,7581226	1,7581226	1,7581226	1,7581226
0,0869754	0,1449674	0,1661567	0,1712963
$l.\sin\frac{1}{2}s = 9,7585913$	9,8080675	9,8255109	9,8296833
9,8455667	9,9530349	9,9916676	0,0009796
Error + 0,1544332	0,0469650	+ 83223	− 9796

Unde *s* continetur intra limites 84°, 53' et 84°, 54'.
Sit ergo

$s = 84°,\ 53'$	$s = 84°,\ 54'$
seu	seu
$s = 5093'$	$s = 5094'$
$\frac{1}{2}s = 42°,\ 26\frac{1}{2}'$	$\frac{1}{2}s = 42°,\ 27'$
$l.\ s = 3{,}7069737$	$3{,}7070589$
subtrahe $3{,}5362739$	$3{,}5362739$
$0{,}1706998$	$0{,}1707850$
$l.\sin.\frac{1}{2}s = 9{,}8292003$	$9{,}8292694$
$0{,}9999001$	$0{,}0000544$
Error + 998	− 544

Hincque oritur

$$\text{arcus } s = AE = 84°,\ 53',\ 38'',\ 51'''$$

et

$$\text{arcus } BE = 5°,\ 6',\ 21'',\ 9'''.\ \text{Q. E. I.}$$

539. Quanquam in primo quadrante omnes arcus sunt suis tangentibus minores, tamen in sequentibus quadrantibus dantur eiusmodi arcus, qui sint aequales suis tangentibus, quos in sequenti problemate methodo ex seriebus petita investigemus.

PROBLEMA IX

Invenire omnes arcus, qui tangentibus suis sint aequales.

SOLUTIO

Primus arcus hac proprietate praeditus est infinite parvus. Tum in secundo quadrante, quia hic tangentes sunt negativae, datur nullus istiusmodi arcus; in tertio vero quadrante dabitur unus 270° aliquanto minor; porro dabuntur eiusmodi arcus in quinto, septimo etc. Ponatur quarta peripheriae pars $= q$, et arcus quaesiti contineantur in hac forma $(2n+1)q - s$, ita ut sit

$$(2n+1)q - s = \cot.s = \frac{1}{\text{tang.}s}$$

Sit $\text{tang.}s = x$; erit

$$s = \frac{1}{x} + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{etc.}$$

ideoque

$$(2n+1)q = \frac{1}{x} + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{etc.}$$

Patet autem, ob *s* arcum eo minorem, quo maior fuerit numerus *n*, fore *x* quantitatem valde parvam ideoque proxime

$$x = \frac{1}{(2n+1)q} \text{ seu } \frac{1}{x} = (2n+1)q\,;$$

propius autem invenitur

$$\frac{1}{x} = (2n+1)q - s = (2n+1)q - \frac{1}{(2n+1)q} - \frac{1}{3(2n+1)^3 q^3}$$
$$- \frac{13}{15(2n+1)^5 q^5} - \frac{146}{105(2n+1)^7 q^7} - \frac{2343}{945(2n+1)^9 q^9} - \text{etc.}$$

Cum ergo sit

$$q = \frac{\pi}{2} = 1,5707963267948,$$

erit arcus quaesitus

$$= (2n+1)1,57079632679 - \frac{1}{(2n+1)}0,63661977$$
$$- \frac{0,17200818}{(2n+1)^3} - \frac{0,09062598}{(2n+1)^5} - \frac{0,05892837}{(2n+1)^7} - \frac{0,04258548}{(2n+1)^9} - \text{etc.}$$

Vel si isti termini, qui in partibus radii exprimuntur, ad mensuram arcuum reducantur, erit arcus quaesitus in genere consideratus

$$= (2n+1)\ 90° - \frac{131313''}{2n+1} - \frac{35479''}{(2n+1)^3} - \frac{18693''}{(2n+1)^5} - \frac{12155''}{(2n+1)^7} - \frac{8784''}{(2n+1)^9}.$$

Arcus ergo quaestioni satisfacientes ordine sunt:

I.	1.	90° - 90°
II.	3.	90°−12°,32',48"
III.	5.	90° −7°, 22', 32",
IV.	7.	90° −5°, 14', 22",
V.	9.	90° −4°, 3', 59",
VI.	11.	90° −3°, 19', 24",
VII.	13.	90° −2°, 48', 37",
VIII.	15.	90° −2°, 26', 5",
IX.	17.	90° −2°, 8', 51",
X.	19.	90° −1°, 55', 16" .

540. Huiusmodi quaestiones plures non propono, cum methodus eas resolvendi ex his exemplis clare perspiciatur. Ceterum haec problemata in hunc finem potissimum sunt excogitata, ut circuli natura, cuius quadratura omnibus methodis adhuc usitatis frustra fuit tentata, penitius inspiciatur. Si enim accidisset, ut in solutione cuiuspiam problematis vel arcus cum tota circumferentia commensurabilis vel eius sinus tangensve per radium construibilis prodiisset, tum utique species quaedam quadraturae circuli haberetur.
Scilicet si in solutione problematis VI. sinus *DE* (Fig. 116), qui prodit $=0,6665578$, inventus fuisset $=0,6666666=\frac{2}{3}$, elegans certe circuli proprietas innotesceret, arcus quippe *AE* construi posset lineae rectae

$$AD+DE=1+\frac{3}{3}+\sqrt{\frac{5}{9}}$$

aequalis. Nulla vero etiamnum ratio patet, quae huiusmodi circuli quadraturam impossibilem esse evincat, atque, si talis detur, nulla alia via praeter hanc, quam hoc capite aperuimus, ad eam investigandam magis apta videtur.

FINIS LIBRI SECUNDI