CHAPTER XXI

CONCERNING TRANSCENDING CURVES

506. So far we have been concerned with algebraic curves, which have been prepared thus, so that, with abscissas taken on some axis, the corresponding applied lines may be expressed by algebraic functions of the abscissas or what returns the same, in which a certain relation shall be able to be expressed between the abscissas and the applied lines by an algebraic equation. And thus hence it follows at once, if the value of the applied line may be unable to be explained by an algebraic function of the abscissa, it is not possible to enumerate a curved line algebraically. But curved lines of this kind, which are not algebraic, are usually called *transcending*. Therefore a transcending curved line may be defined thus, so that it may be called a curve of this kind, in which the relation between the abscissas and the applied line may not be able to be expressed by an algebraic equation. Therefore in just as many ways as the applied line y may be equal to a transcending function of the abscissa x, in so many ways will the curved line be required to be referred to a kind of transcending curve.

507. In the above section we have been occupied mainly with two kinds of transcending quantities, one of which included logarithms and the other the arcs of circles or angles. But if therefore the applied line y shall be equal either to the logarithm of its abscissa x or to the arc of a circle, of which either the sine, cosine, or tangent is expressed by the abscissa x, thus so that there shall be either $y = lx$ or $y = A\cdot \sin. x$, $y = A\cdot \cos. x$, or $y = A\cdot \text{tang}. x$, or if values of this kind may be present only in an equation between x and y, then the curve will be transcending. But these curves are only a kind of transcendence ; for besides these innumerable other transcending expressions are given, the origin of which may be set out more fully in the analysis of infinite quantities, thus so that the number of transcending curves will exceed by far the number of algebraic curves.

508. Whichever function is not algebraic, that is transcending and thus is a curve in the equation of which a transcendence is returned. But an algebraic equation either is rational and it may contain no exponents besides whole numbers, or it is irrational and it may include fractional exponents ; but in this latter case the equation can be returned to rationality always. Therefore an equation expressing the relation between the coordinates x and y prepared thus, so that it shall be neither rational nor able to be led to rationality, that always is transcending. But if therefore in an equation of this kind powers occur, the exponents of which shall be neither whole numbers nor fractions, in no manner will the equation be able to be let back to rationality and thus curves expressed by such equations will be transcending. Hence the first kind and as if the simplest of transcending curves arise, in the equations of which irrational exponents shall be present ; which because they involve neither logarithms nor the arcs of circles, but arise from the notion of irrational numbers only, in a certain way may be considered to belong more to common geometry

and on this account have been called *interscending* by Leibnitz, as if they hold a middle position between algebraic and transcending curves.

509. Therefore there will be an intersending curve of this kind, which may be expressed by the equation $y = x^{\sqrt{2}}$; for to whatever powers taken this equation may be raised, in no manner can it be returned to rationality. Moreover in no way can such a curve be constructed geometrically. Indeed no other powers can be shown geometrically, unless the exponents shall be rational numbers, and on this account curves of this kind disagree especially with algebraic curves. For if we wish to show the exponent $\sqrt{2}$ truly only approximately, by putting any one from these fractions in its place

$$\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{96}{70},$$

which will express the value $\sqrt{2}$ approximately, indeed certain algebraic curves will be produced approaching closely to the sought curve, but they will be of the third, seventh, 17^{th}, or 41^{st} order, etc. Whereby, since $\sqrt{2}$ cannot be expressed rationally except by a fraction, of which the numerator and denominator shall be infinitely great numbers, this curve will be found by increasing the orders of the lines by infinitesimals, and thus cannot be had algebraically. Here it is agreed, that $\sqrt{2}$ may involve a double value, the one positive and the other negative, from which y will result in a double value and thus a double curve will come about.

510. Then truly if we should wish to construct this curve exactly, we are unable to put this in place without the aid of logarithms. For since there shall be $y = x^{\sqrt{2}}$, with logarithms taken there will be $ly = \sqrt{2} \cdot lx$, therefore the logarithm of any abscissa multiplied by $\sqrt{2}$ will give the logarithm of the applied line, from which from some abscissa x the corresponding applied line will be assigned from the table of logarithms. Thus, if there were $x = 0$, there will be $y = 0$, if $x = 1$, there will be $y = 1$; which values are readily found from the equation, but if $x = 2$, there will be
$ly = \sqrt{2} \cdot l2 = \sqrt{2} \cdot 0,3010300$ and on account of $\sqrt{2} = 1,41421356$ there will be
$ly = 0,4257207$ and thus $y = 2,665144$ approximately and if $x = 10$, there will be
$ly = 1,41421356$ and hence $y = 25,954554$. Therefore in this manner for all the individual abscissas the applied lines can be computed and thus the curve will be able to be constructed, if indeed positive values may be attributed to the abscissa x. But if the abscissa x may possess negative values, then it is said with difficulty, whether the values of y shall become real or imaginary ; for if there shall be $x = -1$, and which shall be $(-1)^{\sqrt{2}}$, will not be able to be defined, because no approximations to the value $\sqrt{2}$ can bring help.

511. Much less will be in doubt, if the equations in which the imaginary exponents are being found, ought to be referred to a kind of transcending function. But generally it must arise, so that the expression containing the imaginary exponents may show a real and determined value. Examples of this situation have occurred now above, from which it may suffice here to advance this example

$$2y = x^{+\sqrt{-1}} + x^{-\sqrt{-1}} ,$$

in which, even if each member $x^{+\sqrt{-1}}$ and $x^{-\sqrt{-1}}$ shall be an imaginary quantity, yet the sum of both has a real value. For there shall be $lx = v$, with *e* taken for the number, the hyperbolic logarithm of which is $=1$, and there will be $x = e^{v}$, with which value substituted for *x* there becomes

$$2y = e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}} .$$

But we have seen in the above section §138 to be

$$\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2} = \cos.\mathrm{A}.v ,$$

from which there becomes :

$$y = \cos.\mathrm{A}.v = \cos.\mathrm{A}.lx.$$

Evidently with some value of *x* proposed numerically, the hyperbolic logarithm of which may be taken, then in the circle, the radius of which $=1$, an arc may be cut off equal to that logarithm, and of which the cosine of the arc will give the value of the applied line *y*. Thus, if $x = 2$ may be taken, so that the equation becomes

$$2y = 2^{+\sqrt{-1}} + 2^{-\sqrt{-1}} ,$$

there will be

$$y = \cos.\mathrm{A}.l2 = \cos.\mathrm{A}.0,6931471805599 .$$

But this arc itself equals *l*2, since the arc 3,1415926535 etc. shall contain 180°, by the golden rule may be found to become 39°, 42', 51'', 52''', 8'''', the cosine of which is 0,76923890136400, and this number gives the value of the applied line *y* corresponding to the abscissa $x = 2$. Therefore since in this manner the expressions involve both logarithms and circular arcs, by right they are referred to transcending functions.

512. Therefore transcending curves hold the first place, of which the equations involve logarithms in addition to algebraic quantities, and the simplest of these will be, which shall be expressed by this equation

$$l.\frac{y}{a}=\frac{x}{b} \text{ or } x=b\,l.\frac{y}{a},$$

where it is likewise, whatever kind of logarithm may be taken, because by the multiplication of the constant *b* all systems of logarithms may be returned to the same. Therefore the letter *l* shall denote hyperbolic logarithms, and the curve expressed by the equation $x=bl\frac{y}{a}$ is known generally under the name of *logarithm*. Let *e* be the number, of which the logarithm is $=1$, thus so that there shall be $e=2,718281828459045235360 28$, and there becomes :

$$e^{x:b}=\frac{y}{a} \quad \text{or} \quad y=ae^{x:b}$$

from which equation the nature of the logarithmic curve is most easily recognised. For if in place of *x* successively values may be substituted proceeding in an arithmetic progression, the values of the applied lines *y* will maintain values between themselves in a geometric progression. So that which may be applied more easily to the construction, there may be put

$$e=m^n \quad \text{and} \quad b=nc,$$

and thus

$$y=am^{x:c}$$

where *m* can indicate some positive number greater than one.
Therefore if there shall be

$$x=0,\ c,\ 2c,\ 3c,\ 4c,\ 5c,\ 6c \text{ etc.},$$

the above equation becomes

$$y=a,\ am,\ amm,\ am^3,\ am^4,\ am^5,\ am^6 \text{ etc.};$$

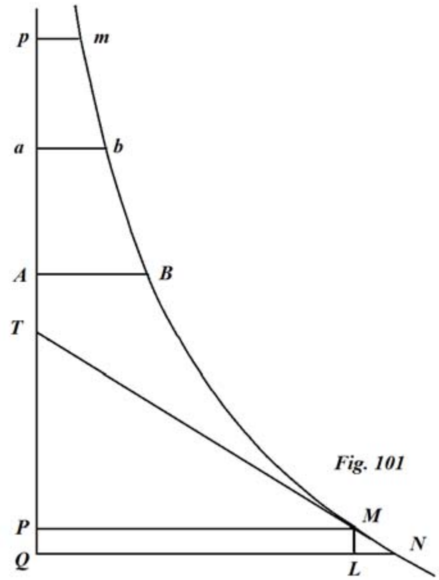
and with negative values attributed to *x*, if here may be put

$$x=-c,\ -2c,\ -3c,\ -4c,\ -5c \text{ etc.},$$

there will be

$$y=\frac{a}{m},\ \frac{a}{mm},\ \frac{a}{m^3},\ \frac{a}{m^4},\ \frac{a}{m^5} \text{ etc.}$$

513. Hence it is apparent (Fig. 101) the applied lines y have positive values everywhere and indeed increasing indefinitely with the positive abscissas x increasing to infinity, but deceasing from infinity from the other side of the axis, thus so that hence the axis shall be the asymptote Ap of the curve. Evidently with A taken for the beginning of the abscissas in this place there will be the applied line $AB = a$ and with the abscissa taken $AP = x$ the applied line will be $PM = y = am^{x:c} = ae^{x:b}$ and thus



$$l \cdot \frac{y}{a} = \frac{x}{b}.$$

From which the abscissa AP divided by the constant b will express the logarithm of the ratio $\frac{PM}{AB}$. If the start of the abscissas may be placed at some other point a on the axis, the equation remains similar. For there shall be $Aa = f$, and on putting $aP = t$ on account of $x = t - f$ there will be

$$y = ae^{(t-f):b} = ae^{t:b} : e^{f:b}.$$

The constant may be called $ae^{f:b} = g$, there will be $y = ge^{t:b}$. Hence on account of $ab = g$, the equation may be understood to become :

$$\frac{aP}{b} = l \cdot \frac{PM}{ab}$$

and thus with any two applied lines PM and pm drawn, the interval Pp in turn from these distance themselves, and there will be

$$\frac{Pp}{b} = l \cdot \frac{PM}{pm}$$

and the constant b, on which the relation depends, will be an example of the parameter of the logarithm.

514. The tangent of this logarithmic curve will be able to be defined easily at some point M. For since on putting $AP = x$ there shall be $PM = ae^{x:b}$, some other applied line QN may be drawn put in place at the interval $PQ = u$ from the first and there shall be

$$QN = ae^{(x+u):b} = ae^{x:b} \cdot e^{u:b};$$

and with *ML* drawn parallel to the axis there will be

$$LN = (QN - PM) = ae^{x:b}(e^{u:b} - 1).$$

The right line *NMT* may be drawn through the points *M* and *N* crossing with the axis at the point *T*, there shall be

$$LN : ML = PM : PT$$

and hence

$$PT = u : (e^{u:b} - 1).$$

Truly, as we have shown in the above section, by the infinite series there is

$$e^{u:b} = 1 + \frac{u}{b} + \frac{uu}{2bb} + \frac{u^3}{6b^3} + \text{etc}.$$

and thus

$$PT = \frac{1}{\frac{1}{b} + \frac{u}{2bb} + \frac{uu}{6b^3} + \text{etc.}}.$$

Now the interval $PQ = u$ may vanish; and on account of the points *M* and *N* coinciding the line *NMT* becomes a tangent to the curve and then it will have the sub tangent $PT = b$ and thus constant, which is the most remarkable property of the logarithmic curve. Therefore the parameter *b* of the logarithm likewise is the same constant sub tangent, of the same magnitude everywhere.

515. Here a question arises, whether the whole logarithmic curve shall be described in this way and whether besides this branch *MBm* departing to infinity on both sides may have no other parts. For we have seen above no given asymptote, to which the two branches may not converge. Therefore we may put in place some logarithm from two similar parts agreed to be placed on each side of the axis, thus so that likewise the asymptote shall become a diameter. Truly the equation $y = ae^{x:b}$ minimally shows this property ; for as often as $\frac{x}{b}$ either is a whole number of a fraction having an odd denominator, then *y* has a single real value and that positive. But if the fraction $\frac{x}{b}$ may have an even denominator, [*i.e.* becoming a square] then the applied line *y* may lead to a twin value, the one positive and the other negative, and here the point of the curve will be shown on the other side of the asymptote; from which logarithm below the asymptote will have innumerable discrete points, which do not constitute a continuous curve, even if on account of infinitely small intervals the continued curve may be a deception ; which is a paradox not finding a place with algebraic lines. Hence also another much more

miraculous paradox arises. For since the logarithms of negative numbers shall be imaginary (which both becomes apparent from itself, as well as thence understood, as $l.-1$ may have a finite ratio to $\sqrt{-1}$
[For from De Moivre's Theorem:
$\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i=\sqrt{-1}=e^{i\frac{\pi}{2}}$; hence $l.\sqrt{-1}=\sqrt{-1}\frac{\pi}{2}$; first established by Johann Bernoulli; see his works vol.1, p. 399]),

$l.-n$ will be an imaginary quantity, which shall be $=i$ [not in the modern sense in the above note]; but since the logarithm of a square shall be equal to twice the logarithm of the root, there will be $l.(-n)^2=l.n^2=2i$. But $l.nn$ is a real quantity, $2l.n$, from which it follows that the real quantity *l.n* and the imaginary quantity *i* to be half of the same real quantity *l.nn.* Hence again some number may be considered to have two halves, the one real and the other imaginary ; and likewise a number may be considered as three third parts, in a four-fold manner as four quarters, and thus so on, of which yet only one part shall be real, which may not be apparent, in whatever way they may be tried to be reconciled with the idea of whole quantities,.

516. Therefore from these conceded, which we have assumed, it may follow for the number *a* one half to become equal to $\frac{a}{2}+l.-1$, and the other simply to $\frac{a}{2}$: for twice the first number is

$$a+2l.-1=a\ l.(-1)^2=a+l.1=a\,,$$

where it is to be observed that

$$+l.-1=-l.-1.$$

even if there shall not be $l.1=0$; for since there shall be $-1=\frac{+1}{-1}$, there becomes

$$l.-1=l.+1-l.-1=-l.-1.$$

In the same manner, since $\sqrt[3]{1}$ shall not only be equal to 1 but also equal to $\frac{-1\pm\sqrt{-3}}{2}$, there will be

$$3l.\frac{-1\pm\sqrt{-3}}{2}=l.1=0\,,$$

and thus the third part quantities of the same *a* will be

$$\frac{a}{3},\quad \frac{a}{3}+l.\frac{-1+\sqrt{-3}}{2}\quad \text{and}\quad \frac{a}{3}+l.\frac{-1-\sqrt{-3}}{2};$$

for the triple of these individual expressions produce the same quantity *a*. Towards removing these doubts, which in no way may be considered to be understood, it will be necessary to put in place another paradox : clearly, endless logarithms are to be given of each number, among which more than one real number may not be given. Thus, even if the logarithm of one is $=0$, yet in addition all the innumerable imaginary logarithms of unity may be given, which are

$$2l.-1,\quad 3l.\frac{-1\pm\sqrt{-3}}{2},\quad 4l.-1\ \text{ et }\ 4l.\pm\sqrt{-1},$$

and innumerable others, which the extraction of the roots shows. But this consideration is more plausible, then the above : for on putting $x=l.a$ there will be $a=e^{x}$ and thus

$$a=1+x+\frac{xx}{2}+\frac{x^{3}}{6}+\frac{x^{4}}{24}+\text{ etc.};$$

which since it shall be an equation of infinite dimensions, it is no wonder, if *x* may have boundless roots. But just as thus we have resolved the latter paradox, yet the first retains its strength, by which we have shown that innumerable discrete points belong to the logarithm below the axis.

5I7. But much more evidence of this kind of infinite discrete points existing can be shown by this equation $y=(-1)^{x}$; for as of the as *x* is either a whole even number or a fraction having an even denominator, y will be $=1$; but if x shall be either a number or a fraction, of which both the numerator as well as the denominator shall be odd numbers, y will be $=-1$; in all the remaining cases, in which *x* is a fraction having an even denominator or thus an irrational number, the value of y will be imaginary. Therefore the equation $y=(-1)^{x}$ will show innumerable discrete points placed on each side of the axis at a distance $=1$, of which no two shall be touching ; yet this does not prevent any two being placed on the same side of the axis nearby so close to each other that the interval shall be smaller than any quantity assigned. For between two nearby values of the abscissa not only one but an infinite number of fractions can be shown, the denominators of which are odd, moreover from these the individual points arise relating to the equation proposed ; therefore these points spread out give the appearance of two right lines parallel to the axis each at a distance from that $=1$; indeed in these lines no interval is able to be shown, in which not one, but rather an infinitude of points may be able to be assigned expressed by the equation $y=(-1)^{x}$. This same anomaly comes about in the equation

$y = (-a)^x$ and with all the similarities to this, where a negative quantity is raised to an indeterminate exponent. Therefore paradoxes of this kind, which can have a place only in transcendental equations, it was necessary to be exposed here.

518. Therefore all the equations belonging to this kind of curves depending on logarithms, in which not only logarithms occur, but also variable exponents, which originate evidently from logarithms progressing to numbers, from which these curves also are accustomed to be called *exponentials*. Therefore a curve of this kind, which may be expressed by this equation $y = x^x$ or $ly = xlx$. Therefore on putting $x = 0$, there will be $y = 1$; if $x = 1$, there will be $y = 1$; if $x = 2$, then $y = 4$; if $x = 3$, then $y = 27$ etc. From which (Fig. 102) *BDM* may express the form of this curve related to the axis *AP*, thus so that by taking $AC = 1$ there shall be $AB = CD = 1$. But between *A* and *C* the applied lines will be less than one ; if indeed $x = \frac{1}{2}$, there will be

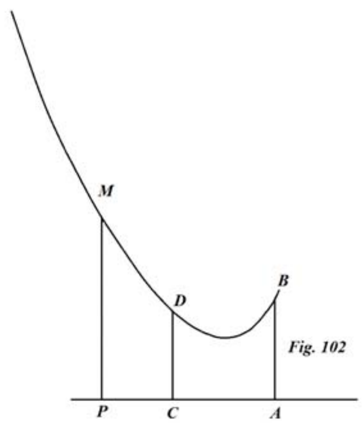


Fig. 102

$$y = \frac{1}{\sqrt{2}} = 0,7071068 ;$$

truly there will be the minimum applied line, if the abscissa may be taken

$$x = \frac{1}{e} = 0,36787944 ,$$

and then the applied line becomes $y = 0,6922005$, as will be shown in the following. But in order that this curve may be prepared beyond *B* as we may consider, it is required to make the abscissa *x* negative and there shall be $y = \frac{1}{(-x)^x}$, from which that part will be constructed from the individual discrete points by converging to the axis as an asymptote. But these points fall on each side of the axis, just as *x* should be an even or odd number. So also infinitely many points of this kind fall below the axis *AP*, if an even fraction may be taken for *x* having an even denominator ; for on putting $x = \frac{1}{2}$ there will be both

$$y = +\frac{1}{\sqrt{2}} \text{ and } y = -\frac{1}{\sqrt{2}}.$$

Therefore the curve *MDB* continued at *B* may end at once, opposite to the nature of algebraic curved lines, but in place of a continuation these discrete points will be had ; from which it may be considered more clearly that the reals of these points will be as if joined together. For unless these be conceded to be present, all the points to be placed on the curve must cease at once at the point *B*, which shall be contrary to the law of continuation and thus absurd.

519. Among the infinitely many other curves of this kind, the construction of which can be effected by logarithms, curves of this kind are given, the construction of which may not appear so easy, yet which may be resolved with the aid of a suitable substitution. Such is the curve expressed by the equation $x^y = y^x$; from which it is seen at once that the applied line *y* may be equal to the abscissa *x*, thus so that a right line inclined to the axis at half a right angle shall satisfy the equation. Yet meanwhile it is evident this equation be made more general, than the equation for the right line $y = x$, nor therefore to exhaust the power of this equation $x^y = y^x$; for it will be possible for this equation to be satisfied, even if there shall not be $x = y$, because, if $x = 2$, it is possible also for $y = 4$. Therefore besides the right line *EAF* the proposed equation may include other parts ; and thus towards finding which we may put $y = tx$ and thus showing the whole line expressed by the equation (Fig.103), so that there shall be $x^{tx} = t^x x^x$, from which with the root of the power *x* extracted there will be

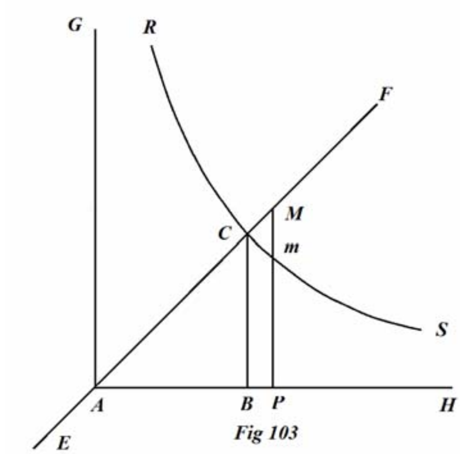


Fig 103

$$x^t = t^x \text{ and } x^{t-1} = t;$$

and thus there will be had

$$x = t^{\frac{1}{t-1}} \text{ and } y = t^{\frac{t}{t-1}}.$$

Or on putting $t - 1 = \frac{1}{u}$ there will be

$$x = (1+\frac{1}{u})^u \text{ and } y = \left(1+\frac{1}{u}\right)^{u+1}.$$

Hence the curve will have the branch *RS* converging at last to the asymptotes *AG* and *AH* besides the right line *EAF*, of which the right line *AF* will be a diameter. But the curve *AF* will cut the right line at the point *C*, thus so that there shall be $AB = BC = e$, with e denoting the number, the logarithm of which is unity. But in addition the equation provided innumerable discrete points, which since the right line *EF* and the curve *RCS* exhaust the equation. Hence therefore innumerable pairs of the two numbers *x* and *y* are able to be shown, so that there shall be $x^y = y^x$, indeed such numbers will be in ratios

$$\begin{array}{ll} x = 2 & y = 4 \\ x = \frac{3^2}{2^2} = \frac{9}{4} & y = \frac{3^3}{2^3} = \frac{27}{8} \\ x = \frac{4^3}{3^3} = \frac{64}{27} & y = \frac{4^4}{3^4} = \frac{256}{81} \\ x = \frac{5^4}{4^4} = \frac{625}{256} & y = \frac{5^5}{4^5} = \frac{3125}{1024} \\ \text{etc.} & \text{etc.} \end{array}$$

clearly of which two numbers, the one raised to the power of the other produces the same quantity ; thus there will be

$$2^4 = 4^2 = 16$$
$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left(\frac{27}{8}\right)^{\frac{9}{4}} = \left(\frac{3}{2}\right)^{\frac{27}{4}},$$
$$\left(\frac{64}{27}\right)^{\frac{256}{81}} = \left(\frac{256}{81}\right)^{\frac{64}{27}} = \left(\frac{4}{3}\right)^{\frac{256}{27}}$$
etc.

520. Although infinitely many points are able to be determined algebraically from these and in similar other curves, yet they are able to be enumerated minimally by algebra, because countless other points are extant, which cannot be shown algebraically in any way. Therefore we may move on to another kind of transcendence, which will require circular arcs ; but here the radius of the circle, the arcs of which enter into the construction, I will express by one always, lest the calculation be disturbed by more characters. But algebraic curves relating to this kind cannot be shown easily, even if the impossibility of the quadrature of the circle has not yet been brought about. Indeed we may consider only this simplest equation of this kind $\frac{y}{a} = \text{A.sin.}\frac{x}{c}$, thus so that the applied line y shall be proportional to the arc of a circle, the sine of which is $\frac{x}{c}$ [*i.e.* the arcsine or inverse sine function.] Because indeed innumerable arcs are agreed upon for the same sine $\frac{x}{c}$, the applied line y will be a an 'infinitinomial' function and thus both the curve itself as well as other right lines will be cut at an infinite number of points, which property itself distinguishes that curve from algebraic curves. Let s be the smallest

agreeing arc of the sine $\frac{x}{c}$ and π shall denote the semi-circumference of the circle, the following values of $\frac{y}{a}$ will be

$$s,\quad \pi - s,\quad 2\pi + s,\quad 3\pi - s,\quad 4\pi + s,\quad 5\pi - s \text{ etc.}$$
$$-\pi - s,\quad -2\pi + s,\quad -3\pi - s, -4\pi + s,\quad -5\pi - s \text{ etc.}$$

Therefore with the right line *CAB* taken for the axis and *A* for the beginning of the abscissas (Fig. 104), in the first place on putting $x = 0$ the applied lines will be :

$$AA^1 = \pi a,\ AA^2 = 2\pi a,\ AA^3 = 3\pi a \text{ etc.}$$

Likewise on the other side :

$$AA^{-1} = \pi a,\ AA^{-2} = 2\pi a,\ AA^{-3} = 3\pi a \text{ etc.}$$

and the curve will pass through these individual points of the curve. Truly with the abscissa taken $AP = x$, the applied lines will cut the curve at in infinite number of points *M* and there will be :

$$PM^1 = as,\ PM^2 = a(\pi - s),\ PM^3 = a(2\pi + s) \text{ etc.}$$

Therefore the whole curve will be composed from the infinitude of similar portions $AE^1A^1,\ A^1F^1A^2,\ A^2E^2A^3,\ A^3F^2A^4$ etc . ; thus so that the individual right lines parallel to the axis *BC*, quae per which may be drawn through the points *E* and *F*, shall become the future diameters of the curve. Truly there will be $AC = AB = c$ and the intervals $E^1E^2,\ E^2E^3,\ E^1E^{-1},\ E^{-1}E^{-2}$ likewise $F^1F^2,\ F^1F^{-1},\ F^{-1}F^{-2}$ will be each equal to $2a\pi$. This curve has been called by Leibnitz *the line of the sine*, because with its aid each arc of the sine can be found easily. For since there shall be

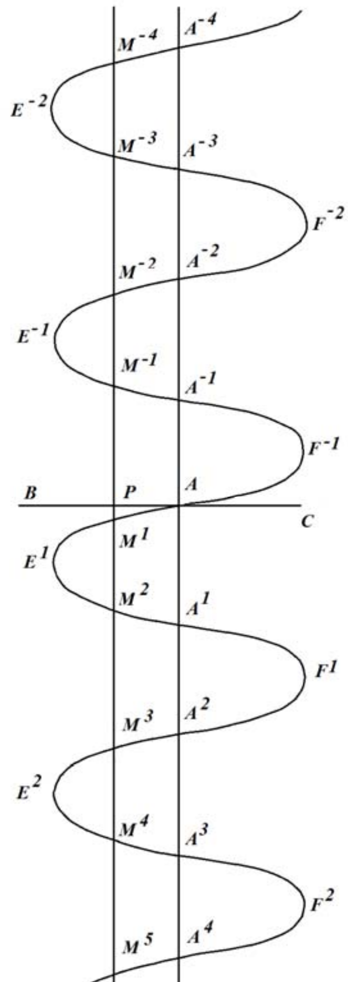


Fig. 104

$$\frac{y}{a} = \text{A. sin.}\frac{x}{c},$$

in turn there will be

$$\frac{x}{c}=\sin.\mathrm{A}.\frac{y}{a}.$$

If there may be put

$$\frac{y}{a}=\frac{1}{2}\pi-\frac{x}{a}$$

there becomes

$$\frac{x}{c}=\cos.\mathrm{A}.\frac{z}{a};$$

and thus likewise *the line of the cosine* may be had.

521. From this consideration in a similar manner arises *the line of the tangent*, the equation of which will be $y=\mathrm{A.tang.}x$, for brevity therefore putting $a=1$ and $c=1$; hence on converting it becomes

$$x=\mathrm{tang.A.}y=\frac{\sin.y}{\cos.y},$$

the figure of the curve is readily deduced from the nature of the tangent. Moreover it will have an infinitude of asymptotes parallel to each other. In a similar manner *the line of the secant* can be described from the equation

$$y=\mathrm{A.sec.}x \text{ or } x=\mathrm{sec.A.}y=\frac{1}{\cos.y},$$

which also has an infinitude of branches extending to infinity. Truly especially from this kind of curves the *cycloid* or *trochoid* becomes known, which is described by a point on the periphery of a circle on moving forwards by rotating on a straight line, the equation of which between orthogonal coordinates is

$$y=\sqrt{(1-xx)}+\mathrm{A.cos.}x.$$

This curve both on account of the ease of description as well as on account of many conspicuous properties which it enjoys, is especially worthy of note. But because most are unable to be explained without the analysis of the infinites, here we will consider briefly only particular ones, which follow immediately from the description.

522. Therefore the circle *ACB* may rotate on the line *EA* (Fig. 105) *;* and, so that the investigation may extend more broadly, not a point of the periphery *B* but rather some point of the diameter produced *D* will describe the curved line *Dd* in some manner. Let the radius of this circle be $CA = CB = a$, the distance $CD = b$, and indeed at this position the place of the point *D* may possess the maximum height. Upon rotating the circle arrives at the position *aQbR;* and on putting the distance $AQ = z$, the arc $aQ = z$,which divided by the radius *a* will give the angle $acQ = \frac{z}{a}$, and the describing point will be at *d,* so that there shall be $cd = b$, the angle

$$dcQ = \pi - \frac{z}{a},$$

and *d* will be a point on the curve sought. In the first place the normal *dp* is drawn from *d* to the line *AQ,* then the normal *dn* to the line *QR* ; there will be

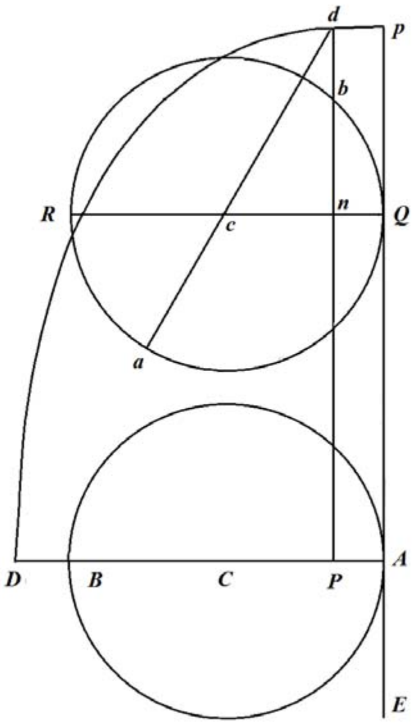


Fig. 105

$$dn = b \cdot \sin.\frac{z}{a} \text{ and } cn = -b.\cos.\frac{z}{a}$$

therefore

$$Qn = dp = a + b \cdot \cos.\frac{z}{a}.$$

Produce *dn,* then the right line *AD* will cross at *P,* and the coordinates may be called

$$DP = x, \quad Pd = y;$$

there will be

$$x = b + cn \text{ or } x = b - b \cdot \cos.\frac{z}{a}$$

and

$$y = AQ + dn = z + b \cdot \sin.\frac{z}{a}.$$

Therefore since there shall be

$$b \cdot \cos.\frac{z}{a} = b - x$$

there will be

$$b \cdot \sin.\frac{z}{a} = \sqrt{(2bx - xx)}$$

and

$$z = aA.\cos.\left(1-\frac{x}{b}\right) = aA.\sin.\frac{\sqrt{(2bx-xx)}}{b};$$

from which with the values substituted, there will be,

$$y = \sqrt{(2bx-xx)} + aA.\sin.\frac{\sqrt{(2bx-xx)}}{b}.$$

Or, if the abscissas may be computed from the centre on the axis *AD* and there may be called $b - x = t$, there will be

$$\sqrt{(2bx-xx)} = \sqrt{(bb-tt)}$$

and this equation will be had between *t* and *y*

$$y = \sqrt{(bb-tt)} + a\text{A.cos.}\frac{t}{b},$$

which equation gives the *ordinary* cycloid, if there were $b = a$ *;* but if either *b* shall be greater than *a* or *b* less than *a*, the curve is called either a *shortened* or *elongated* cycloid. But always *y* will be an 'infiniplex' function of x or *t;* or in whatever manner a right line parallel to the base *AQ* will be cut at infinitely many points, unless its distance x or *t* were so great, so that $\sqrt{(2bx-xx)}$ or $\sqrt{(bb-tt)}$ become imaginary quantities.

523. Among curves of this kind, which are known especially, the *epicycloids* and *hypocycloids* must be referred to (Fig.106), which arise, if the circle *ACB* is rotating on the periphery of another circle *OAQ* and between these some point *D* moving either outside or inside the circle taken, will describe the curve *Dd*. The radius of the immobile circle *OA* may be put $= c$, the radius of the moving circle $CA = CB = a$ and the distance of the describing point $CD = b$; but the right line *OD* may be

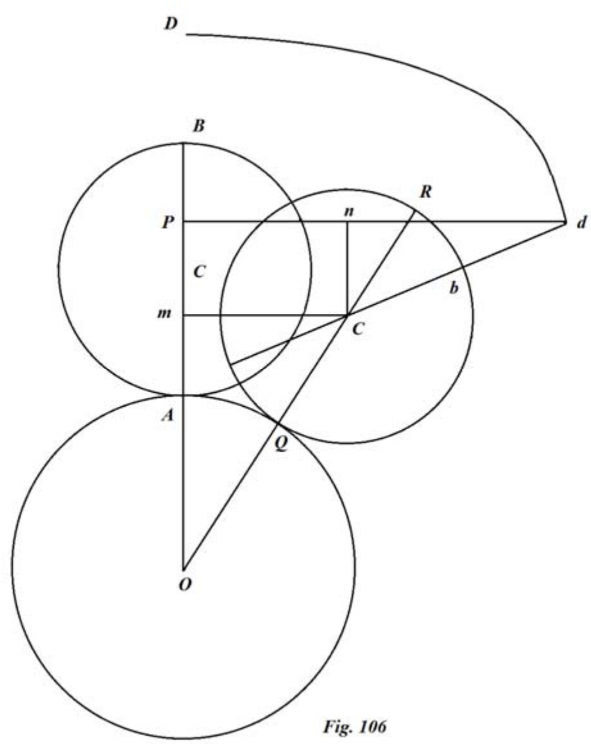


Fig. 106

taken for the axis of the curve sought *Dd*. With this initial situation, in which the points *O, C, D* may lie on a line, the mobile circle may precede to the position *QcR*, with the arc described $AQ = z$, thus so that the angle shall be $AOQ = \frac{z}{c}$. Therefore the arc will be $Qa = AQ = z$ and hence the angle $acQ = \frac{z}{a} = Rcd$ and, with the right line taken $cd = CD = b$, the point *d* will be on the curve *Dd*. From that the perpendicular *dP* may be sent to the axis and likewise from *c* the perpendicular *cm* and *cn* parallel to the axis *OD*. Therefore on account of the angle

$$Rcn = AOQ = \frac{z}{c}$$

the angle *dcn* will become

$$dcn = \frac{z}{c} + \frac{z}{a} = \frac{(a+c)z}{ac}.$$

From which there is obtained

$$dn = b \cdot \sin.\frac{(a+c)z}{ac}$$

and

$$cn = b \cdot \cos.\frac{(a+c)z}{ac}$$

Then on account of $OC = Oc = a + c$ there will be

$$cm = (a+c) \cdot \sin.\frac{z}{c}$$

and

$$Om = (a+c) \cdot \cos.\frac{z}{c}$$

Therefore with the coordinates called $OP = x$ and $Pd = y$ there will be

$$x = (a+\mathrm{c}) \cdot \cos.\frac{z}{c} + b \cdot \cos.\frac{(a+c)z}{ac}$$

and

$$y = (a+\mathrm{c}) \cdot \sin.\frac{z}{c} + b \cdot \sin.\frac{(a+c)z}{ac}.$$

Hence it is apparent, if $\frac{a+c}{a}$ were a rational number, then on account of the commensurability of the angles $\frac{z}{c}$ and $\frac{(a+c)z}{ac}$ the unknown z can be eliminated and thus an algebraic equation can be found between x and y. In the remaining cases the curve described in this way will be transcending.

Moreover here it is required to be noted, if a shall be negative, then a hypocycloid will be produced, with the moving circle falling within the fixed circle.

[Note that in all these constructions, the rate of rotation of the moving circle is constant].

Indeed generally b is put equal to the radius a and thus proper epicycloids and hypocycloids thus are said to come about. Therefore here the curves appear more general and, because the equations are not more difficult, it has been considered to add this condition. If the squares xx and yy are added, there will be

$$xx+yy=(a+c)^2+b^2+2b(a+c)\cdot\cos.\frac{z}{a},$$

with the help of which equation the elimination of z from that may be brought about more easily, as often indeed as the quantities a and c were commensurable.

524. Besides the case, in which the radii of both the circles a and c are commensurable between each other and the curves become algebraic, this case deserves to be observed, in which $b=-a-c$ or in which the point of the curve D falls on the centre of the fixed circle O. Therefore let there be $b=-a-c$ and the equation becomes

$$xx+yy=2(a+c)^2\left(1-\cos.\frac{z}{a}\right)=4(a+c)^2\left(\cos.\frac{z}{2a}\right)^2;$$

from which there becomes

$$\cos.\frac{z}{2a}=\frac{\sqrt{(xx+yy)}}{2(a+c)}.$$

Then, since there shall be

$$x=(a+c)(\cos.\frac{z}{c}-\cos.\frac{(a+c)z}{ac}) \text{ and } y=(a+c)(\sin.\frac{z}{c}-\sin.\frac{(a+c)z}{ac}),$$

there will be

$$\frac{x}{y}=-\tan.\frac{(2a+c)z}{2ac} \text{ and } \sin.\frac{(2a+c)z}{2ac}=\frac{x}{\sqrt{xx+yy}}$$

and

$$\cos.\frac{(2a+c)z}{2ac}=\frac{-y}{\sqrt{xx+yy}}.$$

Whereby, since there shall be

$$\sqrt{xx+yy}=2(a+c)\cos.\frac{z}{2a},$$

there becomes

$$x=2(a+c)\cos.\frac{z}{2a}\cdot\sin.\frac{(2a+c)z}{2ac}$$

and

$$y=-2(a+c)\cos.\frac{z}{2a}\cdot\cos.\frac{(2a+c)z}{2ac}.$$

For example let there be $c=2a$; the equations become

$$x=6a\cdot\cos.\frac{z}{2a}\cdot\sin.\frac{z}{a}\ \text{ and }\ y=-6a\cdot\cos.\frac{z}{2a}\cdot\cos.\frac{z}{a}$$

and

$$\sqrt{xx+yy}=6a\cdot\cos.\frac{z}{2a}.$$

We may put

$$\cos.\frac{z}{2a}=q,$$

there will become

$$\sin.\frac{z}{2a}=\sqrt{(1-qq)}\ ,\ \sin.\frac{z}{a}=2q\sqrt{(1-qq)}\ \text{ and }\ \cos.\frac{z}{a}=2qq-1,$$

from which there becomes

$$q=\frac{\sqrt{xx+yy}}{6a}$$

and

$$y=-6aq(2qq-1)=(1-2qq)\sqrt{(xx+yy)}=(1-\frac{xx-yy}{18aa})\sqrt{(xx+yy)}$$

or

$$18aay=(18aa-xx-yy)\sqrt{(xx+yy)}.$$

Putting $18aa = ff$, and with the square taken, this equation of the sixth order will be found :

$$\left(xx+yy\right)^3 - 2ff\left(xx+yy\right)^2 + f^4xx = 0.$$

Truly because here by us it is proposed to consider transcending curves rather than algebraic ones, we may move on from these presented to curves of this kind, the construction of which likewise requires both logarithms as well as circular arcs.

525. Truly above now we have a curve of this kind arising from the equation (Fig. 107) :

$$2y = x^{+\sqrt{-1}} + x^{-\sqrt{-1}},$$

which we have changed into this $y = \cos.\mathrm{A}.lx$.
Truly this will be changed further into

$$\mathrm{A}.\cos.y = lx \text{ and } x = e^{A.\cos.y}.$$

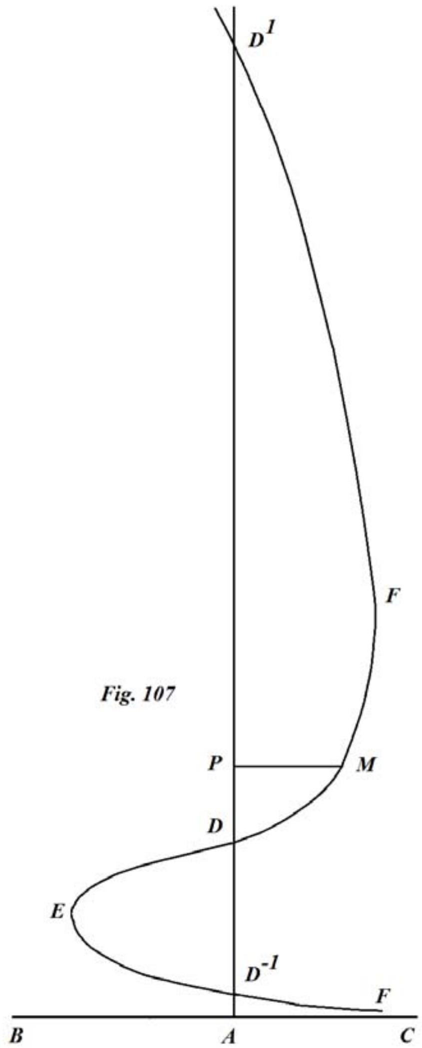


Fig. 107

Therefore with the right line *AP* taken for the axis and on which *A* is taken for the start of the abscissas, in the first place it is apparent in the region of negative abscissas beyond *A* of the curve no continued part is to be given, but the axis *AP* will be intersected at an infinite number of points *D*, of which the distances of the points from *A* constitute a geometric progression, it will be namely

$$AD = e^{\frac{\pi}{2}}, AD^1 = e^{\frac{3\pi}{2}}, AD^2 = e^{\frac{5\pi}{2}}, AD^3 = e^{\frac{7\pi}{2}} \text{ etc.,}$$

then indeed an infinitude of intersections will be given approaching closer to *A*

$$AD^{-1} = e^{\frac{-\pi}{2}}, \ AD^{-2} = e^{\frac{-3\pi}{2}}, \ AD^{-3} = e^{\frac{-5\pi}{2}} \text{ etc.,}$$

Then this curve departs from the axis on both sides to the distances $AB = AC = 1$ and there the right lines parallel to the axis touch at an infinitude of points *E* and *F*, of which the distance from *B* and *C* equally constitute a geometric progression. Therefore the curve approaches towards the right line *BC* with infinitude of turning points and finally therefore may be confused with that. Therefore a property of this singular curve consists in that not an infinite right line but the finite line *BC* shall be the asymptote of the curve, from which the nature of this curve itself is distinguished mainly from algebra.

526. For transcending curves, the construction of which requires angles either alone or jointly with logarithms, reference also must be made to the innumerable kinds of *spirals*. But spirals may be regarded with respect of some fixed point *C* as the centre (Fig.108), about which generally an infinitude of turns are circumscribed. The nature of these curves may be explained most conveniently by an equation between each point *M* of the curve, at a distance *CM* from the centre *C* and at an angle *ACM*, which this right line *CM* makes with the given right line *CA* in position. Therefore the angle shall be $ACM = s$ or *s* shall be the arc of the circle described with radius $=1$, which shall be the measure of the angle ACM, the on putting the right line $CM = z$. But if now some equation may be given between the variable *s* and *z*, the curve of a spiral will result. For since the angle *ACM* may be able to be expressed in an infinite number of ways besides *s*, because the angles $2\pi + s, 4\pi + s, 6\pi + s$ etc., and likewise $-2\pi + s, -4\pi + s,$ etc. will show the same position of the right line *CM* ; and with these values substituted in place of *s* into an equation, an infinite number of diverse distances *CM* will be obtained and thus the right line *CM* produced will cut the curve in an infinite number of points, unless the quantity *z* becomes imaginary from these values.

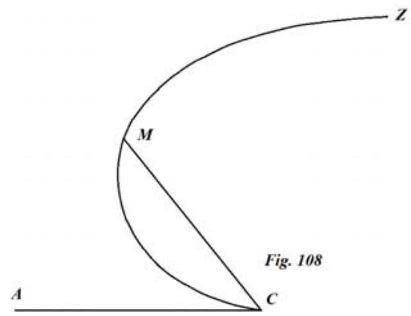


Fig. 108

Therefore we may begin from the simplest case, in which there is $y = as$; and for the same position of the right line *CM* these will be the values of *y*: $a(2\pi + s),\ a(4\pi + s),\ a(6\pi + s)$ etc., and likewise $-a(2\pi - s), -a(4\pi - s), -a(6\pi - s)$ etc. But indeed also, if for *s* there may be put $\pi + s$, the position of the right line *CM* will remain the same, and in addition, because the value of *z* may be taken negative, hence to the values of *z* assigned it will be required to have added these values $-a(\pi + s),\ -a(3\pi + s),\ -a(5\pi + s)$ etc. and besides those $a(\pi - s),\ a(3\pi - s),\ a(5\pi - s)$ etc. Therefore the form of the curve will be such (Fig.109), as may be shown in the figure [relegated to the margin in the original]; clearly the right line *AC* is a tangent at *C* and hence with two branches, with infinite turns going around the centre *C* on both sides, and by themselves mutually crossing on the right line *BC* always normal to *AC*, is extended to infinity;

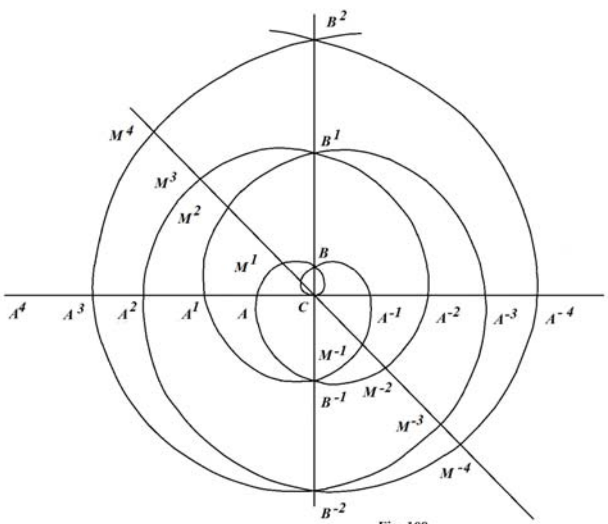


Fig. 109

and the right line *BCB* will be its diameter. Moreover it is usual to call this curve after its discoverer, the spiral of Archimedes ; and if once it has been describe exactly, it will serve for any angle in whatever parts requiring to be cut, as may be apparent at once from the equation $z = as$.

527. In the same manner as the equation $z = as$, which, if *z* and *s* were orthogonal coordinates, would be for a straight line, has given rise to the Archimedes spiral, thus if other algebraic equations may be admitted between *z* and *s*, infinitely many other spiral lines will be produced, if indeed the equation may be prepared thus, so that the real values of *z* may correspond to the individual values of *s*. Thus this equation $z = \frac{a}{s}$, which is similar to the equation for the hyperbola related to asymptotes, will provide a spiral, which has been called the *hyperbolic spiral* by the celebrated Johan. Bernoulli ; and, after it shall have emerged from the centre *C* by an infinite number of gyrations, finally it approaches towards the right line *AA* as an asymptote [Diagram not provided]. But if the equation $z = a\sqrt{s}$ may be proposed, no distance of the real *z* will correspond to angles with *s* taken negative ; but twin values of *z* will correspond to positive values of the individual *s*, the one positive and the other negative, yet an infinite spiral will be resolved about the centre *C*. But if the equation between *z* and *s* were of this kind $z = a\sqrt{(nn - ss)}$, no real value will be had of the variable *z*, unless *s* may be contained within the limits $+n$ and $-n$; and thus in this case the curve will be finite. Clearly, if (Fig. 110) the right lines *EF* may be inclined on both sides to the axis *ACB* through the centre *CB, EF* making an angle with the axis $= n$, these will be tangents of the curve themselves crossing at *C*, and the curve will have the form *ACBCA* of a *lemniscates* [*i.e*. ribbon]. Moreover in a similar manner innumerable other forms of transcending lines may be obtained, which may be set out in great abundance.

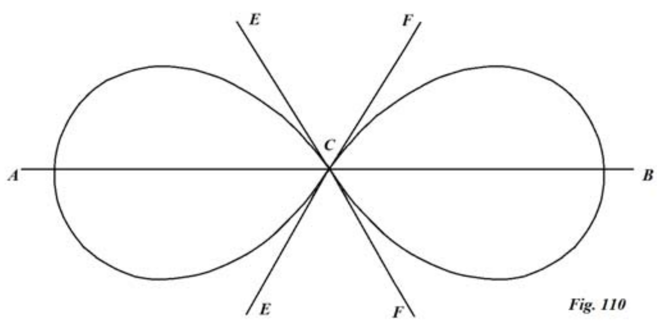
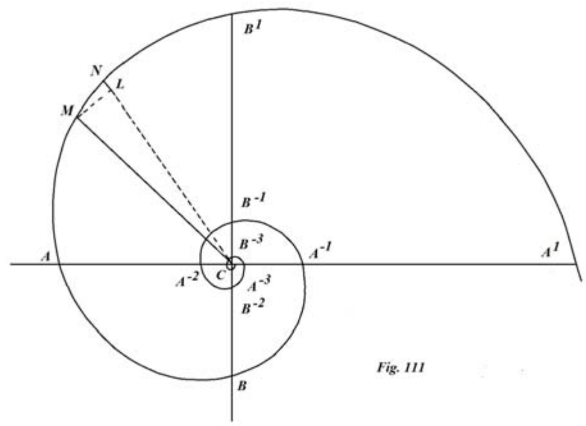


Fig. 110

528. This treatment again may be expanded on greatly, if transcending equations thus may be admitted between *z* and *s* instead of algebraic equations. From which kind before the rest this curved line is noteworthy, which is expressed by this equation $s = nl.\frac{z}{a}$, in which namely the angles *s* are proportional to the logarithms of the distances *z* ; on which account this curve is called the *logarithmic spiral* and on account of many characteristic properties has been noted especially. The most distinguished property of this curve is (Fig. 111), that all the right lines drawn from the centre *C* will intersect the curve at equal angles. Towards that the angle being drawn from the equation shall be

$ACM = s$ and the right line $CM = z$, and there will be

$$s = nl.\frac{z}{a} \text{ and } z = ae^{\frac{s}{n}},$$



then a greater angle may be taken $ACN = s + v$, there will be the right line

$$CN = ae^{\frac{s}{n}}e^{\frac{v}{n}},$$

and thus with centre *C* with the arc *ML* described, which will be $= zv$, *LN* will become

$$LN = ae^{\frac{s}{n}}(e^{\frac{v}{n}} - 1) = ae^{\frac{s}{n}}(\frac{v}{n} + \frac{v^2}{2n^2} + \frac{v^3}{6n^3} + \text{ etc.}).$$

Hence there will be

$$\frac{ML}{LN} = \frac{v}{\frac{v}{n} + \frac{v^2}{2n^2} + \frac{v^3}{6n^3} + \text{ etc.}} = \frac{n}{1 + \frac{v}{2n} + \frac{v^2}{6nn} + \text{ etc.}}.$$

But with the difference of the angles vanishing, $MCN = v$, the tangent of the angle will become $\frac{ML}{LN}$, which the radius *CM* makes with the curve ; from which on making $v = 0$ the tangent of this angle *AMC* will be *n* and thus equal to this constant angle. If there were $n = 1$, this angle will be half of a right angle and in this case the logarithmic spiral may be called semi-rectangular.

CAPUT XXI

DE LINEIS CURVIS TRANSCENDENTIBUS

506. Hactenus de Lineis curvis algebraicis egimus, quae ita sunt comparatae, ut, sumtis abscissis in axe quocunque, applicatae respondentes exprimantur per functiones algebraicas abscissarum seu, quod eodem redit, in quibus relatio inter abscissas et applicatas exprimi possit per aequationem algebraicam. Hinc itaque sponte sequitur, si valor applicatae per functionem algebraicam abscissae explicari nequeat, lineam curvam algebraicis annumerari non posse. Huiusmodi autem lineae curvae, quae algebraicae non sunt, *transcendentes* vocari solent. Linea igitur transcendens ita definitur, ut eiusmodi curva esse dicatur, in qua relatio inter abscissas et applicatas aequatione algebraica exprimi nequeat. Quoties ergo applicata y functioni transcendenti ipsius abscissae x aequatur, toties linea curva ad genus transcendentium erit referenda.

507. In superiori sectione duas potissimum species quantitatum transcendentium evolvimus, quarum altera logarithmos altera arcus circulares seu angulos complectebatur. Quodsi ergo applicata y sit aequalis vel logarithmo ipsius abscissae x vel arcui circuli, cuius sinus seu cosinus seu tangens per abscissam x exprimitur, ita ut sit $y = lx$ vel $y = A \cdot \sin. x$ vel $y = A \cdot \cos. x$ vel $y = A \cdot \text{tang.} x$ vel si huiusmodi valores tantum in aequationem inter x et y ingrediantur, tum curva erit transcendens. Sunt autem hae curvae tantum species transcendentium; praeter istas enim dantur innumerabiles aliae expressiones transcendentes, quarum origo in analysi infinitorum fusius exponetur, ita ut numerus curvarum transcendentium longe superet numerum curvarum algebraicarum.

508. Quaecunque functio non est algebraica, ea est transcendens ideoque curvam, in cuius aequationem ingreditur, reddit transcendentem. Aequatio autem algebraica vel est rationalis nullosque exponentes praeter numeros integros continet vel est irrationalis atque exponentes fractos complectitur; hoc autem posteriori casu semper ad rationalitatem revocari potest. Cuius igitur curvae aequatio relationem inter coordinatas x et y exprimens ita est comparata, ut neque sit rationalis neque ad rationalitatem perduci possit, ea semper est transcendens. Quodsi ergo in aequatione eiusmodi potestates occurrant, quarum exponentes neque sint numeri integri neque fracti, ad rationalitatem nullo modo perduci poterit ideoque curvae talibus aequationibus contentae erunt transcendentes. Hinc nascitur prima species et quasi simplicissima curvarum transcendentium, in quarum aequationibus insunt exponentes irrationales; quae quia neque logarithmos neque arcus circulares involvunt, sed ex sola numerorum irrationalium notione nascuntur, magis quodammodo ad geometriam communem pertinere videntur et hanc ob rem ab

LEIBNITIO *interscendentes* sunt appellatae , quasi medium tenerent inter algebraicas et transcendentes.

509. Huiusmodi ergo curva interscendens erit, quae continetur aequatione $y = x^{\sqrt{2}}$; quomodocunque enim haec aequatio potestatibus sumendis evehatur, nunquam ad rationalitatem perducetur. Talis aequatio autem nulla via geometrica construi potest. Geometrice enim nullae aliae potestates exhiberi possunt, nisi quarum exponentes sint numeri rationales, hancque ob causam istiusmodi curvae ab algebraicis maxime discrepant. Si enim exponentem $\sqrt{2}$ tantum vero proxime exhibere velimus, eius loco ponendo aliquam ex his fractionibus

$$\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{96}{70},$$

quae valorem $\sqrt{2}$ proxime exprimunt, curvae quidem algebraicae prodibunt ad quaesitam proxime accedentes, at ordinis erunt vel tertii vel septimi vel decimi septimi vel quadragesimi primi etc. Quare, cum $\sqrt{2}$ rationaliter exprimi nequeat nisi per fractionem, cuius numerator et denominator sint numeri infinite magni, haec curva ordini linearum infinitesimo erit accensenda ideoque pro algebraica haberi non poterit. Hue accedit, quod $\sqrt{2}$ duplicem involvat valorem, alterum affirmativum alterum negativum, ex quo *y* duplicem perpetuo sortietur valorem sicque gemina curva resultabit.

510. Deinde vero si hanc curvam exacte construere velimus, id sine logarithmorum beneficio praestare non possumus. Cum enim sit $y = x^{\sqrt{2}}$, erit logarithmis sumendis $ly = \sqrt{2} \cdot lx$, cuiusvis ergo abscissae logarithmus per $\sqrt{2}$ multiplicatus dabit logarithmum applicatae, unde ad quamvis abscissam *x* respondens applicata ex canone logarithmorum assignabitur. Sic, si fuerit $x = 0$, erit $y = 0$, si $x = 1$, erit $y = 1$; qui valores ex aequatione facillime fluunt, at si $x = 2$, erit $ly = \sqrt{2} \cdot l2 = \sqrt{2} \cdot 0,3010300$ et ob $\sqrt{2} = 1,41421356$ erit $ly = 0,4257207$ ideoque proxime $y = 2,665144$ et si $x = 10$, erit $ly = 1,41421356$ hincque $y = 25,954554$. Hoc igitur modo ad singulas abscissas applicatae supputari atque adeo curva construi poterit, siquidem abscissae *x* valores affirmativi tribuantur. Sin autem abscissa *x* valores obtineat negativos, tum difficile est dictu, utrum valores ipsius *y* futuri sint reales an imaginarii; sit enim $x = -1$, et quid sit $(-1)^{\sqrt{2}}$, definiri non poterit, quoniam approximationes ad valorem $\sqrt{2}$ nihil adiumenti afferunt.

511. Multo minus erit dubitandum, quin aequationes, in quibus adeo exponentes imaginarii reperiuntur, ad genus transcendentium referri debeant. Fieri autem omnino potest, ut expressio continens exponentes imaginarios valorem realem atque

determinatum exhibeat. Huius rei exempla supra iam occurrerunt, unde hic sufficiat unum exemplum attulisse hoc

$$2y = x^{+\sqrt{-1}} + x^{-\sqrt{-1}} \;,$$

in quo, etiamsi utrumque membrum $x^{+\sqrt{-1}}$ et $x^{-\sqrt{-1}}$ sit quantitas imaginaria, tamen summa amborum valorem habet realem. Sit enim $lx = v$, sumto *e* pro numero, cuius logarithmus hyperbolicus est $=1$, erit $x = e^{v}$, quo valore loco *x* substituto erit

$$2y = e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}$$

Vidimus autem in sectione superiori § 138 esse

$$\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2} = \cos.\mathrm{A}.v \,,$$

unde fiet

$$y = \cos.\mathrm{A}.v = \cos.\mathrm{A}.lx.$$

Scilicet proposito quocunque ipsius *x* valore in numeris sumatur eius logarithmus hyperbolicus, tum in circulo, cuius radius $=1$, abscindatur arcus isti logarithmo aequalis, huiusque arcus cosinus dabit valorem applicatae *y*. Sic, si sumatur $x = 2$, ut fit

$$2y = 2^{+\sqrt{-1}} + 2^{-\sqrt{-1}},$$

erit

$$y = \cos.\mathrm{A}.l2 = \cos.\mathrm{A}.0,6931471805599\,.$$

Iste autem arcus ipsi *l2* aequalis, cum arcus 3,1415926535 etc. contineat 180°, per regulam auream invenietur fore 39°, 42', 51'', 52''', 8'''', cuius cosinus est 0,76923890136400, hicque numerus dat valorem applicatae *y* respondentem abscissae $x = 2$. Cum igitur huiusmodi expressiones et logarithmos et arcus circulares involvant, iure ad transcendentes referuntur.

512. Inter curvas ergo transcendentes primum locum tenent, quarum aequationes praeter quantitates algebraicas logarithmos involvunt, atque simplicissima harum erit, quae continetur hac aequatione

$$l.\frac{y}{a} = \frac{x}{b} \;\text{ seu }\; x = b\,l.\frac{y}{a}\,,$$

ubi perinde est, cuiusnam generis logarithmi accipiantur, quia multiplicatione constantis *b* omnia logarithmorum systemata ad idem revocantur. Denotet ergo character *l* logarithmos hyperbolicos, atque curva aequatione $x = bl\frac{y}{a}$ contenta sub nomine *logarithmicae* vulgo est nota. Site numerus, cuius logarithmus est = 1, ita ut sit $e = 2,718281828459045235360 28$, fietque

$$e^{x:b} = \frac{y}{a} \quad \text{seu} \quad y = ae^{x:b}$$

ex qua aequatione natura curvae logarithmicae facillime cognoscitur. Si enim loco *x* successive substituantur valores in arithmetica progressione procedentes, applicatae *y* valores tenebunt inter se progressionem geometricam. Quae quo facilius ad constructionem accomodetur, ponatur

$$e = m^n \quad \text{and} \quad b = nc,$$

eritque

$$y = am^{x:c}$$

ubi *m* numerum quemcunque affirmativum unitate maiorem significare potest.
Si igitur sit

$$x = 0,\ c,\ 2c,\ 3c,\ 4c,\ 5c,\ 6c \text{ etc.},$$

erit

$$y = a,\ am,\ amm,\ am^3,\ am^4,\ am^5,\ am^6 \ \text{ etc.};$$

et tribuendis ipsi *x* valoribus negativis, si ponatur

$$x = -c,\ -2c,\ -3c,\ -4c,\ -5c \ \text{ etc.},$$

erit

$$y = \frac{a}{m},\ \frac{a}{mm},\ \frac{a}{m^3},\ \frac{a}{m^4},\ \frac{a}{m^5} \ \text{ etc.}$$

513. Hinc patet (Fig. 101) applicatas *y* ubique valores habere affirmativos et quidem in infinitum crescentes auctis abscissis *x* affirmative in infinitum, ex altera autem axis parte in infinitum decrescentes, ita ut hinc axis sit curvae asymptota *Ap*. Sumto scilicet *A* pro abscissarum initio erit hoc loco applicata $AB = a$ et sumta abscissa $AP = x$ erit applicata $PM = y = am^{x:c} = ae^{x:b}$ ideoque

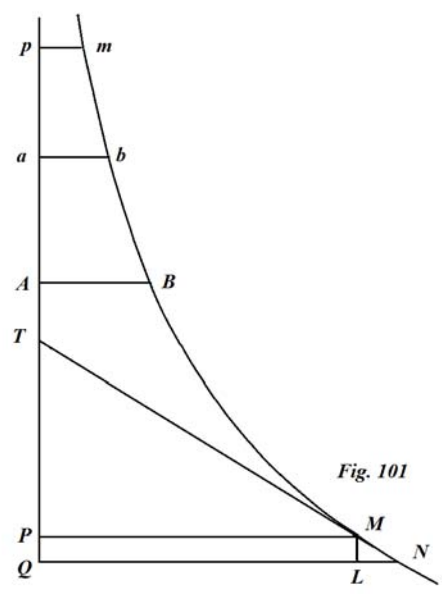


Fig. 101

$$l \cdot \frac{y}{a} = \frac{x}{b}.$$

Unde abscissa *AP* per constantem *b* divisa exprimit logarithmum rationis $\frac{PM}{AB}$. Si abscissarum initium in alio quocunque axis puncto *a* statuatur, aequatio similis manet. Sit enim $Aa = f$, ac posita $aP = t$ ob $x = t - f$ erit

$$y = ae^{(t-f):b} = ae^{t:b} : e^{f:b}.$$

Vocetur constans $ae^{f:b} = g$, erit $y = ge^{t:b}$. Hinc ob $ab = g$ intelligitur fore

$$\frac{aP}{b} = l \cdot \frac{PM}{ab}$$

ideoque ductis duabus quibusvis applicatis *PM* et *pm*, intervallo *Pp* a se invicem distantibus, erit

$$\frac{Pp}{b} = l \cdot \frac{PM}{pm}$$

et constans *b*, a qua ista relatio pendet, erit instar parametri logarithmicae.

514. Tangens huius curvae logarithmicae in quovis puncto *M* etiam facile poterit definiri. Cum enim posita $AP = x$ sit $PM = ae^{x:b}$, ducatur alia quaecunque applicata *QN* a priori intervallo $PQ = u$ dissita eritque

$$QN = ae^{(x+u):b} = ae^{x:b} \cdot e^{u:b};$$

et ducta *ML* axi parallela erit

$$LN = (QN - PM) = ae^{x:b}(e^{u:b} - 1).$$

Per puncta *M* et *N* ducatur recta *NMT* axi occurrens in puncto *T*, erit

$$LN : ML = PM : PT$$

hincque

$$PT = u : (e^{u:b} - 1).$$

Verum, uti in sectione superiori ostendimus, per seriem infinitam est

$$e^{u:b} = 1 + \frac{u}{b} + \frac{uu}{2bb} + \frac{u^3}{6b^3} + \text{etc}.$$

ideoque

$$PT = \frac{1}{\frac{1}{b} + \frac{u}{2bb} + \frac{uu}{6b^3} + \text{etc.}}.$$

Evanescat iam intervallum $PQ = u$; et ob puncta *M* et *N* coincidentia recta *NMT* fiet curvae tangens eritque tum subtangens $PT = b$ ideoque constans, quae est proprietas palmaria curvae logarithmicae. Parameter ergo logarithmicae *b* simul eiusdem est subtangens constantis ubique magnitudinis.

515. Quaestio hic oritur, utrum hoc modo tota curva logarithmica sit descripta et an ea praeter hunc ramum *MBm* utrinque in infinitum excurrentem nullas alias habeat partes. Vidimus enim supra nullam dari asymptotam, ad quam non duo rami convergant. Statuerunt ergo nonnulli logarithmicam ex duabus constare partibus similibus ad utramque axis partem sitis, ita ut asymptota simul futura sit diameter. Verum aequatio $y = ae^{x:b}$ hanc proprietatem minime ostendit; quoties enim est $\frac{x}{b}$ vel numerus integer vel fractio denominatorem habens imparem, tum *y* unicum habet valorem realem eumque affirmativum. Quodsi autem fractio $\frac{x}{b}$ habeat denominatorem parem, tum applicata *y* geminum induet valorem, alterum affirmativum alterum negativum, hicque curvae punctum ad alteram asymptotae partem exhibebit; ex quo logarithmica infra asymptotam innumerabilia habebit puncta discreta, quae curvam continuam non constituunt, etiamsi ob intervalla infinite parva curvam continuam mentiantur; quod est paradoxon in lineis algebraicis locum nullum inveniens. Hinc etiam aliud oritur paradoxon multo magis mirandum. Cum enim numerorum negativorum logarithmi sint imaginarii (quod tum per se patet, tum inde intelligitur, quod $l.-1$ ad $\sqrt{-1}$ rationem habeat finitam), erit $l.-n$ quantitas imaginaria, quae sit $= i$; at cum logarithmus quadrati aequetur duplo logarithmo radicis, erit $l.(-n)^2 = l.n^2 = 2i$. At $l.nn$ est quantitas realis, $2l.n$, unde sequitur et quantitatem realem *l.n* et imaginariam *i* fore semissem eiusdem quantitatis realis *l.nn.* Hinc porro quilibet numerus duplicem habiturus esset semissem, alteram realem alteram imaginariam; similiterque cuiusque numeri triplex daretur triens, quadruplex quadrans et ita porro, quarum tamen partium unica tantum sit realis, quae quomodo cum solita quantitatum notione concillari queant, non liquet.

516. Concessis ergo his, quae assumsimus, sequeretur numeri *a* semissem fore aeque $\frac{a}{2} + l.-1$, ac $\frac{a}{2}$: illius enim duplum est

$$a + 2l.-1 = a\ l.(-1)^2 = a + l.1 = a ,$$

ubi notandum est esse

$$+l.-1 = -l.-1.$$

etiamsi non sit $l.1 = 0$; cum enim sit $-1 = \frac{+1}{-1}$, erit

$$l.-1 = l.+1 - l.-1 = -l.-1.$$

Simili modo, cum sit $\sqrt[3]{1}$ non solum 1 sed etiam $\frac{-1 \pm \sqrt{-3}}{2}$, erit

$$3l.\frac{-1 \pm \sqrt{-3}}{2} = l.1 = 0,$$

ideoque eiusdem quantitatis *a* trientes erunt

$$\frac{a}{3}, \quad \frac{a}{3} + l.\frac{-1+\sqrt{-3}}{2} \quad \text{et} \quad \frac{a}{3} + l.\frac{-1-\sqrt{-3}}{2};$$

tripla enim harum singularum expressionum producunt eandem quantitatem *a.* Ad haec dubia solvenda, quae nullo modo admitti posse videntur, aliud statui oportet paradoxon: scilicet, cuiusque numeri infinitos dari logarithmos, inter quos plus uno reali non detur. Sic, etsi logarithmus unitatis est $= 0$, tamen praeterea innumerabiles alii unitatis dantur logarithmi imaginarii, qui sunt

$$2l.-1, \quad 3l.\frac{-1 \pm \sqrt{-3}}{2}, \quad 4l.-1 \ \text{et} \ 4l.\pm\sqrt{-1},$$

innumerabilesque alii, quos extractio radicum monstrat. Haec autem sententia multo est verisimilior, quam superior: posito enim $x = l.a$ erit $a = e^x$ ideoque

$$a = 1 + x + \frac{xx}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \text{ etc.};$$

quae cum sit aequatio infinitarum dimensionum, mirum non est, si *x* habeat radices infinitas. Quanquam autem sic posterius paradoxon resolvimus, tamen prius suam vim retinet, qua ad logarithmicam infra axem innumerabilia puncta discreta pertinere ostendimus.

5I7. Multo evidentius autem huiusmodi infinitorum punctorum discretorum existentia monstrari potest per hanc aequationem $y = (-1)^x$; quoties enim *x* est numerus vel integer par vel fractus habens numeratorem parem, erit $y = 1$; sin autem *x* sit numerus vel integer impar vel fractus, cuius tam numerator quam denominator sint numeri impares, erit $y=-1$; reliquis casibus omnibus, quibus vel x est fractio denominatorem parem habens vel adeo numerus irrationalis, valor ipsius y erit imaginarius. Aequatio ergo $y=(-1)^x$ exhibebit innumerabilia puncta discreta ad utramque axis partem intervallo $=1$ posita, quorum ne bina quidem sunt contigua; hoc tamen non obstante quaeque bina ad eandem axis partem sita sibi tam erunt propinqua, ut intervallum sit data quavis quantitate assignabili minus. Inter duos enim abscissae valores quantumvis propinquos non solum una sed infinitae fractiones exhiberi possunt, quarum denominatores sint impares, ex his autem singulis nascuntur puncta ad aequationem propositam pertinentia; mentientur ergo haec puncta duas lineas rectas axi parallelas ab eo utrinque intervallo $=1$ dissitas; in his enim lineis nullum intervallum exhiberi potest, in quo non unum, imo infinita puncta aequatione $y=(-1)^x$ contenta assignari queant. Haec eadem anomalia usuvenit in aequatione $y=(-a)^x$ aliisque huic similibus, ubi quantitas negativa ad exponentem indeterminatum elevatur. Huiusmodi ergo paradoxa, quae in curvis tantum transcendentibus locum habere possunt, hic exposuisse necesse erat.

518. Ad hoc ergo genus curvarum a logarithmis pendentium pertinent omnes aequationes, in quibus non solum logarithmi occurrunt, sed etiam exponentes variabiles, quippe qui a logarithmis ad numeros progrediendo oriuntur, unde istae curvae etiam *exponentiales* vocari solent. Huiusmodi ergo curva erit, quae in hac aequatione $y=x^x$ seu $ly=xlx$ continetur. Posito ergo $x = 0$, erit $y=1$; si $x=1$, erit $y=1$; si $x=2$, erit $y=4$; si $x=3$, erit $y=27$ etc. Unde (Fig.102) *BDM* exprimet formam huius curvae ad axem *AP* relatae, ita ut sumta $AC=1$ sit $AB=CD=1$. Intra *A* et *C* autem applicatae erunt unitate minores; si enim sit $x=\frac{1}{2}$, erit

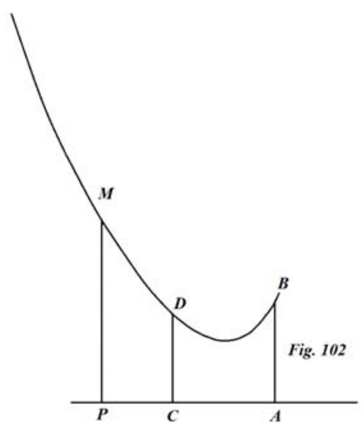


Fig. 102

$$y=\frac{1}{\sqrt{2}}=0,7071068;$$

minima vero erit applicata, si capiatur abscissa

$$x=\frac{1}{e}=0,36787944,$$

fietque tum applicata $y=0,6922005$, uti in sequentibus docebitur. Quemadmodum autem haec curva ultra *B* sit comparata ut videamus, abscissa x facienda est negativa eritque $y=\frac{1}{(-x)^x}$, unde ista pars ex meris punctis discretis constabit ad axem tanquam asymptotam convergentibus. Cadent autem haec puncta ad utramque axis partem, prout x

fuerit numerus vel par vel impar. Quin etiam infra axem *AP* infinita huiusmodi puncta cadent, si pro x sumatur fractio denominatorem habens parem; posito enim $x = \frac{1}{2}$ erit et

$$y = +\frac{1}{\sqrt{2}} \text{ et } y = -\frac{1}{\sqrt{2}}.$$

Curva ergo continua *MDB* in *B* subito terminatur, contra indolem linearum algebraicarum, loco continuationis autem habebit puncta illa discreta; unde realitas istorum punctorum quasi coniugatorum eo luculentius perspicitur. Nisi enim haec adesse concedantur, statui deberet totam curvam in puncto *B* subito cessare, id quod esset legi continuitatis contrarium ideoque absurdum.

519. Inter infinitas alias huius generis curvas, quarum constructio per logarithmos effici potest, dantur eiusmodi, quarum constructio non tam facile patet, quae tamen ope idoneae substitutionis absolvi queat. Talis est curva aequatione $x^y = y^x$ contenta; ex qua quidem statim perspicitur applicatam y perpetuo aequalem esse abscissae x, ita ut recta ad axem sub angulo semirecto inclinata aequationi satisfaciat. Interim tamen manifestum est hanc aequationem latius patere, quam aequationem pro recta $y = x$, neque igitur hanc vim aequationis $x^y = y^x$ exhaurire; satisfieri enim huic aequationi potest, etiamsi non sit $x = y$, quoniam, si $x = 2$, etiam esse potest $y = 4$. Praeter rectam ergo *EAF* aequatio proposita alias complectetur partes; ad quas inveniendas ideoque ad totam lineam (Fig.103) aequatione contentam exhibendam ponamus $y = tx$, ut sit $x^{tx} = t^x x^x$, unde radice potestatis x extrahenda erit

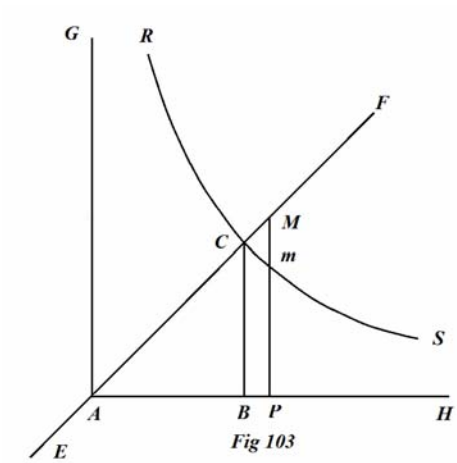


Fig 103

$$x^t = t^x \text{ et } x^{t-1} = t;$$

ideoque habebitur

$$x = t^{\frac{1}{t-1}} \text{ et } y = t^{\frac{t}{t-1}}.$$

Vel posito $t - 1 = \frac{1}{u}$ erit

$$x = (1+\frac{1}{u})^u \text{ et } y = \left(1+\frac{1}{u}\right)^{u+1}.$$

Hinc curva praeter rectam *EAF* habebit ramum *RS* ad rectas *AG* et *AH* tanquam asymptotas convergentem, cuius recta *AF* erit diameter. Secabit autem curva rectam *AF* in puncto *C*, ita ut sit $AB = BC = e$, denotante *e* numerum, cuius logarithmus est unitas. Insuper autem aequatio suppeditat innumerabilia puncta discreta, quae cum recta *EF* et curva *RCS* aequationem exhauriunt. Hinc ergo innumerabilia binorum numerorum *x* et *y* pari exhiberi possunt, ut sit $x^y = y^x$, tales enim numeri in rationalibus erunt

$$\begin{array}{ll} x = 2 & y = 4 \\ x = \frac{3^2}{2^2} = \frac{9}{4} & y = \frac{3^3}{2^3} = \frac{27}{8} \\ x = \frac{4^3}{3^3} = \frac{64}{27} & y = \frac{4^4}{3^4} = \frac{256}{81} \\ x = \frac{5^4}{4^4} = \frac{625}{256} & y = \frac{5^5}{4^5} = \frac{3125}{1024} \\ \text{etc.} & \text{etc.} \end{array}$$

horum scilicet binorum numerorum alter ad alterum elevatus eandem quantitatem producit; sic erit

$$2^4 = 4^2 = 16$$

$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left(\frac{27}{8}\right)^{\frac{9}{4}} = \left(\frac{3}{2}\right)^{\frac{27}{4}},$$

$$\left(\frac{64}{27}\right)^{\frac{256}{81}} = \left(\frac{256}{81}\right)^{\frac{64}{27}} = \left(\frac{4}{3}\right)^{\frac{256}{27}}$$

etc.

520. Quanquam in his similibusque aliis curvis infinita puncta algebraice possunt determinari, minime tamen curvis algebraicis annumerari possunt, quoniam innumerabilia alia extant puncta, quae algebraice nullo modo exhiberi possunt. Transeamus ergo ad alterum curvarum transcendentium genus, quod arcus circulares requirit; hic autem perpetuo radium circuli, cuius arcus constructionem ingrediuntur, unitate exprimo, ne pluribus characteribus calculus perturbetur. Curvas autem ad hoc genus pertinentes non esse algebraicas facile ostendi potest, etiamsi impossibilitas quadraturae circuli nondum sit evicta. Consideremus enim simplicissimam tantum huius generis aequationem hanc $\frac{y}{a} = \text{A. sin.}\frac{x}{c}$, ita ut applicata *y* sit proportionalis arcui circuli, cuius sinus est $\frac{x}{c}$. Quoniam enim eidem sinui $\frac{x}{c}$ innumerabiles arcus conveniunt, applicata *y* erit functio infinitinomia ideoque tam ipsa quam aliae rectae curvam

in infinitis punctis secabunt, quae proprietas istam curvam ab algebraicis clarissime distinguit. Sit *s* minimus arcus sinui $\frac{x}{c}$ conveniens et denotet π semicircumferentiam circuli, erunt valores ipsius $\frac{y}{a}$ sequentes

$$s,\ \pi-s,\ 2\pi+s,\ 3\pi-s,\ 4\pi+s,\ 5\pi-s \text{ etc.}$$
$$-\pi-s,\ -2\pi+s,\ -3\pi-s, -4\pi+s,\ -5\pi-s \text{ etc.}$$

Sumta ergo (Fig. 104) recta *CAB* pro axe et *A* pro abscissarum principio erunt primo posito $x=0$ applicatae $AA^1=\pi a,\ AA^2=2\pi a,\ AA^3=3\pi a$ etc. Itemque ex altera parte

$$AA^{-1}=\pi a,\ AA^{-2}=2\pi a,\ AA^{-3}=3\pi a \text{ etc.}$$

atque per singula haec puncta curva transibit. Sumta vero abscissa $AP=x$, applicata curvam in infinitis punctis *M* secabit eritque

$$PM^1=as,\ PM^2=a(\pi-s),\ PM^3=a(2\pi+s) \text{ etc.}$$

Curva ergo tota ex infinitis portionibus $AE^1A^1,\ A^1F^1A^2,\ A^2E^2A^3,\ A^3F^2A^4$ etc. similibus erit composita; ita ut singulae rectae axi *BC* parallelae, quae per puncta *E* et *F* ducuntur, futurae sint curvae diametri. Erit vero $AC=AB=c$ et intervalla $E^1E^2,\ E^2E^3,\ E^1E^{-1},\ E^{-1}E^{-2}$ itemque $F^1F^2,\ F^1F^{-1},\ F^{-1}F^{-2}$ erunt singula aequalia $2a\pi$.
Curva haec a LEIBNITIO est vocata *linea sinuum*, quoniam eius ope cuiusque arcus sinus facile invenitur. Cum enim sit

$$\frac{y}{a}=\text{A. sin.}\frac{x}{c}$$

erit vicissim

$$\frac{x}{c}=\text{sin.A.}\frac{y}{a}.$$

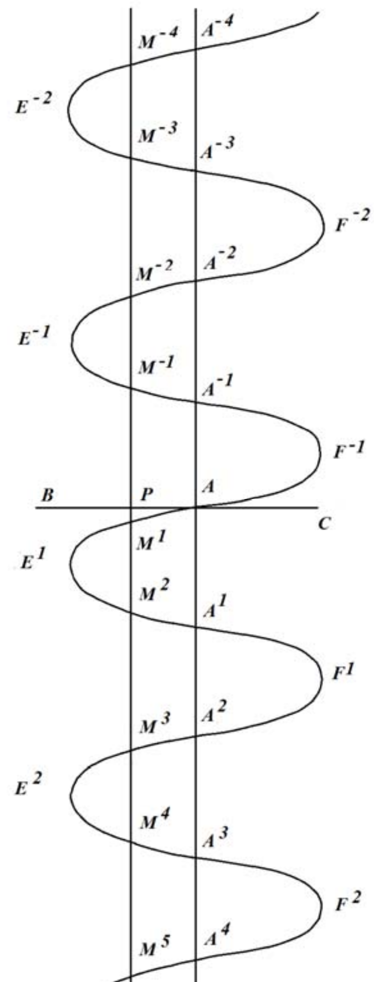


Fig. 104

Si ponatur

$$\frac{y}{a}=\frac{1}{2}\pi-\frac{x}{a}$$

fiet

$$\frac{x}{c}=\text{cos.A.}\frac{z}{a};$$

sicque simul habetur *linea cosinuum.*

521. Simili modo ex hac consideratione oritur *linea tangentium,* cuius aequatio erit $y = \text{A. tang.}\,x$, positis brevitatis ergo $a = 1$ et $c = 1$; hinc ergo convertendo fit

$$x = \text{tang. A.}y = \frac{\sin.y}{\cos.y},$$

cuius curvae figura facile ex natura tangentium colligitur. Habebit autem infinitas asymptotas inter se parallelas. Pari modo describi poterit *linea secantium* ex aequatione

$$y = \text{A.sec.}x \quad \text{seu} \quad x = \text{sec.A.}y = \frac{1}{\cos.y},$$

quae etiam infinitos ramos habet in infinitum excurrentes. Maxime vero ex hoc curvarum genere innotuit *cyclois* seu *trochois*, quae describitur a puncto in peripheria circuli super linea recta rotando progredientis, cuius aequatio inter coordinatas orthogonales est

$$y = \sqrt{(1-xx)} + \text{A.cos.}x.$$

Curva haec cum ob descriptionis facilitatem tum ob plurimas, quibus gaudet, insignes proprietates maxime est notatu digna. Quoniam autem pleraeque sine analysi infinitorum explicari nequeunt, hic tantum praecipuas, quae ex descriptione immediate fluunt, breviter perpendamus.

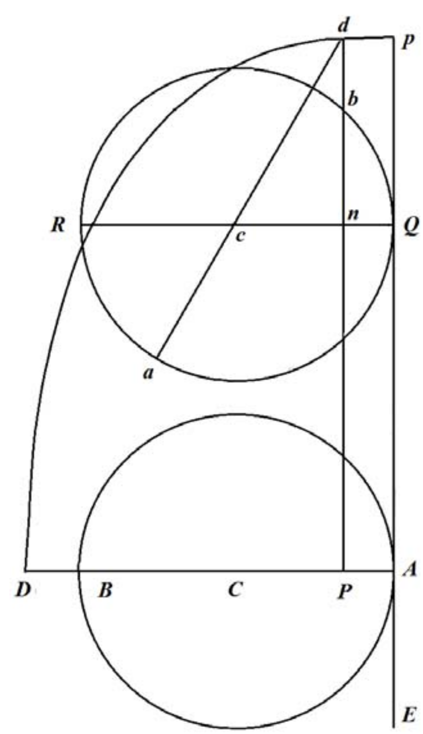


Fig. 105

522. Rotetur ergo (Fig. 105) circulus *ACB* super recta *EA;* atque, ut investigatio latius pateat, non punctum peripheriae *B* sed punctum diametri productae *D* quodcunque describat lineam curvam *Dd.*Sit huius circuli radius $CA = CB = a$, distantia $CD = b$, atque in hoc quidem situ punctum *D* locum obtineat summum. Pervenerit inter rotandum circulus in situm *aQbR;* ac posito spatio $AQ = z$ erit arcus $aQ = z$, qui divisus per radium *a* dabit angulum $acQ = \frac{z}{a}$, et punctum describens erit in *d*, ut sit $cd = b$, angulus

$$dcQ = \pi - \frac{z}{a}$$

et *d* erit punctum in curva quaesita. Ducatur ex *d* primum in rectam *AQ* normalis *dp*, tum in rectam *QR* normalis *dn* ; erit

$$dn = b\cdot\sin.\frac{z}{a} \text{ et } cn = -b.\cos.\frac{z}{a}$$

ergo

$$Qn = dp = a + b\cdot\cos.\frac{z}{a}.$$

Producatur *dn*, donec rectae *AD* occurrat in *P*, ac vocentur coordinatae

$$DP = x,\quad Pd = y\,;$$

erit

$$x = b + cn \text{ seu } x = b - b\cdot\cos.\frac{z}{a}$$

et

$$y = AQ + dn = z + b\cdot\sin.\frac{z}{a}.$$

Cum igitur sit

$$b\cdot\cos.\frac{z}{a} = b - x$$

erit

$$b\cdot\sin.\frac{z}{a} = \sqrt{(2bx - xx)}$$

et

$$z = aA.\cos.\left(1 - \frac{x}{b}\right) = aA.\sin.\frac{\sqrt{(2bx - xx)}}{b}\,;$$

quibus valoribus substitutis, erit

$$y = \sqrt{(2bx - xx)} + aA.\sin.\frac{\sqrt{(2bx - xx)}}{b}.$$

Vel, si abscissae in axe *AD* a centro computentur voceturque $b - x = t$, erit

$$\sqrt{(2bx - xx)} = \sqrt{(bb - tt)}$$

et inter *t* et *y* habebitur aequatio ista

$$y = \sqrt{(bb - tt)} + a\text{A}.\cos.\frac{t}{b},$$

quae aequatio dat cycloidem *ordinariam*, si fuerit $b = a$; sin autem sit vel b maior quam a vel b minor quam a, curva vocatur cyclois vel *curtata* vel *elongate*. Semper autem erit y functio infinitiplex ipsius x vel t; seu quaelibet recta basi AQ parallela curvam in infinitis punctis secabit, nisi eius distantia x vel t fuerit tanta, ut $\sqrt{(2bx - xx)}$ vel $\sqrt{(bb - tt)}$ fiat imaginaria quantitas.

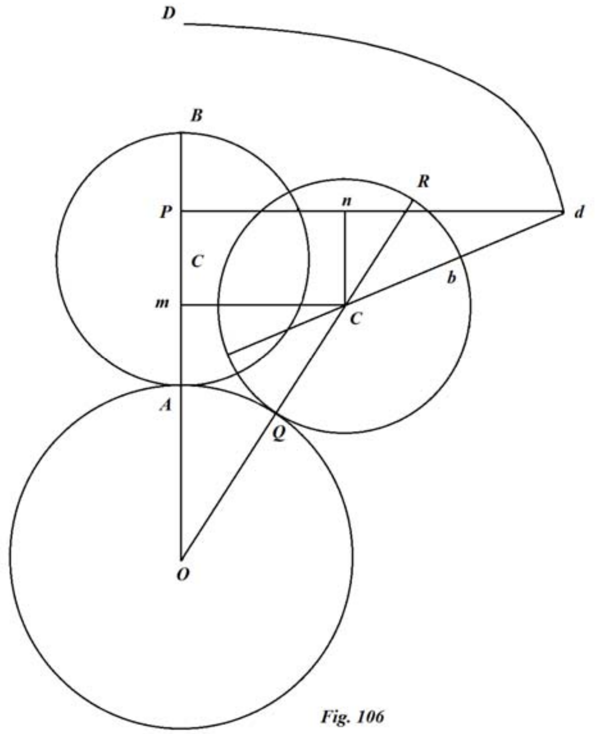


Fig. 106

523. Inter curvas huius generis, quae imprimis sunt cognitae, referri debent *epicycloides* et *hypocycloides* (Fig. 106), quae oriuntur, si circulus ACB super peripheria alterius circuli OAQ rotatur intereaque punctum quodpiam D, vel extra vel intra circulum mobilem sumtum, curvam Dd describit. Ponatur circuli immoti radius $OA = c$, radius circuli mobilis $CA = CB = a$ et distantia puncti describentis $CD = b$; sumatur autem recta OD pro axe curvae quaesitae Dd. A situ hoc initiali, quo puncta O, C, D in directum iacent, processerit circulus mobilis in situm QcR, descripto arcu $AQ = z$, ita ut sit angulus $AOQ = \frac{z}{c}$.
Erit ergo arcus $Qa = AQ = z$ hincque angulus $acQ = \frac{z}{a} = Rcd$ et, sumta recta $cd = CD = b$, erit d punctum in curva Dd. Ex eo in axem demittatur perpendiculum dP itemque ex c perpendiculum cm et en parallela axi OD. Ergo ob angulum

$$Rcn = AOQ = \frac{z}{c}$$

erit angulus

$$dcn = \frac{z}{c} + \frac{z}{a} = \frac{(a+c)z}{ac}.$$

Unde obtinetur

$$dn = b \cdot \sin.\frac{(a+c)z}{ac}$$

et

$$cn = b \cdot \cos.\frac{(a+c)z}{ac}$$

Deinde ob $OC = Oc = a + c$ erit

$$cm = (a+c) \cdot \sin.\frac{z}{c}$$

et

$$Om = (a+c) \cdot \cos.\frac{z}{c}$$

Vocatis ergo coordinatis $OP = x$ et $Pd = y$ erit

$$x = (a+\mathrm{c}) \cdot \cos.\frac{z}{c} + b \cdot \cos.\frac{(a+c)z}{ac}$$

et

$$y = (a+\mathrm{c}) \cdot \sin.\frac{z}{c} + b \cdot \sin.\frac{(a+c)z}{ac}.$$

Hinc patet, si $\frac{a+c}{a}$ fuerit numerus rationalis, tum ob commensurabilitatem angulorum $\frac{z}{c}$ et $\frac{(a+c)z}{ac}$ ipsam incognitam *z* eliminari ideoque aequationem algebraicam inter *x* et *y* inveniri posse. Reliquis casibus curva hoc modo descripta erit transcendens.

Ceterum hic notandum est, si sumatur *a* negativum, tum hypocycloidem esse prodituram, circulo mobili intra circulum immobilem cadente. Vulgo quidem *b* statuitur radio *a* aequalis sicque epicycloides et hypocycloides proprie sic dictae resultant. Hic igitur inventae curvae latius patent et, quia aequationes non sunt difficiliores, hanc conditionem adiicere visum est. Si quadrata *xx* et *yy* addantur, erit

$$xx + yy = (a+c)^2 + b^2 + 2b(a+c) \cdot \cos.\frac{z}{a},$$

cuius aequationis ope eliminatio ipsius *z* eo facilius expedietur, quoties quidem quantitates *a* et *c* fuerint commensurabiles.

524. Praeter casus, quibus amborum circulorum radii *a* et *c* sunt inter se commensurabiles curvaeque fiunt algebraicae, notari meretur iste, quo $b = -a - c$ seu quo punctum curvae *D* in centrum circuli immobilia *O* incidit.
Sit igitur $b = -a - c$ eritque

$$xx + yy = 2(a+c)^2\left(1 - \cos.\frac{z}{a}\right) = 4(a+c)^2\left(\cos.\frac{z}{2a}\right)^2 \;;$$

unde fiet

$$\cos.\frac{z}{2a} = \frac{\sqrt{(xx+yy)}}{2(a+c)}.$$

Deinde, cum sit

$$x = (a+c)(\cos.\frac{z}{c} - \cos.\frac{(a+c)z}{ac}) \text{ et } y = (a+c)(\sin.\frac{z}{c} - \sin.\frac{(a+c)z}{ac}),$$

erit

$$\frac{x}{y} = -\tan.\frac{(2a+c)z}{2ac} \text{ et } \sin.\frac{(2a+c)z}{2ac} = \frac{x}{\sqrt{xx+yy}}$$

atque

$$\cos.\frac{(2a+c)z}{2ac} = \frac{-y}{\sqrt{xx+yy}}.$$

Quare, cum sit

$$\sqrt{xx+yy} = 2(a+c)\cos.\frac{z}{2a},$$

fiet

$$x = 2(a+c)\cos.\frac{z}{2a}\cdot\sin.\frac{(2a+c)z}{2ac}$$

et

$$y = -2(a+c)\cos.\frac{z}{2a}\cdot\cos.\frac{(2a+c)z}{2ac}.$$

Sit exempli gratia $c = 2a$; erit

$$x = 6a\cdot\cos.\frac{z}{2a}\cdot\sin.\frac{z}{a} \text{ et } y = -6a\cdot\cos.\frac{z}{2a}\cdot\cos.\frac{z}{a}$$

et

$$\sqrt{xx+yy} = 6a\cdot\cos.\frac{z}{2a}.$$

Ponamus

$$\cos.\frac{z}{2a} = q,$$

erit

$$\sin.\frac{z}{2a} = \sqrt{(1-qq)} \text{ et } \sin.\frac{z}{a} = 2q\sqrt{(1-qq)} \text{ atque } \cos.\frac{z}{a} = 2qq-1,$$

unde fit

$$q = \frac{\sqrt{xx+yy}}{6a}$$

et

$$y = -6aq(2qq-1) = (1-2qq)\sqrt{(xx+yy)} = (1-\frac{xx-yy}{18aa})\sqrt{(xx+yy)}$$

seu

$$18aay = (18aa-xx-yy)\sqrt{(xx+yy)}.$$

Ponatur $18aa = ff$ et sumtis quadratis habebitur ista aequatio sexti ordinis

$$(xx+yy)^3 - 2ff(xx+yy)^2 + f^4xx = 0.$$

Quoniam vero hic nobis est propositum non curvas algebraicas sed transcendentes contemplari, his missis ad eiusmodi curvas progrediamur, quarum constructio simul tam logarithmos quam arcus circulares requirat.

525. Supra vero iam eiusmodi nacti sumus curvam (Fig. 107) ex aequatione

$$2y = x^{+\sqrt{-1}} + x^{-\sqrt{-1}},$$

quam transmutavimus in hanc $y = \cos.\text{A}.lx$. Haec vero ulterius abit in

$$\text{A}.\cos.y = lx \text{ et } x = e^{A.\cos.y}.$$

Sumta ergo recta AP pro axe in eoque A pro initio abscissarum, primo patet ultra A in regione abscissarum negativarum curvae nullam dari portionem continuam, axis autem AP a curva in infinitis punctis D intersecabitur, quorum punctorum ab A distantiae progressionem geometricam constituent, erit scilicet

$$AD = e^{\frac{\pi}{2}}, AD^1 = e^{\frac{3\pi}{2}}, AD^2 = e^{\frac{5\pi}{2}}, AD^3 = e^{\frac{7\pi}{2}} \text{ etc.,}$$

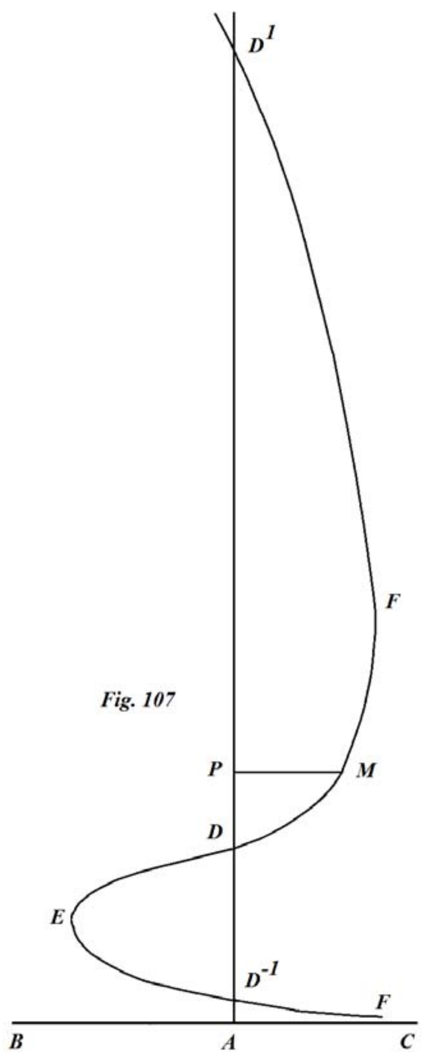


Fig. 107

tum vero dabuntur infinitae intersectiones ad *A* propius accedentes

$$AD^{-1} = e^{\frac{-\pi}{2}},\ \ AD^{-2} = e^{\frac{-3\pi}{2}},\ \ AD^{-3} = e^{\frac{-5\pi}{2}}\ \ \text{etc.},$$

Deinde haec curva utrinque ad axem excurret ad distantias $AB = AC = 1$ ibique rectas axi parallelas tanget in infinitis punctis *E* et *F*, quorum distantiae a *B* et *C* pariter progressionem geometricam constituent. Infinitis ergo flexibus curva ad rectam *BC* accedet atque tandem cum ea prorsus confundetur. Singularis ergo huius curvae proprietas in hoc consistit, quod non recta infinita sed finita *BC* curvae sit asymptota, quo ipso huius curvae indoles ab algebraicis maxime distinguitur.

526. Ad curvas transcendentes, quarum constructio angulos vel solos vel cum logarithmis coniunctos requirit, referri quoque debent innumerabiles *spiralium* species. Respiciunt autem spirales punctum quodpiam fixum *C* tanquam centrum (Fig.108), circa quod plerumque infinitis spiris circumducuntur. Natura harum curvarum commodissime explicatur per aequationem inter cuiusque curvae puncti *M* a centro *C* distantiam *CM* et angulum *ACM*, quem haec recta *CM* cum recta positione data, *CA* constituit. Sit ergo angulus $ACM = s$ seu sit *s* arcus circuli radio $=1$ descripti, qui sit anguli *ACM* mensura, ac ponatur recta $CM = z$. Quodsi nunc detur aequatio quaecunque inter variabiles *s* et *z*, curva resultabit spiralis. Cum enim angulus *ACM* praeter *s* infinitis modis exprimi queat, quoniam anguli $2\pi + s, 4\pi + s, 6\pi + s$ etc., item $-2\pi + s, -4\pi + s,$ etc. eandem positionem rectae *CM* exhibent, his valoribus loco *s* in aequatione substitutis distantia *CM* infinitos diversos obtinebit valores ideoque recta *CM* producta curvam in infinitis punctis secabit, nisi ex his valoribus quantitas *z* fiat imaginaria. Incipiamus ergo a casu simplicissimo, quo est $y = as$; eruntque pro eadem rectae *CM* positione valores ipsius *y* isti $a(2\pi + s),\ a(4\pi + s),\ a(6\pi + s)$ etc., itemque

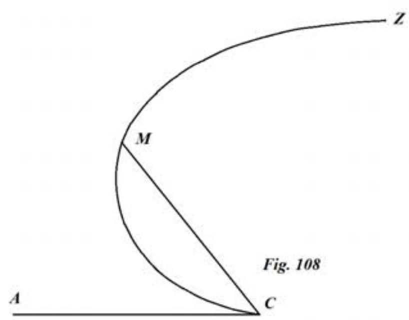


Fig. 108

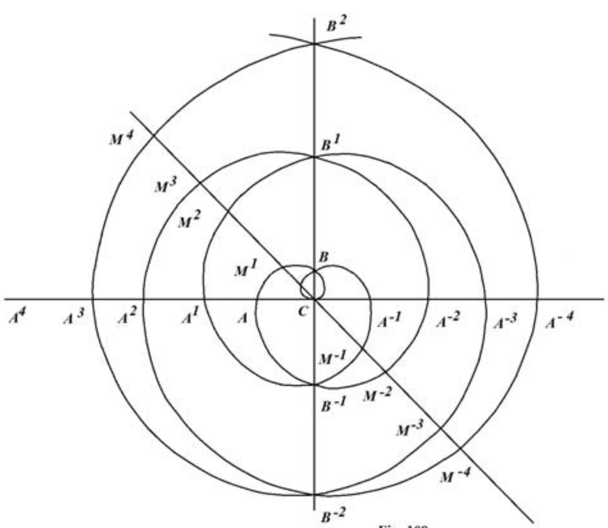


Fig. 109

$-a(2\pi - s),\ -a(4\pi - s),\ -a(6\pi - s)$ etc. Quin etiam, si pro *s* ponatur $\pi + s$, eadem rectae *CM* manebit positio, praeterquam, quod valor ipsius *z* capi debeat negative, hinc ad valores ipsius *z* assignatos addi oportet hos $-a(\pi + s),\ -a(3\pi + s),\ -a(5\pi + s)$ etc. praetereaque istos $a(\pi - s),\ a(3\pi - s),\ a(5\pi - s)$ etc. Curvae ergo (Fig.109) huius forma erit talis, qualis in figura ad marginem allegata repraesentatur; rectam scilicet *AC* in *C* tangit hincque duobus ramis, utrinque infinitis gyris centrum *C* ambientibus et se mutuo in recta *BC* ad *AC* normali perpetuo decussantibus, in infinitum extenditur; eritque recta *BCB* eius diameter. Vocari autem haec curva ab inventore solet *spiralis* ARCHIMEDEA; atque, si semel est exacte descripta, inservit ad quemvis angulum in quotcunque partes secandum, uti ex eius aequatione $z = as$ sponte patet.

527. Quemadmodum aequatio $z = as$, quae, si *z* et *s* essent coordinatae orthogonales, foret pro linea recta, praebuit spiralem ARCHIMEDEAM, ita si aliae aequationes algebraicae inter *z* et *s* accipiantur, infinitae aliae prodibunt lineae spirales, siquidem aequatio ita sit comparata, ut singulis ipsius *s* valoribus respondeant valores reales ipsius *z*. Ita haec aequatio $z = \frac{a}{s}$, quae similis est aequationi pro hyperbola ad asymptotas relata, praebet spiralem, quae a cel. JOHANNE BERNOULLIO vocata est *spiralis hyperbolica;* atque, postquam ex centro *C* infinitis gyris exiisset, tandem in distantia infinita ad rectam *AA* tanquam asymptotam accedit. Quodsi proponatur aequatio $z = a\sqrt{s}$, angulis *s* negative sumtis nulla respondebit distantia realis *z* ; valoribus autem affirmativis singulis ipsius *s* gemini valores ipsius *z* respondebunt, alter affirmativus alter negativus, spirae tamen circa *C* absolventur infinitae.

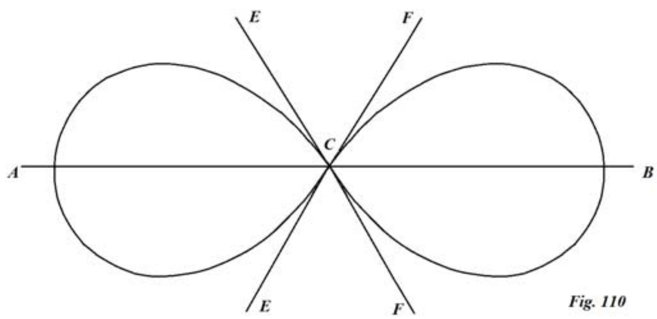


Fig. 110

Sin autem aequatio inter *z* et *s* fuerit huiusmodi $z = a\sqrt{(nn - ss)}$, variabilis *z* nullum habebit valorem realem, nisi *s* contineatur intra hos limites $+n$ et $-n$; ideoque hoc casu curva erit finita. Scilicet, si (Fig. 110) ad axem *ACB* per centrum *C* B utrinque inclinentur rectae *EF*, *EF* cum axe angulum $= n$ constituentes, hae erunt curvae sese in *C* decussantis tangentes ipsaque curva habebit *lemniscatae* formam *ACBCA.* Simili autem modo innumerabiles aliae obtinebuntur linearum transcendentium formae, quas evolvere nimis foret prolixum.

528. Haec tractatio porro in immensum amplificari posset, si inter *z* et *s* non aequationes algebraicae sed adeo transcendentes accipiantur. Ex quo genere prae reliquis notari meretur ea linea curva, quae hac aequatione $s = nl.\frac{z}{a}$ exprimitur, in qua scilicet anguli

s sunt logarithmis distantiarum *z* proportionales ; ob quam causam haec curva *spiralis logarithmica* appellatur atque ob plurimas insignes proprietates maxime est nota. Huius curvae (Fig. 111) primaria proprietas est, quod omnes rectae ex centro *C* eductae curvam sub aequalibus angulis intersecent. Ad eam ex aequatione educendam sit angulus $ACM = s$ et recta $CM = z$, eritque

$$s = nl.\frac{z}{a} \text{ et } z = ae^{\frac{s}{n}},$$

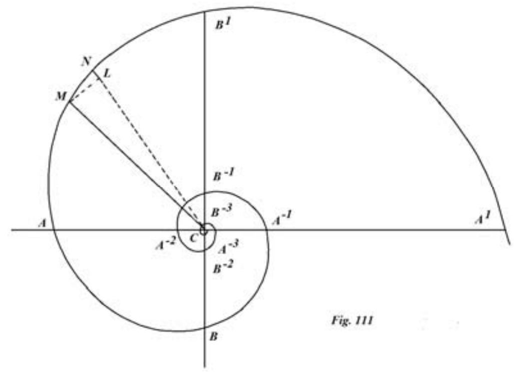


Fig. 111

tum capiatur angulus maior $ACN = s + v$, erit recta

$$CN = ae^{\frac{s}{n}}e^{\frac{v}{n}},$$

ideoque centro *C* descripto arcu *ML*, qui erit $= zv$, fiet

$$LN = ae^{\frac{s}{n}}(e^{\frac{v}{n}} - 1) = ae^{\frac{s}{n}}(\frac{v}{n} + \frac{v^2}{2n^2} + \frac{v^3}{6n^3} + \text{ etc.}).$$

Hinc erit

$$\frac{ML}{LN} = \frac{v}{\frac{v}{n} + \frac{v^2}{2n^2} + \frac{v^3}{6n^3} + \text{ etc.}} = \frac{n}{1 + \frac{v}{2n} + \frac{v^2}{6nn} + \text{ etc.}}.$$

At evanescente angulorum differentia $MCN = v$ fiet $\frac{ML}{LN}$ tangens anguli, quem radius *CM* cum curva constituit; unde facto $v = 0$ istius anguli *AMC* tangens erit *n* ideoque iste angulus constans. Si fuerit $n = 1$, iste angulus erit semirectus hocque casu spiralis logarithmica vocatur semirectangula.