INTRODUCTION
TO
THE ANALYSIS OF INFINITE QUANTITIES.

BOOK TWO

Containing

The theory of curved lines, together with an appendix on surfaces.

CHAPTER I

ABOUT CURVED LINES IN GENERAL

1. Because a variable quantity is a magnitude including in general all the quantities determined among themselves, in geometry a variable quantity of this kind will be represented most conveniently by an indefinite straight line (Fig. 1) *RS.* For since

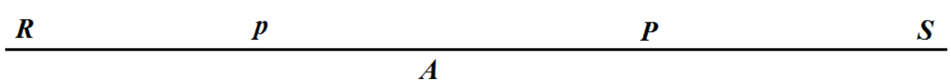

Fig. 1

it may be permitted to remove some determined quantity from the line, and that equally presents to the mind the same variable quantity. Therefore in the first place a point *A* must be assumed on the indefinite line *RS,* from which determined magnitudes may be considered to be removed in the first place ; and thus the determined part *AP* will represent the determined value taken by the variable quantity.

2. Therefore let *x* be a variable quantity, which may be represented by the indefinite right line *RS*, and it is evident that all the values of *x* determined, which likewise shall be real, are able to be represented by parts cut from the line *RS*. Clearly, if the point *P* may be taken by the point *A* itself, then the interval *AP* vanishing will show the value $x = 0$; so that also the more the point *P* may be removed from *A*, in that case the greater value determined of *x* will be represented by interval *AP*.

Moreover these intervals *AP* are called *abscissas.* [The Latin word *abscissa* in the singular is retained, meaning a cut or here perhaps a notch.]

And thus the abscissa show the values determined of the variable.

3. Truly because the indefinite right line *RS* departs from *A* to infinity on both sides, also on both sides all the values of *x* will be able to be cut off. But if moreover we may cut off the positive values of *x* to be progressing to the right of *A*, the negative abscissa values of *x* will be shown by the intervals *Ap* to the left. Since indeed, from that a greater interval *AP* will indicate a greater value of *x*, by which the point *P* may be distant from *A* to the left, thus in turn, from that the greater value of *x* may be diminished, by which the point *P* may be moved more to the left ; and, if *P* may arrive at *A*, there becomes $x = 0$. On this account, if *P* may be moved further to the left, the values of *x* become smaller than zero, that is negative, and thus intervals *Ap* to the left of *A* will show negative values of *x*, if indeed the intervals to the right of *AP* may be agreed to bear positive values taken. But there is a choice, whichever region may be selected designating the positive values of *x*: for the opposite will contain always the negative values of *x*.

4. Therefore since an indefinite right line may show the variable quantity *x*, we may see, how some function of *x* may be represented as conveniently geometrically. Let *y* be some function of *x*, which therefore may adopt a determined value, if a determined value may be substituted for *x*. With the indefinite right line (Fig. 2) *RAS* denoting the values of

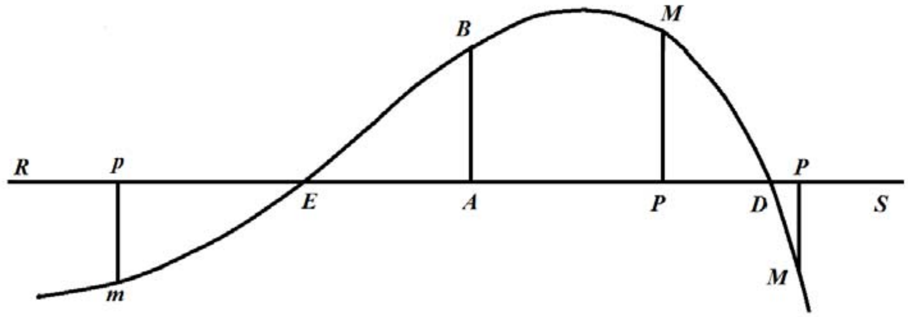

Fig. 2

x, for any value of *x* determined *AP* the right line *PM* may be put in place normally equal to the corresponding value of *y*. Clearly, if the value of *y* may be produced positive, this may be put in place above the right line *RS*, but if a negative value of *y* may arise, this may be attached normally below the right line *RS*. For with positive values of *y* taken they will fall above the right line *RS*, the vanishing values fall on *RS* itself, and negative values below that.

5. A figure of this kind therefore shows a function of x for y, which on putting $x = 0$ may adopt a positive value $= AB$; but if there may be taken $x = AP$, it becomes $y = PM$; if $x = AD$, it becomes $y = 0$ and, if there may be taken $x = AP$, the function y takes a negative value and thus the applied line *PM* falls normally below the right line *RS*. In a similar manner the values of y, which correspond to negative values of x, may be represented by applied lines put in place above *RS*, if they shall be positive ; but the opposite such as *pm* must be put in place below the right line *RS* ; but if for some with the value of x, so that $-x = AE$, $y = 0$ arises, then the length of the applied line vanishes here.

6. If therefore for all the values of x determined the corresponding values of y may be defined in this manner, for the individual points *P* of the right line *RS*, the applied right lines *PM* expressing the values of the function y ; and of these applied lines *PM* the one end *P* is incident on the right line *RS*, truly the other end *M* either will be placed higher or above *RS*, if the value of y were positive, or below if it were negative, or also fall on the line *RS* itself, if it should vanish, as arises at the points *D* and *E*. Therefore the individual extremities of the applied lines *M* will represent some line either right or curved, which therefore will be determined by the function y in this manner. Whereby a certain function of x itself, changed into geometry in this manner, will determine a certain line either straight or curved, the nature of which will depend on the nature of the function y.

7. Moreover in this manner a curved line is known perfectly, which may result from a function y, because all the points of this may be determined from the function y ; for from the individual points *P* the length of the applied normal *PM* may be considered, the end point *M* of which shall be put on the curved line, and thus all the points of the curved line may be found. But in whatever manner a curved line may have been prepared, from the individual points of this, normals can be drawn to the right line *RS*, and thus the interval *AP* will be obtained, which show the values of the variable x, and the lengths of the applied lines *PM*, which represent values of the function y. Hence no point stands out from the curve, because it is not defined by the function y in this manner.

8. Although many curved lines are able to be described by the continued motion of a point mechanically, by which the whole curved line composed may be viewed at the same time, yet we will consider the origin of these curved lines here chiefly as arising from functions showing greater analytical width of application, and more suited to calculation. Therefore some function of x will supply the needs either of certain straight or curved lines, from which in turn the curved lines will be permitted to be called functions. Therefore the nature of each curved line may be expressed by a function of x of this kind, which may always show the true length of this applied line *MP*, while the interval *AP* to which the perpendiculars *MP* may be sent from the individual points of the curve *M* towards the right line *RS*, are indicated by the variable x.

9. From this idea of curved lines the division of these follows at once into *continuous* and *discontinuous* or *mixed* curves. Clearly a continuous curved line has been prepared thus, so that its nature may be expressed by a single defined function of *x*. But if moreover the curved line may be prepared thus, so that the various parts of this *BM, MD, DM* etc. may be expressed by various functions of *x*, thus so that, after the part *BM* had been described from a single function, then from another function the part *MD* may be described, we will call curved lines of this kind *discontinuous*, or *mixed* and *irregular*, on account of which they may not be formed from a single constant law, and instead are composed from parts of various curves.

10. But the talk in geometry is chiefly about a continuous curve and it may be shown below, curves which may be described mechanically from a uniform motion following a certain constant rule, the same may be described also expressed by a single function, and thus to be continuous. Therefore let *mEBMDM* [fig. 2] be a continuous curved line, the nature of which may contain a certain function of *x*, which shall be *y*; and it is evident, from the values of *x* determined taken on the line *RS* from the fixed point *A*, then the corresponding values of *y* provide the length of the applied normals *PM*.

11. Certain names are to be considered in this exposition of curved lines, the most frequent use of which appears in the teaching of curved lines. In the first place therefore the right line *RS*, from which the values of *x* are cut, is called the *axis* or the right line *directrix.*

The point *A*, from which the values of *x* are measured, is called the *beginning of the abscissa.* [Which of course is now called the origin.]

Moreover the parts of the axis *AP*, from which the values of *x* are indicated, are accustomed to be called the *abscissa.*

And the perpendiculars *PM* reaching from the ends of the abscissas *P* to the curved lines are given the name *applied lines*.

Moreover in this case the applied lines are called *normals* or *orthogonals*, because they constitute a right angle with the axis; indeed since in a similar manner the applied lines *PM* may constitute an oblique angle with the axis, in this case the applied lines are called oblique-angled ; truly here we will express the nature of the curves constantly by orthogonal applied lines, unless the contrary may be indicated in words.

12. Therefore if some abscissa *AP* may be expressed by the variable *x*, so that there shall be $AP = x$, then the function *y* will indicate the magnitude of the applied line *PM* and there will be $PM = y$. Therefore the nature of the curved line, if indeed it were continuous, may be expressed by the nature of the function *y* or by the ratio, by which *y* may be composed from *x* and from constant quantities. Therefore on the axis *RS* the part *AS* will be the location of positive abscissas, and the part *AR* the place of negative abscissas ; then indeed the region of positive applied lines exists above the axis *RS*, and moreover below will be the region of negative applied lines.

13. Therefore since a continuous curved line may arise from any function of *x*, this also can become known and described from that known function. For in the first place the positive values of *x* can be granted from 0 and progressing as far as ∞, and for the individual values the corresponding values of the function *y* may be sought, which may be represented by the applied lines either above or below the extended line, provided they may have either positive or negative values; and thus a part of the curve *BMM* will emerge. Then in a similar manner all the negative values of x may be granted from 0 by progressing to $-\infty$, and the corresponding values of *y* will determine a part of the curve *Bem*, and thus the whole curved line satisfied by the function will be shown.

14. Because *y* is a function of *x*, either y will be equal to an explicit function of *x*, or an equation will be given between *x* and *y,* by which *y* is defined by *x* : in either case an equation will be found, which is said to express the nature of the curve. On this account the nature of each curve will be shown by an equation between the two variables *x* and *y*, of which the one *x* will denote the abscissas taken from a given starting point *A* on the axis, truly the other will denote the applied lines *y* normal to the axis. Moreover the abscissa and the applied lines considered together are called the *orthogonal coordinates* ; and hence the nature of the curved lines is said to be defined by an equation between the orthogonal coordinates, and if a determining equation may be found, such shall be the function *y* of *x*.

15. Therefore since a knowledge of curved lines leads to functions, we may consider that there are just as many kinds of curved lines present as there are functions above. Therefore curved lines may be divided most conveniently into *algebraic* and *transcending* according to the manner of functions. Clearly a curved line will be algebraic, if the applied line *y* were an algebraic function of the abscissa *x;* or, since the nature of the curved line may be expressed by an algebraic equation between the coordinates *x* and *y,* any curved lines of this kind are accustomed to be called also *geometric*. But a curved line is *transcending*, the nature of which is expressed by a transcending equation between *x* and *y,* or from which *y* is a transcending function of *x*. And this is the particular division of continued curved lines, since these are either *algebraic* or *transcending.*

16. Moreover the curved line for which the applied line *y* is expressed by a given function of *x*, the nature of the function to be described has to be considered properly, whether it shall be of one form, or of several forms. In the first place we may put *y* to be a uniform function of *x*, or is $y = P$, with *P* denoting some uniform function of *x*, and, because by granting *x* some determined value, the applied line also receives some single determined value, for which each of the abscissa will correspond to a single applied line and on that account the curve itself may be prepared, so that, if at some point *P* of the axis *RS* a normal *PM* may be drawn to itself, that always cuts the curve and that in a single point *M*. Therefore the individual points of the axis will correspond to the points of the curve and, since the axis may be extended to infinity in each direction, the curve also may depart to infinity in each direction. Or the curve arising from such a function will be drawn out

continually to infinity in each direction with the axis stretched out to infinity, figure 2 shows a prolongation, where the curved line *mEBMDM* will be extended to infinity without any interruption.

17. Let *y* be a two-form [*i.e.* two-valued] function of *x*, or, with the uniform functions of *x* denoted by the letters *P* and *Q* let $yy = 2Py - Q$, so that there shall be $y = P \pm \sqrt{(PP - Q)}$. Therefore for each abscissa *x* there will correspond a two-fold applied line *y*, with each proving to be either real or imaginary : the former if $PP > Q$, the latter if $PP < Q$. Therefore as long as each value of y will be real, the abscissa *AP* (Fig. 3) agree on the two-fold applied lines *PM, PM,* or a right line normal to the axis at *P*

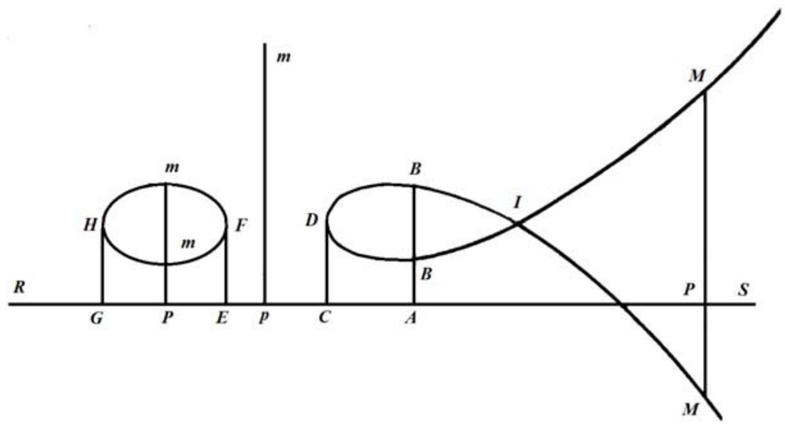

Fig. 3

will cut the curve at the two points *M* and *M*. But where there shall be $PP < Q$, there no longer an applied line present corresponding to the abscissa ; or a normal to the axis from these places on the curve can on no account occur, as arises at *p*. But since before the condition was $PP > Q$, it cannot become $PP < Q$, except by passing through the case $PP = Q$, which will be the boundary between real applied lines and imaginary ones. Therefore where the real applied lines cease, as at *C* or *G*, there it becomes $y = P \pm 0$, or both the applied lines are made equal to each other, and there the path of the curve may be returned on being inflected.

18. Following the figure it is apparent, as long as the negative abscissa $-x$ may be contained between the limits *AC* and *AE*, the applied line *y* becomes imaginary and there is $PP < Q$; beyond *E* by progressing to the left the applied lines again become real, which cannot happen, unless there shall be $PP = Q$ at *E* and thus both the applied lines come together. Then again the two-fold application *Pm, Pm* corresponds to the abscissa *AP*, finally arriving at *G*, where these two applied lines meet and beyond *G* become imaginary anew. Therefore a curved line of this kind will be able to be constructed from the disjoint parts in turn among themselves, from two or more parts, as *MBDBM* and

FmHm ; indeed these lesser parts jointly considered have been agreed to constitute nothing less than a continuous curve or a regular figure, because these individual parts are generated by one and the same function. Therefore these curves have this property, so that, if the right lines *MM* may be produced normally from the individual points of the axis, these always either pass through the curve in two points, or not at all ; unless perhaps the two points of intersection merge into one, which happens if the applied lines may be drawn through the points *D, F, H* or *I*.

19. If y were a triform function [*i.e.* three-valued or cubic function] of x or if y may be defined by an equation of the form $y^3 - Py^2 + Qy - R = 0$, with P, Q and R being uniform functions of x, then for some value of x the applied line y will have three values, which either are all real, or only one, with the remaining two being imaginary. Hence all the applied lines will cut the curve either in three points or in a single point only, unless where the two or even three intersections merge into one point. Therefore since each abscissa at any rate shall meet one real applied line, it is necessary, that the curve extends to infinity in each direction with the axis. Therefore the curve will consist either of a single drawn out line, as in *figure four*, or into two parts separated from each other, as in *figure five*, or into several parts, which still all constitute joined together one single continuous curve.

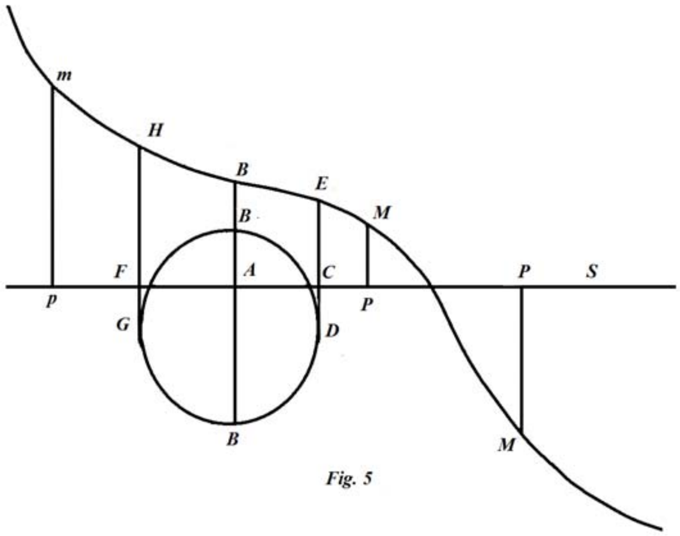

Fig. 5

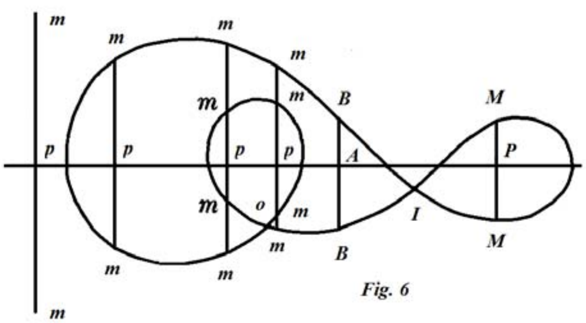

Fig. 6

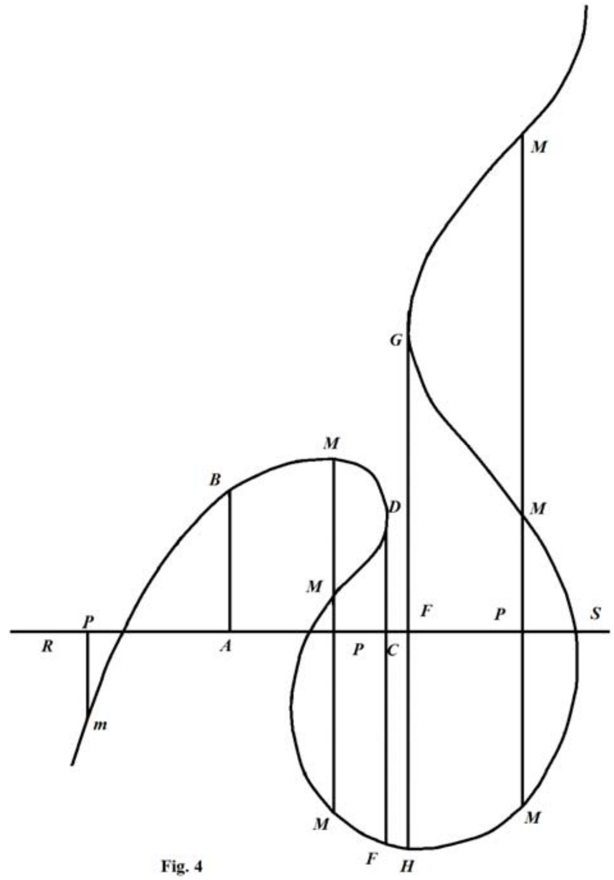

Fig. 4

20. If *y* were a quadriform [*i.e.* a quartic] function or *x* or if *y* may be defined by an equation of this kind $y^4 - Py^3 + Qy^2 - Ry + S = 0$, then for each value of x either four real values of *y* will correspond or only two or none at all. Hence in the curve arising from a quadriform function of this kind the individual applied lines will cut the curve either in four points or in two points only, or none at all, which individual cases *figure six* shows ; but the places *I* and *o* must be observed, where two points of intersection merge into a single point. On this account no branches of the curve extend to infinity, neither on the right nor on the left, neither for two or even four branches. In the first case, so that from neither part no branches are extended to infinity, the curve thence will be closed, as *figure six* shows, and it will enclose a definite space. Hence therefore the nature of the

curved lines can be concluded, which are formed from multiform functions of some designated number.

21. Evidently if *y* were a multi-form function or it will be determined by an equation, in which *n* shall be the exponent of the maximum power of *y*, then the number of real values of y will be either *n*, $n-2$, $n-4$ or $n-6$ etc., therefore in just as many points as the applied line will be allowed to be cut. Thus, if a single applied line may cut a continuous curve in *m* points, all the other applied lines will cut the curve in as many points, the number of which always differs from *m* by an even number ; therefore on no account will the curve be able to be cut in $m+1$, $m-1$, or $m\pm 3$, etc. points. That is, if the number of intersections of one applied line were even or odd, all the remaining applied lines also will be cut either in an even or odd number.

22. Therefore if a single applied line may cut a curve in an odd number of points, then it cannot happen, that any other applied line will never cut the curve : therefore the curve on both sides will have a minimum of one branch extending to infinity and, if from either side several branches may be extending to infinity, the number of these must be odd, because the number of intersections of each applied line cannot be even ; therefore if the branches extending to infinity on both sides may be counted at the same time, the number of these constantly will be even. This likewise has a place, if the applied lines may cut the curve in an even number of points, for then from each side separately either none, two, four, etc. branches will extend to infinity, therefore so that the number of all the branches extending to infinity will be even. Therefore now we have gained some significant properties of continuous and regular curves, so that these can be distinguished from discontinuous and irregular curves.

INTRODUCTIO
IN
ANALYSIN INFINITORUM.
LIBER SECUNDUS
Continens
Theoriam Linearum curvarum, una cum appendice
de Superficiebus,

LIBER SECUNDUS

CAPUT I

DE LINEIS COURVIS IN GENERE

1. Quoniam quantitas variabilis est magnitudo in genere considerata omnes quantitates determinatas in se complectens, in Geometria huiusmodi quantitas variabilis convenientissime repraesentabitur (Fig. 1) per lineam rectam indefinitam *RS*. Cum enim

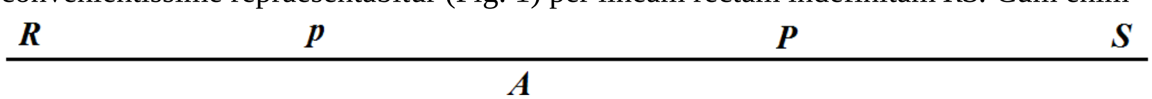

Fig. 1

in linea indefinita magnitudinem quamcunque determinatam abscindere liceat, ea pariter ao quantitas variabilis eandem quantitatis ideam menti offert. Primum igitur in linea indefinita *RS* punctum assumi debet *A*, unde magnitudines determinatae abscindendae initium sumere censeantur; sicque portio determinata *AP* repraesentabit valorem determinatum in quantitate variabili comprehensum.

2. Sit igitur x quantitas variabilis, quae per rectam indefinitam *RS* repraesentetur, atque manifestum est omnes valores determinatos ipsius x, qui quidem sint reales, per portiones in recta *RS* abscindendas repraesentari posse. Scilicet, si punctum *P* in ipso puncto *A*

capiatur, intervallum *AP* evanescens exhibebit valorem $x = 0$; quo magis autem punctum *P* ab *A* removetur, eo maior valor determinatus ipsius *x* intervallo *AP* repraesentabitur.
Vocantur autem haec intervalla *AP abscissae.*
Atque ideo abscissae exhibent variabilis *x* valores determinates.

3. Quia vero recta *RS* indefinita utrinque ab *A* in infinitum excurrit, utrinque etiam omnes ipsius *x* valores abscindi poterunt. Quodsi autem valores affirmativos ipsius *x* ab *A* dextrorsum progrediendo abscindamus, intervalla *Ap* sinistrorsum abscissa valores ipsius *x* negativos exhibebunt. Cum enim, quo longius punctum *P* dextrorsum ab *A* distat, intervallum *AP* eo maiorem valorem ipsius *x* significet, sic vicissim, quo magis punctum *P* sinistrorsum removetur, eo magis valor ipsius *x* diminuetur; atque, si *P* ad *A* perveniat,omnino fiet *x* = o. Hanc ob rem, si *P* ulterius sinistrorsum removeatur, valores ipsius *x* nihilo minores, hoc est negativi, denotabuntur atque ideo intervalla *Ap* ab *A* sinistrorsum abscissa valores ipsius *x* negatives exhibebunt, siquidem intervalla *AP* dextrorsum sumta valores affirmativos praebere censeantur. Arbitrarium autem est, utra plaga ad valores affirmativos ipsius *x* designandos eligatur: semper enim opposita valores ipsius *x* negatives continebit.

4. Cum igitur linea recta indefinita quantitatem variabilem *x* exhibeat, videamus, quomodo functio ipsius *x* quaecunque quam commodissime geometrice repraesentari queat. Sit *y* functio quaecunque ipsius *x*, quae ergo valorem determinatum induat, si pro *x* valor determinatus substituatur. Sumta (Fig. 2) recta indefinita *RAS* ad valores ipsius

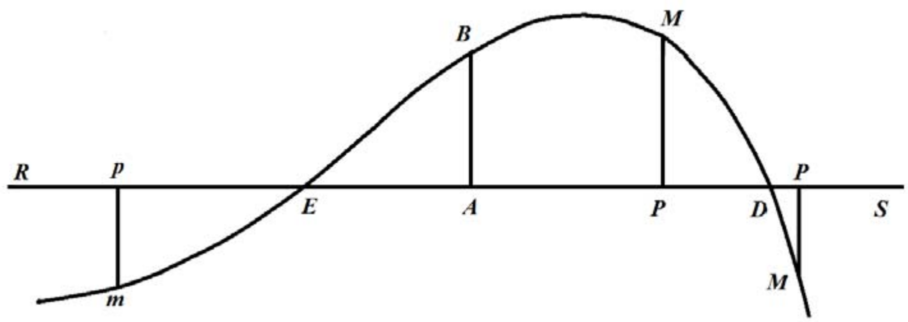

Fig. 2

x denotandos, cuilibet valori ipsius *x* determinate *AP* normaliter applicetur recta *PM* valori ipsius *y* respondenti aequalis. Scilicet, si valor ipsius *y* prodeat affirmativus, is supra rectam *RS* constituatur, sin autem valor ipsius *y* negativus oriatur, is infra rectam *RS* normaliter applicetur. Sumtis enim valoribus ipsius *y* affirmativis supra rectam *RS*, evanescentes in ipsam *RS* et negativi infra eam cadent.

5. Figura ergo eiusmodi functionem ipsius *x* pro *y* exhibet, quae posito $x = 0$ induat valorem affirmativum $= AB$; sin capiatur $x = AP$, fit $y = PM$; si $x = AD$, fit $y = 0$ et, si sumatur $x = AP$, functio *y* accipit valorem negativum ideoque normaliter

applicata *PM* infra rectam *RS* cadit. Simili modo valores ipsius *y*, qui valoribus negativis ipsius *x* respondent, repraesentantur per applicatas supra *RS* positas, si sint affirmativi; contra autem infra rectam *RS* constitui debent, ut *pm;* sin autem pro quopiam ipsius *x* valore, ut $-x = AE$, fiat $y = 0$, tum ibi longitudo applicatae evanescit.

6. Si igitur hoc modo pro omnibus valoribus determinatis ipsius *x* definiantur valores ipsius *y* respondentes, ad singula rectae *RS* puncta *P* constituentur rectae normaliter applicatae *PM* valores functionis *y* exprimentes harumque applicatarum *PM* alteri termini *P* in rectam *RS* incident, alteri vero *M* vel supra *RS* erunt positi, si valores ipsius *y* fuerint affirmativi, vel infra, si sint negativi, vel etiam in ipsam rectam *RS* incident, si evanescant, uti evenit in punctis *D* et *E.* Singulae ergo applicatarum extremitates *M* repraesentabunt lineam quampiam sive rectam sive curvam, quae igitur hoc modo per functionem *y* determinabitur. Quare quaelibet ipsius *x* functio, hoc modo ad Geometriam translata, certam determinabit lineam sive rectam sive curvam, cuius natura a natura functionis *y* pendebit.

7. Hoc autem modo linea curva, quae ex functione *y* resultat, perfecte cognoscitur, quoniam omnia eius puncta ex functione *y* determinantur; in singulis enim punctis *P* constat longitudo applicatae normalis *PM,* cuius extremum punctum *M* in linea curva sit positum, sicque omnia lineae curvae puncta inveniuntur. Quomodocunque autem linea curva fuerit comparata, ex eius singulis punctis rectae normales ad rectam *RS* duci possunt sicque obtinentur intervalla *AP,* quae valores variabilis *x* exhibent, et longitudines applicatarum *PM,* quae valores functionis *y* repraesentant. Hinc nullum curvae extabit punctum, quod non hae ratione per functionem *y* definiatur.

8. Quanquam complures lineae curvae per motum puncti continuum mechanice describi possunt, quo pacto tota linea curva simul oculis offertur, tamen hanc linearum curvarum ex functionibus originem hie potissimum contemplabimur tanquam magis analyticam latiusque patentem atque ad calculum magis accommodatam. Quaelibet ergo functio ipsius *x* suppeditabit lineam quandam sive rectam sive curvam, unde vicissim lineas curvas ad functiones revocare licebit. Cuiusque ergo lineae curvae natura exprimetur per eiusmodi functionem ipsius *x,* quae, dum intervalla *AP,* ad quae perpendicula *MP* ex singulis curvae punctis *M* in rectam *RS* demittuntur, per variabilem *x* indicantur, exhibeat semper veram istius applicatae *MP* longitudinem.

9. Ex hac linearum curvarum idea statim sequitur earum divisio in *continuas* et *discontinuas* seu *mixtas.* Linea scilicet curva *continua* ita est comparata, ut eius natura per unam ipsius *x* functionem definitam exprimatur. Quodsi autem linea curva ita sit comparata, ut variae eius portiones *BM, MD, DM* etc. per varias ipsius *x* functiones exprimantur, ita ut, postquam ex una functione portio *BM* fuerit definita, tum ex alia functione portio *MD* describatur, huiusmodi lineas curvas *discontinuas* seu *mixtas et irregulares* appellamus, propterea quod non secundum unam legem constantem formantur atque ex portionibus variarum curvarum continuarum componuntur.

10. De curvis autem continuis in geometria potissimum est sermo atque infra ostendetur, quae curvae motu uniformi secundum regulam quandam constantem mechanice describuntur, easdem quoque per unicam functionem exprimi atque ideo esse continuas. Sit igitur *mEBMDM* linea curva continua, cuius naturam contineat functio quaepiam ipsius *x*, quae sit *y;* atque manifestum est, sumtis valoribus ipsius *x* determinatis in recta *RS* a puncto fixo *A*, tum valores ipsius *y* respondentes praebere applicatarum normalium *PM* longitudinem.

11. In hac linearum curvarum explicatione nomina quaedam sunt tenenda, quorum frequentissimus usus existit in doctrina de lineis curvis. Primum igitur recta *RS*, in qua valores ipsius *x* abscinduntur, vocatur *axis* seu linea recta *directrix*.

Punctum *A*, a quo valores ipsius *x* mensurantur, dicitur *initium abscissarum*.

Portiones autem axis *AP*, quibus determinati ipsius *x* valores indicantur, vocari solent *abscissae*.

Et perpendiculares *PM* ex terminis abscissarum *P* ad lineam curvam pertingentes nomen *applicatarum* obtinuerunt.

Vocantur autem hoc casu applicatae *normales* seu *orthogonales*, quia cum axe angulum rectum constituunt; cum enim simili modo applicatae *PM* ad angulum obliquum cum axe constitui possint, hoc casu applicatae *obliquangulae* vocantur; hic vero constanter naturam curvarum per applicatas orthogonales explicabimus, nisi expressis verbis contrarium indicetur.

12. Si igitur abscissa quaecunque *AP* insigniatur per variabilem *x*, ut sit $AP = x$, tum functio *y* indicabit magnitudinem applicatae *PM* eritque $PM = y$. Natura igitur lineae curvae, siquidem fuerit continua, exprimetur per qualitatem functionis *y* seu per rationem, qua *y* ex *x* et quantitatibus constantibus componitur. In axe igitur *RS* erit portio *AS* locus abscissarum affirmativarum, portio *AR* locus abscissarum negativarum; tum vero supra axem *RS* existet regio applicatarum affirmativarum, infra autem erit regio applicatarum negativarum.

13. Cum igitur ex qualibet functione ipsius *x* nascatur linea curva continua, haec etiam ex illa functione cognosci atque describi poterit. Tribuantur enim primo ipsi *x* valores affirmativi a 0 ad ∞ usque progrediendo ac pro singulis quaerantur valores functionis *y* respondentes, quae per applicatas sive sursum sive deorsum porrectas repraesententur, prout valores habeant sive affirmatives sive negativos; sicque orietur portio curvae *BMM*. Deinde simili modo ipsi *x* tribuantur omnes valores negativi ab 0 ad $-\infty$ progrediendo, et valores ipsius *y* respondentes determinabunt curvae portionem *BEm* sicque universa linea curva in functione contenta exhibebitur.

14. Quia est *y* functio ipsius *x*, vel *y* aequabitur functioni ipsius *x* explicitae, vel dabitur aequatio inter *x* et *y*, qua *y* per *x* definitur: utroque casu habebitur aequatio, quae dicitur naturam curvae exprimere. Hanc ob rem natura cuiusque lineae curvae per aequationem inter duas variabiles *x* et *y* exhibetur, quarum altera *x* denotet abscissas in axe a dato

principio *A* sumtas, altera vero *y* applicatas ad axem normales. Abscissae autem et applicatae coniunctim consideratae appellantur *coordinatae orthogonales;* hincque natura lineae curvae per aequationem inter coordinatas orthogonales definiri dicitur, si habeatur aequatio determinans, qualis functio ipsius *x* sit *y*.

15. Cum igitur linearum curvarum cognitio ad functiones perducatur, tot varia linearum curvarum existent genera, quot supra functionum esse vidimus. Ad modum ergo functionum lineae curvae aptissime dividuntur in *alqebraicas* et *transcendentes.* Linea curva scilicet erit algebraica, si applicata *y* fuerit functio algebraica ipsius abscissae *x;* seu, cum natura lineae curvae exprimitur per aequationem algebraicam inter coordinatas *x* et *y*, huius generis lineae curvae quoque *geometricae* vocari solent. Linea curva autem *transcendens* est, cuius natura exprimitur per aequationem transcendentem inter *x* et *y*, seu ex qua fit *y* functio transcendens ipsius *x.* Haecque est praecipua linearum curvarum continuarum divisio, qua eae sunt vel *algebraicae* vel *transcendentes.*

16. Ad lineam autem curvam ex data functione ipsius *x*, qua applicata *y* exprimitur, describendam natura functionis, an sit uniformis, an multiformis, probe est attendenda. Ponamus primo *y* esse functionem uniformem ipsius *x*, seu esse $y = P$, denotante *P* functionem quamcunque uniformem ipsius *x*, et, quia ipsi *x* valorem quemvis determinatum tribuendo applicata *y* unum quoque valorem determinatum recipit, unicuique abscissae una respondebit applicata et hanc ob rem curva ita erit comparata, ut, si in quovis axis *RS* puncto *P* ducatur ad ipsum normalis *PM*, ea semper curvam secet idque in unico puncto *M.* Singulis ergo axis punctis singula respondebunt curvae puncta et, cum axis utrinque in infinitum extendatur, curva quoque utrinque in infinitum excurret. Seu curva ex tali functione orta continuo tractu utrinque cum axe in infinitum porrigetur, cuiusmodi tractum figura 2 exhibet, ubi linea curva *mEBMDM* utrinque sine ulla interruptione in infinitum excurrit.

17. Sit *y* functio biformis ipsius *x*, seu denotantibus litteris *P* et *Q* functiones

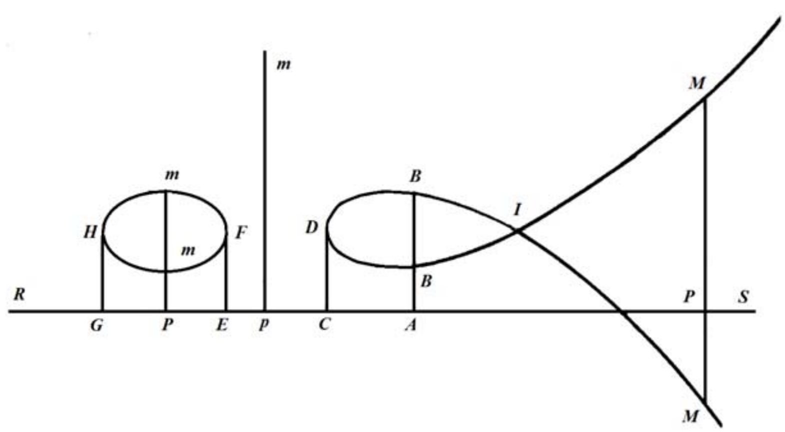

Fig. 3

ipsius x uniformes sit $yy = 2Py - Q$, ut sit $y = P \pm \sqrt{(PP - Q)}$.
Unicuique igitur abscissae x respondebit duplex applicata y, utraque existente vel reali vel imaginaria: prius si $PP > Q$, posterius si $PP < Q$. Quamdiu ergo uterque valor ipsius y erit realis, abscissae *AP* (Fig. 3) duplex conveniet applicata *PM, PM,* seu recta ad axem in *P* normalis curvam in duobus punctis *M* et *M* traiiciet. Ubi autem fit $PP < Q$, ibi abscissae nulla convenit applicata; seu normalis ad axem his in locis curvae nusquam occurret, uti fit in *p*. At cum ante esset $PP > Q$, fieri non poterit $PP < Q$, nisi transeundo per casum $PP = Q$, qui erit limes inter applicatas reales et imaginarias. Ubi ergo applicatae reales desinunt, uti in *C* vel *G*, ibi fit $y = P \pm 0$, seu ambae applicatae inter se fiunt aequales ibique curva cursum inflectendo regredietur.

18. Secundum figuram apparet, dum abscissa negativa $-x$ contineatur intra limites *AC* et *AE,* applicatam y fieri imaginariam esseque $PP < Q$; ultra *E* vero sinistrorsum progrediendo applicatae iterum fiunt reales, quod fieri nequit, nisi in *E* sit $PP = Q$ ideoque ambae applicatae conveniant. Tum rursus abscissis *AP* duplex applicata *Pm, Pm* respondet, donec ad *G* perveniatur, ubi hae duae applicatae conveniunt atque ultra *G* denuo fiunt imaginariae. Huiusmodi ergo linea curva constare poterit ex partibus a se invicem disiunctis, ut *MBDBM* et *FmHm,* duabus pluribusve; nihilo vero minus hae partes coniunctim consideratae unam curvam continuam seu regularem constituere sunt censendae, quia hae singulae partes ex una eademque functione nascuntur. Istae ergo curvae hanc habent proprietatem, ut, si in singulis axis punctis normaliter producantur rectae *MM,* eae semper curvam vel nusquam vel in duobus punctis traiiciant; nisi forte duo intersectionis puncta in unum coaleseant, quod fit, si applicatae per puncta *D, F, H* vel *I* ducantur.

19. Si y fuerit functio triformis ipsius x seu si y per huiusmodi aequationem $y^3 - Py^2 + Qy - R = 0$ definiatur, existentibus *P, Q* et *R* functionibus uniformibus ipsius x, tum pro quovis valore ipsius x applicata y tres habebit valores, qui vel omnes erunt reales, vel unicus tantum, reliquis duobus existentibus imaginariis. Hinc omnes applicatae curvam secabunt vel in tribus punctis vel tantum in unico, nisi ubi duo vel etiam tria intersectionis puncta in unum coalescunt. Cum igitur unicuique abscissae saltem una applicata realis conveniat, necesse est, ut curva utrinque cum axe in infinitum excurrat. Curva ergo vel uno continuo tractu constabit, ut in *figura quarto,* vel duabus partibus a se seiunctis, ut in *figura quinta,* vel pluribus, quae tamen omnes coniunctae unam eandemque curvam continuam constituunt.

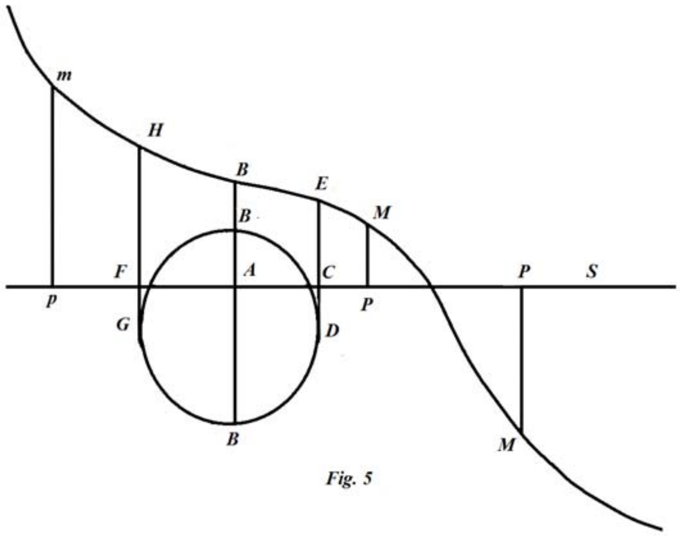

Fig. 5

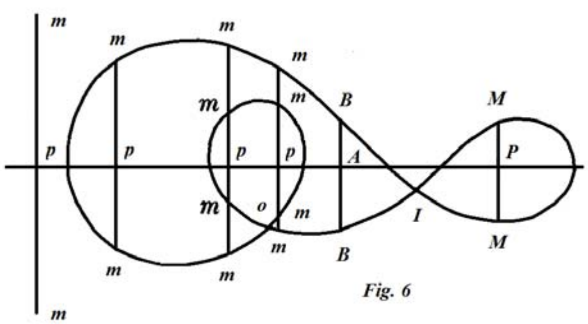

Fig. 6

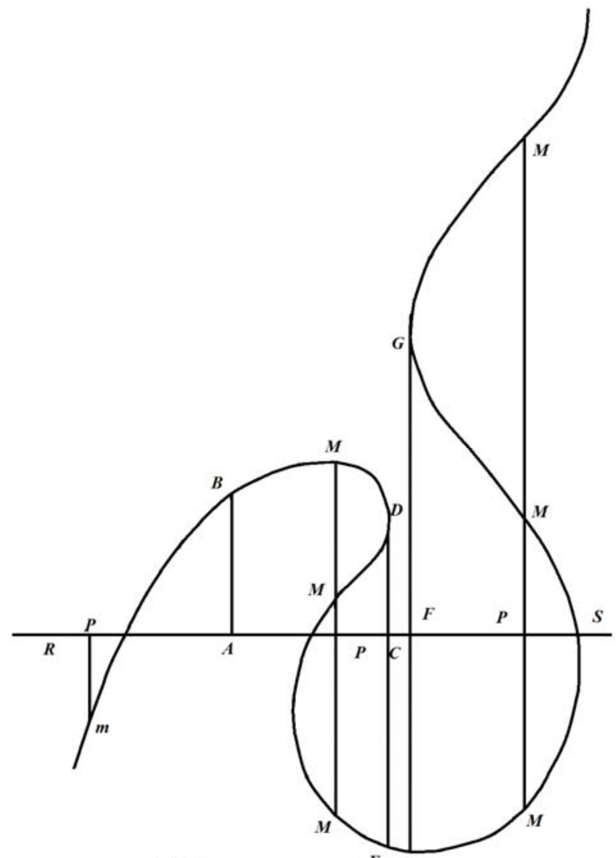

Fig. 4

20. Si *y* fuerit functio quadriformis ipsius *x* seu si *y* per huiusmodi aequationem $y^4 - Py^3 + Qy^2 - Ry + S = 0$ definiatur, tum unicuique valori ipsius *x* vel quatuor respondebunt valores reales ipsius *y* vel duo tantum vel omnino nullus. Hinc in curva ex huiusmodi functione quadriformi orta singulae applicatae curvam secabunt vel in quatuor punctis vel in duobus tantum vel nusquam, quos singulos casus *figura sexta* exhibet; notari autem debent loca *I* et *o*, ubi duo intersectionis puncta in *m m* unum coaleseunt. Hanc ob rem tam dextrorsum quam sinistrorsum vel nulli curvae rami in infinitum excurrunt vel duo vel etiam quatuor. Priori casu, quo ex neutra parte nulli rami in infinitum extenduntur, curva undique erit clausa, ut *figura sexia* indicat, spatiumque

definitum includit. Hinc ergo iam concludi potest indoles linearum curvarum, quae formantur ex functionibus multiformibus quotcunque significatuum.

21. Si scilicet fuerit *y* functio multiformis seu determinetur per aequationem, in qua *n* sit exponens maximae potestatis ipsius *y*, tum numerus valorum realium ipsius *y* erit vel *n* vel $n-2$ vel $n-4$ vel $n-6$ etc., in totidem ergo punctis quaelibet applicata curvam intersecabit. Ita, si una applicata curvam continuam secet in *m* punctis, omnes aliae applicatae curvam secabunt in tot punctis, quorum numerus semper numero pari differat ab *m;* nusquam ergo curva ab applicata secari poterit in $m+1$ vel $m-1$ vel $m\pm 3$ etc. punctis. Hoc est, si numerus intersectionum unius applicatae fuerit par vel impar, omnes quoque applicatae reliquae curvam secabunt in punctorum numero vel pari vel impari.

22. Si igitur una applicata curvam secet in punctorum numero impari, tum fieri nequit, ut ulla alia applicata curvam nusquam intersecet: curva ergo utrinque ad minimum unum habebit ramum in infinitum excurrentem et, si ex alterutra parte plures rami in infinitum extendantur, eorum numerus debet esse impar, quia numerus intersectionum unius cuiusque applicatae non potest esse par; si ergo rami utrinque in infinitum excurrentes simul numerentur, eorum numerus constanter erit par. Hoc idem locum habet, si applicatae curvam intersecent in punctorum numero pari, tum enim ex utraque parte seorsim vel nullus vel duo vel quatuor etc. rami in infinitum excurrent, unde ergo quoque omnium ramorum in infinitum excurrentium numerus erit par. Iam igitur adepti sumus aliquot insignes proprietates curvarum continuarum et regularium, unde eas a curvis discontinuis et irregularibus discernere licet.