CHAPTER XVII

FINDING CURVES FROM OTHER PROPERTIES

391. The questions which we have resolved in the previous chapter were prepared thus, so that they could be reduced easily to an equation between the coordinates, either rectangular or oblique-angled. Now therefore we will consider other properties of this kind, which are not concerned immediately with the properties between parallel applied lines themselves, just as if some characteristic of the right lines drawn from some given point to the curve may be proposed (Fig. 81). Let *C* be the point, from which the right lines *CM, CN* may be drawn to the curve, and some property may have been proposed regarding these lines ; thus it may be agreed to depart from the manner adopted up to this point for expressing curves by coordinates, so that these right lines may be introduced into an equation. [Recall that Euler favored a left-handed *x*-axis, while *A* is the origin ; there was a reason for this !]

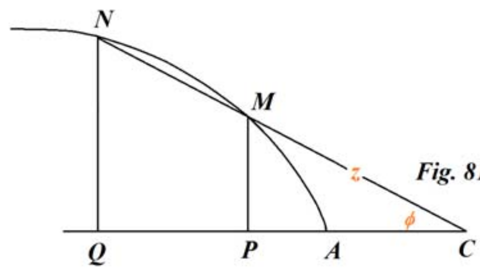

Fig. 81

392. Therefore since the nature of curved lines which may be formed between the two variables can be understood in a number of other ways, in the present circumstances the magnitude of the right line *CM* drawn from a given point *C* to the curve may determine the position of the other variable. Then truly there will be a need for another variable, by which the position of the right line *CM* may be defined ; hence finally a certain right line *CA* may be taken for the axis drawn through the point *C*, and the angle *ACM* or a magnitude depending on this, most conveniently will be taken in turn for the other variable. Therefore let the right line $CM = z$ and the angle $ACM = \varphi$, the sine or tangent of which may be present in the equation; and it is evident, if some equation may be given between *z* and $\sin.\varphi$ or $\operatorname{tang}.\varphi$, the nature of the curve *AMN* can be determined by that ; for indeed however the angle *ACM* and the length of the right line *CM* may be defined, the point *M* of the curve is determined.

393. Moreover we may consider more carefully the manner required for expressing these curved lines. And indeed in the first place the distance *z* will be equal to some function of the sine of the angle φ; if which function were uniform, the right line *CM* may be considered to cross the curve at the single point *M*, because the angle $ACM = \varphi$ corresponds to a single value of the right line *CM*. Truly if the angle φ may be increased by two right angles, the same right line *CM* will remain in place drawn through

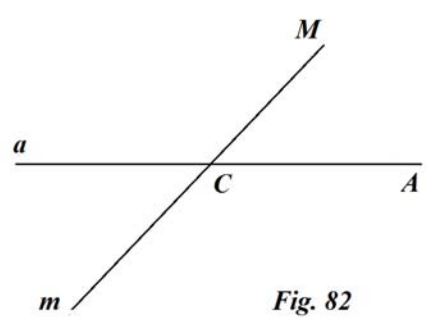

Fig. 82

the point *C*, yet with this distinction, that in Fig. 82 it may be directed in the opposite direction ; and thus another intersection of the right line *CM* will be produced with the curve, even if *z* were equal to a uniform function of the sine of the angle φ. Evidently *P* shall be that function of the sine of the angle φ, so that there shall be thus $z = P$, from which (Fig. 82) the point of the curve *M* may arise [needless to say, this function *P* is not to be confused with the point *P* on the *x* axis]; now the angle φ may be increased by two right angles or the sine may be put in place negative, with which done the function *P* may become *Q*, so that there shall be $z = Q$; hence therefore a new intersection *m* of the same right line *CM* will be produced with the curve, on taking $Cm = Q$.

394. Therefore although *P* shall be a uniform function of the sine of the angle φ, yet the right line *CM* drawn within the given angle $ACM = \varphi$ through the point *C*, will cross the curve at the two points *M* and *m*, unless there shall be $Q = -P$. But if therefore each single right line *CM* must cross the curve in a single point, that quantity *P* is required to be an odd function of the sine of the angle φ. Moreover, this also comes about in use, if *P* were an odd function of the cosine of the angle φ. On account of which all the curves, which intersect individual right lines drawn from *C* at a single point, will be satisfied by this equation $z = P$, if indeed *P* were an odd function both of the sine as well as of the cosine of the angle $ACM = \varphi$.

395. Therefore since the curves (Fig. 81), which may be cut at a single point by a right line drawn from the point *C*, may be satisfied by the equation $z = P$, if *P* were some odd function of the sine and cosine of the angle φ, or a function of this kind, which may adopt a negative value, if both the sine as well as the cosine of the angle φ may be made negative, hence it will be easy for the equation of curves of this kind to be found between orthogonal coordinates. For with the perpendicular *MP* sent from the point *M* to the axis *CA*, if on calling $CP = x,\ PM = y,$ there will be $\frac{y}{z} = \sin.\varphi$ and $\frac{x}{z} = \cos.\varphi$, from which, if P were an odd function of $\frac{x}{z}$ and $\frac{y}{z}$, all these curves will be satisfied by this equation $z = P$. Therefore beginning from the simplest [odd functions] there will be :

$$z = \frac{\alpha x}{z} + \frac{\beta y}{z} + \frac{\gamma z}{x} + \frac{\delta z}{y}$$

and by ascending to higher powers there will be :

$$z = \frac{\alpha x}{z} + \frac{\beta y}{z} + \frac{\gamma z}{x} + \frac{\delta z}{y} + \frac{\varepsilon x^3}{z^3} + \frac{\zeta xxy}{z^3} + \frac{\eta xyy}{z^3} + \frac{\theta y^3}{z^3} + \frac{\iota xx}{yz} + \frac{\chi yy}{xz} + \frac{\lambda yz}{xx} + \text{etc.}$$

[*i.e.* the right hand side is a general odd algebraic expansion of the two variables *x* and *y*, of which the dimension of each quotient in *x* and *y* is zero.]

396. If this equation may be divided by z, then only even powers of z will occur everywhere and thus, since there shall be $z=\sqrt{(xx+yy)}$, with z eliminated no irrational numbers will remain in the equation, and a rational equation will be produced between x and y. Therefore the general equation may be prepared thus, so that 1 or some constant quantity may be equal to a function of x and y of dimensions -1.
[It is important to note this point, as it is used to develop the general algebraic form.]
If P may be a function of this kind, and there may be $C=P$ and thus $\frac{1}{C}=\frac{1}{P}$; but $\frac{1}{P}$ is a function of one dimension of x and y; from which, if some function of one dimension of x and y were equal to a constant, it would be the equation of the curve, which the right lines drawn through the point C may intersect in a single point.

397. Let P be a function of n dimensions of x and y and Q a function of $n+1$ dimensions; then $\frac{Q}{P}$ will be a function of one dimension and thus all the curves, which we shall consider here, will be present in the equation $\frac{Q}{P}=c$ or $Q=cP$ [,for some constant c].
Therefore with n denoting some number, the general equation for these curves will be

$$\alpha x^{n+1}+\beta x^{n}y+\gamma x^{n-1}yy+\delta x^{n-2}y^{3}+\varepsilon x^{n-3}y^{4}+\text{ etc.}=c(Ax^{n}+Bx^{n-1}y+Cx^{n-2}yy+Dx^{n-3}y^{3}+\text{ etc.})$$

From which the order of the individual lines, which may be cut in a single point by a right line drawn from the point C, will be satisfied by the following equations [for $n=0, 1, 2$ etc.]:

I.
$$\alpha x+\beta y=c,$$

II.
$$\alpha xx+\beta xy+\gamma yy=c(Ax+By),$$

III.
$$\alpha x^{3}+\beta xxy+\gamma xyy+\delta y^{3}=c(Axx+Bxy+Cyy),$$

IV.
$$\alpha x^{4}+\beta x^{3}y+\gamma xxyy+\delta xy^{3}+\varepsilon y^{4}=c\left(Ax^{3}+Bxxy+Cxyy+Dy^{3}\right)$$
etc.

398. Therefore the first right line is satisfactory, which certainly agrees with the other right lines drawn through a given point, as it can be cut only at a single point by any other right line. The second equation is for the general conic section, provided the conic section itself may pass through the point C, which intersection shall be in common with all the

right lines drawn from *C*, and so need not be calculated ; hence as conic sections can be cut by some straight line in two points only, all the right lines taken passing through the point *C* will provide only a single [additional] point of intersection. But all the curved lines of the higher orders will themselves pass through the point *C* ; which intersection with all the common right lines drawn through *C* equally need not be computed. And therefore from the higher orders in the equations shown only these will be satisfied, which the right lines drawn through *C* intersect in a single point. Thus we have enumerated all the algebraic curves, which therefore are cut at a single point by a right line drawn through the given point *C* only.

399. Now we may progress to investigating these curves, which individual right lines drawn through the point *C* may cut in two points or not at all, if indeed the roots of the equation indicating a two-fold intersection may become imaginary. Therefore since for some angle $ACM = \varphi$, a two-fold value may be chosen for the right line $CM = z$, that may be defined by a quadratic equation. And thus there shall be

$$zz - Pz + Q = 0,$$

where *P* and *Q* shall be functions of the angle φ either of its sine or cosine [*i.e.* of zero dimensions]. Truly because the right line *CM* must cut at only two points *M* and *N*, not only is it necessary for *p* and *Q* to be uniform functions of the angle φ, but also no new intersections may arise with the angle increased by two right angles φ, that which comes about, if *P* were an odd function of the sine and of the cosine of the angle φ, thus so that it may adopt a negative value, if both the sine and cosine may be taken negative : but *Q* must be an even function of the same sine and cosine.

400. Moreover with the orthogonal coordinates put in place $CP = x$ and $PM = y$ there will be

$$\frac{y}{z} = \sin.\varphi \text{ and } \frac{x}{z} = \cos.\varphi;$$

and thus *P* will be required to be an odd function of $\frac{x}{z}$ and $\frac{y}{z}$ and *Q* an even function of $\frac{x}{z}$ and $\frac{y}{z}$. From these $\frac{P}{z}$ is deduced to be a rational function of x and y and thus is a homogeneous function of dimensions -1. In a similar manner $\frac{Q}{zz}$ will be a rational function of x and y of dimensions -2. But if *L* were some homogeneous function of *x* and *y* of $n+2$ dimensions, likewise *M* can be a homogeneous function of $n+1$

dimensions, and N a homogeneous function of n dimensions, then the fraction $\frac{M}{L}$ will exhibit a function agreeing with $\frac{P}{z}$, and $\frac{N}{L}$ a function agreeing with $\frac{Q}{zz}$.

Whereby, since there shall be $zz - Pz + Q = 0$, there will be $1 - \frac{P}{z} + \frac{Q}{zz} = 0$, from which the general equation for curves, which are cut by right lines through the point C in two points, will be

$$1 - \frac{M}{L} + \frac{N}{L} = 0 \text{ or } \mathrm{L} - M + N = 0,$$

where there is

$$P = \frac{Mz}{L} \text{ and } Q = \frac{Nzz}{L} = \frac{N(xx+yy)}{L};$$

and thus P will be an irrational function of x and y, on account of $z = \sqrt{(xx+yy)}$, and Q is a rational function, each of no dimensions.
[Thus, the general equation to be solved for two points of intersection between the right line and the curve, is the algebraic equation $\mathrm{L} - M + N = 0$, with dimensions $n+2, n+1, n$; related to P and Q by

$L - M + N = 0$ giving, $z^2L - z^2M + z^2N = 0$ and hence $z^2 - z\frac{Mz}{L} + \frac{z^2N}{L} = 0,$

or : $z^2 - Pz + Q = 0,$ as previously.]

401. Hence now it will be easy to show these for lines of any order, which will be intersected in two points or not at all by right lines drawn through the given point C. Evidently for curves of the second order there becomes $n = 0$, and the most general equation of a conic section will be produced

$$\alpha xx + \beta xy + \gamma yy - \delta x - \varepsilon y + \zeta = 0.$$

Therefore with the point C taken anywhere, all the right lines drawn through that will intersect the conic section either at two points or no points at all. Yet meanwhile it can come about, that a right line may intersect a certain single curve in one point only ; which since either with only one or two that may come into use among these infinitely many lines drawn through C, this exception will be of no consequence ; since this paradox can be explained thus, so that the other intersection may go off to infinity ; on account of this cause this exception can be agreed to carry no force in our assertion.

402. But so that it may become apparent, in which cases this exception may have a place, we may reduce the equation between x and y to an equation between z and the angle $ACM = \varphi;$ which on account of $y = z \cdot \sin.\varphi$ and $x = z \cdot \cos.\varphi$ will be changed into this equation :

$$zz(\alpha(\cos.\varphi)^2 + \beta \cdot \sin.\varphi \cdot \cos.\varphi + y(sin.\varphi)^2) - z(\delta \cdot \cos.\varphi + \varepsilon \cdot \sin.\varphi) + \zeta = 0\,;$$

from which it will be apparent, if the coefficient of zz were equal to zero, only a single intersection will have a place ; which therefore comes about, if there were

$$\alpha + \beta \cdot \text{tang.}\varphi + \gamma \left(\text{tang.}\varphi\right)^2 = 0\,.$$

But if this equation therefore may have two real roots, in two cases the right line drawn through C will cut the curve only in a single point. Truly because the roots of the same equation indicate the asymptotes of the curve, it is evident the hyperbola be cut in a single point only by a right line parallel to the other asymptote, only two right lines of this kind are given passing through the point C. In the parabola truly this exception will be allowed with a single right line parallel to the axis. Indeed if the section of the cone were an ellipse, the point C may be taken anywhere, every right line drawn through that point will cut the curve either in two points or not at all.
[Coincident points are of course still allowed in this last case.]

403. Lines of the third order enjoy this property, on putting $n = 1$, and will be satisfied by this equation

$$\alpha x^3 + \beta xxy + \gamma xyy + \delta y^3 - \varepsilon xx - \zeta xy - \eta yy + \theta x + \iota y = 0,$$

which certainly enfolds within itself all the lines of the third order, which therefore will pertain to this, provided the point C may be taken on the curve itself. Indeed with $x = 0$, the value y likewise maintains a vanishing value. In a similar manner for curves of the fourth order with the question satisfied the point C must not only be on the curve but likewise must be a double point of this ; therefore all the lines of the fourth order may be satisfied by a provided double point, as long as a double point may be put in place at the point C. But if C thus were a triple point of the curve, then every right line drawn through that curve will intersect at a single point and will relate to the first case considered. In a like manner lines of the fifth order will be satisfied, if the point C may be set up at a triple point of these, and thus so on. But always it is to be observed, if the right line drawn through C becomes parallel to some asymptote line or parallel to the asymptote axis of a parabola, then a single intersection is given, with the other departing to infinity.

404. These agree uncommonly well with the nature of the lines of each order : because indeed the lines of each order can be intersected in just as many points, as the exponent of the order contains units, (and actually may be intersected in just as many points, unless some intersections become either imaginary or depart to infinity), and because here we may compute all the intersections equally, whether they be real, made at infinity, or imaginary, and we exclude only these which happen at the point *C*, it is evident, since a line of the order *n* may be cut at *n* points by some right line, the point *C* must be deduced to be at a point of as great a multiplicity, as the number of times the number $n-2$ contains units, so that a two-fold intersection may be produced.

405. With these noted it will be easy to show problems which are accustomed to be proposed concerning the relation between each two values of *z* , *CM* and *CN*, either to be resolved, or to show the inconvenience of finding a solution. For since the two values of *z, CM* and *CN,* shall be the roots of this equation $zz-Pz+Q=0$, the sum of these will be $=P$ and the rectangle of these $CM\cdot CN=Q$. Whereby, if in the first place curves of this kind may be required, in which everywhere the sum $CM+CN$ shall be constant, the function P is required to be a constant quantity. But when from the nature of the question, each single right line drawn through *C* must cross the curve in two points only, it is necessary that there shall be (§ 400),

$$P=\frac{Mz}{L}=\frac{M\sqrt{(xx+yy)}}{L},$$

which irrational quantity can never be constant. And therefore no curve can be given satisfying this question properly.

406. But if moreover this condition may be omitted, by which only two intersections with the curve of each right line drawn through the point *C* may be postulated, and and curves of this kind are sought, which certainly may show more than two intersections, among these moreover two shall be present *M* and *N* of this kind, so that $CM=CN$ shall be a constant quantity, innumerable such curves will be able to be shown by putting *P* to be equal to that constant quantity $CM+CN=a$. For there will be $zz-az+Q=0$ with *Q* denoting some function $\frac{Nzz}{L}$; and because this equation at this stage is troubled with irrationality, with that removed there will be

$$aazz=(zz+Q)^2 \text{ or } aa=zz\left(1+\frac{N}{L}\right)^2$$

or

$$aaLL=(xx+yy)(LL+2LN+NN),$$

in which L will be a homogenous function of $n+2$ dimensions but N a homogeneous function of n dimensions of x and y. Therefore the simplest curve resolving the question in this sense will be had, if there may be put

$$L = xx + yy \text{ et } N = \pm bb,$$

and there will be

$$aa\left(xx+yy\right) = (xx+yy \pm bb)^2,$$

which is for a complex line of the fourth order ; for it involves two circles concentric at C. But the simplest continuous curves satisfying what is sought will be of the sixth order, for on putting

$$L = \alpha xx + \beta xy + \gamma yy \text{ and } N = \pm bb,$$

for with which the equation will be

$$aa(\alpha xx + \beta xy + \gamma yy)^2 = (xx+yy)(\alpha xx + \beta xy + \gamma yy \pm bb)^2.$$

Let $\alpha = 1$, $\beta = 0$ and $\gamma = 0$, there will be

$$yy + xx = \frac{aax^4}{x^4 \pm 2bbxx + b^4}$$

or

$$y = \frac{x\sqrt{(aaxx - x^4 \mp 2bbxx - b^4)}}{xx \pm bb}.$$

407. But if solutions of this kind, in which right lines drawn through C may intersect in more than two points may be excluded, as the nature of the condition may be seen to require, in short no curves are said to satisfy the question and therefore no continued line will be given, which thus with the right lines drawn through C will be cut in only two points M and N, so that the sum $CM + CN$ shall be constant. But truly if these intersections may be postulated of this nature, so that the rectangle $CM \cdot CN$ must be constant, which property thus is agreed on in the circle, the point C may be taken anywhere, infinitely many other curved lines can be found, which maintain the same property. For Q will need to be a constant quantity, certainly equal to that rectangle $CM \cdot CN$, which shall be $= aa$; which position, since there shall be $Q = \frac{Nzz}{L}$, and therefore a rational function of x and y, one would not dispute.

408 . Therefore there shall be $\frac{Nzz}{L}=aa$ or $L=\frac{Nzz}{aa}=\frac{N(xx+yy)}{aa}$, and all the curves satisfying the question will be satisfied by this equation

$$\frac{N(xx+yy)}{aa}-M+N=0 \text{ or } Maa=N(xx+yy+aa),$$

where M denotes some homogeneous function of $n+1$ dimensions, N truly a homogeneous function of n dimensions of x and y, thus so that there shall be

$$\frac{M}{N}=\frac{xx+yy+aa}{aa}$$

a function of one dimension of x and y. Therefore this equation embraces all the curves, which by the right lines drawn through C are cut in the two points M and N only, so that the rectangle $CM\cdot CN$ everywhere shall be a constant $=aa$.

409. Therefore since $\frac{M}{N}$ shall be a homogeneous function of one dimension of x and y, the simplest case will be produced, if there is put $\frac{M}{N}=\frac{\alpha x+\beta y}{a}$, from which this equation arises

$$xx+yy-a(\alpha x+\beta y)+aa=0,$$

which is always for a circle, and, since it shall be the general equation for a circle between orthogonal coordinates, it is evident a circle satisfies the question, wherever the point C may be taken, as generally it agrees with the Elements. Therefore besides the circle no other curve from the conic sections satisfies this question. Truly from the following individual sequences an infinite abundance of satisfying lines will be able to be shown and certainly all, which arise from any order will be satisfied. Thus lines of the third order, which enjoy this property, will be satisfied by this equation

$$\frac{\alpha xx+\beta xy+\gamma yy}{a(\delta x+\varepsilon y)}=\frac{xx+yy+aa}{aa}$$

or

$$(\delta x+\varepsilon y)(xx+yy)-a(\alpha xx+\beta xy+\gamma yy)+aa(\delta x+\varepsilon y)=0.$$

And in a similar manner, from all the following orders of lines, these which are satisfactory will be shown.

410. Now this equation shall be proposed, so that between all the curved lines, which may be cut at two points by right lines drawn through the point, these may be defined, in which the sum of the squares $CM^2 + CN^2$ shall be a constant quantity, for example $= 2aa$. Therefore since there shall be

$$CM + CN = P \text{ and } CM \cdot CN = Q, \text{ there will be } CM^2 + CN^2 = PP - 2Q;$$

therefore there must be

$$PP - 2Q = 2aa \text{ or } Q = \frac{PP - 2aa}{3a}.$$

Whereby, on account of $P = \frac{Mz}{L}$ and $Q = \frac{Nzz}{L}$, there will be

$$\frac{2Nzz}{L} = \frac{MMzz}{LL} - 2aa \text{ and thus } N = \frac{MM}{2L} - \frac{aaL}{zz}:$$

which equation, since *L* shall be a function of $n+2$ dimensions, *M* a function of $n+1$ dimensions and *N* a function of n dimensions of x and *y*, involves no difficulty. Therefore with functions of this kind taken for *L* and *M*, there will be

$$N = \frac{MM}{2L} - \frac{aaL}{zz},$$

from which this general equation will result for curves satisfying the equation :

$$L - M + \frac{MM}{2L} - \frac{aaL}{zz} = 0$$

or

$$2LL(xx+yy) - 2LM(xx+yy) + MM(xx+yy) - 2aaLL = 0,$$

which, if there shall be $M = 0$, provides a circle, the centre of which is at *C*, which it is evident to satisfy the question.

411. We may put $n+1=0$, so that *M* shall be a constant quantity $= 2b$ and $L = \alpha x + \beta y$, and a line of the fourth order will arise satisfied by this equation

$$(\alpha x + \beta y)^2 (xx + yy - aa) - 2b(\alpha x + \beta y)(xx + yy) + 2bb(xx + yy) = 0.$$

Another equation of the fourth order is found, if there may be put

$$L = xx + yy \text{ and } M = 2(\alpha x + \beta y)a,$$

then indeed the equation divided by $2xx + 2yy$ will give

$$(xx + yy)^2 - 2a(\alpha x + \beta y)(xx + yy) + 2aa(\alpha x + \beta y)^2 - aa(xx + yy) = 0.$$

But if the division by $xx + yy$ may not follow, the equation found (on putting *2M* in place of *M*), which is

$$LL(xx + yy) - 2LM(xx + yy) + 2MM\left(xx + yy\right) - aaLL = 0,$$

will be always of order $2n + 6$, and thus from any equal order the equation will be found satisfying the curve. Truly in addition, if *L* were divisible by $xx + yy$, clearly if, with N denoting some homogeneous function of *x* and *y* of *n* dimensions, there will be $L = (xx + yy)N$, this other general equation will arise

$$NN(xx + yy)^2 - 2MN(xx + yy) + 2MM - aaNN(xx + yy) = 0,$$

which is of order $2n + 4$, thus so that from the individual equal orders a two-fold equation may arise for the curves proposed enjoying this property. Thus from the sixth order the curves are satisfied by these two equations :

$$\begin{gathered}(\alpha xx + \beta xy + \gamma yy)^2(xx + yy - aa)\\ -2a(\delta x + \varepsilon y)(xx + yy)(\alpha xx + \beta xy + \gamma yy - a(\delta x + \varepsilon y)) = 0\end{gathered}$$

and

$$\begin{gathered}(\delta x + \varepsilon y)^2(xx + yy)(xx + yy - aa)\\ = 2a(\alpha xx + \beta xy + \gamma yy)((\delta x + \varepsilon y)(xx + yy) - a(\alpha xx + \beta yy + \gamma yy)).\end{gathered}$$

Therefore in no odd order of the lines is any line given resolving this question.

412. If now there may not be sought curves of this kind, in which the sum of the squares shall be constant $CM^2 + CN^2$ constant, but in which there shall be

$$CM^2 + CM \cdot CN + CN^2$$

or generally

$$CM^2 + n \cdot CM \cdot CN + CN^2$$

shall be a constant quantity, the problem may permit resolution in a similar manner. For since there shall be

$$CM^2 + n \cdot CM \cdot CN + CN^2 = PP + (n-2)Q,$$

there may be put $PP+(n-2)Q=aa$ and there will be $Q=\frac{aa-PP}{n-2}$, which equation may be worked out without any inconvenience. Therefore since there shall be

$$P=\frac{Mz}{L} \text{ and } Q=\frac{Nzz}{L},$$

the equation becomes

$$\frac{MMzz}{LL}+\frac{(n-2)Nzz}{L}=aa$$

and thus

$$N=\frac{aaL}{(n-2)zz}-\frac{MM}{(n-2)L}.$$

Whereby, since the equation for the curves shall be $L-M+N=0$, by this property there will be had, since $CM^2+ n\cdot CM\cdot CN+CN^2$ must be of a constant magnitude $=aa$, this equation :

$$(n-2)LLzz-(n-2)LMzz+aaLL-MMzz=0$$

or on account of $zz=xx+yy$, it becomes

$$a\alpha LL+(xx+yy)\big((n-2)LL-(n-2)LM-MM\big)=0$$

with the function L being of $m+2$, and M of $m+1$ dimensions of x and y. Let N be some homogeneous function of m dimensions and there may be put $L=(xx+yy)N$, this other general equation will be produced :

$$aa(xx+yy)NN+(n-2)(xx+yy)^2 NN-(n-2)(xx+yy)MN-MM=0.$$

413. If $n=2$ may be put in place, so that there shall be $(CM+CN)^2=aa$, the equation becomes either

$$aaLL=(xx+yy)MM \text{ or } MM=aa(xx+yy)NN.$$

Moreover each equation, since it shall be homogeneous, will contain two or more equations of this form $\alpha y=\beta x$; and thus the question cannot be satisfied, unless by two or more right lines drawn through the point C; but which with that understood, because the question proposed may not be satisfied, it is evident that no solution is admitted to this problem, as we have noted above ; for $CM+CN$ must be equal to a constant *a*. But if truly there may be put $n=-2$, thus so that the square of the difference $(CN-CM)^2$ and thus the difference itself *MN* must be constant, these two equations will arise :

$$aaLL=(xx+yy)\left(2L-M\right)^2,$$

and

$$aa(xx+yy)NN=(2(xx+yy)N-M)^2,$$

from which the simplest solution will arise, if there may be put $N=1$ and $M=2bx$; for there will be

$$aa(xx+yy)=4(xx+yy-bx)^2,$$

or on putting $aa=8cc$ there becomes

$$(xx+yy)^2=2(cc+bx)(xx+yy)-bbxx.$$

Therefore

$$xx+yy=cc+bx\pm c\sqrt{(cc+2bx)}$$

and thus

$$y=\sqrt{(cc+bx-xx\pm c\sqrt{(cc+2bx)})}.$$

414. Therefore innumerable curved lines are given, which thus are cut by right lines drawn through the point *C* in two points *M* and *N*, so that the interval *MN* shall be constant always. And indeed it is apparent in the first place for a circle having the centre at *C*, for indeed then the interval *MN* is equal to the diameter of the circle always ; moreover the circle is produced from the general equation, if there may be put $M=0$. Then truly after the circle lines of the fourth order are satisfied by this equation

$$aa(xx+yy)=4(xx+yy-bx)^2$$

and by this

$$aaxx=(xx+yy)(2x-2b)^2\ ;$$

towards understanding the form of which it will be expedient to return to the equation between *z* and the angle φ. Therefore since there shall be
$xx+yy=zz$ and $x=z\cdot\cos.\varphi$, and also $y=z\cdot\sin.\varphi$, on putting $a=2c$; in the first place there will be

$$cczz=(zz-bz\cdot\cos.\varphi)^2 \text{ or } b\cdot\cos.\varphi\pm c=z,$$

and then

$$cc(\cos.\varphi)^2 = \left(z \cdot \cos.\varphi - b\right)^2 \quad \text{or} \quad z = \frac{b}{\cos.\varphi} \pm c,$$

from which the construction of the curves arises easily.

415. Indeed, for the curve requiring to be constructed (Fig. 83, 84, 85) satisfied by the equation

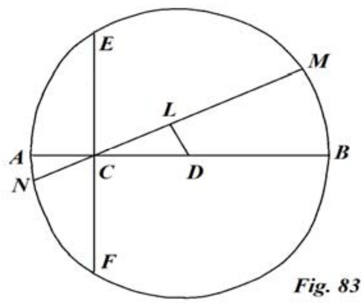

Fig. 83

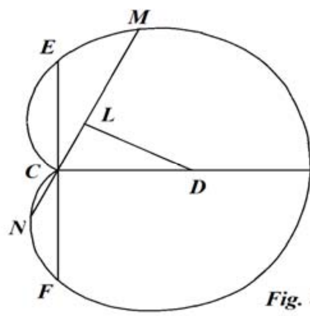

Fig. 84

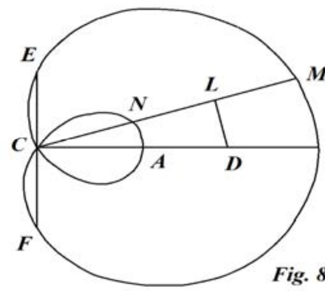

Fig. 85

$z = b \cdot \cos.\varphi \pm c$, the right line *ACB* is drawn through *C,* in which there is taken $CD = b$, and from *D* on each side $DA = DB = c$, and the points *A* and *B* will be the first points on the curve. Then, with some right line *NCM* drawn through *C,* from *D* on that the perpendicular *DL* may be sent and from *L* on each side there is taken $LM = LN = c$, the points *M* and *N* will be on the curve sought and thus likewise the interval $MN = 2c$ always, as the question demands.

Here it is to be observed, if $CD = b$ were less than *c,* the curve (Fig. 83) will have a conjugate point at *C*.

But if there shall be $b = c$, the curve (Fig. 84) at *C* will be provided with a cusp, with the interval *AC* vanishing.

Finally, if *b* shall be greater than *c,* the point *A* (Fig. 85) falls between *C* and *B* and the curve will have a node at *C* or a double point. Moreover the diameter of these curves will be the right line *ACB,* and because *ECF* stands normally to this, it will be $= 2c$.

416. Besides these curves from the fourth order of lines corresponding among themselves they are satisfied also by other lines of the same order extending to infinity, which are satisfied by this equation

$$z=\frac{b}{\cos.\varphi}\pm c.$$

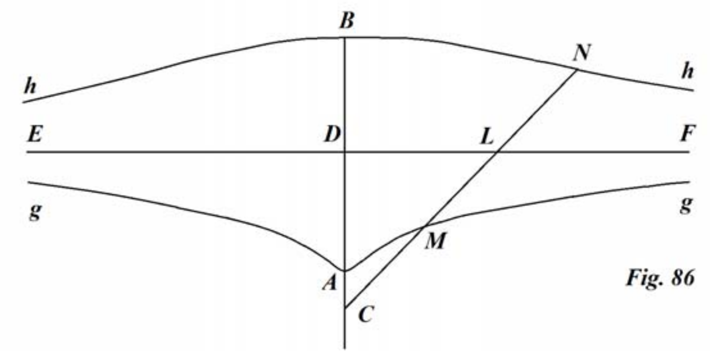

Fig. 86

The construction of which thus will be itself considered (Fig. 86) : draw the principal right line *CAB* through *C,* taking $CD=b$ and there is taken $DA=DB=c$, the points *A* and *B* will be on the curve. Then, the normal *EDF* is drawn through *D* and on drawing some right line *CL* there will be $CL=\frac{b}{\cos.\varphi}$, with the angle called $DCL=\varphi$. Then always $LM=LN=c$ may be cut off, and the points *M* and *N* will determine the curve sought. From this construction it is evident the curve thus described to be the *conchoid* of the ancients, having the pole *C* and the right asymptote *EF,* to which the four branches of the curve converge at infinity. Moreover the part *hBh* is called the *exterior conchoid* and the part *gAg* the *interior,* besides which parts there is the conjugate point present at *C*.

417. These curves from lines of the fourth order are satisfied. but it will be easy to show curves of higher orders, as many as wished. For if *P* were some odd function of the sine and cosine of the angle φ, then that equation $z=bP\pm c$ will provide a continued curve, as all the right lines drawn through *C* thus will be cut in the two points *M* and *N*, so that the interval *MN* shall become a constant $=2c$. But all these curves can be referred to the genus of the conchoids, with any curved line being satisfied by the equation $z=bP$ requiring to be substituted in place of the right directrix *EF*. Moreover above we have seen this equation involving closed curved lines, which will be cut only at a single point by right lines drawn through the point *C*. Whereby, on account of the arbitrary intervals *c*, from any curve $z=bP$, innumerable curves can be described adapted to the present situation.

418. One may take for argument's sake the curve *CEDLF* (Fig. 87), which by all the right lines drawn through the point *C* will be cut at the single points *D, L* only. Then from these individual right lines *CL* produced on each side from *L* equal intervals may be taken $LM = LN = c$ and the points *M* and *N* will be on the curve sought. Thus by this continued motion the curve *AMPCQBNRC* will be described, which thus will be intersected by the individual right lines drawn through *C* at the two points *M* and *N*, so that the interval *MN* shall be $= 2c$ always. Where it is to be observed, if the curve *CEDF* were a circle drawn through the point *C*, then the curve described becomes a line of the fourth order, as we found initially in paragraph 414.

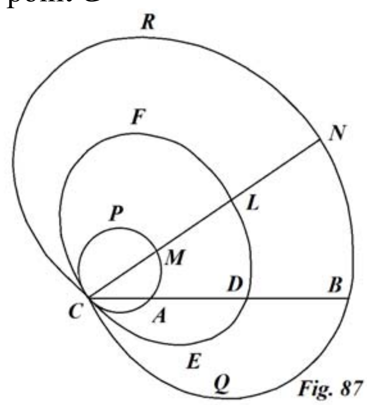

Fig. 87

419. Thus we have therefore satisfied the question, by which the curved line *AMN* will be sought from a right line drawn through the point *C* thus being cut in the two points *M* and *N,* so that $CN - CM$ or $CM^2 - 2CM \cdot CN + CN^2$ shall be a constant property always. Therefore we may set out now in a few words the case (Fig. 81), in which

$$CM^2 + CM \cdot CN + CN^2$$

must be a constant quantity. Therefore in paragraph 412 it is required to put $n = 1$ and thus either this equation will arise

$$aaLL = (xx + yy)(LL - LM + MM)$$

with the function *L* being of $m+1$ dimensions and with the function M of *m* dimensions of *x* and *y,* or this other equation will arise

$$aa(xx + yy)NN = (xx + yy)^2 NN - (xx + yy)MN + MM\ ,$$

in which *M* is a homogeneous function of *x* and *y* higher by one dimension, than the function *N*.

420. Indeed it is evident in the first place, if there is put $M = C$, for a circle to be produced the centre of which shall be set at the point *C* ; which, since all the right lines drawn from *C* to the curve shall be equal, also it is satisfied in all the questions of this kind. But for the present case the most simple curves will be produced after the circle, if there may be put $M = b$ and $L = x$ in the first equation, and it will become

$$aaxx = (xx + yy)(xx - bx + bb)$$

or

$$yy = \frac{xx(aa - bb + bx - xx)}{bb - bx + xx}.$$

But if moreover $N = 1$ and $M = bx$ may be put into the other equation, a line of the fourth order also will be had:

$$aa(xx + yy) = (xx + yy)^2 - bx(xx + yy) + bbxx$$

or

$$xx + yy = \tfrac{1}{2}bx + \tfrac{1}{2}aa \pm \sqrt{(\tfrac{1}{4}a^4 + \tfrac{1}{2}aabx - \tfrac{3}{4}bbxx)},$$

which satisfies the first equation equally.

421. With these questions settled we may consider higher powers of the two values of z which arise from the equation $zz - Pz + Q = 0$ with

$$P \text{ being} = \frac{Mz}{L} \text{ and } Q = \frac{Nzz}{L}$$

where L signifies a homogeneous function of dimensions $n+2, M$, $n+1$ and N, of n dimensions of x and y ; and x is equal to the abscissa CP and y is equal to the applied line PM. Therefore the question shall be proposed, by which the two intersections M and N of this kind are required, so that there shall be $CM^3 + CN^3 = a^3$. Therefore since there shall be from the nature of the equation

$$zz - Pz + Q = 0,$$

$$CM^3 + CN^3 = P^3 - 3PQ,$$

one must have $P^3 - 3PQ = a^3$; which equation cannot have a place, since P^3 and PQ shall be irrational quantities. Therefore it follows that this equation cannot be satisfied in the strictest sense; but if the number of intersections may not be considered, even if from two more may be produced, then indeed satisfying can be found in an infinite number of ways by putting $Q = \frac{P^3 - a^3}{3P}$ and by taking some function of the sine and of the cosine of the angle $ACM = \varphi$ for P.

422. But if curves of this kind may be required, in which there shall be

$$CM^4 + CN^4 = a^4,$$

then it is required to put

$$P^4 - 4PPQ + 2QQ = a^4;$$

which equation, since no irrationality shall be present, involves no contradiction. Therefore there will have to be

$$Q = PP + \sqrt{(\tfrac{1}{2}P^4 + \tfrac{1}{2}a^4)}$$

which function, not withstanding the sign of the root can be considered as uniform, because, if $\sqrt{(\tfrac{1}{2}P^4 + \tfrac{1}{2}a^4)}$ were taken positive, imaginary values occur for z. Therefore there will be

$$\frac{Nzz}{L} = \frac{MMzz}{LL} - \sqrt{\left(\frac{M^4z^4}{2L^4} + \tfrac{1}{2}a^4\right)};$$

and, since there shall be $L - M + N = 0$ for the curve, or

$$zz - \frac{Mzz}{L} + \frac{Nzz}{L} = 0,$$

there will be

$$zz - \frac{Mzz}{L} + \frac{MMzz}{LL} - \sqrt{\left(\frac{M^4z^4}{2L^4} + \tfrac{1}{2}a^4\right)} = 0.$$

Consequently with the irrationality removed there will be

$$\frac{z^4}{L^4}\left(LL - LM + MM\right)^2 = \frac{M^4z^4}{2L^4} + \tfrac{1}{2}a^4$$

or

$$(xx + yy)^2\left(2(LL - LM + MM)^2 - M^4\right) = a^4L^4,$$

which satisfy all the curves included within itself.

423. These and similar questions can be resolved by another easier manner, as above in paragraph 372. For since there shall be $CM \cdot CN = Q$, if either of *CM* and *CN* may be said $= z$, the other will be $= \frac{Q}{z} = \frac{Nz}{L}$ on account of $Q = \frac{Nzz}{L}$. Whereby, if there should be

$$CM^n + CN^n = a^n$$

it becomes

$$z^n + \frac{N^n z^n}{L^n} = a^n \text{ and thus } z^n = \frac{a^n L^n}{L^n + N^n},$$

which equation, if *n* were an even number, now is rational and will satisfy the question. But if *n* shall be an odd number, squares must be taken to remove the irrationality ; with which done, so that the number of intersections may be doubled and thus in that sense an unsatisfactory curve may arise, as the desired result. Thus if there ought to be

$$CM^2 + CN^2 = a^2$$

the equation becomes

$$zz = xx + yy = \frac{aaLL}{LL + NN},$$

which agrees with the result found above (§ 410)

$$xx + yy = \frac{aaLL}{(L-M)^2 + L^2}$$

on account of $L - M + N = 0$. Therefore generally, if there must be $CM^n + CN^n = a^n$ and *n* were an even number, this equation will be obtained :

$$z^n = (xx + yy)^{\frac{n}{2}} = \frac{a^n L^n}{L^n + N^n} = \frac{a^n L^n}{L^n + (L-M)^n}$$

with *L* being a function of $m+2$ dimensions, *M* a function of $m+1$ dimensions and *N* a function of m dimensions of x and y.

424. This same solution is elicited also from a consideration of the sum $CM + CN = P$. For if either of *CM* and *CN* may be put $= z$, the other will be $= P - z$. Hence, if $CM^n + CN^n$ must be constant, there will be $z^n + (P-z)^n = a^n$. But we have seen there must be

$$P = \frac{Mz}{L} \text{ and } Q = \frac{Nzz}{L},$$

thus so that there shall be $L-M+N=0$; from which there shall be
[for $Q=z(P-z)=\frac{Nzz}{L}$ and $(P-z)=\frac{Nz}{L}=\frac{(M-L)z}{L}$,]

$$z^n+\frac{z^n\left(M-L\right)^n}{L^n}=a^n$$

or

$$z^n=\frac{a^nL^n}{L^n+\left(M-L\right)^n} \text{ or } z^n=\frac{a^nL^n}{L^n+N^n},$$

or by eliminating L there will be

$$z^n=\frac{a^n\left(M-N\right)^n}{\left(M-N\right)^n+N^n}$$

These equations, if n were an even number, will fulfill the proposed condition adequately. But if *n* were an odd number, two intersections *M* and *N* indeed will be given, so that there shall be $CM^n+CN^n=a^n$; but besides these two other intersections enjoy the same property, thus so that any right line drawn through *C* involves this proposed property twice.

425. With these set out it will be easy to resolve other greatly more difficult questions ; indeed to find the curve, which thus may be cut in two points *M* and *N* by all the right lines passing through *C*, thus so that

$$CM^n+CN^n+\alpha CM\cdot CN(CM^{n-2}+CN^{n-2})+\beta\cdot CM^2\cdot CN^2\ (CM^{n-4}+CN^{n-4})+\text{ etc.}$$

shall be a constant quantity $=a^n$. The one value may be put $CM=z$, the other will be

$$CN=\frac{Q}{z}=\frac{Nz}{L};\ \text{[as above, } CM\cdot CN=Q\text{;]}$$

with which values substituted this equation will arise, by which the nature of the curve is expressed,

$$z^n\left(L^n+N^n+\alpha LN(L^{n-2}+N^{n-2})+\beta LLNN(L^{n-4}+N^{n-4})+\text{ etc.}\right)=\alpha^nL^n$$

But $L-M+N=0$ and *L*, *M* and *N* are homogeneous functions of *x* and *y* of dimensions $m+2,\ m+1$ and *m*, as we have described above ; from which there will be either

$L = M - N$ or $N = M - L$ and thus hence infinitely many solutions will be deduced.

426. Now we may go on to investigating curves, which will be cut in three points by the individual lines drawn through a fixed point *C*. The nature of curves of this kind may be expressed by a general equation of this kind :

$$z^3 - Pzz + Qz - R = 0,$$

where *z* denotes the distance of a point of this curve from *C*, and *P*, *Q*, *R* are functions either of the sine or cosine of this angle $ACM = \varphi$. Moreover by the same ratios, which we have asserted above, it will be apparent that no more than three intersections will be produced, that *P* and *R* must be odd functions of the $\sin.\varphi$ and $\cos.\varphi$, truly an even function *Q* must be put in place. But if therefore orthogonal coordinates $CP = x$, $PM = y$, may be put in place so that there shall be $xx + yy = zz$, and *K, L, M* and *N* may denote homogeneous functions of $n+3, n+2, n+1$ and *n* dimensions of *x* and *y*, there will be

$$P = \frac{Lz}{K},\ Q = \frac{Mzz}{K},\ \text{and}\ R = \frac{Nz^3}{K};$$

and thus this general equation will be had between the orthogonal coordinates *x* and *y* for curves of this kind :

$$K - L + M - N = 0;$$

from which it is apparent the point *C* becomes a multiple point of the curve, as many times as the index *n* contains units.

427. Therefore in the first place all the lines of the third order belong here, wherever the point *C* may be taken outside the curve. Then, all the lines of the fourth order may be satisfied as long as the point *C* may be taken on the curve. All three lines of the fifth order, in which a double point may be given, here are referred to, but only if the point *C* may be put in place at a double point of these. And in a similar manner lines of higher orders will satisfy this condition, if they may have multiple points of so great exponents, as many times as *n* may contain unity, if $n+3$ may set out the order, to which the equation may pertain.

428. Let *p, q, r* be these three values of *z*, which may be satisfied by the equation

$$z^3 - Pzz + Qz - R = 0$$

for some value of the angle $CAM = \varphi$; and from the nature of the equation there will be

$$P = p + q + r,\ Q = pq + pr + qr\ \text{ et }\ R = pqr.$$

Now since P and R cannot be expressed rationally by x and y [*i.e.* as they always involve a square root term], it is evident curves of this kind cannot be shown, in which either the quantity $p+q+r$ or pqr shall be a constant quantity, nor thus any odd function of p, q and r can be put equal to a constant. But even functions will be able to maintain a constant value without any difficulty. Thus if it may be required, so that there shall be

$$pq+pr+qr=aa, \text{ there will be } Q=\frac{Mzz}{K}=aa$$

and thus $M(xx+yy)=aaK$; which value substituted into the equation $K-L+M-N=0$ will give the general equation with this provided property satisfied among themselves :

$$M(xx+yy)-aaL+aaM-aaN=0$$

or this equation, on eliminating M :

$$(xx+yy)K-(xx+yy)L+aaK-(xx+yy)N=0 .$$

429. In a like manner other similar questions may be resolved easily. As the curve may be sought, which will be cut into three parts by right lines drawn through the point, so that there shall be

$$pp+qq+rr=aa.$$

For since here shall be

$$pp+qq+rr=PP-2Q \text{ and } P=\frac{Lz}{K}$$

and

$$Q=\frac{Mzz}{K},$$

the equation becomes :

$$\frac{LLzz}{KK}-\frac{2Mzz}{K}=aa$$

or

$$(xx+yy)LL-2(xx+yy)KM=aaKK .$$

But for curves allowing three intersections this general equation will be had : $K-L+M-N=0$, the nature of which agrees with this, so that the maximum number of dimensions of x and y exceeds the minimum by three. From which therefore an equation of this kind may be obtained, and likewise there shall be :

$$(xx+yy)LL-2(xx+yy)KM=aaKK ,$$

that equation may be multiplied by $2(xx+yy)K$, so that M may be eliminated, and this general equation for the proposed case satisfying

$$2(xx+yy)KK-2(xx+yy)KL+(xx+yy)LL-aaKK-2(xx+yy)KN=0.$$

For the term, in which the most dimensions are present, is $2(xx+yy)KK$ and it contains $2n+8$ dimensions of x and y; and the least term is $2(xx+yy)KN$ and it contains $2n+5$ dimensions, as the nature of the problem demands.

430. Therefore because neither the greatest nor the least term can vanish, we can put the dimension $n=0$ for the simplest curve requiring to be found, and there shall be $N=b^3$, $K=x(xx+yy)$ and $L=0$, and this equation will be produced

$$2(xx+yy)^3xx-aaxx(xx+yy)^2-2b^3x(xx+yy)^2=0,$$

which divided by $2x(xx+yy)^2$ provides this equation

$$x(xx+yy)-\tfrac{1}{2}aax-b^3=0,$$

which relates to the third order. But if there shall not be $L=0$ but

$$L=2c(xx+yy),$$

this equation of order four will be produced :

$$xx(xx+yy)-2cx(xx+yy)+2cc(xx+yy)-\tfrac{1}{2}aaxx-b^3x=0$$

or

$$xx(xx+yy)+(2c-x)^2(xx+yy)=aaxx+2b^3x.$$

Moreover in a similar manner, many other curves sought from higher orders will be elicited satisfying the question.

431. Then also these curves can be found, in which $p^4+q^4+r^4$ shall be a constant quantity. For since there shall be

$$p^4+q^4+r^4=P^4-4PPQ+2QQ+4PR,$$

one will be able to put

$$P^4 - 4PPQ + 2QQ + 4PR = c^4.$$

Therefore there will be

$$z^4(L^4 - 4KLLM + 2KKMM + 4KKLN) = c^4K^4$$

and thus

$$4KKLNz^4 = c^4K^4 - z^4(L^4 - 4KLLM + 2KKMM),$$

from which the value of *N* substituted into the equation $K - L + M - N = 0$ will give the general equation for curves satisfying this condition.

432. Moreover both this condition $p^4 + q^4 + r^4 = c^4$ and the preceding $pp + qq + rr = aa$ can be satisfied at the same time. For by this there must be

$$zzLL - 2zzKM = aaKK;$$

from which there becomes

$$2zzKM = zzLL - aaKK.$$

Then, since there shall be

$$4KKLNz^4 = c^4K^4 - L^4z^4 + 4KLLMz^4 - 2KKMMz^4,$$

there will be

$$4KKLNz^4 = c^4K^4 + L^4z^4 - 2aaKKLLzz - 2KKMMz^4$$

and

$$4KKLMz^4 = 2KL^3z^4 - 2aaK^3Lzz.$$

These values may be substituted in place of *M* and *N* in the equation $K - L + M - N = 0$, or

$$4K^3Lz^4 - 4KKLLz^4 + 4KKLMz^4 - 4KKLNz^4 = 0,$$

and this equation will be produced for the curve

$$4K^3Lz^4 - 4KKLLz^4 + 2KL^3z^4 - 2aaK^3Lzz - c^4K^4 - L^4z^4 + 2aaKKLLzz + 2KKMMz^4 = 0.$$

But on account of

$$KMzz = \tfrac{1}{2}LLzz - \tfrac{1}{2}aaKK$$

there will be

$$2KKMMz^4 = \tfrac{1}{2}L4z^4 - aaKKLLzz + \tfrac{1}{2}a^4K^4,$$

and thus this general equation will be had for the curve sought :

$$8K^3Lz^4 - 8KKLLz^4 + 4KL^3z^4 - 4aaK^3Lzz - 2c^4K^4 - L^4z^4 + 2aaKKLLzz + a^4K^4 = 0 .$$

433. Because *K* must be a homogeneous function of *x* and *y* higher by one dimension than *L,* the simplest curve, in which three intersections can be shown and likewise both $pp + qq + rr = aa$ and $p^4 + q^4 + r^4 = c^4$, will be produced, if there may be put $K = zz$ and $L = bx$ *;* therefore there will be :

$$8bxz^6 - 8bbxxz^4 + 4b^3x^3zz - 4aabxz^4 - 2c^4z^4 - b^4x^4 + 2aabbxxzz + a^4z^4 = 0,$$

which on account of $zz = xx + yy$ is rational and presents a line of the seventh order, of which *C* is a quadruple point. Moreover another line of the seventh order will be obtained satisfying the relation, if there may be put $K = x$ and $L = b$ *;* indeed there shall be

$$8bx^3z^4 - 8bbxxz^4 + 4b^3xz^4 - 4aabx^3zz - 2c^4x^4 - b^4z^4 + 2aabbxxzz + a^4x^4 = 0$$

or

$$z^4 = \frac{4aabx^3zz - 2aabbxxzz + 2c^4x^4 - a^4x^4}{8bx^3 - 8bbxx + 4b^3x - b^4}$$

From which there becomes

$$zz = \frac{2aabx^3 - aabbxx \pm xxy\sqrt{(2bx - bb)\left(2c^4(bb - 2bx + 4xx) - 2a^4(bb - 2bx + 2xx)\right)}}{b(2x - b)(4xx - 2bx + bb)}.$$

434. Now it would be permitted to progress to higher curves, which will be intersected in four points by right lines drawn through the point *C* ; and from those, these may be found, which shall be provided with given properties. Truly, if we may attend to the precepts treated in the preceding, truly no difficulty will remain and everything, which will be wished for in this kind, with hardly any labour may be either worked out or, if the question does not admit to a genuine solution, this itself at once may become known. On which account I will not linger further on these matters, and proceed to another argument relating to the understanding of curved lines.

CAPUT XVII

DE INVENTIONE CURVARUM EX ALIIS PROPRIETATIBUS

391. Quaestiones, quas in praecedente capite resolvimus, ita erant comparatae, ut ad aequationem inter coordinatas sive rectangulas sive obliquangulas facile revocari possent. Nunc igitur eiusmodi proprietates contemplemur, quae non immediate applicatas inter se parallelas respiciant, veluti si rectarum ex dato quodam puncto ad curvam eductarum indoles quaepiam proponatur (Fig. 81). Sit *C* punctum, unde rectae ad curvam educantur *CM*, *CN* atque proprietas quaepiam has rectas respiciens fuerit proposita; conveniet a modo hactenus usitato naturam curvarum per coordinatas exprimendi ita recedere, ut istae rectae in aequationem introducantur.

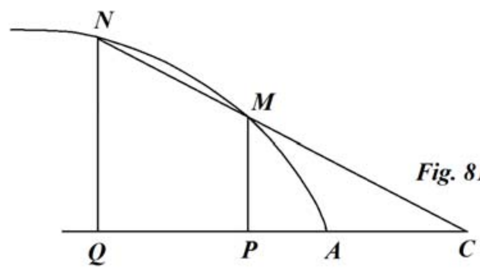

Fig. 81

392. Cum igitur pluribus aliis modis naturae linearum aequationibus comprehendi queant, quae inter duas variabiles formentur, in praesenti negotio quantitas rectae *CM* ex dato puncto *C* ad curvam eductae alterius variabilis locum sustineat. Tum vero alia opus erit variabili, qua situs rectae *CM* definiatur; hunc in finem assumatur recta quaepiam *CA* per punctum *C* ducta pro axe, atque angulus *ACM* seu quantitas, ab hoc angulo pendens commodissime vicem alterius variabilis tenebit. Sit ergo recta $CM = z$ et angulus $ACM = \varphi$, cuius sinus tangensve in aequationem ingrediatur; atque manifestum est, si detur aequatio quaecunque inter z et $\sin.\varphi$ seu $\tang.\varphi$, per eam curvae *AMN* naturam determinari; pro quovis enim angulo *ACM* definitur longitudo rectae *CM* sicque punctum curvae *M* determinatur.

393. Diligentius autem perpendamus hunc lineas curvas exprimendi modum. Ac primo quidem aequetur distantia z functioni cuicunque sinus anguli φ quae functio si fuerit uniformis, videatur recta *CM* curvae in unico puncto *M* occurrere, quia angulo $ACM = \varphi$ unicus valor rectae *CM* respondet. Verum si angulus φ duobus rectis augeatur, eadem manebit rectae *CM* per punctum *C* ductae positio, hoc tantum discrimine, quod in Fig. 82 plagam oppositam dirigatur; sicque alia eiusdem rectae *CM* intersectio cum curva prodibit, etiamsi z aequetur functioni uniformi sinus anguli φ. Scilicet sit P functio illa sinus anguli φ, ita ut sit

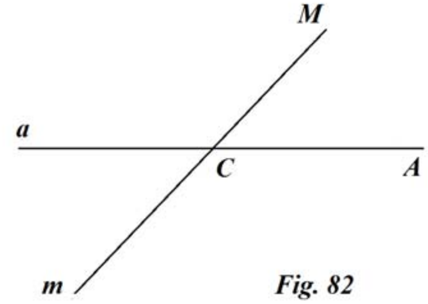

Fig. 82

$z = P$, unde oriatur (Fig. 82) punctum curvae M; augeatur nunc angulus φ duobus rectis seu eius sinus statuatur negativus, quo facto abeat P in Q, ut sit $z = Q$; hinc ergo prodibit nova intersectio eiusdem rectae CM productae cum curva m sumendo $Cm = Q$.

394. Quamvis ergo P sit functio uniformis sinus anguli φ, tamen recta CM sub dato angulo $ACM = \varphi$ per punctum C ducta curvae in duobus punctis M et m occurret, nisi sit $Q = -P$. Quodsi ergo unaquaeque recta CM curvae in unico tantum puncto occurrere debeat, quantitatem illam P functionem esse oportet imparem sinus anguli φ. Hoc idem autem usuvenit, si P fuerit functio impar cosinus anguli φ. Quamobrem omnes curvae, quas singulae rectae ex C eductae in unico puncto intersecant, continebuntur in hac aequatione $z = P$, siquidem P fuerit functio impar cum sinus tum cosinus anguli $ACM = \varphi$.

395. Cum igitur curvae (Fig. 81), quae a rectis ex puncto C ductis in unico puncto secantur, contineantur in aequatione $z = P$, si P fuerit functio impar sinus et cosinus anguli φ, seu eiusmodi functio, quae valorem negativum induat, si tam sinus quam cosinus anguli φ statuatur negativus, hinc facile pro huiusmodi curvis aequatio inter coordinatas orthogonales reperiri poterit. Demisso enim ex puncto M in axem CA perpendiculo MP, si dicatur $CP = x$, $PM = y$, erit $\frac{y}{z} = \sin.\varphi$ et $\frac{x}{z} = \cos.\varphi$, unde, si P fuerit functio impar ipsarum $\frac{x}{z}$ et $\frac{y}{z}$, omnes istae curvae continebuntur in hac aequatione $z = P$. A simplicissimis ergo incipiendo erit

$$z = \frac{\alpha x}{z} + \frac{\beta y}{z} + \frac{\gamma z}{x} + \frac{\delta z}{y}$$

atque ad altiores potestates ascendendo erit

$$z = \frac{\alpha x}{z} + \frac{\beta y}{z} + \frac{\gamma z}{x} + \frac{\delta z}{y} + \frac{\varepsilon x^3}{z^3} + \frac{\zeta xxy}{z^3} + \frac{\eta xyy}{z^3} + \frac{\theta y^3}{z^3} + \frac{\iota xx}{yz} + \frac{\chi yy}{xz} + \frac{\lambda yz}{xx} + \text{etc.}$$

396. Si haec aequatio per z dividatur, ubique pares tantum ipsius z occurrent potestates ideoque, cum sit $z = \sqrt{(xx + yy)}$, eliminando z nulla irrationalitas in aequatione remanebit prodibitque aequatio rationalis inter x et y. Aequatio ergo generalis ita erit comparata, ut unitas seu quantitas constans aequetur functioni -1 dimensionum ipsarum x et y. Cuiusmodi functio si fuerit P, erit $C = P$ ideoque $\frac{1}{C} = \frac{1}{P}$; at $\frac{1}{P}$ erat functio unius dimensionis ipsarum x et y; unde, si functio quaecunque unius dimensionis ipsarum x et y aequetur constanti, aequatio erit pro curva, quam rectae per punctum C eductae in unico puncto intersecant.

397. Sit P functio n dimensionum ipsarum x et y et Q functio $n+1$ dimensionum; erit $\frac{Q}{P}$ functio unius dimensionis ideoque omnes curvae, quas hic contemplamur, continebuntur in aequatione $\frac{Q}{P}=c$ seu $Q=cP$. Denotante ergo n numerum quemcunque, aequatio generalis pro his curvis erit

$$\alpha x^{n+1}+\beta x^{n}y+\gamma x^{n-1}yy+\delta x^{n-2}y^{3}+\varepsilon x^{n-3}y^{4}+\text{ etc.}=c(Ax^{n}+Bx^{n-1}y+Cx^{n-2}yy+Dx^{n-3}y^{3}+\text{ etc.})$$

Ex qua lineae singulorum ordinum, quae a rectis ex puncto C eductis in unico tantum puncto secantur, in sequentibus aequationibus continebuntur:

I.
$$\alpha x+\beta y=c,$$

II.
$$\alpha xx+\beta xy+\gamma yy=c(Ax+By),$$

III.
$$\alpha x^{3}+\beta xxy+\gamma xyy+\delta y^{3}=c(Axx+Bxy+Cyy),$$

IV.
$$\alpha x^{4}+\beta x^{3}y+\gamma xxyy+\delta xy^{3}+\varepsilon y^{4}=c\left(Ax^{3}+Bxxy+Cxyy+Dy^{3}\right)$$
etc.

398. Primum ergo linea recta satisfacit, quam utique constat ab aliis lineis rectis per datum punctum ductis non nisi in uno puncto secari posse. Secunda aequatio est pro sectionibus conicis generalis, dummodo sectio conica per ipsum punctum C transeat, quae intersectio, cum omnibus rectis ex C eductis communis sit, non computatur; quoniam ergo sectiones conicae a recta quacunque nonnisi in duobus punctis secari possunt, omnis recta per punctum C in ipsa curva utcunque sumtum transiens unicam tantum praebebit intersectionem. Lineae autem curvae sequentium ordinum omnes per ipsum punctum C transeunt, quae intersectio omnibus rectis per C ductis communis pariter non computatur. Atque idcirco ex altioribus ordinibus in aequationibus exhibitis eae tantum continentur, quas rectae per C ductae in unico puncto intersecant. Sic igitur omnes enumeravimus curvas algebraicas, quae a rectis per datum punctum C ductis nonnisi in unico puncto traiiciuntur.

399. Progrediamur iam ad eas curvas investigandas, quas singulae rectae per punctum C ductae vel in duobus punctis intersecant vel nusquam, siquidem radices aequationis duplicem intersectionem indicantis fiant imaginariae. Cum igitur pro quovis angulo

$ACM = \varphi$ recta $CM = z$ duplicem sortiatur valorem, ea per aequationem quadraticam definietur. Sit itaque

$$zz - Pz + Q = 0\,,$$

ubi P et Q sint functiones anguli φ seu eius sinus cosinusve. Quoniam vero recta CM curvam nonnisi in duobus punctis M et N secare debet, non solum p et Q functiones uniformes anguli φ esse oportet, sed etiam aucto angulo φ duobus rectis nullae novae intersectiones oriri debent, id quod evenit, si P fuerit functio impar sinus et cosinus anguli φ, ita ut valorem induat negativum, si sinus et cosinus negative accipiantur Q autem esse debet functio par eiusdem sinus et cosinus.

400. Positis autem coordinatis orthogonalibus $CP = x$ et $PM = y$ erit

$$\frac{y}{z} = \sin.\varphi \text{ et } \frac{x}{z} = \cos.\varphi\,;$$

ideoque P debebit esse functio impar ipsarum $\frac{x}{z}$ et $\frac{y}{z}$ et Q functio par ipsarum $\frac{x}{z}$ et $\frac{y}{z}$. Ex his colligitur fore $\frac{P}{z}$ functio rationalis ipsarum x et y atque adeo functio homogenea -1 dimensionum. Simili modo erit $\frac{Q}{zz}$ functio rationalis ipsarum x et y homogenea -2 dimensionum. Quodsi ergo fuerit L functio homogenea $n+2$ dimensionum, M functio homogenea $n+1$ dimensionum atque N functio homogenea n dimensionum quaecunque ipsarum x et y,

fractio $\frac{M}{L}$ exhibebit functionem convenientem pro $\frac{P}{z}$

et

$\frac{N}{L}$ functionem convenientem pro $\frac{Q}{zz}$

Quare, cum sit $zz - Pz + Q = 0$, erit $1 - \frac{P}{z} + \frac{Q}{zz} = 0$, unde aequatio generalis pro curvis, quae a rectis per punctum C ductis in duobus punctis secantur, erit

$$1 - \frac{M}{L} + \frac{N}{L} = 0 \text{ seu } \mathrm{L} - M + N = 0,$$

ubi est

$$P = \frac{Mz}{L} \text{ et } Q = \frac{Nzz}{L} = \frac{N(xx+yy)}{L};$$

eritque adeo *P* functio irrationalis ipsarum *x* et *y,* ob $z = \sqrt{(xx+yy)}$, et *Q* est functio rationalis nullius dimensionis.

401. Hinc iam facile erit ex quovis linearum ordine eas exhibere, quae a rectis per datum punctum *C* ductis in duobus punctis vel nusquam intersecentur. Pro secundo scilicet ordine fiat $n = 0$, ac prodibit aequatio generalissima sectionum conicarum

$$\alpha xx + \beta xy + \gamma yy - \delta x - \varepsilon y + \zeta = 0.$$

Puncto ergo *C* sumto ubicunque omnis recta per id ducta sectionem conicam vel in duobus punctis vel nusquam intersecabit. Interim tamen fieri potest, ut unaquaepiam recta curvam in uno tantum puncto intersecet; quod cum inter infinitas illas rectas per *C* ductas vel uni vel duabus tantum usuveniat, haec exceptio nullius erit momenti; quin etiam ita hoc paradoxon explicari potest, ut altera intersectio in infinitum abeat; quam ob causam ista exception nostro asserto nullam vim inferre censenda est.

402. Quo autem pateat, quibus casibus ista exceptio locum habeat, aequationem inter *x* et *y* reducamus ad aequationem inter *z* et angulum
$ACM = \varphi$; quae ob $y = z \cdot \sin.\varphi$ et $x = z \cdot \cos.\varphi$ abibit in hanc

$$zz(\alpha(\cos.\varphi)^2 + \beta \cdot \sin.\varphi \cdot \cos.\varphi + y(sin.\varphi)^2) - z(\delta \cdot \cos.\varphi + \varepsilon \cdot \sin.\varphi) + \zeta = 0;$$

ex qua patet, si fuerit coefficiens ipsius *zz* aequalis nihilo, unicam tantum intersectionem locum habere; quod ergo evenit, si fuerit

$$\alpha + \beta \cdot \text{tang.}\varphi + \gamma\left(\text{tang.}\varphi\right)^2 = 0.$$

Quodsi ergo haec aequatio duas habeat radices reales, duobus casibus recta per *C* educta curvam in unico tantum puncto secabit. Quoniam vero eiusdem aequationis radices indicant asymptotas curvae, perspicuum est hyperbolas a rectis alteri asymptotae parallelis in unico tantum puncto secari, cuiusmodi rectae per punctum *C* transeuntes duae tantum dantur. In parabola vero unica recta axi parallela hanc exceptionem patietur. Verum si sectio conica fuerit ellipsis, ubicunque assumatur punctum *C,* omnis recta per id ducta curvam vel nusquam vel in duobus punctis secabit.

403. Lineae tertii ordinis ista proprietate gaudentes, posito $n = 1$, continebuntur in hac aequatione

$$\alpha x^3 + \beta xxy + \gamma xyy + \delta y^3 - \varepsilon xx - \zeta xy - \eta yy + \theta x + \iota y = 0,$$

quae quidem in se complectitur omnes lineas tertii ordinis, quae ergo omnes huc pertinent, dummodo punctum C in ipsa curva capiatur. Facto enim $x=0$ simul y valorem obtinet evanescentem. Simili modo pro curvis quarti ordinis quaesito satisfacientibus punctum C non solum in curva, sed simul eius punctum duplex esse debet; omnis ergo linea quarti ordinis puncto duplici praedita quaesito satisfaciet, dummodo punctum C in puncto duplici statuatur. Sin autem C fuerit adeo curvae punctum triplex, tum omnis recta per id ducta curvam in unico puncto intersecabit pertinebitque ad casum primo consideratum. Pari modo lineae quinti ordinis satisfacient, si punctum C in earum puncto triplici statuatur, atque ita porro. Perpetuo autem notandum est, si recta per C ducta parallela fiat alicui asymptotae rectae seu axi asymptotae parabolicae, tum semper unicam dari intersectionem, altera in infinitum abeunte.

404. Egregie haec conveniunt cum natura linearum cuiusque ordinis: quia enim linea cuiusque ordinis a linea recta in tot punctis intersecari potest, quot exponens ordinis continet unitates, (atque revera in totidem punctis intersecatur, nisi aliquot intersectiones vel fiant imaginariae vel in infinitum abeant), et quia hic omnes intersectiones, sive reales sive in infinito factas sive imaginarias, aeque computamus easque tantum excludimus, quae in ipso puncto C fiunt, manifestum est, cum linea ordinis n in n punctis a quaque linea recta secetur, punctum C in puncto totuplici, quot numerus $n-2$ continet unitates, collocari debere, ut intersectio duplex prodeat.

405. His notatis facile erit problemata, quae circa relationem inter quosque binos ipsius z valores CM et CN proponi solent, vel resolvere vel solutionis inconvenientiam ostendere. Cum enim duo ipsius z valores CM et CN sint radices huius aequationis $zz-Pz+Q=0$, erit ipsorum summa $=P$ et rectangulum eorum $CM\cdot CN=Q$. Quare, si primum requirantur eiusmodi curvae, in quibus ubique sit summa $CM+CN$ constans, functionem P quantitatem constantem esse oporteret. Cum autem ex quaestionis natura unaquaeque recta per C ducta curvae in duobus tantum punctis occurrere debeat, necesse est, ut sit (§ 400),

$$P=\frac{Mz}{L}=\frac{M\sqrt{(xx+yy)}}{L},$$

quae quantitas irrationalitatem involvens nunquam constans esse potest. Atque idcirco nulla datur curva huic quaestioni proprie satisfaciens.

406. Quodsi autem ista conditio, qua duae tantum cuiusque rectae per C ductae intersectiones cum curva postulantur, omittatur atque eiusmodi quaerantur curvae, quae quidem plures duabus intersectiones exhibeant, inter eas autem duae M et N eiusmodi adsint, ut sit $CM=CN$ quantitas constans, tales curvae innumerabiles exhiberi poterunt ponendo P quantitati illi constanti $CM+CN=a$. Erit enim $zz-az+Q=0$ denotante Q functionem $\frac{Nzz}{L}$; et quia haec aequatio adhuc irrationalitate laborat, ea sublata erit

$aazz = (zz+Q)^2$ seu $aa = zz\left(1+\frac{N}{L}\right)^2$

seu

$$aaLL = (xx+yy)(LL+2LN+NN),$$

in qua erit *L* functio homogenea $n+2$ at *N* functio homogenea *n* dimensionum ipsarum *x* et *y*. Simplicissima ergo curva hoc sensu quaestionem resolvens habebitur, si ponatur

$$L = xx+yy \text{ et } N = \pm bb,$$

eritque

$$aa\left(xx+yy\right) = (xx+yy\pm bb)^2,$$

quae est pro linea quarti ordinis complexa; complectitur enim duos circulos in *C* concentricos. Curvae autem continuae simplicissimae quaesito satisfacientes erunt sexti ordinis, ponendo

$$L = \alpha xx+\beta xy+\gamma yy \text{ et } N = \pm bb,$$

pro quibus aequatio erit

$$aa(\alpha xx+\beta xy+\gamma yy)^2 = (xx+yy)(\alpha xx+\beta xy+\gamma yy\pm bb)^2.$$

Sit $\alpha = 1$, $\beta = 0$ et $\gamma = 0$, erit

$$yy+xx = \frac{aax^4}{x^4\pm 2bbxx+b^4}$$

seu

$$y = \frac{x\sqrt{(aaxx-\ x^4\mp 2bbxx-b^4)}}{xx\pm bb}.$$

407. Sin autem huiusmodi solutiones, quibus rectae per *C* ductae curvam in pluribus quam duobus punctis intersecant, excludantur, quam conditionem natura quaestionis requirere videtur, nullae prorsus curvae quaestioni satisfacere sunt dicendae ac propterea nulla dabitur linea continua, quae a rectis per *C* ductis ita in duobus tantum punctis *M* et *N* intersecetur, ut summa $CM+CN$ sit constans. At vero si istae intersectiones huius indolis postulentur, ut rectangulum $CM\cdot CN$ debeat esse constans, quae proprietas in circulum ita competit, ut is satisfaciat, ubicunque punctum *C* capiatur, infinitae aliae lineae curvae inveniri poterunt, quae idem praestent. Debebit enim *Q* esse quantitas constans, aequalis scilicet illi rectangulo $CM\cdot CN$, quod sit $=aa$; quae positio, cum sit $Q=\frac{Nzz}{L}$, ac propterea functio rationalis ipsarum *x* et *y*, non pugnat.

408 . Sit igitur $\frac{Nzz}{L} = aa$ seu $L = \frac{Nzz}{aa} = \frac{N(xx+yy)}{aa}$, atque curvae quaesito satisfacientes omnes continebuntur in hac aequatione

$$\frac{N(xx+yy)}{aa} - M + N = 0 \text{ seu } Maa = N(xx+yy+aa),$$

ubi M denotat functionem quamcunque homogeneam $n+1$ dimensionum, N vero functionem homogeneam n dimensionum ipsarum x et y, ita ut sit

$$\frac{M}{N} = \frac{xx+yy+aa}{aa}$$

functio unius dimensionis ipsarum x et y. Haec ergo aequatio omnes complectitur curvas, quae a rectis per C ductis in duobus tantum punctis M et N ita secantur, ut rectangulum $CM \cdot CN$ sit ubique constans $= aa$.

409. Cum igitur $\frac{M}{N}$ sit functio homogenea unius dimensionis ipsarum x et y, casus simplicissimus prodibit, si ponatur $\frac{M}{N} = \frac{\alpha x + \beta y}{a}$, ex quo orietur haec aequatio

$$xx + yy - a\left(\alpha x + \beta y\right) + aa = 0,$$

quae semper est pro circulo, et, cum sit aequatio pro circulo generalis inter coordinatas orthogonales, manifestum est circulum quaesito satisfacere, ubicunque punctum C accipiatur, omnino uti ex elementis constat. Praeter circulum ergo ex sectionibus conicis nulla alia curva huic quaestioni satisfacit. Verum ex singulis ordinibus linearum sequentibus infinita linearum satisfacientium copia exhiberi potest et quidem omnes, quae ex quolibet ordine satisfaciunt. Sic lineae tertii ordinis, quae ist hac proprietate gaudent, continebuntur in, hac aequatione

$$\frac{\alpha xx + \beta xy + \gamma yy}{a(\delta x + \varepsilon y)} = \frac{xx+yy+aa}{aa}$$

seu

$$(\delta x + \varepsilon y)(xx+yy) - a(\alpha xx + \beta xy + \gamma yy) + aa(\delta x + \varepsilon y) = 0.$$

Atque simili modo ex omnibus sequentibus linearum ordinibus eae, quae satisfaciunt, exhibebuntur.

410. Proposita iam sit haec quaestio, ut inter omnes lineas curvas, quae a rectis per punctum C ductis in duobus punctis secantur, eae definiantur, in quibus sit summa quadratorum $CM^2 + CN^2$ quantitas constans, puta $= 2aa$.
Cum igitur sit $CM + CN = P$ et $CM \cdot CN = Q$, erit $CM^2 + CN^2 = PP - 2Q$;
debebit ergo esse

$$PP-2Q=2aa \text{ seu } Q=\frac{PP-2aa}{3a}.$$

Quare, ob $P=\frac{Mz}{L}$ et $Q=\frac{Nzz}{L}$, erit

$$\frac{2Nzz}{L}=\frac{MMzz}{LL}-2aa \text{ ideoque } N=\frac{MM}{2L}-\frac{aaL}{zz}:$$

quae aequatio, cum sit L functio $n+2$ dimensionum, M functio $n+1$ dimensionum et N functio n dimensionum ipsarum x et y, nullam implicat difficultatem. Sumtis ergo pro L et M eiusmodi functionibus, erit

$$N=\frac{MM}{2L}-\frac{aaL}{zz},$$

unde pro curvis quaesito satisfacientibus ista resultat aequatio generalis

$$L-M+\frac{MM}{2L}-\frac{aaL}{zz}=0$$

seu

$$2LL(xx+yy)-2LM(xx+yy)+MM(xx+yy)-2aaLL=0,$$

quae, si sit $M=0$, praebet circulum, cuius centrum in C, quem quaesito satisfacere per se est perspicuum.

411. Ponamus $n+1=0$, ut sit M quantitas constans $=2b$ et $L=\alpha x+\beta y$, atque orietur linea quarti ordinis hac aequatione contenta

$$(\alpha x+\beta y)^2(xx+yy-aa)-2b(\alpha x+\beta y)(xx+yy)+2bb(xx+yy)=0.$$

Alia aequatio quarti ordinis reperitur, si ponatur

$$L=xx+yy \text{ et } M=2(\alpha x+\beta y)a,$$

tum enim aequatio per $2xx+2yy$ divisa dabit

$$(xx+yy)^2-2a(\alpha x+\beta y)(xx+yy)+2aa(\alpha x+\beta y)^2-aa(xx+yy)=0.$$

Nisi autem divisio per $xx+yy$ succedat, aequatio inventa (ponendo $2M$ loco M), quae est

$$LL(xx+yy)-2LM(xx+yy)+2MM\left(xx+yy\right)-aaLL=0,$$

semper erit ordinis $2n+6$, ideoque ex quolibet ordine pari obtinetur aequatio pro curva satisfaciente. Praeterea vero, si L per $xx+yy$ fuerit divisibilis, scilicet si, denotante N functionem quamcunque homogeneam n dimensionum ipsarum x et y, fuerit $L=(xx+yy)N$, orietur alia aequatio generalis haec

$$NN(xx+yy)^2-2MN(xx+yy)+2MM-aaNN(xx+yy)=0,$$

quae est ordinis $2n+4$, ita ut ex singulis ordinibus paribus duplex nascatur aequatio pro curvis proposita proprietate gaudentibus. Sic ex ordine sexto satisfacient curvae in his duabus aequationibus contentae

$$(\alpha xx+\beta xy+\gamma yy)^2(xx+yy-aa)$$
$$-2a(\delta x+\varepsilon y)(xx+yy)(\alpha xx+\beta xy+\gamma yy-a(\delta x+\varepsilon y))=0$$

et

$$(\delta x+\varepsilon y)^2(xx+yy)(xx+yy-aa)$$
$$=2a(\alpha xx+\beta xy+\gamma yy)((\delta x+\varepsilon y)(xx+yy)-a(\alpha xx+\beta yy+\gamma yy)).$$

In nullo ergo linearum ordine impari ulla datur linea hanc quaestionem resolvens.

412. Si iam non quaerantur eiusmodi curvae, in quibus sit summa quadratorum CM^2+CN^2 constans, sed in quibus sit

$$CM^2+CM\cdot CN+CN^2$$

vel generaliter

$$CM^2+\ n\cdot CM\cdot CN+CN^2$$

quantitas constans, problema simili modo resolutionem admittet. Cum enim sit

$$CM^2+n\cdot CM\cdot CN+CN^2=PP+(n-2)Q,$$

ponatur $PP+(n-2)Q=aa$ eritque $Q=\dfrac{aa-PP}{n-2}$, quae aequatio nullo incommodo laborat. Cum igitur sit

$$P=\frac{Mz}{L}\ \text{ et }\ Q=\frac{Nzz}{L},$$

erit

$$\frac{MMzz}{LL}+\frac{(n-2)Nzz}{L}=aa$$

ideoque

$$N=\frac{aaL}{(n-2)zz}-\frac{MM}{(n-2)L}.$$

Quare, cum aequatio pro curva sit $L-M+N=0$, habebitur pro hac proprietate, qua $CM^2+\ n\cdot CM\cdot CN+CN^2$ debet esse constantis magnitudinis $=aa$, ista aequatio

$$(n-2)LLzz-(n-2)LMzz+aaLL-MMzz=0$$

seu ob $zz=xx+yy$ erit

$$a\alpha LL+(xx+yy)\big((n-2)LL-(n-2)LM-MM\big)=0$$

existente *L* functione $m+2$ et M, $m+1$ dimensionum ipsarum *x* et *y*. Sit *N* functio quaecunque homogenea *m* dimensionum ac ponatur $L=(xx+yy)N$, prodibit alia aequatio generalis haec

$$aa(xx+yy)NN+(n-2)(xx+yy)^2NN-(n-2)(xx+yy)MN-MM=0.$$

413. Si statuatur $n=2$, ut sit $(CM+CN)^2=aa$, fiet vel

$$aaLL=(xx+yy)MM\quad\text{vel}\quad MM=aa(xx+yy)NN.$$

Utraque autem aequatio, cum sit homogenea, continebit duas pluresve aequationes huius formae $\alpha y=\beta x$; ideoque quaesito satisfieri non poterit, nisi duabus pluribusve rectis per punctum *C* ductis; quae autem cum eo sensu, quo quaestio proponitur, non satisfaciant, perspicuum est hoc problema nullam admittere solutionem, uti supra iam notavimus; deberet enim esse $CM+CN=$ constanti *a*. Quodsi vero statuatur $n=-2$, ita ut quadratum differentiae $(CN-CM)^2$ ideoque ipsa differentia *MN* deberet esse constans, orientur hae duae aequationes

$$aaLL=(xx+yy)(2L-M)^2,$$

et

$$aa(xx+yy)NN=(2(xx+yy)N-M)^2,$$

unde simplicissima solutio oritur, si ponatur $N=1$ et $M=2bx$; erit enim

$$aa(xx+yy)=4(xx+yy-bx)^2,$$

seu posito $aa=8cc$ erit

$$(xx+yy)^2=2(cc+bx)(xx+yy)-bbxx.$$

Ergo

$$xx+yy=cc+bx\pm c\sqrt{(cc+2bx)}$$

atque

$$y=\sqrt{(cc+bx-xx\pm c\sqrt{(cc+2bx)})}.$$

414. Dantur ergo innumerabiles lineae curvae, quae a rectis per punctum *C* ductis ita in duobus punctis *M* et *N* secantur, ut intervallum *MN* perpetuo sit constans. Ac primo quidem patet huic conditioni satisfacere circulum in *C* centrum habentem, erit enim tum perpetuo intervallum $MN=$ diametro circuli; prodit autem circulus ex aequationibus generalibus, si ponatur $M=0$. Tum vero post circulum satisfaciunt lineae quarti ordinis hac aequatione

$$aa(xx+yy)=4(xx+yy-bx)^2$$

atque hac

$$aaxx=(xx+yy)(2x-2b)^2$$

contentae; ad quarum formam cognoscendam expediet ad aequationem inter *z* et ang. φ regredi. Cum igitur sit $xx+yy=zz$ et $x=z\cdot\cos.\varphi$ et $y=z\cdot\sin.\varphi$, posito $a=2c$ erit primo

$$cczz=(zz-bz\cdot\cos.\varphi)^2 \text{ seu } b\cdot\cos.\varphi\pm c=z,$$

tumque

$$cc(\cos.\varphi)^2=\left(z\cdot\cos.\varphi-b\right)^2 \quad \text{seu} \quad z=\frac{b}{\cos.\varphi}\pm c,$$

ex quibus facilis curvarum constructio nascitur.

415. Ad curvam (Fig. 83, 84, 85) enim aequatione $z=b\cdot\cos.\varphi\pm c$ contentam construendam per *C* ducatur recta *ACB,* in qua sumatur $CD=b$,

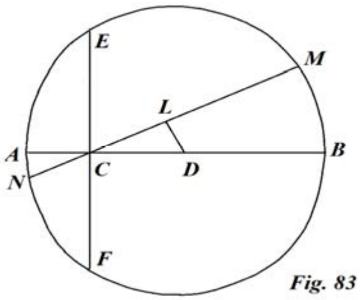

Fig. 83

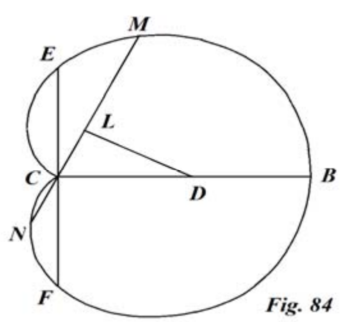

Fig. 84

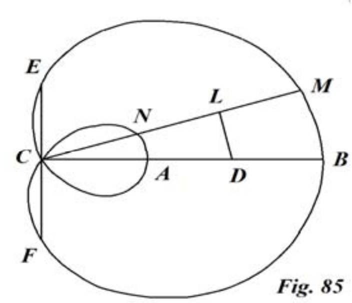

Fig. 85

et ex *D* sumatur utrinque $DA = DB = c$, erunt primo puncta *A* et *B* in curva quaesita. Tum, ducta quavis recta *NCM* per *C,* ex D in eam demittatur perpendiculum *DL* et ab *L* utrinque sumatur $LM = LN = c$, erunt puncta *M* et *N* in quaesita curva ideoque perpetuo intervallum $MN = 2c$, uti quaestio postulat.

Hic notandum est, si fuerit $CD = b$ minor quam *c,* curvam (Fig. 83) in *C* habituram esse punctum coniugatum.

Sin autem sit $b = c$, curva (Fig. 84) in *C* cuspide erit praedita, evanescente intervallo *AC.*

Denique, si sit *b* maior quam *c,* punctum *A* (Fig. 85) inter *C* et *B* cadet curvaque in *C* habebit nodum seu punctum duplex. Ceterum harum curvarum diameter erit recta *ACB,* et quae huic normaliter insistit *ECF,* erit $= 2c$.

416. Praeter has curvas in se redeuntes ex linearum ordine quarto etiam satisfaciunt ex eodem ordine aliae in infinitum excurrentes, quae hac aequatione

$z = \frac{b}{\cos.\varphi} \pm c$ continentur.

Quarum constructio ita se habebit: ducta (Fig. 86) per *C* recta principali *CAB* sumatur $CD = b$ capiaturque $DA = DB = c$, erunt puncta *A* et *B* in curva. Deinde, per *D* ducatur normalis *EDF* et acta recta quacunque *CL* erit $CL = \frac{b}{\cos.\varphi}$, vocato angulo

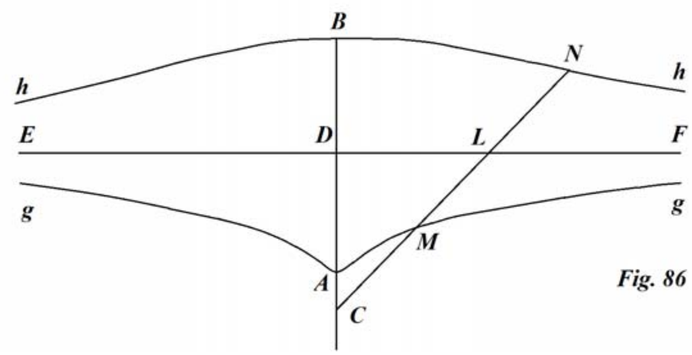

Fig. 86

$DCL = \varphi$. Tum perpetuo abscindatur $LM = LN = c$, atque puncta *M* et *N* determinabunt curvam quaesitam. Ex hac autem constructione perspicuum est curvam sic descriptam esse *conchoidem* Veterum, polum *C* habentem et asymptotam rectam *EF,* ad quam quatuor curvae rami in infinitum convergunt. Vocatur autem portio *hBh conchois exterior* et *gAg interior,* praeter quas partes in *C* punctum existit coniugatum.

417. Hae curvae ex linearum ordine quarto satisfaciunt. Facile autem erit curvas altiorum ordinum, quot libuerit, exhibere. Quodsi enim fuerit *P functio* impar sinus et cosinus anguli φ, tum ista aequatio $z = bP \pm c$ praebebit curvam continuam, quam omnes rectae per *C* ductae ita in duobus punctis *M* et *N* secabunt, ut intervallum *MN* futurum sit constans $= 2c$. Referri autem hae curvae omnes poterunt ad genus conchoidalium, loco rectae *EF* directricis substituendo lineam quamcunque curvam aequatione $z = bP$ contentam. Supra autem vidimus hanc aequationem in se complecti lineas curvas, quae a rectis per punctum *C* ductis nonnisi in uno puncto secentur. Quare, ob intervallum *c* arbitrarium, ex unaquaque curva $z = bP$ innumerabiles curvae ad praesens institutum accommodatae describi poterunt.

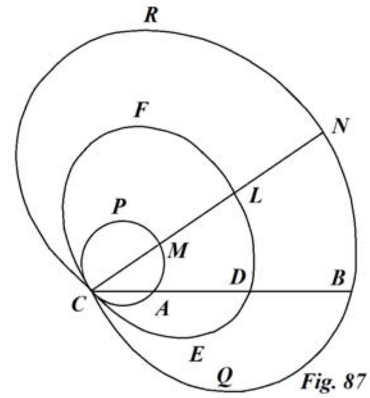

Fig. 87

418. Sumatur scilicet pro lubitu curva *CEDLF* (Fig. 87), quae ab omnibus rectis per punctum *C* ductis in unicis tantum punctis *D, L* secetur. Tum in his singulis rectis *CL* productis utrinque ab *L* capiantur intervalla aequalia $LM = LN = c$ eruntque puncta *M* et *N* in curva quaesita. Sic igitur motu continuo describi poterit curva *AMPCQBNRC,* quae a singulis rectis per *C* ductis in binis punctis *M* et *N* ita intersecabitur, ut intervallum *MN* sit perpetuo $= 2c$. Ubi notandum est, si curva *CEDF* fuerit circulus per punctum *C* ductus, tum curvam descriptam fore eandem lineam quarti ordinis, quam primo invenimus paragrapho 414.

419. Sic igitur satisfecimus quaestioni, qua quaerebantur lineae curvae *AMN* a rectis per punctum *C* ductis ita secandae in duobus punctis *M* et *N,* ut esset $CN - CM$ seu $CM^2 - 2CM \cdot CN + CN^2$ perpetuo quantitas constans. Paucis igitur adhuc evolvamus casum (Fig. 81), quo

$$CM^2 + CM \cdot CN + CN^2$$

debet esse quantitas constans. In paragrapho 412 ergo poni debet $n = 1$ sicque nascetur vel ista aequatio

$$aaLL = (xx + yy)(LL - LM + MM)$$

existente *L* functione $m+1$ et *M* functione *m* dimensionum ipsarum *x* et *y*, vel orietur haec altera aequatio

$$aa(xx + yy)NN = (xx + yy)^2 NN - (xx + yy)MN + MM\ ,$$

in qua *M* est functio homogenea una dimensione superior ipsarum *x* et *y,* quam function.

420. Primum quidem perspicuum est, si ponatur $M = C$, prodire circulum, cuius centrum in puncto *C* sit constitutum; qui, cum omnes rectae ex *C* ad curvam ductae sint aequales, etiam omnibus huius generis quaestionibus satisfacit. Pro praesenti autem casu post circulum curvae simplicissimae prodibunt, si in priori aequatione ponatur $M = b$ et $L = x$, eritque

$$aaxx = (xx + yy)(xx - bx + bb)$$

sive

$$yy = \frac{xx(aa - bb + bx - xx)}{bb - bx + xx}.$$

Quodsi autem in altera aequatione ponatur $N=1$ et $M=bx$, habebitur quoque linea quarti ordinis

$$aa(xx+yy)=(xx+yy)^2-bx(xx+yy)+bbxx$$

seu

$$xx+yy=\tfrac{1}{2}bx+\tfrac{1}{2}aa\pm\sqrt{(\tfrac{1}{4}a^4+\tfrac{1}{2}aabx-\tfrac{3}{4}bbxx)}\,,$$

quae pariter ac prior quaestioni satisfaciet.

421. His expeditis quaestionibus consideremus altiores potestates binorum ipsius z valorum ex aequatione $zz-Pz+Q=0$ existente

$$P=\frac{Mz}{L} \text{ et } Q=\frac{Nzz}{L}$$

ubi L functionem homogeneam $n+2, M$, $n+1$ et N, n dimensionum ipsarum x et y significat; estque $x=$ abscissae CP et $y=$ applicatae PM. Proposita igitur sit quaestio, qua binae intersectiones M et N eius indolis requiruntur, ut sit $CM^3+CN^3=a^3$. Cum ergo sit ex aequationis

$$zz-Pz+Q=0$$

natura,

$$CM^3+CN^3=P^3-3PQ\,,$$

debebit esse $P^3-3PQ=a^3$; quae aequatio, cum P^3 et PQ sint quantitates irrationales, locum habere nequit. Huic ergo quaestioni in stricto sensu prorsus satisfieri non potest; sin autem numerus intersectionum non spectetur, etiamsi duabus plures prodeant, tum quidem infinitis modis curvae satisfacientes inveniri possunt ponendo $Q=\dfrac{P^3-a^3}{3P}$ et pro P capiendo functionem quamcunque sinus et cosinus anguli $ACM=\varphi$.

422. Sin autem eiusmodi curvae requirantur, in quibus sit

$$CM^4+CN^4=a^4\,,$$

tum poni debebit

$$P^4-4PPQ+2QQ=a^4\,;$$

quae aequatio, cum nulla insit irrationalitas, nullam involvit contradictionem.

Debebit ergo esse

$$Q = PP + \sqrt{(\tfrac{1}{2}P^4 + \tfrac{1}{2}a^4)}$$

quae functio non obstante signo radicali tanquam uniformis spectari potest, quia, si $\sqrt{(\tfrac{1}{2}P^4 + \tfrac{1}{2}a^4)}$ sumeretur positive, pro z valores imaginarii resultarent. Erit ergo

$$\frac{Nzz}{L} = \frac{MMzz}{LL} - \sqrt{\left(\frac{M^4z^4}{2L^4} + \tfrac{1}{2}a^4\right)};$$

et, cum pro curva sit $L - M + N = 0$ seu

$$zz - \frac{Mzz}{L} + \frac{Nzz}{L} = 0,$$

erit

$$zz - \frac{Mzz}{L} + \frac{MMzz}{LL} - \sqrt{\left(\frac{M^4z^4}{2L^4} + \tfrac{1}{2}a^4\right)} = 0.$$

Consequenter sublata irrationalitate erit

$$\frac{z^4}{L^4}\left(LL - LM + MM\right)^2 = \frac{M^4z^4}{2L^4} + \tfrac{1}{2}a^4$$

seu

$$(xx + yy)^2\left(2(LL - LM + MM)^2 - M^4\right) = a^4L^4,$$

quae omnes curvas satisfacientes in se complectitur.

423. Alio faciliori modo, uti supra paragrapho 372, haec et similes quaestiones resolvi poterunt. Cum enim sit $CM \cdot CN = Q$, si altera ipsarum CM et CN dicatur $= z$, erit altera $= \dfrac{Q}{z} = \dfrac{Nz}{L}$ ob $Q = \dfrac{Nzz}{L}$. Quare, si debeat esse

$$CM^n + CN^n = a^n$$

fiet

$$z^n + \frac{N^nz^n}{L^n} = a^n \text{ ideoque } z^n = \frac{a^nL^n}{L^n + N^n},$$

quae aequatio, si fuerit n numerus par, iam est rationalis quaesitoque satisfaciet.

At si sit *n* numerus impar, ad irrationalitatem tollendam quadrata sumi debent; quo fit, ut numerus intersectionum duplicetur sicque curva oriatur non eo sensu satisfaciens, uti desideratur. Sic si debeat esse

$$CM^2 + CN^2 = a^2$$

fiet

$$zz = xx + yy = \frac{aaLL}{LL + NN},$$

quae convenit cum supra (§ 410) inventa

$$xx + yy = \frac{aaLL}{(L - M)^2 + L^2}$$

ob $L - M + N = 0$. Generaliter ergo, si debeat esse $CM^n + CN^n = a^n$ fueritque *n* numerus par, obtinebitur ista aequatio

$$z^n = (xx + yy)^{\frac{n}{2}} = \frac{a^n L^n}{L^n + N^n} = \frac{a^n L^n}{L^n + (L - M)^n}$$

existentibus *L* functione $m + 2$ dimensionum, *M* functione $m + 1$ dimensionum et *N* functione *m* dimensionum ipsarum *x* et *y*.

424. Haec eadem solutio etiam ex consideratione summae $CM + CN = P$ eruitur. Si enim altera ipsarum *CM* et *CN* ponatur $= z$, erit altera $= P - z$. Hinc, si $CM^n + CN^n$ debeat esse constans, erit $z^n + (P - z)^n = a^n$. Vidimus autem esse debere

$$P = \frac{Mz}{L} \text{ et } Q = \frac{Nzz}{L},$$

ita ut sit $L - M + N = 0$; ex quo erit

$$z^n + \frac{z^n (M - L)^n}{L^n} = a^n$$

seu

$$z^n = \frac{a^n L^n}{L^n + (M - L)^n} \text{ vel } z^n = \frac{a^n L^n}{L^n + N^n},$$

vel eliminando *L* erit

$$z^n = \frac{a^n\left(M-N\right)^n}{\left(M-N\right)^n + N^n}$$

Hae aequationes, si fuerit *n* numerus par, conditionem propositam adaequate adimplent. At si *n* sit numerus impar, dabuntur quidem duae intersectiones *M* et *N*, ut sit $CM^n + CN^n = a^n$; ; at praeter has habebuntur duae aliae intersectiones eadem proprietate gaudentes, ita ut quaelibet recta per *C* ducta bis proprietatem propositam involvat.

425. His expositis facile erit quaestiones alias maxime difficiles resolvere; debeat enim curva inveniri, quae ab omnibus rectis per *C* ductis ita in duobus punctis *M* et *N* secetur, ut sit

$$CM^n + CN^n + \alpha CM \cdot CN(CM^{n-2} + CN^{n-2}) + \beta \cdot CM^2 \cdot CN^2 \ (CM^{n-4} + CN^{n-4}) + \text{ etc.}$$

quantitas constans $= a^n$. Ponatur alter valor $CM = z$, erit alter

$$CN = \frac{Q}{z} = \frac{Nz}{L};$$

quibus valoribus substitutis orietur ista aequatio, qua natura curvae exprimitur,

$$z^n\left(L^n + N^n + \alpha LN(L^{n-2} + N^{n-2}) + \beta LLNN(L^{n-4} + N^{n-4}) + \text{ etc.}\right) = \alpha^n L^n$$

Est autem $L - M + N = 0$ atque *L, M* et *N* sunt functiones homogeneae ipsarum *x* et *y* dimensionum $m+2,\ m+1$ et *m,* uti supra descripsimus; unde erit vel $L = M - N$ vel $N = M - L$ sicque infinitae solutiones hinc deducentur.

426. Pergamus iam ad curvas investigandas, quae a singulis rectis per punctum fixum *C* ductis in tribus punctis secentur. Huiusmodi ergo curvarum natura exprimetur hac aequatione generali

$$z^3 - Pzz + Qz - R = 0,$$

ubi *z* denotat distantiam cuiusque curvae puncti a *C,* et *P, Q, R* sunt functiones anguli $ACM = \varphi$ eiusve sinus et cosinus. Per easdem autem rationes, quas supra allegavimus, apparet, ne plures quam tres intersectiones prodeant, *P* et *R* esse debere functiones impares ipsius $\sin.\varphi$ et $\cos.\varphi$, verum *Q* statui debere functionem parem. Quodsi ergo ponantur coordinatae orthogonales $CP = x,\ PM = y$, ut sit $xx + yy = zz$, atque denotent *K, L, M* et *N* functiones homogeneas $n+3, n+2, n+1$ et *n* dimensionum ipsarum *x* et *y,* fore

$$P = \frac{Lz}{K},\ Q = \frac{Mzz}{K},\ \text{et}\ R = \frac{Nz^3}{K};$$

ideoque inter coordinatas orthogonales *x* et *y* habebitur pro huiusmodi curvis ista aequatio generalis

$$K - L + M - N = 0;$$

ex qua patet punctum *C* fore curvae punctum totuplex, quot index *n* contineat unitates.

427. Primum ergo huc pertinent omnes lineae tertii ordinis, ubicunque punctum *C* extra curvam capiatur. Deinde, in hac aequatione continentur omnes lineae quarti ordinis, dummodo punctum *C* in ipsa curva accipiatur. Tertio omnes lineae quinti ordinis, in quibus datur punctum duplex, huc referuntur, si modo punctum *C* in earum puncto duplici constituatur. Similique modo lineae altiorum ordinum hanc conditionem implebunt, si habeantur puncta multiplicia tanti exponentis, quot *n* contineat unitates, si $n+3$ exponat ordinem, ad quem aequatio pertineat.

428. Sint *p, q, r* tres illi valores ipsius *z,* quos obtinet ex aequatione

$$z^3 - Pzz + Qz - R = 0$$

pro quovis valore anguli $CAM = \varphi$; eritque ex natura aequationum

$$P = p+q+r,\ Q = pq+pr+qr\ \text{ et }\ R = pqr.$$

Cum iam *P* et *R* per *x* et *y* rationaliter exprimi nequeant, manifestum est eiusmodi curvas exhiberi non posse, in quibus sit vel $p+q+r$ vel pqr quantitas constans, neque adeo ulla functio impar ipsarum *p, q* et *r* constanti aequalis poni poterit. Pares autem functiones sine ulla difficultate constantem valorem obtinere poterunt. Sic si requiratur, ut sit

$$pq+pr+qr = aa,\ \text{erit}\ Q = \frac{Mzz}{K} = aa$$

ideoque $M(xx+yy) = aaK$*;* qui valor in aequatione $K - L + M - N = 0$ substitutus dabit aequationem generalem omnes curvas hac proprietate praeditas in se complectentem:

$$M(xx+yy) - aaL + aaM - aaN = 0$$

vel eliminando *M* hanc

$$(xx+yy)K - (xx+yy)L + aaK - (xx+yy)N = 0.$$

429. Pari modo aliae similes quaestiones facile resolventur. Uti quaeratur curva, quae a rectis per C ductis ita in tribus punctis secetur, ut sit

$$pp+qq+rr=aa.$$

Cum enim sit

$$pp+qq+rr=PP-2Q \text{ et } P=\frac{Lz}{K}$$

atque

$$Q=\frac{Mzz}{K},$$

fiet

$$\frac{LLzz}{KK}-\frac{2Mzz}{K}=aa$$

seu

$$(xx+yy)LL-2(xx+yy)KM=aaKK.$$

At pro curvis tres intersectiones admittentibus habetur haec aequatio generalis $K-L+M-N=0$, cuius natura in hoc consistit, ut maximus ipsarum x et y dimensionum numerus ternario superet minimum. Quo igitur huiusmodi obtineatur aequatio, simulque sit

$$(xx+yy)LL-2(xx+yy)KM=aaKK,$$

multiplicetur illa aequatio per $2\left(xx+yy\right)K$, ut eliminari possit M, atque prodibit haec aequatio generalis casui proposito satisfaciens

$$2\left(xx+yy\right)KK-2\left(xx+yy\right)KL+\left(xx+yy\right)LL-aaKK-2\left(xx+yy\right)KN=0.$$

Membrum enim, in quo plurimae insunt dimensiones, est $2\left(xx+yy\right)KK$ continetque $2n+8$ dimensiones ipsarum x et y; atque membrum infimum est $2\left(xx+yy\right)KN$ et continet $2n+5$ dimensiones, uti natura rei postulat.

430. Quoniam ergo neque summum neque imum membrum evanescere potest, ponamus ad curvam simplicissimam inveniendam $n=0$ sitque $N=b^3$, $K=x(xx+yy)$ et $L=0$, atque prodibit haec aequatio

$$2(xx+yy)^3xx-aaxx(xx+yy)^2-2b^3x(xx+yy)^2=0,$$

quae per $2x(xx+yy)^2$ divisa praebet hanc

$$x(xx+yy)-\tfrac{1}{2}aax-b^3=0,$$

quae pertinet ad ordinem tertium. Sin autem non sit $L=0$ sed

$$L=2c(xx+yy),$$

prodibit aequatio ordinis quarti

$$xx(xx+yy)-2cx(xx+yy)+2cc(xx+yy)-\tfrac{1}{2}aaxx-b^3x=0$$

seu

$$xx(xx+\ yy)+(2c-\ x)^2(xx+yy)=aaxx+2b^3x.$$

Simili autem modo ex altioribus ordinibus plurimae aliae curvae quaestioni satisfacientes eruentur.

431. Deinde etiam curvae inveniri poterunt eae, in quibus sit $p^4+q^4+r^4$ quantitas constans. Cum enim sit

$$p^4+q^4+r^4=P^4-4PPQ+2QQ+4PR,$$

poni debebit

$$P^4-4PPQ+2QQ+4PR=c^4.$$

Erit ergo

$$z^4(L^4-4KLLM+2KKMM+4KKLN)=c^4K^4$$

ideoque

$$4KKLNz^4=c^4K^4-z^4(L^4-4KLLM+2KKMM),$$

unde valor ipsius N in aequatione $K-L+M-N=0$ substitutus dabit aequationem generalem pro curvis huic conditioni satisfacientibus.

432. Poterit autem simul et huic conditioni $p^4+q^4+r^4=c^4$ et praecedenti $pp+qq+rr=aa$ satisfieri. Per hanc enim esse debet

$$zzLL-2zzKM=aaKK;$$

unde fit

$$2zzKM=zzLL-aaKK.$$

Deinde, cum sit

$$4KKLNz^4=c^4K^4-L^4z^4+4KLLMz^4-2KKMMz^4,$$

erit

$$4KKLNz^4 = c^4K^4 + L^4z^4 - 2aaKKLLzz - 2KKMMz^4$$

et

$$4KKLMz^4 = 2KL^3z^4 - 2aaK^3Lzz.$$

Substituantur hi valores loco M et N in aequatione $K - L + M - N = 0$ seu

$$4K^3Lz^4 - 4KKLLz^4 + 4KKLMz^4 - 4KKLNz^4 = 0,$$

atque prodibit haec aequatio pro curva

$$4K^3Lz^4 - 4KKLLz^4 + 2KL^3z^4 - 2aaK^3Lzz - c^4K^4 - L^4z^4 + 2aaKKLLzz + 2KKMMz^4 = 0.$$

At ob

$$KMzz = \tfrac{1}{2}LLzz - \tfrac{1}{2}aaKK$$

erit

$$2KKMMz^4 = \tfrac{1}{2}L4z^4 - aaKKLLzz + \tfrac{1}{2}a^4K^4,$$

ideoque pro curvis quaesitis habebitur haec aequatio generalis

$$8K^3Lz^4 - 8KKLLz^4 + 4KL^3z^4 - 4aaK^3Lzz - 2c^4K^4 - L^4z^4 + 2aaKKLLzz + a^4K^4 = 0.$$

433. Quia K debet esse functio homogenea ipsarum x et y una dimensione altior quam L, curva simplicissima, in qua tres intersectiones exhibeant simul $pp + qq + rr = aa$ et $p^4 + q^4 + r^4 = c^4$, prodibit, si ponatur $K = zz$ et $L = bx$; erit ergo

$$8bxz^6 - 8bbxxz^4 + 4b^3x^3zz - 4aabxz^4 - 2c^4z^4 - b^4x^4 + 2aabbxxzz + a^4z^4 = 0,$$

quae ob $zz = xx + yy$ est rationalis praebetque lineam ordinis septimi, cuius C est punctum quadruplex. Alia autem linea septimi ordinis satisfaciens obtinebitur, si ponatur $K = x$ et $L = b$; erit enim

$$8bx^3z^4 - 8bbxxz^4 + 4b^3xz^4 - 4aabx^3zz - 2c^4x^4 - b^4z^4 + 2aabbxxzz + a^4x^4 = 0$$

seu

$$z^4 = \frac{4aabx^3zz - 2aabbxxzz + 2c^4x^4 - a^4x^4}{8bx^3 - 8bbxx + 4b^3x - b^4}$$

Unde fit

$$zz = \frac{2aabx^3 - aabbxx \pm xxy\sqrt{(2bx-bb)\left(2c^4(bb-2bx+4xx) - 2a^4(bb-2bx+2xx)\right)}}{b(2x-b)(4xx-2bx+bb)}.$$

434. Iam ulterius progredi liceret ad curvas, quae a rectis per punctum C ductis in quatuor punctis intersecentur; atque ex iis illae inveniri possent, quae datis proprietatibus sint praeditae. Verum, si ad praecepta in praecedentibus tradita attendamus, nulla prorsus supererit difficultas omniaque, quae in hoc genere desiderari poterunt, sine ullo fere labore vel expedientur vel, nisi quaestio solutionem genuinam admittat, hoc ipsum statim cognoscetur. Quamobrem huic materiae amplius non immorabor, ad aliud argumentum ad cognitionem linearum curvarum pertinens progressurus.