CHAPTER XVI

FINDING CURVES FROM THE GIVEN PROPERTIES OF APPLIED LINES

364. *P* and *Q* shall be any rational functions of the abscissa *x* and the nature of the curve may be expressed by this equation $yy - Py + Q = 0$. Hence therefore either none or two applied lines will correspond to whichever abscissa *x*; but the sum of these two applied lines will be $= P$ and the product $= Q$. Therefore if *P* were a constant quantity, the sum of the two corresponding applied lines will be a constant quantity for individual abscissas and the curve will have a diameter; but this also arises, if there were $P = a + nx$; for then the right line present in this equation $z = \frac{1}{2}a + \frac{1}{2}nx$ will be a diameter, with this name taken in a wider sense, thus so that oblique coordinates may not be excluded. But if *Q* were a constant quantity, then the rectangle of the two applied lines will be constant everywhere ; therefore the axis will never be able to be cut by the curve. But if there shall be $Q = \alpha + \beta x + \gamma xx$ and this expression may have two real factors, then the axis will be cut by the curve in two points and *Q* will be a multiple of the rectangle formed from the parts of the axis, and thus the rectangle formed by the applied lines will itself be in a constant ratio to the parts of the axis.

365. Therefore these properties, which we have observed to arise above in conic sections, come about in innumerable other curved lines. Thus the constant magnitude of the rectangles formed by two applied lines corresponding to the same abscissas, which relation we have seen the hyperbola to entertain to the asymptote, itself is common to all the curves present in this equation $yy - Py \pm aa = 0$. Then, with the right line *EF* taken (Fig. 19) for the axis cut in the two points *E* and *F*, since with conic sections the rectangle $PM \cdot PN$ shall have a constant ratio to the rectangle $PE \cdot PF$, this property for conic sections will be common to all the curves described by this equation :

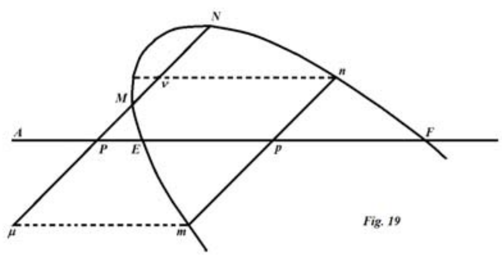

Fig. 19

$$yy - Py + ax - nxx = 0.$$

But there will be $PM \cdot PN = PE \cdot PF$ or $pm \cdot pn = Ep \cdot pF$, if there were $yy - Py = ax - xx$. Therefore these properties, which a given circle is to be composed from its parts, not only is itself common with an infinitude of curves of higher order, but also can be applied to the remaining conic sections. For let there be $P = b + nx$, and the equation

$$yy - nxy + xx = ax + by,$$

which is for a circle if $n = 0$ and the angle *EPM* is right, will contain the ellipse also, if *nn* shall be less than 4, and the hyperbola, if *nn* shall be greater than 4, and the parabola, if $nn = 4$.

[For at once we have from the *y* discriminant :
$(b+nx)^2 - 4(x^2 - ax) = x^2(n^2-4) + 2x(bn+2a) + b^2$.]

366. Hence we conclude in every conic section *AEBF*, the axes or the principal diameters of which shall be *AB, EF*, if any two right lines *pq* and *mn* (Fig. 77) may be drawn, which may be inclined to the principal axes at half a right angle, thus these to be mutually intersecting at *h*, so that there shall be $mh \cdot nh = ph \cdot qh$. Which indeed is evident from the first properties known : for if the right lines *PQ* and *MN* may be drawn through the centre *C* at half a right angle to the principal axis, they will be equal to each other and thus $MC \cdot NC = PC \cdot QC$; whereby, since by the same law all the right lines may themselves be cut by these parallel, there will be also $mh \cdot nh = ph \cdot qh$. But indeed also this is understood, only if the right lines *MN* and *PQ* may be drawn thus, so that they may be inclined equally to the same principal axis, or so that there shall be $PCA = NCA$, on account of $CP = CN$ all the right lines parallel to these thus mutually cut each other, so that the rectangles of the parts shall be equal, evidently so that there shall be $mh \cdot hn = ph \cdot hq$.

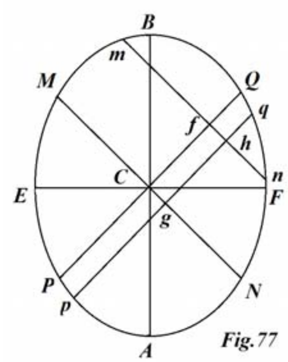

Fig.77

367. From these premises we may consider other questions about the two applied lines corresponding to this abscissa from the equation $yy - Py + Q = 0$. Let AP (Fig. 78) be the abscissa *x*, to which the two applied lines *PM* and *PN* may correspond *;* and in the first place all the curves of this nature, so that $PM^2 + PN^2$ shall be a constant quantity $= aa$. Since there shall be

$$PM + PN = P \text{ and } PM \cdot PN = Q,$$

there will be

$$PM^2 + PN^2 = PP - 2Q$$

and the condition sought is satisfied, if there were

$$PP - 2Q = aa \text{ or } Q = \frac{PP - aa}{2};$$

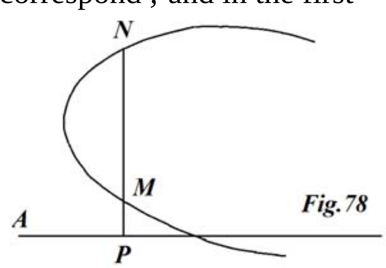

Fig.78

from which this equation will be obtained for the curve sought :

$$yy - Py + \frac{PP - aa}{2} = 0 .$$

But if on putting $P = 2nx$, a conic section will be produced satisfying

$$yy - 2nxy + 2nnxx - \tfrac{1}{2}aa = 0 ,$$

which is the equation for an ellipse, with the abscissas computed from the centre.

368. Hence this elegant property of ellipses follows. If the parallelogram *GHIK* may be described about some two conjugate diameters *AB* and *EF* of an ellipse (Fig. 79), of which the sides are tangents to the ellipse at the points *A*, *B*, *E*, *F*, the diagonals *GK* and *HI* of which parallelogram thus will cut all the chords *MN* parallel to either diameter *EF* at *P* and *p*, so that the sum of the squares $PM^2 + PN^2$ or $pM^2 + pN^2$ shall be constant always, certainly equal to $2CE^2$. In a similar manner with the chord *RS* drawn parallel to the other diameter *AB* there will be

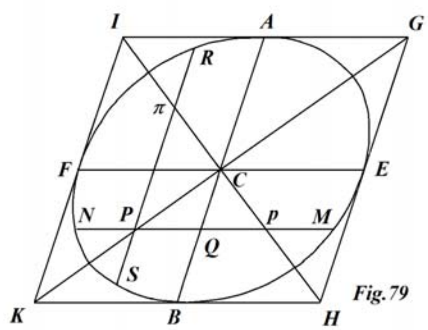

Fig. 79

$$PR^2 + PS^2 = \pi R^2 + \pi S^2 = 2CA^2 .$$

For on putting

$CA = CB = a,\ CE = CF = b,\ CQ = t,\ QM = u$, there will be

$$aauu + bbtt = aabb .$$

Now there is $a : b = CQ(t) : PQ$ and *GP* to *CQ* is in a given ratio, such as *m*:1. Whereby, on putting $CP = x,\ PM = y$, there will be

$$x = mt \text{ and } y = u + \frac{bt}{a}$$

or

$$t = \frac{x}{m} \text{ and } u = y - \frac{bx}{ma}$$

with which values substituted this equation will arise

$$aayy - \frac{2abxy}{m} + \frac{2bbxx}{mm} = aabb.$$

Let $\frac{b}{ma} = n$, there will be

$$yy - 2nxy + 2nnxx = bb,$$

which is the equation found before showing the constant magnitude $PM^2 + PN^2$.

369. Now curves may be sought, in which the sum of the cubes $PM^3 + PN^3$ shall always be a constant quantity (Fig. 78). Since there shall be $PM + PN = P$, there will be

$$PM^3 + PN^3 = P^3 - 3P \cdot Q;$$

whereby, if on putting $PM^3 + PN^3 = a^3$, there will be $Q = \frac{P^3 - a^3}{3P}$, and thus the general equation for these curves will be the :

$$yy - Py + \frac{1}{3}PP - \frac{a^3}{3P} = 0,$$

were it is permitted to substitute any general function of *x*. Therefore the simplest curve having this property will be a line of the third order, which on putting $P = 3nx$ and $a = 3nb$, will be expressed by this equation

$$xyy - 3nxxy + 3nnx^3 - 3nnb^3 = 0,$$

which belongs to the second kind, following the enumeration made above.

370. In a similar manner, if it must be effected that $PM^4 + PN^4$ shall be constant, because there is

$$PM^4 + PN^4 = P^4 - 4P \cdot P \cdot Q + 2Q \cdot Q,$$

[Recall that, from the roots of the quadratic in *y*,

α and β, $\alpha^4 + \beta^4 = (\alpha+\beta)^4 - 4\alpha^3\beta - 6\alpha^2\beta^2 - 4\alpha^3\beta = (\alpha+\beta)^4 - 4(\alpha+\beta)^2\alpha\beta + 2\alpha^2\beta^2$;]

the quantity *Q* thus must be determined by *P*, so that there shall be

$$P^4 - 4PPQ + 2QQ = a^4 \text{ or } Q = PP + \sqrt{(\tfrac{1}{2}P^4 + \tfrac{1}{2}a^4)}.$$

Truly because both *P* and *Q* must be rational or uniform functions of *x*, lest *y* shall be induced to have more than two values for whatever the abscissa *x*, the quantity $\sqrt{(\frac{1}{2}P^4+\frac{1}{2}a^4)}$ must be rational ; since that cannot arise, the function *Q* will be always of two forms and thus the applied line *y* will return a four-fold function. Truly from the equation $yy-Py+Q=0$ is elicited :

$$y=\tfrac{1}{2}P\pm\sqrt{\left(-\tfrac{3}{4}PP\pm\sqrt{(\tfrac{1}{2}P^4+\tfrac{1}{2}a^4)}\right)},$$

from which it is apparent the applied line *y* cannot be real, unless $\sqrt{(\frac{1}{2}P^4+\frac{1}{2}a^4)}$ may be taken positive; whereby, not withstanding the biform nature of the function *Q*, the applied line *y* nowhere will have more than two values, of which the sum of the squares will be constant, just as the nature of the question required.

371. But if again a curve of this kind may be required, so that the sum of the corresponding fifth powers of the two values of *y* may be considered constant for each of the abscissas *x* or so that there shall be $PM^5+PN^5=a^5$, the equation must become

$$P^5-5P^3Q+5PQQ=a^5.$$

Therefore since from the equation for the curve $yy-\ Py+Q=0$ there shall be $Q=-yy+Py$, there will be

$$P^5-5P^4y+10P^3yy-10PPy^3+5Py^4=a^5$$

or

$$\left(P-y\right)^5+y^5=a^5.$$

In the same manner there will be found, if there must be $PM^6+PN^6=a^6$, this equation

$$\left(P-y\right)^6+y^6=a^6$$

And generally, if the curve is sought, in which there shall be $PM^n+PN^n=a^n$, this equation itself will be found :

$$\left(P-y\right)^n+y^n=a^n,$$

where for *P* some uniform function of *x* can be taken as wished. But the account of this equation is brought out ; for since the sum of the two applied lines shall be $=P$, if one shall be *y*, the other will be $=P-y$, from which at once there becomes

$$(P-y)^n + y^n = a^n.$$

372. But if P may be eliminated in place of Q, by putting into the equations, by which the relation between P and Q may be satisfied, $P = \frac{yy+Q}{y}$, the equation will arise for $PM^n + PN^n = a^n$: namely

$$y^n + \frac{Q^n}{y^n} = a^n.$$

For since the product of the applied right lines shall be $= Q$, if one may be put $= y$, the other will be $= \frac{Q}{y}$; from which the equation found at once flows. Therefore for the curve, in which there shall be $PM^n + PN^n = a^n$, we have obtained two general equations, the one $(P-y)^n + y^n$ the other $y^n + \frac{Q^n}{y^n} = a^n$, from the latter of which there emerges

$$y^{2n} = a^n y^n - Q^n \quad \text{and} \quad y^n = \tfrac{1}{2}a^n \pm \sqrt{(\tfrac{1}{4}a^{2n} - Q^n)},$$

thus so that there shall be

$$y = \sqrt[n]{\left(\tfrac{1}{2}a^n \pm \sqrt{(\tfrac{1}{4}a^{2n} - Q^n)}\right)},$$

which is a function of two forms only and for whatever abscissa does not show more than two applied lines, provided Q^n were a rational or uniform function x. But the first equation $y^n + (P-y)^n = a^n$ enjoys this prerogative, as the number of dimensions shall be less.

373. Truly not only do these equations solve the question, if n shall be a positive whole number, but also if it shall be either negative or a fraction. Thus

if there should be :	this equation is found :
$\frac{1}{PM}+\frac{1}{PN}=\frac{1}{a}$,	$aP=Py-yy$ or $aQ+ayy=Qy$
$\frac{1}{PM^2}+\frac{1}{PN^2}=\frac{1}{aa}$	$aayy+aa(P-y)^2=yy(P-y)^2$ or $aaQQ+aay^4=QQyy$
$\frac{1}{PM^3}+\frac{1}{PN^3}=\frac{1}{a^3}$	$a^3y^3+a^3(P-y)^3=y^3(P-y)^3$ or $a^3Q^3+a^3y^6=Q^3y^3$

etc.

Moreover, for fractional exponents, thus these will be found :

if there should be :	this equation is found :
$\sqrt{PM}+\sqrt{PN}=\sqrt{a}$	$\sqrt{y}+\sqrt{P-y}=\sqrt{a}$ or $y=\sqrt{ay}-\sqrt{Q}$, which reduced to being rational provides $yy-Py+\frac{1}{4}(a-P)^2=0$ or $yy-\left(a-2\sqrt{Q}\right)y+Q=0$
$\sqrt[3]{PM}+\sqrt[3]{PN}=\sqrt[3]{a}$	$\sqrt[3]{y}+\sqrt[3]{(P-y)}=\sqrt[3]{a}$ either $yy-Py+\frac{1}{27a}(a-P)^3=0$ or $\sqrt[3]{y}+\sqrt[3]{\frac{Q}{y}}=\sqrt[3]{a}$ or $yy-(a-3\sqrt[3]{aQ})y+Q=0$

etc.

Therefore in this manner all the algebraic curves, in which there shall be everywhere

$$PM^n + PN^n = a^n$$

are able to be considered by one general equation, whether *n* shall be a positive or negative whole number, or a fraction.

374. Here the matters concerned with the condition which have been explained regarding two applied right lines corresponding to the one abscissa *x*, can be transferred to three applied right lines corresponding to the individual abscissa by the same method. Moreover a general equation for curves, which the individual applied lines cut in three points, is this

$$y^3 - Pyy + Qy - R = 0\,,$$

with the letters *P*, *Q* and *R* denoting any uniform functions of *x*. Let *p*, *q*, *r* be the three applied right lines corresponding to the abscissa *x*, of which one indeed is real always, truly here we consider chiefly these curves in place, in which all three applied right lines are real. But from the nature of the equations there will be $P = p+q+r,\ Q = pq+pr+qr$ and $R = pqr$. Whereby, if the curve may be desired, in which either $p+q+r,\ pq+pr+qr$ or pqr shall be a constant quantity, nothing other is required to be done, except that either *P*, *Q* or *R* may be set up to be constant, with the two remaining staying arbitrary.

375. Hence also the curves can be found, in which the quantity $p^n + q^n + r^n$ shall be constant everywhere ; for indeed by that, which has been discussed in the first book,

$$\begin{aligned}
&p+q+r = P,\\
&p^2+q^2+r^2 = P^2 - 2Q,\\
&p^3+q^3+r^3 = P^3 - 3PQ \quad + 3\mathrm{R},\\
&p^4+q^4+r^4 = P^4 - 4PPQ + 2QQ \quad + 4PR,\\
&p^5+q^5+r^5 = P^5 - 5P^3Q \quad + 5PQQ + 5PPR - 5QR, \text{ etc.}
\end{aligned}$$

Then, if *n* shall be a negative number, putting $z = \dfrac{1}{y}$; there will be

$$z^3 - \frac{Qzz}{R} + \frac{Pz}{R} - \frac{1}{R} = 0$$

and the three roots of this equation are $\dfrac{1}{p},\ \dfrac{1}{q},\ \dfrac{1}{r}$. Hence in a similar manner there will be :

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{Q}{R},$$
$$\frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2}=\frac{QQ-2PR}{RR},$$
$$\frac{1}{p^3}+\frac{1}{q^3}+\frac{1}{r^3}=\frac{Q^3-3PQR+3\mathrm{RR}}{R^3},$$
$$\frac{1}{p^4}+\frac{1}{q^4}+\frac{1}{r^4}=\frac{Q^4-4PQQR+4QRR+2PPRR}{R^4}, \text{ etc.}$$

Therefore an expression put in place equal to the constant quantity will provide a suitable relation between the functions *P, Q* and *R.* And, if with the aid of this equation from the equation $y^3-Pyy+Qy-R=0$ one of these functions *P, Q* or *R* may be eliminated, the equation will be had for the curve sought. Thus, if the curve may be sought, in which there shall be $p^3+q^3+r^3=a^3$, there becomes $P^3-3PQ+3R=a^3$ and, on account of $R=y^3-Pyy+Qy$, this equation will be had

$$3y^3-3Pyy+3Qy+P^3-3PQ=a^3,$$

with satisfaction given for the curves sought.

376. Therefore if n shall be a positive or negative whole number, the solution will be found readily from the formulas given ; but a greater difficulty occurs, if *n* were a fraction. The curved line may be proposed requiring to be found, in which there shall be

$$\sqrt{p}+\sqrt{q}+\sqrt{r}=\sqrt{a}\,.$$

The square on both sides may be taken, and on account of $p + q + r = P$, there will be had

$$P+2\sqrt{pq}+2\sqrt{pr}+2\sqrt{qr} \;=a$$

or

$$\frac{a-P}{2}=\sqrt{pq}+\sqrt{pr}+\sqrt{qr}.$$

The squares may be taken again, and on account of $pq+pr+qr=Q$ the equation becomes

$$\frac{(a-P)^2}{4} = Q + 2\sqrt{ppqr} + 2\sqrt{pqqr} + 2\sqrt{pqrr}$$
$$= Q + 2\ (\sqrt{p} + \sqrt{q} + \sqrt{r})\ \sqrt{pqr} = 2\sqrt{a}R + Q,$$

from which it becomes

$$(a-P)^2 = 4Q + 8\sqrt{a}R \text{ or } Q = \frac{(a-P)^2}{4} - 2\sqrt{a}R.$$

Whereby the curve sought will be contained in this equation :

$$y^3 - Pyy + (\tfrac{1}{4}(a-P)^2 - 2\sqrt{a}R)y - R = 0$$

or (with the irrationality removed, on account of $R = \frac{(aa-2aP+PP-4Q)^2}{64a}$), in this equation

$$y^3 - Pyy + Qy - \frac{(aa-2aP+PP-4Q)^2}{64a} = 0\,.$$

377. But this operation may become exceedingly troublesome, if the higher powers of roots may be proposed; therefore another way will have to be undertaken, which will be seen from this example. Truly the curve may be sought, in which there shall be

$$\sqrt[3]{p} + \sqrt[3]{q} + \sqrt[3]{r} = \sqrt[3]{a}\,.$$

Putting

$$\sqrt[3]{pq} + \sqrt[3]{pr} + \sqrt[3]{qr} = v$$

and, since there shall be $\sqrt[3]{pqr} = \sqrt[3]{R}$, the first equation becomes [on squaring]

$$\sqrt[3]{pp} + \sqrt[3]{qq} + \sqrt[3]{rr} = \sqrt[3]{aa} - 2v$$

and [on cubing]

$$p + q + r = a - 3v\sqrt[3]{a} + 3\sqrt[3]{R} = P\,.$$

[Recall that the powers of the roots may be expressed in terms of the coefficients of the equation, which are symmetric functions of the roots :

$$(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3=a^3+3a^2b+3ab^2+b^3+3a^2c+3b^2c+6abc+3ac^2+3bc^2$$
$$=a^3+b^3+c^3+3(a+b+c)(ab+bc+ca)-3abc.$$

Hence : $a^3+b^3+c^3=(a+b+c)^3-3(a+b+c)(ab+bc+ca)+3abc,$ etc.]

Then

$$\sqrt[3]{ppqq}+\sqrt[3]{pprr}+\sqrt[3]{qqrr}=vv-2\sqrt[3]{a}R$$

and

$$pq+pr+qr=Q=v^3-3v\sqrt[3]{aR}+3\sqrt[3]{RR}\,.$$

Now with suitable values found for P and Q, on taking some function of x for v, this equation will be obtained for the curve sought

$$y^3-(a-3v\sqrt{a}+3\sqrt[3]{R})yy+(v^3-3v\sqrt[3]{aR}+3\sqrt[3]{RR})y-R=0\,.$$

378. Yet a general solution can be put in place without these difficulties standing in the way. For since from the equation $y^3-Pyy+Qy-R=0$, y may denote the three applied lines p, q and r, putting $p=y$, there will be

$$P=y+q+r \text{ and } Q=qy+ry+qr\,,$$

or

$$q+r=P-y \text{ and } qr=Q-y(q+r)=Q-Py+yy\,.$$

Hence the equations will produce

$$q-r=\sqrt{(PP+2Py-3yy-4Q)}$$

and thus

$$q=\tfrac{1}{2}(P-y)+\tfrac{1}{2}\sqrt{(PP+2Py-3yy-4Q)}$$

and

$$r=\tfrac{1}{2}(P-y)-\tfrac{1}{2}\sqrt{(PP+2Py-3yy-4Q)}\,.$$

Therefore when the curve is sought, in which there shall be $p^n+q^n+r^m=a^n$, it will satisfy this equation

$$y^n + (\tfrac{1}{2}(P-y) + \tfrac{1}{2}\sqrt{(PP+2Py-3yy-4Q)})^n$$
$$+ (\tfrac{1}{2}(P-y) - \tfrac{1}{2}\sqrt{(PP+2Py-3yy-4Q)})^n = a^n,$$

which will solve the question equally, whether n were a whole number or a fraction.

379. Innumerable other questions about the condition of these three applied lines can be resolved by the same method: just as, if some function of x may be assumed for a^n ; then also indeed, besides the sum of some powers, other functions of p, q and r can be proposed, provided these quantities thus shall be present equally, so that no variation will arise from an interchange of these. Thus, these three applied lines p, q and r corresponding to the same abscissa x are able to be defined thus, so that the triangle which is formed from these, may have a constant area. For the area of this triangle will be [from Heron's formula]

$$= \tfrac{1}{4}\sqrt{(2ppqq + 2pprr + 2qqrr - p^4 - q^4 - r^4)},$$

which may be put $= aa$. Therefore since there shall be

$$p^4 + q^4 + r^4 = P^4 - 4PPQ + 4PR + 2QQ$$

and

$$ppqq + pprr + qqrr = QQ - 2PR,$$

the above equation becomes

$$16a^4 = 4PPQ - 8PR - P^4 \quad \text{and} \quad R = \tfrac{1}{2}PQ - \tfrac{1}{8}P^3 - \frac{2a^4}{P};$$

and thus this equation will be provided :

$$y^3 - Pyy + Qy - \tfrac{1}{2}PQ + \tfrac{1}{8}P^3 + \frac{2a^4}{P} = 0.$$

If P may be taken constant $= 2b$, the perimeter of all these above triangles above becomes constant. Whereby, if on taking $Q = mxx + nbx + kaa$, a line of the third order will be produced expressed by this equation

$$y^3 + mxxy - 2byy + nbxy - mbxx + kaay - nbbx + \frac{a^4}{b} - kaab + b^3 = 0,$$

of which this will be a property, that in the first place the sum of the three applied right lines *p, q* and *r* of the corresponding abscissa shall be the constant *2b,* then truly the area of the triangle formed by the sides *p, q* and *r* shall be the same everywhere *aa.*

380. Similar questions can be resolved with the help of the same method concerning four or more applied lines corresponding to the same abscissa ; in which business no further difficulty arises, so we may move on to other questions in which the applied lines do not correspond to the same abscissa but to different ones that will be compared among themselves. Certainly some relation may be proposed between the applied lines *PM* and *QN* (Fig. 80), one of which may correspond to the abscissa $AP = +x$, the other to the abscissa $AQ = -x$. $y = X$ shall be the equation for this curve with *X* being some function of *x,* and this function *X* will give the applied line *PM;* but if indeed $-x$ were put in place of $+x$ everywhere, the same function *X* will give the other applied line *QN.* Therefore if *X* should be an even function of *x,* for example *P,* there becomes $QN = PM$, but if *X* shall be an odd function of *x,* for example $= Q$, there will be $QN = -PM$. And if *P* and *R* may specify even functions, but *Q* and *S* odd functions of *x,* and if the equation for the curve were $y = \frac{P+Q}{R+S}$, there will be

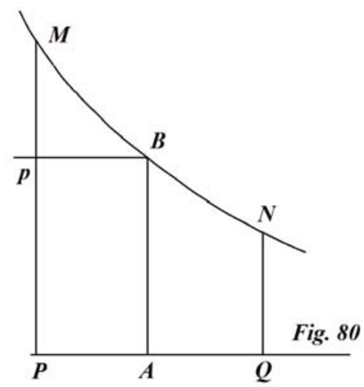

Fig. 80

$$PM = \frac{P+Q}{R+S} \text{ and } QN = \frac{P-Q}{R-S}.$$

381. If a curve of this kind is required to be found, so that $PM + QN$ shall be a constant quantity, evidently $2AB = 2a$. And it is evident the equation $y = a + Q$ satisfies this question, with *Q* being some odd function of *x* ; for there will be $PM = a + Q$ and $QN = a - Q$ and thus $PM + QN = 2a$, as required. But if therefore one may put $y - a = u$, there will be $u = Q$, which will be the equation for the same curve, with the right line *Bp* taken for the axis and with the point *B* for the starting point of the abscissas *x,* thus so that there shall be $Bp = x$ and $pM = u$. But the equation $u = Q$ signifies a curve with equal parts set down on both sides around the centre *B,* from one part to the other. Therefore any curve described of this kind *MBN* and with some right line *PQ* taken for the axe, thus satisfies the question, as with the perpendicular *BA* sent from the centre *B* to this axis and with equal abscissas taken on each side $AP = AQ$ the sum $PM + QN$ will always become constant $= 2AB$.

382. But for curves, which have two equal parts set out alternately about the centre *B*, the two equations we have found above, which are between the coordinates *x* and *u* :

I.
$$0 = \alpha x + \beta u + \gamma x^3 + \delta xxu + \varepsilon xuu + \zeta u^3 + \eta x^5 + \theta x^4 u + \text{etc}.$$

II.
$$0 = \alpha + \beta xx + \gamma xu + \delta uu + \varepsilon x^4 + \zeta x^3 u + \eta xxuu + \theta xu^3 + \text{etc}.$$

Whereby, if $u = y - a$ is put into each of these equations, two general equations will be had between the coordinates *x* and *y* by satisfying the proposed question of the algebraic curve. Therefore in the first place it satisfies all the right lines drawn through the point *B* , then also it solves the question of any conic section having its centre at the point *B*. Because truly in the latter case each of the abscissas *AP* and *AQ* correspond to two applied lines (unless for the case where the curve is a hyperbola, the applied lines may be taken parallel to the other asymptote), the two equal abscissas will have the same sum of the applied lines.

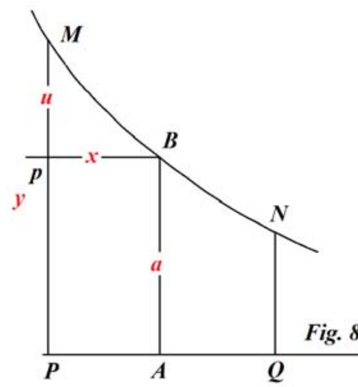

Fig. 80

383. If the curve *MBN* may be sought, in which not the sum of two applied lines *PM* and *QN*, but the sum of any powers of these shall be constant, the solution may be resolved in a similar manner. Indeed the equation may be needed to become $PM^n + QN^n = 2a^n$, and it is evident for this condition to be satisfied by this equation $y^n = a^n + Q$, with *Q* being some odd function of *x;* for there will be

$$PM^n = a^n + Q \quad \text{and} \quad QN^n = a^n - Q$$

and thus $PM^n + QN^n = 2a^n$. There may be put $y^n - a^n = u$, and the equation $u = Q$ will express the nature of the curve provided between the coordinates *x* and *u* from the equal parts set down one alternately about the centre *B*. On account of which, if in the equations given in the preceding paragraphs $y^n - a^n$ may be written everywhere in place of *u*, the general equations will be produced for satisfying the curve sought.

384. Therefore since questions of this kind may have no difficult ties, this question shall be proposed, by which the curve *MBN* may be sought thus so that, on the axis from a fixed point *A*, if on both sides equal abscissas may be taken *AP*, *AQ*, the rectangle of the applied lines $PM \cdot QN$ shall be of constant magnitude, for example $= aa$. Several particular solutions of this question can be given, the more outstanding ones of which, before we may inquire in general, we set out here. *P* shall be an even function and *Q* an

odd function of its abscissa $AP = x$ and the applied line may be put in place $PM = y = P + Q$; from which, with x taken negative, there becomes $QN = P - Q$. Therefore it is necessary to become

$$PM \cdot QN = PP - QQ = aa \text{ or } P = \sqrt{(aa + QQ)},$$

which expression $\sqrt{(aa + QQ)}$, because QQ is an even function of x and therefore also that shows an even function, provides a fitting value for P. Hence this equation will be had for the curve sought $y = Q + \sqrt{(aa + QQ)}$, on taking some odd function of x for Q.

385. But since the sign for the root by itself involves an ambiguity, each of the abscissas x will correspond to a pair of applied right lines, the one positive and the other negative ; thus the abscissas AP will correspond to the applied lines

$$Q + \sqrt{(aa + QQ)} \text{ and } Q - \sqrt{(aa + QQ)};$$

but the abscissas AQ will agree with the applied lines

$$-Q + \sqrt{(aa + QQ)} \quad \text{and} \quad -Q - \sqrt{(aa + QQ)};$$

from which the curve will have equal parts about the point A as the centre on opposite sides. Truly nor is this ambiguity arising from the sign permitted to be removed by taking for Q an odd function of this kind such as $\frac{aa}{4x} - x$, from which $aa + QQ$ becomes a square; indeed there will become $\sqrt{(aa + QQ)} = \frac{aa}{4x} + x$ and thus an odd function, which cannot be substituted in place of P. On account of which an odd function of x of this kind must be taken for Q, so that $aa + QQ$ does not become a square.

[Recall in all of these that Euler uses a left-handed x-axis, so that positive values are to the left, and negative values are to the right. Also, the points P and Q on the diagram represent x coordinates w.r.t. A as the origin, while in the equations given, P and Q refer to even and odd functions of x at these points.]

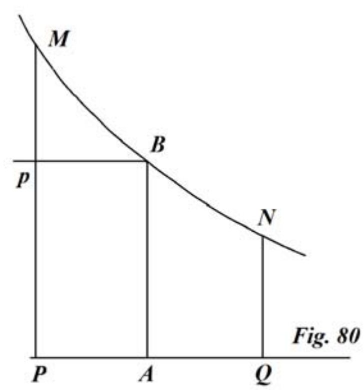

Fig. 80

386. In a similar manner, if the equation is put in place $y=(P+Q)^n$, making $QN=(P-Q)^n$, and thus there must become $\left(P^2-Q^2\right)^n=aa$. Hence there is formed $P^2=a^{\frac{2}{n}}+Q^2$ and $P=\sqrt{(a^{\frac{2}{n}}+Q^2)}$, which quantity, provided it were irrational, can be taken for *P*. Whereby for the curve satisfying the question, this equation will be obtained

$$y=\left(Q+\sqrt{(a^{\frac{2}{n}}+\ Q^2)}\right)^n.$$

But the construction of these curve will be easy : some curve is described having two similar and equal parts placed on opposite sides about the centre *A*, and the applied line of this curve corresponding to the abscissa $AP=x$ may be put $=z$; z will be an odd function of *x* and thus can be substituted in place of *Q*. But from the equation found there arises

$$y^{\frac{1}{n}}-Q=\sqrt{(a^{\frac{2}{n}}+Q^2)}$$

and thus

$$Q=z=\frac{y^{\frac{2}{n}}-a^{\frac{2}{n}}}{2y^{\frac{1}{n}}}$$

Putting $\frac{1}{n}=m$, and, if in the equation between *z* and *x* given everywhere $z=\frac{y^{2m}-a^{2m}}{2y^m}$, the equation will be obtained between *x* and *y* for the curve sought. Therefore since we have found the two equations between *z* and *x*, clearly either [even]

$$0=\alpha+\beta xx+\gamma xz+\delta zz+\varepsilon x^4+\zeta x^3z+\eta xxzz+\theta xz^3+\text{etc}.$$

or [odd]

$$0=\alpha x+\beta z+\gamma x^3+\delta xxz+\varepsilon xzz+\zeta z^3+\eta x^5+\theta x^4z+\text{etc},$$

if $z=y^m-\frac{a^{2m}}{y^m}$ may be put into these two equations (we ignore the divisor 2, because any multiple of *z* can be taken for *Q*), the two general equations arise satisfying the curve sought.

387. In addition to P there may be some even function R as well as besides Q some odd function S of x, and this equation may be put in place for the curve sought

$$y=\frac{P+Q}{R+S}=PM\ ;$$

therefore there will be $QN=\dfrac{P-Q}{R-S}$ and the equation becomes $PM\cdot QN=\dfrac{PP-QQ}{RR-SS}=aa$, to which condition the equation will be most easily satisfied by putting $y=\dfrac{P+Q}{P-Q}a$ or also on putting $y=\left(\dfrac{P+Q}{P-Q}\right)^n a$. In this way the first inconvenience may be avoided, that two or more applied lines will correspond to this abscissa, and curves of this kind may be found, so that only a single applied line may correspond to the individual abscissas. Hence the simplest curve satisfying this equation will be a line of the second order contained in this equation $y=\dfrac{b+x}{b-x}a$ and thus a hyperbola. Truly a hyperbola can also satisfy the first equation found $y=Q+\sqrt{(aa+QQ)}$; for on putting $Q=nx$, the equation will become $yy-2nxy=aa$. From which it is seen that this problem can be satisfied in two ways by a hyperbola.

388. From these premises it is seen how the equation for the curve sought must be prepared, thus so that, if $-x$ may be put in place of x and $\dfrac{aa}{y}$ in place of y, no change is apparent. Formulas of this kind are

$$\left(y^n+\frac{a^{2n}}{y^n}\right)P \text{ and } \left(y^n-\frac{a^{2n}}{y^n}\right)Q,$$

if indeed P may denote an even function and Q an odd function of x. But if therefore an equation may be formed, which were composed from some formulas of this kind, that will be by satisfying the curve sought. But if therefore M, P, R, T etc. denote some even functions of x and N, Q, S, V etc. some odd functions, the following general equation for the curve will be had [*i.e.* unchanged on making the above interchange of coordinates]

$$\begin{aligned}0=M&+\left(\frac{y}{a}+\frac{a}{y}\right)P+\left(\frac{yy}{aa}+\frac{aa}{yy}\right)R+\left(\frac{y^3}{a^3}+\frac{a^3}{y^3}\right)T+\text{etc.}\\&+\left(\frac{y}{a}-\frac{a}{y}\right)Q+\left(\frac{yy}{aa}-\frac{aa}{yy}\right)S+\left(\frac{y^3}{a^3}-\frac{a^3}{y^3}\right)V+\text{etc.},\end{aligned}$$

which if it may be multiplied by an odd function of x, the even functions will be changed into odd functions and the odd functions into even functions, from which also an equation of this kind will be satisfied

$$0 = N + \left(\frac{y}{a} + \frac{a}{y}\right)Q + \left(\frac{yy}{aa} + \frac{aa}{yy}\right)S + \left(\frac{y^3}{a^3} + \frac{a^3}{y^3}\right)V + \text{etc.}$$
$$+\left(\frac{y}{a} - \frac{a}{y}\right)P + \left(\frac{yy}{aa} - \frac{aa}{yy}\right)R + \left(\frac{y^3}{a^3} - \frac{a^3}{y^3}\right)T + \text{etc.},$$

which equations free from fractions will give these rational equations for the indefinite order n:

I.

$$0 = a^n y^n M + a^{n-1}y^{n+1}\left(P+Q\right) + a^{n-2}y^{n+2}(R+S) + a^{n-3}y^{n+3}\left(T+V\right) + \text{etc.};$$
$$+a^{n-1}y^{n-1}\left(P-Q\right) + a^{n-2}y^{n-2}(R-S) + a^{n-3}y^{n-3}\left(T-V\right) + \text{etc.}$$

II.

$$0 = a^n y^n N + a^{n-1}y^{n+1}\left(P+Q\right) + a^{n-2}y^{n+2}(R+S) + a^{n-3}y^{n+3}\left(T+V\right) + \text{etc.}$$
$$-a^{n-1}y^{n-1}\left(P-Q\right) - a^{n-2}y^{n-2}(R-S) - a^{n-3}y^{n-3}\left(T-V\right) - \text{etc.}$$

389. Truly in the formulas

$$\left(y^n + \frac{a^{2n}}{y^n}\right)P \text{ and } \left(y^n - \frac{a^{2n}}{y^n}\right)Q,$$

it is permitted to write some fractional number in place of n. Whereby, of for n the numbers $\frac{1}{2}$, $\frac{3}{2}$, $\frac{7}{2}$ etc. may be written, hence irrationalities will vanish at once from the general equation; for the equation will be had

$$0 = \frac{y+a}{\sqrt{ay}}P + \frac{y^3+a^3}{ay\sqrt{ay}}R + \frac{y^5+a^5}{aayy\sqrt{ay}}T + \text{etc.}$$
$$+\frac{y-a}{\sqrt{ay}}Q + \frac{y^3-a^3}{ay\sqrt{ay}}S + \frac{y^5-a^5}{aayy\sqrt{ay}}V + \text{etc.}$$

or this equation

$$0=\frac{y+a}{\sqrt{ay}}Q+\frac{y^3+a^3}{ay\sqrt{ay}}S+\frac{y^5+a^5}{aayy\sqrt{ay}}V+\text{etc.}$$
$$+\frac{y-a}{\sqrt{ay}}P+\frac{y^3-a^3}{ay\sqrt{ay}}R+\frac{y^5-a^5}{aayy\sqrt{ay}}T+\text{etc.},$$

which free from fractions changes into these :

[III]

$$0=+a^ny^{n+1}\left(P+Q\right)+a^{n-1}y^{n+2}(R+S)+a^{n-2}y^{n+3}\left(T+V\right)+\text{etc.}$$
$$+a^{n+1}y^n\left(P-Q\right)+a^{n+2}y^{n-1}(R-S)+a^{n+3}y^{n-2}\left(T-V\right)+\text{etc.}$$

and

[IV.]

$$0=+a^ny^{n+1}\left(P+Q\right)+a^{n-1}y^{n+2}(R+S)+a^{n-2}y^{n+3}\left(T+V\right)+\text{etc.}$$
$$-a^{n+1}y^n\left(P-Q\right)-a^{n+2}y^{n-1}(R-S)-a^{n+3}y^{n-2}\left(T-V\right)-\text{etc.}$$

390. Now from these four equations from the individual orders of the lines these may be easily found which resolve the problem. And indeed in the first place from the first order a right line parallel to the axis *AP* and passing through the point *B* will be satisfactory. From the second order the first two equations on making $n=1$ give $\alpha axy+yy-aa=0$, which is produced from the second order by putting $P=1$ and $Q=0$; for the first order gives no curved line. The two latter equations give on making $n=0$

$$y(\alpha+\beta x)\pm a\left(\alpha-\beta x\right)=0\,.$$

From the third order the two first equations give on making $n=1$

$$0=ay(\alpha+\beta xx)+yy(\gamma+\delta x)+aa(\gamma-\delta x)$$

and

$$0=\alpha ayx+yy(\gamma+\delta x)-aa(y-\delta x)\,,$$

but the two latter equations give on putting $n=0$ and $n=1$

$$0=ay(\alpha+\beta xx)+yy(\gamma+\delta x)+aa(\gamma-\delta x)$$

and

$$0=\alpha ayx+yy(\gamma+\delta x)-aa(\gamma-\delta x),$$

and in a similar manner from the following orders, all the lines sought satisfying the equation may be found.

CAPUT XVI

DE INVENTIONE CURVARUM EX DATIS APPLICATARUM PROPRIETATIBUS

364. Sint P et Q functiones quaecunque rationales abscissae x atque natura curvae exprimatur hac aequatione $yy - Py + Q = 0$. Hinc ergo unicuique abscissae x vel nulla vel duplex respondebit applicata; erit autem harum duarum applicatarum summa $= P$ et productum $= Q$. Si igitur P fuerit quantitas constans, summa binarum applicatarum singulis abscissis respondentium erit constans atque curva habe bit diametrum; hoc idem autem evenit, si fuerit $P = a + nx$; tum enim linea recta hac aequatione $z = \frac{1}{2}a + \frac{1}{2}nx$ contenta erit diameter, in latiore significatione hoc nomine accepto, ita ut obliquitas non excludatur. Sin autem fuerit Q quantitas constans, tum rectangulum binarum applicatarum erit ubique constans; axis ergo a curva nusquam secari poterit. At si sit $Q = \alpha + \beta x + yxx$ haecque expressio duos habeat factores reales, axis a curva in duobus punctis traiicietur atque Q erit multiplum rectanguli ex partibus axis, ideoque rectangulum applicatarum se habebit ad rectangulum partium axis in constanti ratione.

365. Hae igitur proprietates, quas supra sectionibus conicis convenire observavimus, in innumerabiles alias lineas curvas competunt. Sic constans magnitudo rectangulorum ex binis applicatis eidem abscissae respondentibus formatorum, qua hyperbolam ad asymptotam relatam gaudere vidimus, communis ipsi est cum omnibus curvis hac aequatione $yy - Py \pm aa = 0$ contentis. Deinde, sumta recta EF (Fig. 19) curvam in duobus punctis E et F secante pro axe, cum in sectionibus conicis rectangulum $PM \cdot PN$ ad rectangulum $PE \cdot PF$ constantem habeat rationem, haec proprietas sectionibus conicis communis erit cum omnibus curvis in hac aequatione

$$yy - Py + ax - nxx = 0$$

contentis. Erit autem $PM \cdot PN = PE \cdot PF$ seu $pm \cdot pn = Ep \cdot pF$, si fuerit $yy - Py = ax - xx$. Haec igitur proprietas, qua circulum praeditum esse ex elementis constat, non solum ipsi communis est cum infinitis curvis altiorum ordinum, sed etiam in reliquas sectiones conicas cadit. Sit enim $P = b + nx$, atque aequatio

$$yy - nxy + xx = ax + by,$$

quae est pro circulo, si $n = 0$ et angulus EPM rectus, complectetur quoque ellipsin, si nn minor quam 4, et hyperbolam, si nn maior quam 4, atque parabolam, si $nn = 4$.

366. Hinc concludimus in omni sectione conica *AEBF*, cuius axes seu diametri principales sint *AB*, *EF*, si binae ducantur rectae quaecunque *pq* et *mn* (Fig. 77), quae ad axes principales sub angulo semi-recto inclinentur, eas in *h* se mutuo ita esse secturas, ut sit $mh \cdot nh = ph \cdot qh$. Quod quidem manifestum est ex proprietatibus palmariis: si enim per centrum *C* ducantur rectae *PQ* et *MN* sub angulis semirectis ad axes principales, erunt inter se aequales ideoque $MC \cdot NC = PC \cdot QC$; quare, cum omnes rectae his parallelae eadem lege se secent, erit quoque $mh \cdot nh = ph \cdot qh$. Quin etiam hinc intelligitur, si modo rectae *MN* et *PQ* ita ducantur, ut ad eundem axem principalem aequaliter inclinentur seu ut sit $PCA = NCA$, ob $CP = CN$ omnes rectas his parallelas se mutuo ita secare, ut rectangula partium sint aequalia, scilicet ut sit $mh \cdot hn = ph \cdot hq$.

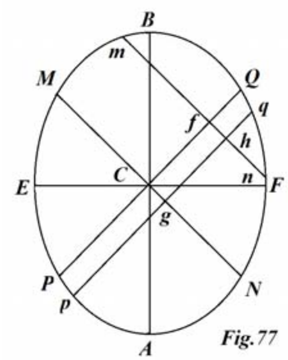

Fig. 77

367. His praemissis contemplemur alias quaestiones circa binas applicatas cuique abscissae respondentes ex aequatione $yy - Py + Q = 0$. Sit *AP* (Fig. 78) abscissa *x*, cui respondeant duae applicatae *PM, PN;* ac primo quaerantur omnes curvae huius indolis, ut sit $PM^2 + PN^2$ quantitas constans $= aa$. Cum sit

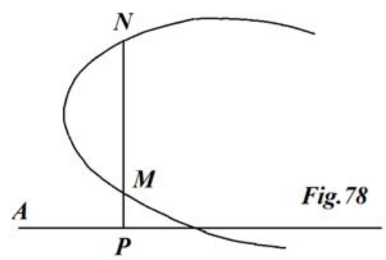

Fig. 78

$$PM + PN = P \text{ et } PM \cdot PN = Q,$$

erit

$$PM^2 + PN^2 = PP - 2Q$$

et quaesito satisfiet, si fuerit

$$PP - 2Q = aa \text{ seu } Q = \frac{PP - aa}{2};$$

unde pro curvis desideratis obtinebitur ista aequatio

$$yy - Py + \frac{PP - aa}{2} = 0.$$

Quodsi ponatur $P = 2nx$, prodibit sectio conica proprietate proposita gaudens

$$yy - 2nxy + 2nnxx - \tfrac{1}{2}aa = 0,$$

quae aequatio est pro ellipsi, abscissis a centro computatis.

368. Hinc sequitur non inelegans ellipsium proprietas ista. Si (Fig. 79) circa ellipseos duas quasvis diametros coniugatas *AB* et *EF* describatur parallelogrammum *GHIK*, cuius latera ellipsin tangent in punctis *A*, *B*, *E*, *F*, huius parallelogrammi diagonales *GK* et *HI* omnes chordas *MN* alterutri diametro *EF* parallelas ita secabunt in *P* et *p*, ut sit quadratorum summa $PM^2 + PN^2$ vel $pM^2 + pN^2$ perpetuo constans, nempe aequalis $2CE^2$. Similique modo ducta chorda *RS* diametro alteri *AB* parallela erit

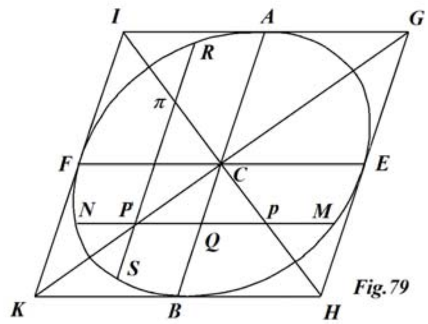

Fig. 79

$$PR^2 + PS^2 = \pi R^2 + \pi S^2 = 2CA^2.$$

Positis enim

$CA = CB = a,\ CE = CF = b,\ CQ = t,\ QM = u$, erit

$$aauu + bbtt = aabb.$$

Iam est $a : b = CQ(t) : PQ$ et *GP* ad *CQ* ratione data, puta *m*:1. Quare, posita $CP = x,\ PM = y$, erit

$$x = mt \text{ et } y = u + \frac{bt}{a}$$

seu

$$t = \frac{x}{m} \text{ et } u = y - \frac{bx}{ma}$$

quibus valoribus substitutis orietur ista aequatio

$$aayy - \frac{2abxy}{m} + \frac{2bbxx}{mm} = aabb.$$

Sit $\dfrac{b}{ma} = n$, erit

$$yy - 2nxy + 2nnxx = bb,$$

quae est aequatio ante inventa indicans esse $PM^2 + PN^2$ magnitudinem constantem.

369. Quaerantur nunc curvae, in quibus sit summa cuborum $PM^3 + PN^3$ perpetuo quantitas constans (Fig. 78). Cum sit $PM + PN = P$, erit

$$PM^3 + PN^3 = P^3 - 3PQ\,;$$

quare, si ponatur $PM^3 + PN^3 = a^3$, erit $Q = \frac{P^3 - a^3}{3P}$ ideoque pro his curvis erit aequatio generalis

$$yy - Py + \frac{1}{3}PP - \frac{a^3}{3P} = 0\,,$$

ubi pro functionem quamcunque rationalem ipsius x substituere licet. Simplicissima ergo curva hanc habens proprietatem erit linea tertii ordinis, quae, ponendo $P = 3nx$ et $a = 3nb$, hac aequatione exprimetur

$$xyy - 3nxxy + 3nnx^3 - 3nnb^3 = 0\,,$$

quae pertinet ad speciem secundam secundum enumerationem supra factam.

370. Simili modo, si effici debeat, ut sit $PM^4 + PN^4$ constans, quia est

$$PM^4 + PN^4 = P^4 - 4PPQ + 2QQ\,,$$

quantitas Q per P ita determinari debet, ut sit

$$P^4 - 4PPQ + 2QQ = a^4 \text{ seu } Q = PP + \sqrt{(\tfrac{1}{2}P^4 + \tfrac{1}{2}a^4)}\,.$$

Quia vero tam P quam Q debent esse functiones rationales seu uniformes ipsius x, ne y plures quam duos valores pro quavis abscissa x induere possit, quantitas $\sqrt{(\tfrac{1}{2}P^4 + \tfrac{1}{2}a^4)}$ deberet esse rationalis; quod cum fieri nequeat, functio Q semper erit biformis ideoque applicatam y reddet functionem quadriformem. Verum ex aequatione $yy - Py + Q = 0$ elicitur

$$y = \tfrac{1}{2}P \pm \sqrt{\left(-\tfrac{3}{4}PP \pm \sqrt{(\tfrac{1}{2}P^4 + \tfrac{1}{2}a^4)}\right)}\,,$$

unde patet applicatam y realem esse non posse, nisi $\sqrt{(\frac{1}{2}P^4+\frac{1}{2}a^4)}$ affirmative sumatur; quare, non obstante functionis Q biformitate, applicata y nunquam plures duobus valores habebit, quorum biquadratorum summa erit constans, sicut natura quaestionis requirit.

371. Quodsi porro eiusmodi requiratur curva, ut binorum ipsius y valorum cuique abscissae x respondentium potestates quintae summam constantem constituant seu ut sit $PM^5+PN^5=a^5$, debebit esse

$$P^5-5P^3Q+5PQQ=a^5.$$

Cum igitur ex aequatione pro curva $yy-Py+Q=0$ sit $Q=-yy+Py$, erit

$$P^5-5P^4y+10P^3yy-10PPy^3+5Py^4=a^5$$

seu

$$(P-y)^5+y^5=a^5.$$

Eodem modo reperietur, si debeat esse $PM^6+PN^6=a^6$, haec aequatio

$$(P-y)^6+y^6=a^6$$

Atque generaliter, si quaeratur curva, in qua sit $PM^n+PN^n=a^n$, obtinebitur ista aequatio

$$(P-y)^n+y^n=a^n,$$

ubi pro P functio quaecunque uniformis ipsius x pro lubitu accipi potest. Ratio autem huius aequationis in promtu est; cum enim summa ambarum applicatarum sit $=P$, si altera sit y, altera erit $=P-y$, unde statim fit

$$(P-y)^n+y^n=a^n.$$

372. Quodsi autem loco Q eliminetur P ponendo in aequationibus, quibus relatio inter P et Q continetur, $P=\frac{yy+Q}{y}$, orietur pro $PM^n+PN^n=a^n$ haec aequatio

$$y^n+\frac{Q^n}{y^n}=a^n.$$

Cum enim applicatarum productum sit $=Q$, si una ponatur $=y$, erit altera $=\frac{Q}{y}$; unde aequatio inventa statim fluit. Pro curvis ergo, in quibus sit

$PM^n + PN^n = a^n$, duas nacti sumus aequationes generales, alteram $(P-y)^n + y^n$ alteram $y^n + \dfrac{Q^n}{y^n} = a^n$, ex quarum posteriori emergit

$$y^{2n} = a^n y^n - Q^n \quad \text{et} \quad y^n = \tfrac{1}{2}a^n \pm \sqrt{(\tfrac{1}{4}a^{2n} - Q^n)},$$

ita ut sit

$$y = \sqrt[n]{\left(\tfrac{1}{2}a^n \pm \sqrt{(\tfrac{1}{4}a^{2n} - Q^n)}\right)}$$

quae est functio tantum biformis atque pro quavis abscissa plures duabus applicatas non exhibet, dummodo Q^n fuerit functio rationalis seu uniformis ipsius *x*. Prior autem aequatio $y^n + (P-y)^n = a^n$ hac gaudet praerogativa, ut numerus dimensionum sit minor.

373. Neque vero hae aequationes solum quaestionem solvunt, si *n* sit numerus integer affirmativus, sed etiam, si sit vel negativus vel fractus. Sic

si debeat esse	habebitur haec aequatio
$\dfrac{1}{PM} + \dfrac{1}{PN} = \dfrac{1}{a}$	$aP = Py - yy$ seu $aQ + ayy = Qy$
$\dfrac{1}{PM^2} + \dfrac{1}{PN^2} = \dfrac{1}{aa}$	$aayy + aa(P-y)^2 = yy(P-y)^2$ seu $aaQQ + aay^4 = QQyy$
$\dfrac{1}{PM^3} + \dfrac{1}{PN^3} = \dfrac{1}{a^3}$	$a^3y^3 + a^3(P-y)^3 = y^3(P-y)^3$ seu $a^3Q^3 + a^3y^6 = Q^3y^3$

etc.

Pro exponentibus autem fractis ita res se habebit:

si debeat esse	habebitur haec aequatio
$\sqrt{PM} + \sqrt{PN} = \sqrt{a}$	$\sqrt{y} + \sqrt{P-y} = \sqrt{a}$ seu $y = \sqrt{ay} - \sqrt{Q}$, quae ad rationalitatem reductae praebent $yy - Py + \tfrac{1}{4}(a-P)^2 = 0$ seu

	$yy-\left(a-2\sqrt{Q}\right)y+Q=0$
$\sqrt[3]{PM}+\sqrt[3]{PN}=\sqrt[3]{a}$	$\sqrt[3]{y}+\sqrt[3]{(P-y)}=\sqrt[3]{a}$ vel $yy-Py+\frac{1}{27}(a-P)^3=0$ seu $\sqrt[3]{y}+\sqrt[3]{\frac{Q}{y}}=\sqrt[3]{a}$ vel $yy-(a-3\sqrt[3]{aQ})y+Q=0$

etc.

Hoc igitur modo omnes curvae algebraicae, in quibus ubique sit

$$PM^n+PN^n=a^n$$

una aequatione generali comprehendi possunt, sive n sit numerus integer affirmativus sive negativus sive fractus.

374. Quae hic de conditione duarum applicatarum unicuique abscissae x respondentium sunt exposita, eadem methodo transferri possunt ad ternas applicatas singulis abscissis respondentes. Aequatio autem generalis pro curvis, quas singulae applicatae in tribus punctis secant, est haec

$$y^3-Pyy+Qy-R=0\,,$$

denotantibus litteris P, Q et R functiones quascunque uniformes ipsius x. Sint p, q, r tres applicatae abscissae x respondentes, quarum una quidem semper est realis, verum hic ad ea potissimum curvae loca spectamus, in quibus omnes tres applicatae sint reales. Erit autem ex natura aequationum $P=p+q+r$, $Q=pq+pr+qr$ et $R=pqr$. Quare, si curva desideretur, in qua sit vel $p+q+r$ vel $pq+pr+qr$ vel pqr quantitas constans, nil aliud est faciendum, nisi ut vel P vel Q vel R quantitas constituatur constans, binis reliquis manentibus arbitrariis.

375. Hinc quoque curva inveniri poterunt, in quibus sit $p^n+q^n+r^n$ quantitas constans ubique; est enim per ea, quae in superiori libro sunt tradita,

$$p+q+r=P,$$
$$p^2+q^2+r^2=P^2-2Q,$$
$$p^3+q^3+r^3=P^3-3PQ\quad+3R,$$
$$p^4+q^4+r^4=P^4-4PPQ+2QQ\quad+4PR,$$
$$p^5+q^5+r^5=P^5-5P^3Q\quad+5PQQ+5PPR-5QR,\ \text{etc.}$$

Deinde, si n sit numerus negativus, ponatur $z=\frac{1}{y}$; erit

$$z^3-\frac{Qzz}{R}+\frac{Pz}{R}-\frac{1}{R}=0$$

et huius aequationis tres radices sunt $\frac{1}{p},\ \frac{1}{q},\ \frac{1}{r}$. Hinc similar modi erit

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{Q}{R},$$
$$\frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2}=\frac{QQ-2PR}{RR},$$
$$\frac{1}{p^3}+\frac{1}{q^3}+\frac{1}{r^3}=\frac{Q^3-3PQR+3RR}{R^3},$$
$$\frac{1}{p^4}+\frac{1}{q^4}+\frac{1}{r^4}=\frac{Q^4-4PQQR+4QRR+2PPRR}{R^4},\ \text{etc.}$$

Huiusmodi ergo expressio quantitati constanti aequalis posita praebebit relationem idoneam inter functiones P, Q et R. Atque, si huius aequationis ope ex aequatione $y^3-Pyy+Qy-R=0$ una harum functionum P, Q vel R eliminetur, habebitur aequatio pro curva quaesita. Sic, si quaeratur curva, in qua sit $p^3+q^3+r^3=a^3$, fiet $P^3-3PQ+3R=a^3$ et, ob $R=y^3-Pyy+Qy$, habebitur haec aequatio

$$3y^3-3Pyy+3Qy+P^3-3PQ=a^3$$

pro curvis quaesito satisfacientibus.

376. Sive igitur n sit numerus affirmativus sive negativus integer, solutio per datas formulas facile expedietur; at maior difficultas occurrit, si n fuerit numerus fractus. Proponatur quaerenda linea curva, in qua sit

$$\sqrt{p}+\sqrt{q}+\sqrt{r}=\sqrt{a}\,.$$

Sumantur utrinque quadrata atque ob $p + q + r = P$ habebitur

$$P+2\sqrt{pq}+2\sqrt{pr}+2\sqrt{qr}\ \ =a$$

seu

$$\frac{a-P}{2}=\sqrt{pq}+\sqrt{pr}+\sqrt{qr}.$$

Sumantur denuo quadrata atque ob $pq+pr+qr=Q$ erit

$$\frac{(a-P)^2}{4}=Q+2\sqrt{ppqr}+2\sqrt{pqqr}+2\sqrt{pqrr}$$
$$=Q+2\ (\sqrt{p}+\sqrt{q}+\sqrt{r})\ \sqrt{pqr}=2\sqrt{a}R+Q,$$

unde oritur

$$(a-P)^2=4Q+8\sqrt{a}R \text{ seu } Q=\frac{(a-P)^2}{4}-2\sqrt{a}R.$$

Quare curvae quaesitae continebuntur in hac aequatione

$$y^3-Pyy+(\tfrac{1}{4}(a-P)^2-2\sqrt{a}R)y-R=0$$

seu (sublata irrationalitate, ob $R=\dfrac{(aa-2aP+PP-4Q)^2}{64a}$) in hac aequatione

$$y^3-Pyy+Qy-\frac{(aa-2aP+PP-4Q)^2}{64a}=0\,.$$

377. Haec autem operatio nimis fit molesta, si radices altiorum potestatum proponantur; alia ergo via erit ineunda, quae ex hoc exemplo perspicietur. Quaeratur nempe curva, in qua sit

$$\sqrt[3]{p}+\sqrt[3]{q}+\sqrt[3]{r}=\sqrt[3]{a}\,.$$

Ponatur

$$\sqrt[3]{pq}+\sqrt[3]{pr}+\sqrt[3]{qr}=v$$

et, cum sit $\sqrt[3]{pqr}=\sqrt[3]{R}$, fiet

$$\sqrt[3]{pp}+\sqrt[3]{qq}+\sqrt[3]{rr}=\sqrt[3]{aa}-2v$$

et

$$p+q+r=a-3v\sqrt[3]{a}+3\sqrt[3]{R}=P.$$

Deinde

$$\sqrt[3]{ppqq}+\sqrt[3]{pprr}+\sqrt[3]{qqrr}=vv-2\sqrt[3]{a}R$$

et

$$pq+pr+qr=Q=v^3-3v\sqrt[3]{aR}+3\sqrt[3]{RR}.$$

Inventis iam pro *P* et *Q* idoneis valoribus, sumendo pro *v* functionem quamcunque ipsius *x*, pro curvis quaesitis obtinebitur haec aequatio

$$y^3-(a-3v\sqrt{a}+3\sqrt[3]{R})yy+(v^3-3v\sqrt[3]{aR}+3\sqrt[3]{RR})y-R=0.$$

378. His tamen difficultatibus non obstantibus solutio generalis concinnari poterit. Cum enim ex aequatione $y^3-Pyy+Qy-R=0$, *y* denotet has tres applicatas *p*, *q* et *r*, ponatur $p=y$, erit

$$P=y+q+r \text{ et } Q=qy+ry+qr,$$

seu

$$q+r=P-y \text{ et } qr=Q-y(q+r)=Q-Py+yy.$$

Hinc prodit

$$q-r=\sqrt{(PP+2Py-3yy-4Q)}$$

ideoque

$$q=\tfrac{1}{2}(P-y)+\tfrac{1}{2}\sqrt{(PP+2Py-3yy-4Q)}$$

et

$$r=\tfrac{1}{2}(P-y)-\tfrac{1}{2}\sqrt{(PP+2Py-3yy-4Q)}.$$

Quando ergo quaeritur curva, in qua sit $p^n+q^n+r^m=a^n$, satisfaciet haec aequatio

$$y^n + (\tfrac{1}{2}(P-y) + \tfrac{1}{2}\sqrt{(PP+2Py-3yy-4Q)})^n$$
$$+ \ (\tfrac{1}{2}(P-y) - \tfrac{1}{2}\sqrt{(PP+2Py-3yy-4Q)})^n = a^n,$$

quae aeque quaestionem solvit, sive n fuerit numerus integer sive fractus.

379. Innumerabiles aliae quaestiones circa conditionem harum trium applicatarum eadem methodo resolvi possunt: velut, si pro a^n functio quaecunque ipsius x assumatur; tum vero etiam, praeter summam quarumcunque potestatum, aliae functiones ipsarum p, q et r proponi possunt, dummodo hae quantitates ita aequaliter insint, ut earum permutatione nulla variatio oriatur. Sic, istae tres applicatae p, q et r eidem abscissae x respondentes ita definiri poterunt, ut triangulum, quod ex iis formatur, constantem habeat aream. Huius enim trianguli area erit

$$= \tfrac{1}{4}\sqrt{(2ppqq + 2pprr + 2qqrr - p^4 - q^4 - r^4)},$$

quae ponatur $= aa$. Cum igitur sit

$$p^4 + q^4 + r^4 = P^4 - 4PPQ + 4PR + 2QQ$$

et

$$ppqq + pprr + qqrr = QQ - 2PR,$$

fiet

$$16a^4 = 4PPQ - 8PR - P^4 \quad \text{et} \quad R = \tfrac{1}{2}PQ - \tfrac{1}{8}P^3 - \frac{2a^4}{P};$$

ideoque habebitur ista aequatio

$$y^3 - Pyy + Qy - \tfrac{1}{2}PQ + \tfrac{1}{8}P^3 + \frac{2a^4}{P} = 0.$$

Si P capiatur constans $= 2b$, fiet insuper perimeter omnium horum triangulorum constans. Quare, si sumatur $Q = mxx + nbx + kaa$, prodibit linea tertii ordinis hac aequatione expressa

$$y^3 + mxxy - 2byy + nbxy - mbxx + kaay - nbbx + \frac{a^4}{b} - kaab + b^3 = 0,$$

cuius haec erit proprietas, ut trium applicatarum p, q et r singulis abscissis respondentium primum summa sit constans $2b$, tum vero area trianguli ex lateribus p, q et r formati sit ubique eadem aa.

380. Similes quaestiones eiusdem methodi ope resolvi possunt circa quatuor pluresve applicatas eidem abscissae respondentes; in quo negotio cum nulla amplius occurrat difficultas, ad alias progrediamur quaestiones, in quibus applicatae non eidem abscissae, sed diversis respondentes inter se comparentur. Proposita scilicet relatio quaedam inter applicatas PM et QN (Fig. 80), quarum altera abscissae $AP = +x$, altera abscissae $AQ = -x$ respondeat. Sit $y = X$ aequatio pro hac curva existente X functione quacunque ipsius x, atque haec functio X dabit applicatam PM; quodsi vero loco $+x$ ubique ponatur $-x$, eadem functio X dabit alteram applicatam QN. Si ergo X esset functio par ipsius x, puta P, foret $QN = PM$, sin autem sit X functio impar ipsius x, puta $= Q$, erit $QN = -PM$. Atque si P et R denotent functiones pares, at Q et S functiones impares ipsius x fueritque aequatio pro curva $y = \dfrac{P+Q}{R+S}$, erit

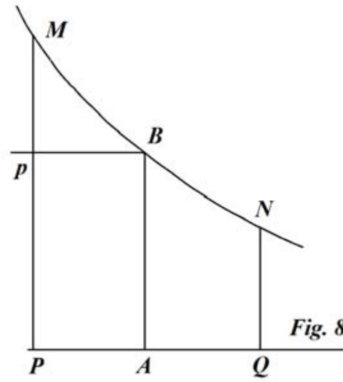

Fig. 80

$$PM = \frac{P+Q}{R+S} \text{ et } QN = \frac{P-Q}{R-S}.$$

381. Quaerenda sit curva huius indolis, ut sit $PM + QN$ quantitas constans, nempe $2AB = 2a$. Atque manifestum est huic quaestioni satisfacere aequationem $y = a + Q$ existente Q functione impari ipsius x; erit enim $PM = a + Q$ et $QN = a - Q$ ideoque $PM + QN = 2a$, uti requiritur. Quodsi ergo ponatur $y - a = u$, erit $u = Q$, quae erit aequatio pro eadem curva, sumta recta Bp pro axe et puncto B pro abscissarum x initio, ita ut sit $Bp = x$ et $pM = u$. Aequatio autem $u = Q$ indicat curvam partibus aequalibus utrinque circa centrum B alternatim dispositis praeditam. Descripta ergo huiusmodi curva quacunque MBN sumtaque recta quacunque PQ pro axe, quaestioni ita satisfiet, ut demisso in hunc axem ex centro B perpendiculo BA sumtisque utrinque abscissis aequalibus $AP = AQ$ semper future sit summa $PM + QN$ constans $= 2AB$.

382. Pro curvis autem, quae duas habent partes aequales circa centrum B alternatim dispositas, duas invenimus supra aequationes, quae inter coordinatas x et u sunt

I.
$$0 = \alpha x + \beta u + \gamma x^3 + \delta xxu + \varepsilon xuu + \zeta u^3 + \eta x^5 + \theta x^4 u + \text{etc}.$$

II.
$$0 = \alpha + \beta xx + \gamma xu + \delta uu + \varepsilon x^4 + \zeta x^3 u + \eta xxuu + \theta xu^3 + \text{etc}.$$

Quare, si in utraque harum aequationum ponatur $u = y - a$, habebuntur duae aequationes generales inter coordinatas x et y pro curvis algebraicis quaestioni propositae satisfacientibus. Satisfacit ergo primo omnis linea recta per punctum *B* ducta, deinde quoque omnis sectio conica centrum habens in puncto *B* quaestionem solvet. Quia vero hoc posteriori casu utrique abscissae *AP* et *AQ* gemina applicata respondet (nisi curva existente hyperbola, applicatae alteri asymptotae parallelae capiantur), bina habebuntur applicatarum paria eandem summam constituentia.

383. Si quaeratur curva *MBN*, in qua non summa binarum applicatarum *PM* et *QN*, sed summa quarumcunque potestatum earum sit constans, solutio simili modo absolvetur. Oporteat enim esse $PM^n + QN^n = 2a^n$, atque perspicuum est huic conditioni satisfieri hac aequatione $y^n = a^n + Q$ existente *Q* functione quacunque impari ipsius *x;* erit enim

$$PM^n = a^n + Q \text{ et } QN^n = a^n - Q$$

ideoque $PM^n + QN^n = 2a^n$. Ponatur $y^n - a^n = u$, atque aequatio $u = Q$ exprimet naturam curvae duabus partibus aequalibus alternis circa centrum *B* dispositis praeditam inter coordinatas *x* et *u.* Quamobrem, si in aequationibus paragrapho praecedenti datis ubique loco *u* scribatur $y^n - a^n$, prodibunt aequationes generales pro curvis quaesito satisfacientibus.

384. Cum igitur huiusmodi quaestiones nihil habeant difficultatis, proposita sit haec quaestio, qua quaeritur curva *MBN*, ita ut in axe a puncto fixo *A* si sumantur utrinque abscissae *AP*, *AQ* aequales, rectangulum applicatarum $PM \cdot QN$ futurum sit magnitudinis constantis, puta $= aa$. Huius quaestionis plures dari possunt solutiones particulares, quarum praecipuas, antequam in generalem inquiramus, hic evolvamus. Sit *P* functio par et *Q* functio impar ipsius abscissae $AP = x$ ac ponatur applicata $PM = y = P + Q$; ex qua, sumta *x* negativa, fiet $QN = P - Q$. Oportet ergo esse $PM \cdot QN = PP - QQ = aa$ seu $P = \sqrt{(aa + QQ)}$, quae expressio $\sqrt{(aa + QQ)}$, quia *QQ* est functio par ipsius *x* ac propterea quoque ipsa functionem parem exhibet, convenientem valorem pro *P* praebet. Hinc pro curva quaesita habebitur ista aequatio $y = Q + \sqrt{(aa + QQ)}$, sumendo pro *Q* functionem quamcunque imparem ipsius *x.*

385. Cum autem signum radicale per se ambiguitatem involvat, unicuique abscissae *x* gemina respondebit applicata, altera affirmativa altera negativa; sic abscissae *AP* respondebunt applicatae

$$Q + \sqrt{(aa + QQ)} \quad \text{et} \quad Q - \sqrt{(aa + QQ)};$$

at abscissae *AQ* convenient applicatae

$$-Q + \sqrt{(aa + QQ)} \quad \text{et} \quad -Q - \sqrt{(aa + QQ)};$$

unde curva partes habebit aequales circa punctum *A* tanquam centrum alternatim positas. Neque vero hanc ambiguitatem a signo ortam tollere licet sumendo pro *Q* eiusmodi functionem imparem uti $\frac{aa}{4x}-x$, qua fiat $aa+QQ$ quadratum; fieret enim $=\sqrt{(aa+QQ)}=\frac{aa}{4x}+x$ ideoque functio impar, quae in locum ipsius *P* substitui non posset. Quocirca pro *Q* eiusmodi functio impar ipsius *x* sumi debet, ut $aa+QQ$ non fiat quadratum.

386. Simili modo, si ponatur $y=(P+Q)^n$, fiet $QN=(P-Q)^n$, ideoque esse debebit $\left(P^2-Q^2\right)^n=aa$. Hinc fiet $P^2=a^{\frac{2}{n}}+Q^2$ et $P=\sqrt{(a^{\frac{2}{n}}+Q^2)}$, quae quantitas, dummodo fuerit irrationalis, pro *P* assumi poterit. Quare pro curva quaestioni satisfaciente obtinebitur haec aequatio

$$y=(Q+\sqrt{(a^{\frac{2}{n}}+\ Q^2)})^n.$$

Constructio autem harum curvarum erit facilis: describatur curva quaecunque duas partes similes et aequales habens alternatim circa centrum *A* positas, huiusque curvae applicata abscissae $AP=x$ respondens ponatur $=z$; erit *z* functio impar ipsius *x* ideoque in locum ipsius *Q* substitui poterit. At ex aequatione inventa oritur

$$y^{\frac{1}{n}}-Q=\sqrt{(a^{\frac{2}{n}}+Q^2)}$$

ideoque

$$Q=z=\frac{y^{\frac{2}{n}}-a^{\frac{2}{n}}}{2y^{\frac{1}{n}}}$$

Ponatur $\frac{1}{n}=m$, atque, si in aequatione inter *z* et *x* data ubique ponatur $z=\frac{y^{2m}-a^{2m}}{2y^m}$, obtinebitur aequatio inter *x* et *y* pro curva quaesita. Cum igitur inter *z* et *x* binas invenerimus aequationes, scilicet vel

$$0=\alpha+\beta xx+\gamma xz+\delta zz+\varepsilon x^4+\zeta x^3z+\eta xxzz+\theta xz^3+\text{etc}.$$

vel

$$0=\alpha x+\beta z+\gamma x^3+\delta xxz+\varepsilon xzz+\zeta z^3+\eta x^5+\theta x^4z+\text{etc},$$

si in his aequationibus ponatur $z = y^m - \frac{a^{2m}}{y^m}$ (divisorem 2 negligimus, quia pro Q quodcunque multiplum ipsius z sumi potest), duae orientur aequationes generales pro curvis quaesito satisfacientibus.

387. Sit praeter P quoque R functio par et praeter Q quoque S functio impar ipsius x ac statuatur pro curvis quaesitis haec aequatio

$$y = \frac{P+Q}{R+S} = PM\ ;$$

erit ergo $QN = \frac{P-Q}{R-S}$ fietque $\frac{PP-QQ}{RR-SS} = aa$, cui conditioni facillime satisfit ponendo $y = \frac{P+Q}{P-Q}a$ vel etiam statuendo $y = \left(\frac{P+Q}{P-Q}\right)^n a$. Hoc modo prius incommodum, quod cuique abscissae duae pluresve applicatae respondebant, evitatur atque eiusmodi curvae inveniuntur, ut singulis abscissis unica tantum applicata respondeat. Hinc curva simplicissima satisfaciens erit linea secundi ordinis hac aequatione $y = \frac{b+x}{b-x}a$ contenta atque ideo hyperbola. Hyperbola vero etiam satisfacit aequationi prius inventae $y = Q + \sqrt{(aa+QQ)}$ ponendo $Q = nx$, erit enim $yy - 2nxy = aa$. Unde huic problemati duplici modo per hyperbolam satisfieri potest.

388. His praemissis perspicuum est aequationem pro curva quaesita ita comparatam esse debere, ut ea, si loco x ponatur $-x$ et $\frac{aa}{y}$ loco y, nullam alterationem patiatur. Huiusmodi formulae sunt

$$\left(y^n + \frac{a^{2n}}{y^n}\right)P \text{ et } \left(y^n - \frac{a^{2n}}{y^n}\right)Q,$$

siquidem P functionem parem et Q imparem ipsius x denotet. Quodsi ergo aequatio formetur, quae ex quotcunque huiusmodi formulis fuerit composita, ea erit pro curva quaestioni satisfaciente. Quodsi ergo M, P, R, T etc. denotent functiones quascunque pares ipsius x atque N, Q, S, V etc. functiones impares, sequens aequatio generalis habebitur

$$0 = M + \left(\frac{y}{a} + \frac{a}{y}\right)P + \left(\frac{yy}{aa} + \frac{aa}{yy}\right)R + \left(\frac{y^3}{a^3} + \frac{a^3}{y^3}\right)T + \text{etc.}$$
$$+ \left(\frac{y}{a} - \frac{a}{y}\right)Q + \left(\frac{yy}{aa} - \frac{aa}{yy}\right)S + \left(\frac{y^3}{a^3} - \frac{a^3}{y^3}\right)V + \text{etc.},$$

quae si multiplicetur per functionem imparem ipsius x, functiones pares in impares et vicissim permutabuntur, unde etiam huiusmodi aequatio satisfaciet

$$0 = N + \left(\frac{y}{a} + \frac{a}{y}\right)Q + \left(\frac{yy}{aa} + \frac{aa}{yy}\right)S + \left(\frac{y^3}{a^3} + \frac{a^3}{y^3}\right)V + \text{etc.}$$
$$+\left(\frac{y}{a} - \frac{a}{y}\right)P + \left(\frac{yy}{aa} - \frac{aa}{yy}\right)R + \left(\frac{y^3}{a^3} - \frac{a^3}{y^3}\right)T + \text{etc.},$$

quae aequationes a fractionibus liberatae dabunt has aequationes rationales ordinis indefiniti n:

I.

$$0 = a^n y^n M + a^{n-1} y^{n+1}(P+Q) + a^{n-2} y^{n+2}(R+S) + a^{n-3} y^{n+3}(T+V) + \text{etc.}$$
$$+ a^{n-1} y^{n-1}(P-Q) + a^{n-2} y^{n-2}(R-S) + a^{n-3} y^{n-3}(T-V) + \text{etc.}$$

II.

$$0 = a^n y^n N + a^{n-1} y^{n+1}(P+Q) + a^{n-2} y^{n+2}(R+S) + a^{n-3} y^{n+3}(T+V) + \text{etc.}$$
$$- a^{n-1} y^{n-1}(P-Q) - a^{n-2} y^{n-2}(R-S) - a^{n-3} y^{n-3}(T-V) - \text{etc.}$$

389. In formulis vero

$$\left(y^n + \frac{a^{2n}}{y^n}\right)P \text{ et } \left(y^n - \frac{a^{2n}}{y^n}\right)Q$$

loco n quoque numeros fractos scribere licet. Quare, si pro n scribantur numeri $\frac{1}{2}$, $\frac{3}{2}$, $\frac{7}{2}$ etc., ex aequationibus generalibus hinc oriundis irrationalitas sponte evanescet; habebitur enim

$$0 = \frac{y+a}{\sqrt{ay}}P + \frac{y^3+a^3}{ay\sqrt{ay}}R + \frac{y^5+a^5}{aayy\sqrt{ay}}T + \text{etc.}$$
$$+ \frac{y-a}{\sqrt{ay}}Q + \frac{y^3-a^3}{ay\sqrt{ay}}S + \frac{y^5-a^5}{aayy\sqrt{ay}}V + \text{etc.}$$

vel haec aequatio

$$0 = \frac{y+a}{\sqrt{ay}}Q + \frac{y^3+a^3}{ay\sqrt{ay}}S + \frac{y^5+a^5}{aayy\sqrt{ay}}V + \text{etc.}$$
$$+ \frac{y-a}{\sqrt{ay}}P + \frac{y^3-a^3}{ay\sqrt{ay}}R + \frac{y^5-a^5}{aayy\sqrt{ay}}T + \text{etc.},$$

quae a fractionibus liberatae abeunt in has:

[III]

$$0 = +a^n y^{n+1}\left(P+Q\right) + a^{n-1} y^{n+2}(R+S) + a^{n-2} y^{n+3}\left(T+V\right) + \text{etc.}$$
$$+ a^{n+1} y^n \left(P-Q\right) + a^{n+2} y^{n-1}(R-S) + a^{n+3} y^{n-2}\left(T-V\right) + \text{etc.}$$

et

[IV.]

$$0 = +a^n y^{n+1}\left(P+Q\right) + a^{n-1} y^{n+2}(R+S) + a^{n-2} y^{n+3}\left(T+V\right) + \text{etc.}$$
$$- a^{n+1} y^n \left(P-Q\right) - a^{n+2} y^{n-1}(R-S) - a^{n+3} y^{n-2}\left(T-V\right) - \text{etc.}$$

390. Ex his quatuor aequationibus iam ex singulis linearum ordinibus eae, quae problema resolvant, facile invenientur. Ac primo quidem ex primo ordine satisfacit linea recta axi *AP* parallela ac per punctum *B* transiens. Ex ordine secundo binae aequationes priores faciendo $n=1$ dant $\alpha axy + yy - aa = 0$, quae ex secunda nascitur ponendo $N = \alpha x$ et $P=1$ et $Q=0$; prima enim nullam dat lineam curvam. Binae posteriores aequationes dant faciendo $n=0$

$$y(\alpha+\beta x) \pm a\left(\alpha-\beta x\right) = 0 .$$

Ex ordine tertio binae aequationes priores dant faciendo $n=1$

$$0 = ay(\alpha+\beta xx) + yy(\gamma+\delta x) + aa(\gamma-\delta x)$$

et

$$0 = \alpha ayx + yy(\gamma+\delta x) - aa(y-\delta x)\,,$$

binae autem aequationes posteriores dant ponendo $n=0$ et $n=1$

$$0 = ay(\alpha+\beta xx) + yy(\gamma+\delta x) + aa(\gamma-\delta x)$$

et

$$0 = \alpha ayx + yy(\gamma+\delta x) - aa(\gamma-\delta x),$$

similique modo ex sequentibus ordinibus omnes lineae quaesito satisfacientes reperientur.