CHAPTER XV

CONCERNING CURVES WITH ONE OR SEVERAL DIAMETERS

336. With the lines of the second order above we have seen all these to have at least one orthogonal diameter, which may cut the whole curve into two similar and equal parts. Clearly the parabola has one diameter of this kind and therefore it is composed from two equal and similar parts. But the ellipse and hyperbola have two diameters of this kind crossing each other normally at the centre ; and thus in these four arcs or branches are given equal and similar to each other. Truly the circle, because it may be divided into two equal and similar parts by all the right lines drawn through the centre, will have innumerable equal parts, evidently all arcs, which are subtending equal chords, likewise equal and similar between themselves.

337. Therefore here we will consider carefully that similitude of two or more parts of the same curve from the given work and these curves, of which two or more parts are similar

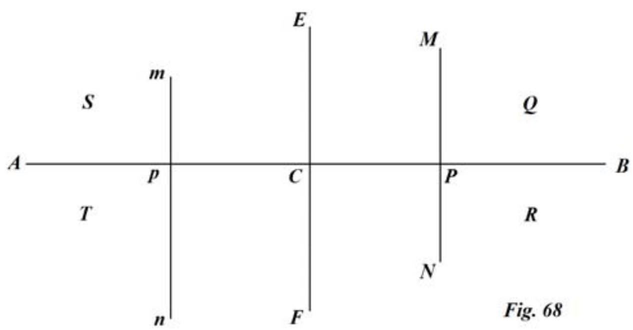


Fig. 68

between themselves, according to the general equations we will recall. And indeed in the first place, if we consider the equation between the orthogonal coordinates x and y, with the whole space (Fig. 68) divided into four equal regions by the right lines *AB, EF* cutting each other normally at *C*, indicated by the letters *Q, R, S, T* , a part of the curve arises situated in the region *Q* , with x and y taken positive ; but with the abscissa x positive and the applied line y negative, the part of the curve placed in the region *R* arises; but if x may be place negative with y remaining positive, the part of the curve positioned in the region *S* will be produced ; finally the part situated in the region *T* will be found with each coordinate y and x negative.

338. Therefore the parts situated in the regions *Q* and *R* will be equal and similar to each other, if the equation were prepared thus, so that it will not be changed even if $-y$ may be written in place of y. Therefore since the powers of all the even exponents of y can make use of this property, it is apparent, if in the equation for the curve no odd powers of

y itself may occur, the parts of the curve situated in the regions *Q* and *R* are equal and similar to each other and thus the right line *AB,* on which the abscissas $CP = x$ are taken, becomes a diameter of the curve. Therefore curves of this kind, if indeed they were algebraic, are all contained in this general equation

$$0 = \alpha + \beta x + \gamma xx + \delta yy + \varepsilon x^3 + \zeta xyy + \eta x^4 + \theta xxyy + \iota y^4 + \text{etc.},$$

which expression thus can be described, so that it shall be a rational function of x and *yy* themselves. But if therefore *Z* were some rational function of *x* and *yy,* then the equation $Z = 0$ expresses a curved line, which will be bisected by the right line *AB* into two equal and similar parts ; therefore the parts placed in the regions *S* and *T* also will be equal and similar amongst themselves.

339. Truly the parts in the regions *Q* and *S* will be equal and similar, if the equation were prepared thus, so that it will not be changed on putting $-x$ in place of x ; whereby, if Z were some rational function of *xx* and *y* themselves*,* then the equation $Z = 0$ may express a curve, which may be bisected by the right line *EF* into two similar and equal parts. Therefore the equation for these curves will be of the kind

$$0 = \alpha + \beta y + \gamma xx + \delta yy + \varepsilon xxy + \zeta y^3 + \eta x^4 + \theta xxyy + \iota y^4 + \text{etc.}$$

Therefore by this equation the part of the curve situated in *S* will be equal and similar to the portion in *Q,* and in a like manner the part in *T* likewise to the part in *R.*

340. But the parts in the opposite regions *Q* and *T* or *R* and *S* will be similar and equal, if the equation between the coordinates *x* and *y* were prepared thus, so that each *x* and *y* were made negative no change would come to mind. Let $Z = 0$ be the equation for these curves, and initially it is apparent, if *Z* were a function of *x* and *y* of even dimensions or if it were the sum from some number of functions of homogeneous functions of even dimensions, then the equation $Z = 0$ will enjoy the prescribed property. Then truly, if *Z* were the sum of however many homogeneous functions of odd dimensions, with *x* and *y* taken negative Z will be changed into $-Z$ and thus, since there shall be $Z = 0$, there will be also $-Z = 0$. Hence therefore a twofold general equation arises for curves, which in the opposite regions *Q* and *T* and likewise in *R* and *S* have equal and similar parts, evidently the one will be

$$0 = \alpha + \beta xx + \gamma xy + \delta yy + \varepsilon x^4 + \zeta x^3 y + \eta xxyy + \theta xy^3 + \iota y^3 + \chi x^6 + \text{etc.},$$

and truly the other will be

$$0 = \alpha x + \beta y + \gamma x^3 + \delta xxy + \varepsilon xyy + \zeta y^3 + \eta x^5 + \theta x^4 y + \iota x^3 yy + \text{etc.}$$

341. Therefore the curves, which have two equal and similar parts, are of two kinds : for thus either these two parts are set out on both sides around a right line, so that all the orthogonal coordinates likewise may be cut into two parts according to that right line, in which case that right line is called an orthogonal *diameter* of the curve, to which belong the equations treated in paragraphs 338 and 339. Or these two similar and equal parts fall in the regions opposite Q and R or opposite T and S , thus so that every right line drawn through the point C divides the curve alternately into two equal parts, curves of this kind are contained in the equations shown in the previous paragraph. Therefore hence we describe the different position of the equal parts thus, so that these, which belong to the first kind, we may call *diametrically equal ,* truly those according to the latter kind, we may call *alternately equal*. Because truly in the latter kind the point C is given, through which all the right lines produced to the curve on each side likewise are bisected, it will be agreed to give this point the name *centre*, thus so that the two parts of the curve alternatively may be said to have equal parts alternatively from the *centre provided* ; truly these curves, which have equal parts diametrically, may be said to be *provided with the diameter*.

342. Since the equation $Z = 0$ may provide these curves, the diameter of which is the right line AB, if the y coordinates may maintain even dimensions only in the function Z, and the same equation $Z = 0$ may indicate the diameter EF of the curve, if the other co-odinates x may have even exponents everywhere, it follows, if Z were a function of this kind of x and y, so that all the exponents both of x as well as of y shall be even numbers, then each right line AB and EF becomes an orthogonal diameter of the curve ; and thus the four parts situated in the regions Q, R, B and T become similar and equal to each other. Therefore all the curves of this kind will be contained in this general equation :

$$0 = \alpha + \beta xx + \gamma yy + \delta x^4 + \varepsilon xxyy + \zeta y^4 + \eta x^6 + \theta x^4 yy + \text{ etc.}$$

343. Therefore curves contained in this equation will have the two orthogonal diameters AB and EF mutually intersecting each other normally at C. Therefore all these curves belong to the second, fourth, sixth, etc. orders of lines, thus so that in no odd order of the lines may any given curved line be contained with two diameters mutually intersecting each other normally. Then, because this equation itself also is contained in the equation in the previous paragraph 340, these curves likewise will have a centre at the point C, thus so that every right line through that produced on both sides to the curve likewise may be cut into two equal parts. Therefore curves of this kind which have a double diameter, will then be represented by the equation $Z = 0$, if indeed Z were some rational function of xx and yy.

344. Therefore because in this way we have been led to lines provided with two diameters, we may inquire into equations for curved lines which may have more diameters. And indeed in the first place it may be shown readily, if a certain curve may have only two diameters, these are required to be normal to each other, thus so that no curve may be given provided with only two diameters, which may not be represented in

the manner found. For we may consider two diameters *AB* and *EF* of a certain curved line (Fig. 69) not crossing each other normally at *C*. Therefore since *EC* shall be a diameter, both sides of the curve will be prepared equally around that ; whereby, since its nearer part may have a right line *AC* for a diameter, also the further part will have the diameter *GC* at the same point *C* with *EC* putting the angle $GCE = ACE$ in place. In a similar manner, since *GC* shall be a diameter, also the right line *IC,* with $GCI = GCE$ present*,* is a diameter of the same kind, as is *EC.* Again also the right line *LC* will be a diameter*,* with the angles taken $ICL = ICG$ *;* and thus by progressing continually new diameters may be found, until they may fall again on the first *AC* ; which comes about, if the angle *ACE* may have a rational ratio to the right angle.

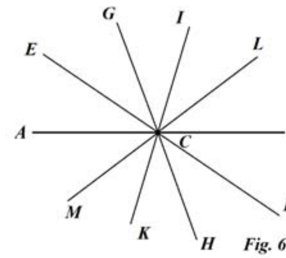


Fig. 69

345. Therefore unless the angle *ACE* may have a rational ratio to the right angle, the number of diameters will be infinite, in which case the curve will be a circle, obviously in which every right line drawn through the centre is an orthogonal diameter ; for here we restrict the name of diameter to orthogonal diameters alone, because curves are divided into two similar and equal parts with these alone. From these it is understood that no algebraic curve can have two diameters parallel to each other ; indeed on account of the alleged ratios, if two parallel diameters were had, likewise there would have to be an infinitude parallel and equally distant from each other, and thus there would be a right line cutting a curve of this kind at an infinite number of points, which property does not agree with algebraic curved lines.

346. But if therefore a certain curved line should have several diameters, all these will intersect each other mutually at the same point *C* and will be distant from each other within the same angles. Truly these diameters will be progressing from two kinds alternately ; indeed the diameter *CG* will be of this same kind as the diameter *CA* ; and the equation for the curve with the diameter *CG* taken for the axis, will agree with the equation for the curve with the diameter *CA* taken for the axis ; therefore the alternate diameters *CA, CG, CL* etc. affect the same curve equally, and in a similar manner the diameters *CE, CI* etc. will relate with the same ratio to the curve. On which account, if the number of diameters were finite, then the angle *ACG* will be equal to a certain part of four right angles, or the angle *ACE* will be some amount of the angle of 180 degrees or the semi-perimeter, that we may call $= \pi$.

347. If the angle were $ACE = 90° = \frac{1}{2}\pi$ (Fig. 70)*,* now the case we have examined above emerges, so that the curve has two diameters normal to each other. We will investigate curves of this kind anew, but by a method different from the start, which may be able to be applied to the discovery of more diameters. Therefore a curve with two diameters *AB* and *EF* shall be provided ; in that some point *M* may be taken and with the right line *CM*

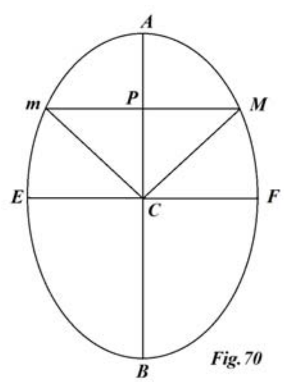


Fig. 70

drawn from the centre *C*, putting $CM = z$ and the angle $ACM = s$ an equation is sought between *z* and *s*. And indeed initially it is understood, because the right line *AC* is a diameter, *z* must be a function of *s* of this kind, which may remain the same, even if $-s$ may be put in place of *s* ; for with the angle $ACM = s$ with the negative *Acm* taken, the right line *Cm* must become $= CM$. Truly cos.*s* is a function of *s* of this kind, which remains the same on putting $-s$ in place of $+s$, for this requirement will be satisfied on account of which, if *z* were some rational function of cos.*s*.

348. Putting the abscissa $CP = x$, the applied line $PM = y$, there will be $z = \sqrt{(xx + yy)}$ and $\cos.s = \frac{x}{z}$; and $Z = 0$ shall be the equation for the curve, of which the right line *CA* shall be a diameter; and *Z* will have to be a rational function of *z* and $\frac{x}{z}$ or of *z* and *x* themselves, or on account of the rationality, of $xx + yy$ and *x*. But if *Z* were a function of $xx + yy$ and *x*, it will be also a function of *yy* and *x*. Indeed there shall be $xx + yy = u$; because *Z* must be a function a function of *x* and *u*, putting $u = t + xx$, so that there shall be $t = yy$, *Z* becomes a function of *t* and *x*, that is of *yy* and *x* themselves. Therefore as often as *Z* shall be a rational function of *yy* and *x*, so the right line *CA* also will be a diameter of the curve, which is the same property enjoyed by curves with one diameter, that we have found above.

349. But the curve sought is required to be provided with the two diameters *AB* and *EF* ; from which *CB* will be a diameter of the same nature as *CA*. Whereby, if the right line $CM = z$ may be referred to the diameter *CB*, on account of the angle $BCM = \pi - s$ it is necessary that *z* shall be a function of *s* of the same kind, which may not be changed, even if $\pi - s$ may be put in place of *s*. Indeed a function of this kind must become sin.*s*, *i.e.* $\sin.s = \sin.(\pi - s)$: but the preceding condition is not satisfied in this manner. Hence an expression of this kind must be found, which may pertain equally to the angles $s, -s$ and $\pi - s$; such is $\cos.2s$, indeed there is :

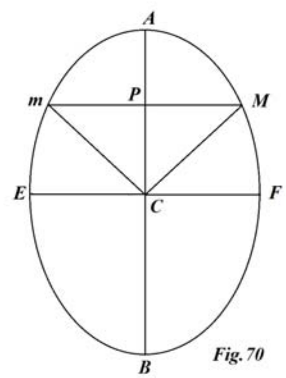


Fig. 70

$$\cos.2s = \cos.-2s = \cos.\ 2(\pi - s).$$

On account of which the equation $Z = 0$ will be for a curve with two diameters *AB* and *EF* provided, if *Z* were a rational function of *z* and cos.*2s* themselves. Truly there is

$$\cos.2s = \frac{xx - yy}{zz}.$$

From which Z will become a function of $xx+yy$ and $xx-yy$ or of xx and yy, as we have found above.

350. We may progress to investigating curves provided with three diameters (Fig. 71) AB, EF and GH ; which diameters mutually cut each other at the same point C at the angles ACE, ECG, $GCB = 60° = \frac{1}{3}\pi$, and the alternate diameters CA, CG, CF will be of the same nature. Whereby, if there may be put $CM = z$ and the angle $ACM = s$, on account of $GCM = \frac{2}{3}\pi - s$ the equation for the curve $Z = 0$ thus must be prepared, so that Z shall be a rational function of z and of a certain quantity w, which thus may depend on s, so that it may remain the same, if in place of s there may be put $-s$, or $\frac{2}{3}\pi - s$. Therefore there will be $w = \cos.3s$; indeed there is

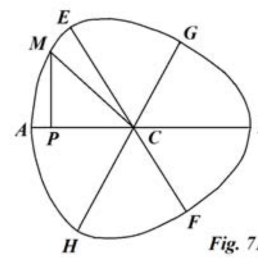


Fig. 71

$$\cos.3s = \cos.-3s = \cos.(2\pi - 3s).$$

But, with the coordinates put in place $CP = x, PM = y$, it becomes

$$\cos.3s = \frac{x^3 - 3xyy}{z^3},$$

and thus Z must be a rational function of $xx+yy$ and $x^3 - 3xyy$ themselves.

351. But if therefore there may be put $xx+yy=t$ and $x^3-3xyy=u$, this will be the general equation for curves provided with three diameters :

$$0 = \alpha + \beta t + \gamma u + \delta tt + \varepsilon tu + \zeta uu + \eta t^3 + \text{etc.},$$

which provides this equation between x and y:

$$0 = \alpha + \beta(xx+yy) + \gamma x(xx-3yy) + \delta(xx+yy)^2 + \text{etc.}$$

Therefore since the equation $0 = \alpha + \beta xx + \beta yy$ shall be for a circle, which having an infinitude of diameters, also satisfies the question for three diameters, the most simple curve having three diameters will be a line of the third order expressed by this equation :

$$x^3 - 3xyy = axx + ayy + b^3,$$

which consists of an asymptotic equilateral triangle, at the midpoint of which the point C is present ; and the individual asymptotes are of the kind $u = \frac{A}{tt}$.
Therefore these curves belong to the fifth kind following the enumeration made by us earlier [see Ch. 9 above].

352. If the curve (Fig. 72) should have four diameters *AB, EF, GH* and *IK* intersecting each other mutually at the point *C* at the semi-right angles $= \frac{1}{4}\pi$, then the diameters *CA, CG, CB* and *CH* will be of the same nature. Whereby on putting $CM = z$ and the angle $ACM = s$ some function of *s* must be sought, which will not change, whether $-s$ or $\frac{2}{4}\pi - s$ is put in place of *s*. But cos.4*s* is such a function. Whereby, if *Z* were a function of z and cos.4*s* or, which returns to the same, of $xx + yy$ and $x^4 - 6xxyy + y^4$, then the equation $Z = 0$ will give a curve provided with four diameters. Therefore *Z* will be a function of *t* and *u* by putting

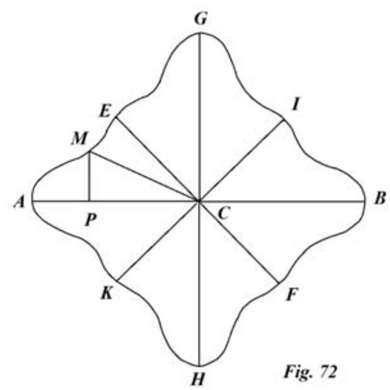


Fig. 72

$$t = xx+yy \text{ and } u = x^4 - 6xxyy + y^4 .$$

But there may be put $v = tt - u$, and *Z* will be a function of t and *v*, that is of $xx + yy$ and $xxyy$ themselves. Or *Z* can be defined thus also, so that is shall be a function of these two quantities $xx + yy$ and $x^4 + y^4$.

353. So that a curve expressed by the equation $Z = 0$ shall have five diameters, it is necessary, that *Z* shall be a function of z and cos.5*s*. Whereby, with the orthogonal coordinates taken *x* and *y*, on account of

$$\cos.5s = \frac{x^5 - 10x^3yy + 5xy^4}{z^5},$$

Z must be a rational function of these expressions

$$xx + yy \text{ and } x^5 - 10x^3yy + 5xy^4 .$$

Therefore the simplest curve, which besides the circle may have five diameters, is a line of the fifth order and may be expressed by this equation

$$x^5 - 10x^3yy + 5xy^4 = a(xx + yy)^2 + b(xx + yy) + c .$$

Therefore this curve, on account of all the real factors of the above first member, will have five asymptotes with their intersections forming a regular pentagon, in the middle of which shall be the centre *C*.

354. Now from these generally it is apparent a curve expressed by the equation $Z = 0$ having *n* diameters, of which the angle taken between two neighbouring diameters $= \frac{\pi}{n}$, if *Z* were a function of z and cos.*ns* or, between the orthogonal coordinates, some rational function of these expressions $xx + yy$ and

$$x^n - \frac{n(n-1)}{1 \cdot 2} x^{n-2} yy + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} x^{n-4} y^4 - \text{etc.}$$

Or this equation

$$0 = \alpha + \beta t + \gamma u + \delta tt + \varepsilon tu + \zeta uu + \eta t^3 + \theta ttu + \text{etc.}$$

will provide a curve with curve with *n* given diameters, if there may be put $t = xx + yy$ and

$$u = x^n - \frac{n(n-1)}{1 \cdot 2} x^{n-2} yy + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} x^{n-4} y^4 - \text{etc.}$$

Hence curves can be found, as many as the number desired, which have diameters mutually intersecting each other in equal angles at the same point *C*. Truly likewise all these equations generally contain algebraic curves, which shall be provided with the given number of diameters.

355. Curves of this kind with several diameters provided have similar and equal parts among themselves. Thus a curve (Fig. 70) with two diameters provided has four similar and equal parts *AE, BE, AF* et *BF.* Moreover a curve (Fig. 71) provided with three diameters has six similar and equal parts *AE, GE, GB, FB, FH* and *AH.* And a curve (Fig. 72) with four diameters provided has eight similar and equal parts *AE, AK, GE, GI, BI, BF, HF* and *HK;* and in a similar manner the number of equal parts is always twice as great as the number of diameters. Furthermore as we have considered above curves to be given having two equal and similar parts, which yet may be without a diameter, so also many curves may be given having equal and similar parts, which yet are lacking diameters.

356. We may begin (Fig. 73) with two equal parts themselves from the region *BKF* placed opposite *AME,* which case we have now treated above. Because if a curve indeed should have two equal parts only, unavoidably they must be opposite each other, which will be more apparent, when we will consider several equal parts. Therefore we may put, as before, $CM = z$ and the angle $ACM = s$, and it is evident with the angles s and $\pi + s$ are required to agree on the same value of z ; therefore with the angle $ACM = \pi + s$ there becomes $z = CK$, moreover there must be $CK = CM$; therefore the common expression of the angles s and $\pi + s$ is required to be found, tang. s is of this kind; for there is $\text{tang.}s = \text{tang.}(\pi + s)$. Therefore the equation $Z = 0$ will be for the curve, such as we seek, if Z were a function of z and tang.s or a function of $xx + yy$ and $\frac{x}{y}$ themselves. We may put $\frac{x}{y} = t$ and there will be $xx + yy = yy(1 + tt)$. Whereby Z will be a function of t and $yy(1 + tt)$, that is of t and yy, from which the same equations result, which we have found above.

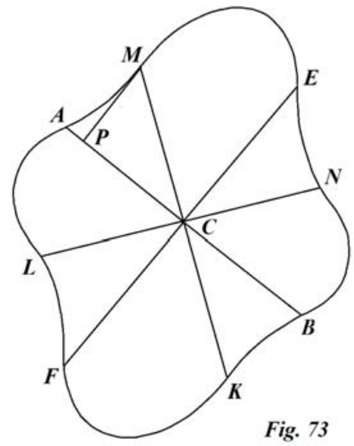


Fig. 73

357. But so that we may avoid fractions, with which tangents labour, we will be able to put the same calculations in place through sines and cosines. For since there shall be

$$\text{both } \sin.2s = \sin.2(\pi + s) \text{ and } \cos.2s = \cos.2(\pi + s),$$

what is sought will be found, if Z may be taken some rational function of these three formulas z, sin.2s and cos.2s or of $xx + yy$, $2xy$ and $xx - yy$ themselves. Where it is to be observed, if either of the expressions sin.2s and cos.2s may be omitted, a curve with the above diameter is required to be had. Therefore the solution may be returned here, so that Z may become a rational function of xx, yy and xy themselves, from which an equation of this kind may arise :

$$0 = \alpha + \beta xx + \gamma xy + \delta yy + \varepsilon x^4 + \zeta x^3 y + \eta xxyy + \theta xy^3 + \iota y^4 + \text{ etc .}$$

And if the terms in which x is not present may vanish, the whole equation will be able to be divided by x and it will produce :

$$0 = \beta x + \gamma y + \varepsilon x^3 + \zeta xxy + \eta xyy + \theta y^3 + \chi x^5 + \text{etc.},$$

which are both these equations which we have found above. [See §340.]

358. Now a curve may be sought (Fig. 74), which may have only three equal and similar parts *AM, BN* and *DL.* Therefore this will be prepared thus, so that with the three right lines drawn *CM, CN* and *CL* drawn from the midpoint *C* with equal angles these shall always be equal to each other henceforth. Therefore on putting the angle $ACM = s$ and with the right line $CM = z$, the right line *z* thus may be defined by *s*, so that from these three angles

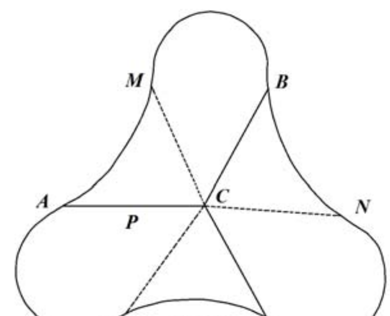


Fig. 74

$$s,\quad \tfrac{2}{3}\pi + s \quad \text{and} \quad \tfrac{4}{3}\pi + s,$$

the same value of *z* may be come upon ; indeed there is $MCN = NCL = \frac{2}{3}\pi$. But of these three angles these expressions sin.3*s* and cos.3*s* are common. Whereby, if *Z* were a rational function of the three quantities $xx + yy,\ 3xxy - y^3$ and $x^3 - 3xyy$, the equation $Z = 0$ will give all the curves sought. Therefore a general equation of this kind will arise :

$$0 = \alpha + \beta\left(xx + yy\right) + \gamma(3xxy - y^3) + \delta(x^3 - 3xyy) + \varepsilon(xx + yy)^2$$
$$+ \zeta(xx + yy)(3xxy - y^3) + \eta(xx + yy)\left(x^3 - 3xyy\right) + \text{etc.}$$

Therefore lines of the third order provided with this property will be contained in this equation

$$0 = \alpha + \beta xx + \beta yy + \delta x^3 + 3\gamma xxy - 3\delta xyy - \gamma y^3.$$

359. If the curve (Fig. 73) must have four equal parts *AM, EN, BK* and *FL,* thus so that with some four right lines *CM, CN, CK* et *CL* drawn from the midpoint *C* with equal angles, these shall become equal angles on putting the angle *ACM*= *s* and the right line $CM = z$*;* and on account of the angles $MCN = NCK = KCL = 90° = \frac{1}{2}\pi$, the right line *z* must be expressed by this angle *s* thus, so that the same value may correspond to these angles. Hence truly the expressions sin.4*s* and cos.4*s* have the property, whereby the equation $Z = 0$ will give a curve provided with four equal parts, if *Z* were some rational function of these three quantities

$$xx + yy,\ 4x^3y - 4xy^3 \text{ and } x^4 - 6xxyy + y^4.$$

Hence the general equation for curves of this kind will be

$$0 = \alpha + \beta xx + \beta yy + \gamma x^4 + \delta x^3 y + \varepsilon xxyy - \delta xy^3 + \gamma y^4 + \text{etc.}$$

360. It is evident in a similar manner, if a curve is required without diameters, which yet may have five equal and similar parts, in the equation $Z=0$, Z must be a rational function of these three quantities

$$xx+yy, \quad 5x^4y \ -10xxy^3+y^5 \quad \text{and} \quad x^5-10x^3yy+5xy^4$$

and, if the number of equal parts should be $=n$, then Z must be a rational function of these three :

$$xx \ + \ yy,$$

$$nx^{n-1}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^4+ \ \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^{n-5}y^5-\text{etc. and}$$

$$x^n-\frac{n(n-1)}{1\cdot 2}x^{n-2}+\frac{n(n-1)(n-2)(\text{n}-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}y^4-\text{etc.}$$

But if either of the latter expressions may not be present in the equation, the curve will have as many diameters as there are units in the number n.

[The latter two equations arise from expressing the multiple sines and cosines in terms of single sines and cosines of an angle. In the following sections, curves with cusps are described.]

361. In this twofold enumeration of curves having some number of equal parts, which either shall be from diameters provided or with these lacking, generally all the algebraic curves will be contained, which indeed may have two or more similar and equal parts. So that which may be made clear, a continued curve may have two parts (Fig. 75) *OAa, OBb* equal and similar to each other.

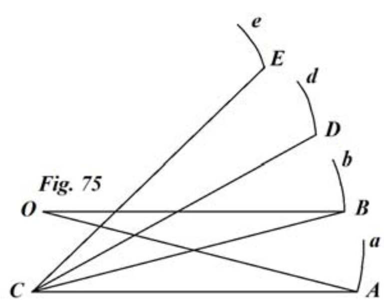


Fig. 75

[We are to understand that *Aa* and *Bb*, etc. are corresponding arcs of a curve, but these come from different parts of the complete curve, no diameter is necessarily assumed to be present. Consider, for example, the arcs formed in Fig. 74, where the arcs lie on the same side of *CA*.]
AB may be joined and on this line as a base an isosceles triangle *ACB* may be constructed, of which the angle *C* shall be equal to the angle *O*. Now, because the angles *OAC* and *OBC* shall be equal, also the parts of the curve *CAa* and *CBb* will be similar and equal, and on account of the law of continuity, if the angles *BCD, DCE* etc. may be taken equal to the individual angle *ACB* and $CD=CE=CA=CB$, the curve will have besides these similar and equal parts *Aa, Bb*, the individual similar and equal parts of right lines *Dd, Ee* etc. Therefore unless the ratio of the angle *ACB* to 360° were irrational, the

number of equal parts will be a finite number, but otherwise infinite, nor therefore ending among the algebraic lines. Therefore this curve always will be an algebraic curve contained among these with the diameters missing, which we have investigated previously.

362. But if the two similar and equal parts fall in the opposite sense from the right lines (Fig. 76) *AO* and *BO*, thus so that the part *OAa* shall be the similar and equal of the part *OBb,* then the right lines *AR* and *BS* are drawn on both sides, so that there shall be $OAR = OBS = \frac{1}{2} AOB$; and the right lines *AR* and *BS* will be parallel to each other. *AB* may be joined, and through the midpoint *C*, with *AR* and *BS* themselves drawn parallel to *CV,* the parts *aA, bB* will be equal and similar with respect to the right line *CV*. Therefore unless there shall be $ba = 0$, [the case considered below], since the arc *bB* on progressing from *b* to *a*, will correspond to the other similar and equal part of the arc *aA,* thus also for this by progressing from *a* to *e* through the interval $ae = ba$, which in turn will correspond with the other similar and equal part of arc *eE,* and for this again the arc *dD,* thus so that this curve will have an infinite number of similar and equal parts each arranged around the right line *CV*. Therefore a curve of this kind cannot be algebraic [*i.e.* it does not have a finite number of similar and equal parts, and in addition is composed of many disjointed such parts.]

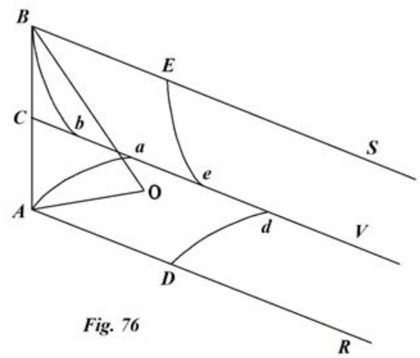


Fig. 76

363. Thus this itself arises above, if the right line *AB* were oblique to the parallel lines *AR* and *BS* or (which amounts to the same), if in the triangle *AOB* the sides *AO* and *BO* were unequal. But if there were $AO = BO$, then likewise the right line *AB* will be perpendicular to the parallel lines *AR* and *BS* and to *CV,* which likewise will pass through *O.* [We will now show that the curve does have a finite number of similar and equal parts and diameters]. Therefore in this case the points *b* and *a* will merge. And because the portions *aA* and *bB* not only are equal and similar, but also arranged equally about each side of the right line *CV*, this right line *CV* will be a diameter of the curve ; which case relates to the former curves examined having a diameter. On which account, according to the cases examined in this chapter, generally all the algebraic curves which have two or more similar and equal parts are reported on. [These latter curves described here have cusps ; this distinction is not made by Euler.]

CAPUT XV

DE CURVIS UNA PLURIBUSVE DIAMETRIS PRAEDITIS

336. De lineis secundi ordinis supra vidimus eas omnes unam ad minimum habere diametrum orthogonalem, quae totam ourvam in duas partes similes et aequales secet. Parabola scilicet eiusmodi unam habet diametrum ac propterea duabus constat partibus aequalibus et similibus. Ellipsis autem atque hyperbola duas eiusmodi habent diametros se mutuo in centro normaliter decussantes; ideoque in iis quatuor dantur arcus seu rami inter se aequales et similes. Circulus vero, quia ab omni recta per centrum ducta in duas partes similes et aequales dividitur, innumeras habebit partes aequales, omnes scilicet arcus, qui aequalibus chordis subtenduntur, simul inter se sunt aequales et similes.

337. Hanc igitur duarum pluriumve partium eiusdem curvae similitudinem hic data opera perpendemus easque curvas, quarum duae pluresve partes inter se sunt similes, ad

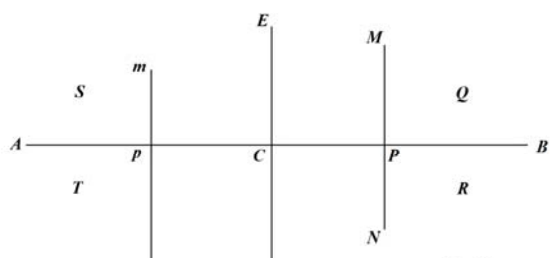


Fig. 68

aequationes generales revocabimus. Ac primo quidem, si consideremus aequationem inter coordinatas orthogonales x et y, diviso (Fig. 68) universo spatio in quatuor regiones litteris Q, R, S, T indicatas per rectas AB, EF se mutuo in C normaliter secantes, sumtis x et y affirmativis portio curvae in regione Q sita oritur; sumta autem abscissa x affirmativa at applicata y negativa, portio curvae in regione R sita oritur; sin autem x negativa ponatur manente y affirmativa, portio curvae in regione S sita prodibit; portio denique in regione T sita invenitur posita utraque coordinata y et x negativa.

338. Portiones ergo in regionibus Q et R sitae inter se erunt aequales et similes, si aequatio ita fuerit comparata, ut non mutetur, etiamsi $-y$ loco y scribatur. Cum igitur omnis potestas parium exponentium ipsius y hac gaudeat proprietate, patet, si in aequatione pro curva nullae potestates impares ipsius y occurrant, curvae portiones in regionibus Q et R sitas inter se fore aequales et similes ideoque rectam AB, in qua abscissae $CP = x$ capiuntur, fore curvae diametrum. Huiusmodi ergo curvae, siquidem fuerint algebraicae, omnes in hac aequatione generali continebuntur

$$0 = \alpha + \beta x + \gamma xx + \delta yy + \varepsilon x^3 + \zeta xyy + \eta x^4 + \theta xxyy + \iota y^4 + \text{etc.},$$

quae expressio ita describi potest, ut sit functio rationalis ipsarum *x* et *yy.* Quodsi ergo *Z* fuerit functio quaecunque rationalis ipsarum *x* et *yy,* tum aequatio $Z = 0$ exprimet lineam curvam, quae a recta *AB* in duas partes similes et aequales bisecabitur; erunt ergo quoque portiones in regionibus *S* et *T* sitae inter se aequales et similes.

339. Portiones vero in regionibus *Q* et *S* erunt aequales et similes, si aequatio ita fuerit comparata, ut posito - *x* loco *x* non immutetur; quare, si *Z* fuerit functio quaecunque rationalis ipsarum *xx* et *y,* tun aequatio $Z = 0$ exprimet curvam, quae per rectam *EF* in duas partes similes et aequales bisecabitur. Aequatio ergo pro his curvis erit huiusmodi

$$0 = \alpha + \beta y + \gamma xx + \delta yy + \varepsilon xxy + \zeta y^3 + \eta x^4 + \theta xxyy + \iota y^4 + \text{etc.}$$

Per hanc ergo aequationem portio curvae in *S* sita similis et aequalis erit portioni in *Q* similique modo portio in *T* portioni in *R.*

340. Portiones autem in regionibus oppositis *Q* et *T* seu *R* et *S* erunt similes et aequales, si aequatio inter coordinatas *x* et *y* ita fuerit comparata, ut posita utraque *x* et *y* negativa nullam mutationem subeat. Sit $Z = 0$ aequatio pro his curvis, ac primo patet, si *Z* fuerit functio ipsarum *x* et *y* parium dimensionum seu si fuerit aggregatum ex quotcunque functionibus homogeneis parium dimensionum, tum aequationem $Z = 0$ praescripta gaudere proprietate. Tum vero, si *Z* fuerit aggregatum quotcunque functionum homogenearum imparium dimensionum, sumtis *x* et *y* negativis *Z* abibit in $-Z$ ideoque, cum esset $Z = 0$, erit quoque $-Z = 0$. Hinc ergo duplex nascitur aequatio generalis pro curvis, quae in regionibus oppositis *Q* et *T* itemque in *R* et *S* portiones habent aequales et similes, alterea scilicet erit

$$0 = \alpha + \beta xx + \gamma xy + \delta yy + \varepsilon x^4 + \zeta x^3 y + \eta xxyy + \theta xy^3 + \iota y^3 + \chi x^6 + \text{etc.},$$

altera vero erit

$$0 = \alpha x + \beta y + \gamma x^3 + \delta xxy + \varepsilon xyy + \zeta y^3 + \eta x^5 + \theta x^4 y + \iota x^3 yy + \text{etc.}$$

341. Curvae ergo, quae duas habent partes similes et aequales, duplicis sunt generis: vel enim hae duae partes utrinque circa lineam rectam ita sunt dispositae, ut omnes ordinatae orthogonales ad illam rectam simul bifariam secentur, quo casu illa recta *diameter* curvae *orthogonalis* appellatur, quorsum pertinent aequationes paragraphia 338 et 339 traditae. Vel binae illae partes similes et aequales in regiones oppositas *Q* et *R* seu *T* et *S* cadunt, ita ut omnis recta per punctum *C* ducta curvam dividat in duas partes alternatim aequales, cuiusmodi curvae continentur in aequationibus in paragrapho praecedente exhibitis. Hanc igitur partium aequalium diversam positionem ita describemus, ut eas, quae ad priorem speciem pertinent, *diametraliter aequales,* quae vero ad posteriorem, *alternatim*

aequales appellemus. Quia vero in posteriore specie datur punctum *C,* per quod omnis recta utrinque ad curvam producta simul bisecatur, hoc punctum *centri* nomine appellari convenit, ita ut curvae binas partes alternatim aequales habentes *centro praeditae* dicantur; illae vero curvae, quae duas partes diametraliter aequales habent, *diametro praeditae* vocentur.

342. Cum aequatio $Z = 0$ praebeat curvas, quarum diameter est recta *AB,* si coordinata y pares tantum obtineat dimensiones in functione *Z,* atque eadem aequatio $Z = 0$ rectam *EF* curvae diametrum indicet, si altera coordinata x ubique pares habeat exponentes, sequitur, si *Z* eiusmodi fuerit functio ipsarum x et $y,$ ut omnes exponentes tam ipsius x quam ipsius y sint numeri pares, tum utramque rectam *AB* et *EF* fore curvae diametrum orthogonalem; ideoque quatuor partes in regionibus *Q, R, B* et *T* sitas inter se fore aequales et similes. Huiusmodi ergo curvae omnes in hac generali aequatione continebuntur:

$$0 = \alpha + \beta xx + \gamma yy + \delta x^4 + \varepsilon xxyy + \zeta y^4 + \eta x^6 + \theta x^4 yy + \text{ etc.}$$

343. Curvae ergo in hac aequatione contentae duas habebunt diametros orthogonales *AB* et *EF* se mutuo in *C* normaliter intersecantes. Pertinent ergo hae curvae omnes ad linearum ordines vel secundum vel quartum vel sextum etc., ita ut in nullo linearum ordine impari ulla contineatur linea curva duabus diametris se mutuo normaliter intersecantibus praedita. Deinde, quia ista aequatio quoque continetur in aequatione priori paragraphi 340, hae curvae simul centrum habebunt in puncto *C,* ita ut omnis recta per id utrinque ad curvam producta in eo simul bifariam secetur. Huiusmodi igitur curvas duplici diametro gaudentes praebebit $Z = 0$, siquidem fuerit *Z* functio quaecunque rationalis ipsarum xx et $yy.$

344. Quia igitur hoc modo deducti sumus ad lineas curvas duabus diametris praeditas, inquiramus in aequationes pro lineis curvis, quae plures habeant diametros. Ac primo quidem facile ostendetur, si quaepiam curva duas tantum habeat diametros, eas inter se normales esse oportere, ita ut nulla curva duabus diametris tantum praedita detur, quae non in aequatione modo inventa contineatur. Ponamus enim (Fig. 69) cuiuspiam lineae curvae duas esse diametros et *EF* sese in *C* non normaliter decussantes. Cum igitur *EC* sit diameter, curva utrinque circa eam aequaliter erit comparata ; quare, cum eius pars citerior rectam *AC* pro diametro habeat, etiam pars ulterior diametrum habebit *GC* in eodem puncto *C* cum *EC* angulum $GCE = ACE$ constituentem. Simili modo, cum *GC* sit diameter, debebit quoque recta *IC,* existente $GCI = GCE$, esse diameter eiusdem indolis, cuius est *EC.* Porro diameter quoque erit recta *LC,* sumto angulo $ICL = ICG$; sicque progrediendo continuo novae diametri reperientur, donec in primam *AC* recidant; quod evenit, si angulus *ACE* ad angulum rectum habeat rationem rationalem.

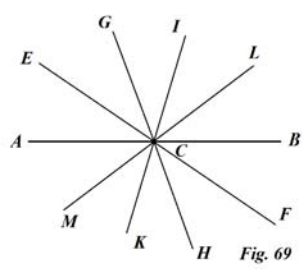


Fig. 69

345. Nisi ergo angulus ACE ad angulum rectum habeat rationem rationalem, numerus diametrorum erit infinitus, quo casu curva erit circulus, quippe in quo omnis recta per centrum ducta est diameter orthogonalis; hic enim diametri nomen ad solas diametros orthogonales restringimus, quia his solis curvae in duas partes similes et aequales dividuntur. Ex his intelligitur nullam curvam algebraicam duas habere posse diametros inter se parallelas ; ob rationes enim allegatas, si duas haberent diametros parallelas, simul infinitas inter se parallelas et aequaliter distantes habere deberent ideoque linea recta huiusmodi curvam in infinitis punctis secare posset, quae proprietas in lineas curvas algebraicas non cadit.

346. Quodsi ergo quaepiam linea curva plures habeat diametros, eae omnes se mutuo in eodem puncto C intersecabunt atque a se invicem sub aequalibus angulis distabunt. Erunt vero hae diametri duplicis generis alternatim progredientes; diameter enim CG eiusdem erit indolis, cuius est diameter CA ; atque aequatio pro curva, sumta diametro CG pro axe, conveniet cum aequatione pro curva, sumta diametro CA pro axe; diametri ergo alternae CA, CG, CL etc. aequaliter ad curvam erunt affectae, similique modo diametri CE, CI etc. eadem ratione ad curvam pertinebunt. Quamobrem, si numerus diametrorum fuerit finitus, tum angulus ACG erit pars aliquota quatuor rectorum seu angulus ACE erit pars aliquota anguli 180 graduum seu semiperipheriae, quam vocemus $=\pi$.

347. Si fuerit (Fig. 70) angulus $ACE = 90° = \frac{1}{2}\pi$, casus existit iam supra tractatus, quo curva duas habet diametros inter se normales. Huiusmodi ergo curvas denuo investigemus, at methodo, diversa a priori, quae aeque ad inventionem plurium diametrorum accomodari queat. Sit igitur curva duabus diametris AB et EF praedita; sumatur in ea quodcunque punctum M et ducta ex centro C recta CM ponatur $CM = z$ et angulus $ACM = s$ quaeraturque aequatio inter z et s. Ac primo quidem intelligitur, quia recta AC est diameter, z esse debere eiusmodi functionem ipsius s, quae maneat eadem, etiamsi $-s$ loco s ponatur; sumto enim angulo $ACM = s$ negativo ACm, recta Cm debet esse $= CM$. Verum B cos.s est eiusmodi functio ipsius s, quae manet eadem posito $-s$ loco $+s$, quamobrem huic requisito satisfiet, si fuerit z functio quaecunque rationalis ipsius cos.s.

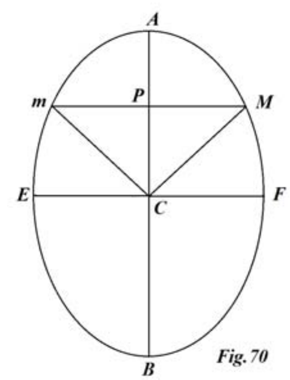


Fig. 70

348. Ponatur abscissa $CP = x$, applicata $PM = y$, erit $z = \sqrt{(xx + yy)}$ et cos.$s = \dfrac{x}{z}$; sitque $Z = 0$ aequatio pro curva, cuius recta CA sit diameter; atque esse debebit Z functio rationalis ipsarum z et $\dfrac{x}{z}$ vel ipsarum z et x vel, ob rationalitatem, ipsarum $xx + yy$ et x. At si Z fuerit functio ipsarum $xx + yy$ et x, erit quoque functio ipsarum yy et x. Sit enim $xx + yy = u$; quia Z debet esse functio ipsarum x et u, posito $u = t + xx$, ut sit $t = yy$, fiet

Z functio ipsarum t et x, hoc est ipsarum yy et x. Quoties ergo Z fuerit functio rationalis ipsarum yy et x, toties recta CA curvae erit diameter, quae est eadem proprietas curvarum una diametro gaudentium, quam supra invenimus.

349. At curvam quaesitam duabus diametris AB et EF praeditam esse oportet; unde CB erit diameter eiusdem indolis ac CA. Quare, si recta $CM = z$ ad diametrum CB referatur, ob angulum $BCM = \pi - s$ necesse est, ut z eiusmodi sit functio ipsius s, quae non varietur, etiamsi loco s ponatur $\pi - s$. Huiusmodi functio quidem foret sin.s, est $\sin.s = \sin.(\pi - s)$: sed hoc modo praecedenti conditioni non satisfit. Hinc eiusmodi expressio inveniri debet, quae ad angulos $s, -s$ et $\pi - s$ aequaliter pertineat; talis est $\cos.2s$, est enim

$$\cos.2s = \cos.-2s = \cos.\,2(\pi - s).$$

Quocirca aequatio $Z = 0$ erit pro curva duabus diametris AB et EF praedita, si Z fuerit functio rationalis ipsarum z et cos.$2s$. Est vero

$$\cos.2s = \frac{xx - yy}{zz}.$$

Ex quo Z debebit esse functio ipsarum $xx + yy$ et $xx - yy$ vel ipsarum xx et yy, uti supra invenimus.

350. Progrediamur (Fig. 71) ad curvas tribus diametris AB, EF et GH praeditas investigandas; quae diametri in eodem puncto C ad angulos $ACE,\ ECG,\ GCB = 60° = \frac{1}{3}\pi$ se mutuo secabunt, atque diametri alternae CA, CG, CF eiusdem erunt indolis. Quare, si ponatur $CM = z$ et angulus $ACM = s$, ob $GCM = \frac{2}{3}\pi - s$ aequatio pro curva $Z = 0$ ita debebit esse comparata, ut Z sit functio rationalis ipsius z et quantitatis cuiuspiam w, quae ab s ita pendeat, ut maneat eadem, sive loco s ponatur $-s$, sive $\frac{2}{3}\pi - s$. Erit ergo $w = \cos.3s$; est enim

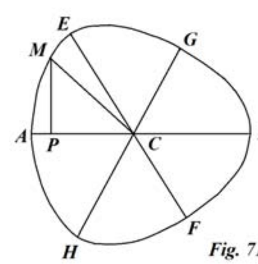


Fig. 71

$$\cos.3s = \cos.-3s = \cos.(2\pi - 3s).$$

At, positis coordinatis $CP = x, PM = y$, erit

$$\cos.3s = \frac{x^3 - 3xyy}{z^3},$$

ideoque Z esse debet functio rationalis ipsarum $xx + yy$ et $x^3 - 3xyy$.

351. Quodsi ergo ponatur $xx + yy = t$ et $x^3 - 3xyy = u$, haec erit aequatio generalis pro curvis tribus diametris praeditis

$$0 = \alpha + \beta t + \gamma u + \delta tt + \varepsilon tu + \zeta uu + \eta t^3 + \text{etc.},$$

quae praebet hanc inter x et y:

$$0 = \alpha + \beta(xx + yy) + \gamma x(xx - 3yy) \ + \delta(xx + yy)^2 + \text{etc.}$$

Cum igitur aequatio $0 = \alpha + \beta xx + \beta yy$ sit pro circulo, qui, habens infinitas diametros, etiam quaestioni de tribus diametris satisfacit, simplicissima curva tres habens diametros erit linea tertii ordinis hac aequatione expressa

$$x^3 - 3xyy = axx + ayy + b^3,$$

quae tres habet asymptotas triangulum aequilaterum comprehendentes, in cuius medio existit punctum C ; et singulae asymptotae sunt speciei $u = \frac{A}{tt}$.
Pertinent ergo hae curvae ad speciem quintam secundum enumerationem a nobis supra factam.

352. Si curva (Fig. 72) habeat quatuor diametros *AB, EF, GH* et *IK* se mutuo in puncto C ad angulos semirectos $= \frac{1}{4}\pi$ intersecantes, tum diametri *CA, CG, CB* et *CH* eiusdem erunt naturae. Quare posita $CM = z$ et angulo $ACM = s$ quaeri debet functio quaedam ipsius s, quae non mutetur, sive loco s ponatur $-s$ sive $\frac{2}{4}\pi - s$. Talis autem functio est cos.4s. Quare, si Z fuerit functio ipsarum z et cos.4s seu, quod eodem redit, ipsarum $xx + yy$ et $x^4 - 6xxyy + y^4$, tum aequatio $Z = 0$ dabit curvam quatuor diametris praeditam. Erit ergo Z functio ipsarum t et u positis

$$t = xx + yy \ \text{ et } \ u = x^4 - 6xxyy + y^4.$$

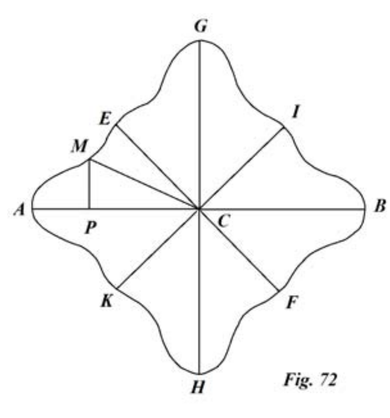


Fig. 72

Ponatur autem $v = tt - u$, eritque Z functio ipsarum t et v, hoc est ipsarum $xx + yy$ et $xxyy$. Vel etiam Z ita definiri potest, ut sit functio harum duarum quantitatum $xx + yy$ et $x^4 + y^4$.

353. Ut curva aequatione $Z=0$ expressa habeat quinque diametros, oportet, ut Z sit functio ipsarum z et cos.5s. Quare, sumtis coordinatis orthogonalibus x et y, ob

$$\cos.5s=\frac{x^5-10x^3yy+5xy^4}{z^5},$$

debebit esse Z functio rationalis harum expressionum

$$xx+yy \quad et \quad x^5-10x^3yy+5xy^4.$$

Curva igitur simplicissima, quae praeter circulum quinque habeat diametros, est linea quinti ordinis atque hac aequatione exprimetur

$$x^5-10x^3yy+5xy^4=a(xx+yy)^2+b(xx+yy)+c.$$

Haec ergo curva, propter omnes factores supremi membri reales, habebit quinque asymptotas suis intersectionibus pentagonum regulare, in cuius medio sit centrum C, formantes.

354. Ex his iam generaliter patet curvam aequatione $Z=0$ expressam habituram esse n diametros, quarum binae proximae angulum $=\frac{\pi}{n}$ comprehendant, si fuerit Z functio ipsarum z et cos. ns seu, inter coordinatas orthogonales, functio quaecunque rationalis harum expressionum $xx+yy$ et

$$x^n-\frac{n(n-1)}{1\cdot 2}x^{n-2}yy+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}y^4-\text{etc.}$$

Seu aequatio haec

$$0=\alpha+\beta t+\gamma u+\delta tt+\varepsilon tu+\zeta uu+\eta t^3+\theta ttu+\text{etc.}$$

praebebit curvam n diametris praeditam, si ponatur $t=xx+yy$ et

$$u=x^n-\frac{n(n-1)}{1\cdot 2}x^{n-2}yy+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}y^4-\text{etc.}$$

Hinc ergo curvae inveniri possunt, quae tot, quot lubuerit, habeant diametros se mutuo in angulis aequalibus in eodem puncto C intersecantes. Simul vero hae aequationes in se complectuntur omnes omnino curvas algebraicas, quae dato diametrorum numero sint praeditae.

355. Huiusmodi curvae pluribus diametris praeditae duplo plures habent partes inter se similes et aequales. Sic curva (Fig. 70) duabus diametris praedita quatuor habet partes similes et aequales *AE, BE, AF* et *BF.* Curva autem (Fig. 71) tribus diametris praedita habet sex partes similes et aequales *AE, GE, GB, FB, FH* et *AH.* Atque curva (Fig. 72) quatuor diametris praedita octo habet partes similes et aequales *AE, AK, GE, GI, BI, BF, HF* et *HK;* similique modo numerus partium aequalium semper duplo maior est quam numerus diametrorum. Quemadmodum autem supra vidimus dari curvas duas partes similes habentes, quae tamen diametro careant, ita dabuntur quoque curvae plures partes similes et aequales habentes, quae tamen diametris destituantur.

356. Incipiamus (Fig. 73) a duabus partibus aequalibus sibi e regione oppositis *AME, BKF,* quem quidem casum supra iam tractavimus. Quodsi enim curva duas tantum habere debeat partes aequales, necessario sibi oppositae esse debent, quod clarius patebit, quando plures partes aequales contemplabimur. Ponamus ergo, ut ante, $CM = z$ et angulum $ACM = s$, ac manifestum est angulis s et $\pi + s$ eundem valorem ipsius z convenire oportere; sumto enim angulo $ACM = \pi + s$ fiet $z = CK$, at esse debet $CK = CM$; quaerenda ergo est expressio communis angulis s et $\pi + s$, cuiusmodi est tang. s; est enim $\text{tang.}s = \text{tang.}(\pi + s)$. Aequatio igitur $Z = 0$ erit pro tali curva, qualem quaerimus, si fuerit Z functio ipsarum z et tang.s seu functio ipsarum $xx + yy$ et $\frac{x}{y}$. Ponamus $\frac{x}{y} = t$ eritque $xx + yy = yy(1 + tt)$. Quare Z debebit esse functio ipsarum t et $yy(1 + tt)$, hoc est ipsarum t et yy, unde eaedem aequationes resultant, quas supra invenimus.

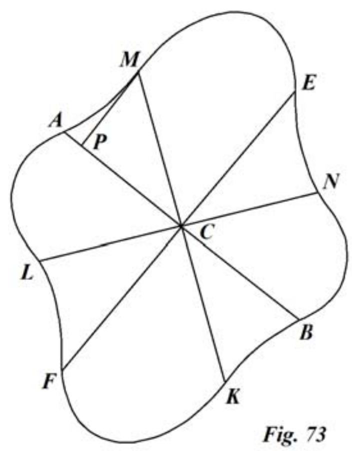


Fig. 73

357. Quo autem fractiones, quibus tangentes laborant, evitemus, idem negotium per sinus et cosinus expedire poterimus. Cum enim sit

$$\text{et } \sin.2s = \sin.\ 2(\pi + s) \text{ et } \cos.2s = \cos.2(\pi + s),$$

quaesitum obtinebitur, si Z capiatur functio quaecunque rationalis trium harum formularum z, sin.$2s$ et cos.$2s$ seu ipsarum $xx + yy$, $2xy$ et $xx - yy$. Ubi notandum est, si expressionum sin.$2s$ et cos.$2s$ altera omittatur, curvam insuper diametrum esse habituram. Solutio ergo huc redibit, ut Z capiatur functio ipsarum xx, yy et xy rationalis, unde huiusmodi orietur aequatio

$$0 = \alpha + \beta xx + \gamma xy + \delta yy + \varepsilon x^4 + \zeta x^3 y + \eta xxyy + \theta xy^3 + \iota y^4 + \text{ etc}.$$

Atque si termini, in quibus non inest x, evanescant, tota aequatio dividi poterit per x et prodibit

$$0 = \beta x + \gamma y + \varepsilon x^3 + \zeta xxy + \eta xyy + \theta y^3 + \chi x^5 + \text{etc.},$$

quae sunt ambae illae aequationes, quas supra invenimus.

358. Quaeratur nunc curva (Fig. 74), quae tres tantum habeat partes similes et aequales AM, BN et DL. Haec ergo ita erit comparata, ut eductis ex puncto medio C tribus rectis CM, CN et CL in angulis aequalibus eae semper inter se futurae sint aequales. Positis ergo angulo $ACM = s$ et recta $CM = z$, recta z per s ita definietur, ut his tribus angulis

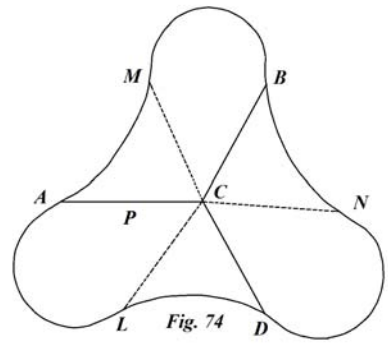


Fig. 74

$$s, \quad \tfrac{2}{3}\pi + s \quad \text{et} \quad \tfrac{4}{3}\pi + s$$

idem valor ipsius z conveniat; est enim $MCN = NCL = \frac{2}{3}\pi$. Horum autem trium angulorum communes sunt hae expressiones sin. $3s$ et cos.$3s$. Quare, si Z fuerit functio rationalis harum trium quantitatum $xx + yy$, $3xxy - y^3$ et $x^3 - 3xyy$, aequatio $Z = 0$ dabit curvas quaesitas omnes. Huiusmodi ergo orietur aequatio generalis

$$\begin{aligned}0 = \alpha + \beta\left(xx + yy\right) + \gamma(3xxy - y^3) + \delta(x^3 - 3xyy) + \varepsilon(xx + yy)^2 \\ + \zeta(xx + yy)(3xxy - y^3) + \eta(xx + yy)\left(x^3 - 3xyy\right) + \text{etc.}\end{aligned}$$

Lineae igitur tertii ordinis hac proprietate praeditae continentur in hac aequatione

$$0 = \alpha + \beta xx + \beta yy + \delta x^3 + 3\gamma xxy - 3\delta xyy - \gamma y^3.$$

359. Si curva (Fig. 73) quatuor habere debeat partes aequales AM, EN, BK et FL, ita ut ex puncto medio C eductis quatuor rectis quibusvis CM, CN, CK et CL sub angulis aequalibus eae futurae sint aequales, ponatur angulus $ACM = s$ et recta $CM = z$; atque ob angulos $MCN = NCK = KCL = 90° = \frac{1}{2}\pi$, recta z per angulum s ita debet exprimi, ut his angulis idem respondeat valor. Hanc vero proprietatem habent expressiones sin.$4s$ et cos.$4s$, quare aequatio $Z = 0$ dabit curvam quatuor eiusmodi partibus aequalibus praeditam, si fuerit Z functio quaecunque rationalis harum trium quantitatum

$$xx + yy, \; 4x^3y - 4xy^3 \; \text{et} \; x^4 - 6xxyy + y^4.$$

Hinc aequatio generalis pro istiusmodi curvis erit

$$0=\alpha+\beta xx+\beta yy+\gamma x^4+\delta x^3y+\varepsilon xxyy-\delta xy^3+\gamma y^4+\text{etc.}$$

360. Simili modo apparet, si quaeri debeat curva diametris destituta, quae tamen quinque habeat partes aequales et similes, in aequatione $Z=0$ esse debere Z functionem rationalem harum trium quantitatum

$$xx+yy,\quad 5x^4y\ -10xxy^3+y^5\quad \text{et}\quad x^5-10x^3yy+5xy^4$$

atque, si numerus partium aequalium esse debeat $=n$, tum Z esse debet functio rationalis harum trium

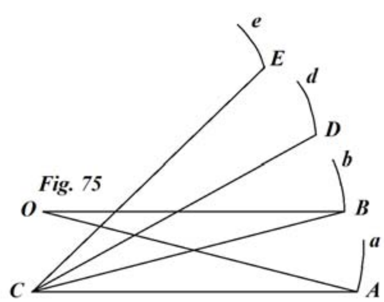
$$xx+\ yy,$$

$$nx^{n-1}-\frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^{n-3}y^4+\ \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot2\cdot3\cdot4\cdot5}x^{n-5}y^5-\text{etc. et}$$

$$x^n-\frac{n(n-1)}{1\cdot2}x^{n-2}+\frac{n(n-1)(n-2)(\mathrm{n}-3)}{1\cdot2\cdot3\cdot4}x^{n-4}y^4-\text{etc.}$$

Quodsi alterutra posteriorum expressionum non ingrediatur in aequationem, curva habebit tot diametros, quot numerus *n* continet unitates.

361. In duplici hac enumeratione curvarum aliquot partes aequales habentium, quae vel diametris sint praeditae vel iis careant, continentur omnino omnes curvae algebraicae, quae quidem duas pluresve habeant partes similes et aequales. Quod ut ostendatur, habeat curva continua (Fig. 75) duas partes *OAa, OBb* inter se similes et aequales. Iungatur *AB* superque ea tanquam basi constituatur triangulum isoscele *ACB,* cuius angulus *C* aequalis sit angulo *C.* Iam, quia anguli *OAC* et *OBC* sunt aequales, erunt quoque curvae partes *CAa* et *CBb* similes et aequales, atque ob legem continuitatis, si capiantur anguli *BCD, DCE* etc. aequales singuli angulo *ACB* et $CD=CE=CA=CB$, habebit curva praeterea ad has singulas rectas partes *Dd, Ee* etc. similes et aequales partibus *Aa, Bb.* Nisi ergo ratio anguli *ACB* ad 360° fuerit irrationalis, partium aequalium numerus erit finitus, contra autem infinitus, neque adeo in lineas algebraicas cadens. Semper ergo curva ista continetur in iis, quas ante investigavimus, diametris carentes.

Fig. 75


362. Sin autem (Fig. 76) duae partes similes et aequales in plagas oppositas rectarum

AO et *BO* cadant, ita ut sit pars *OAa* similis et aequalis parti *OBb,* tum utrinque ducantur rectae *AR* et *BS,* ut sit $OAR = OBS = \frac{1}{2} AOB$; eruntque rectae *AR* et *BS* inter se parallelae. Iungatur *AB* et per punctum medium *C* agatur ipsis *AR* et *BS* parallela *CV,* erunt partes *aA, bB* respectu rectae *CV* similes et aequales. Nisi igitur sit $ba = 0$, quia arcui *bB,* a *b* ad *a* progrediendo, respondet ex altera parte arcus similis et aequalis *aA,* ita quoque huic ab *a* ad *e* per spatium $ae = ba$ progrediendo respondebit ex altera parte arcus similis et aequalis *eE,* huicque porro arcus *dD,* ita ut haec curva habitura sit infinitas partes similes et aequales utrinque circa rectam *CV* dispositas. Huiusmodi ergo curva algebraica esse nequit.

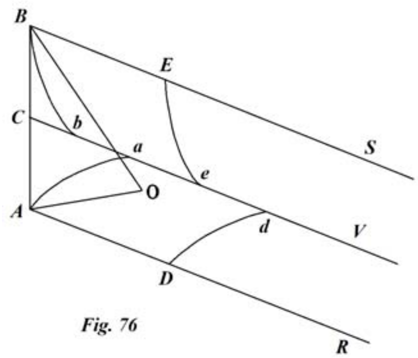


Fig. 76

363. Hoc ita se habet, si recta *AB* fuerit obliqua ad parallelas *AR* et *BS* vel (quod eodem redit), si in triangulo *AOB* latera *AO* et *BO* fuerint inaequalia. Sin autem fuerit $AO = BO$, tum simul recta *AB* erit perpendicularis ad parallelas *AR* et *BS* et ad *CV,* quae simul per *O* transibit. Hoc ergo casu puncta *b* et *a* congruent. Et quia portiones *aA* et *bB* non solum erunt aequales et similes, sed etiam utrinque circa rectam *CV* aequaliter dispositae, haec recta *CV* erit curvae diameter; qui casus ad priores curvas expositas diametro gaudentes pertinent. Quocirca ad casus in hoc capite expositos referuntur omnes omnino curvae algebraicae, quae duas pluresve partes habent similes et aequales.