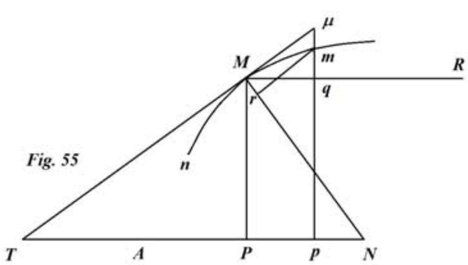
CHAPTER XIV

CONCERNING THE CURVATURE OF CURVED LINES

304. Just as in the chapter above we have investigated right lines, which will indicate the direction of this curved line at some point, thus here we will investigate simpler curved lines, which may agree so precisely with a proposed curve, that as if perhaps in the smallest space they may be merged together. For thus from the known nature of the simpler curve the nature of the proposed curve likewise may be deduced. Clearly here we will proceed by a similar method, as we have used above in examining the nature of the branches extending to infinity; that is to say at first by investigating the right line, which touches the curve, then truly a simpler curved line, which may meet the proposed curve much more than and which may not only touch, but will be in a much better contact. Moreover the contact of the smallest arcs of curved lines of this kind is accustomed to be called *osculating.*

305. Therefore let some equation be proposed between the orthogonal coordinates x and y, and the nature of the minimum part of the curve Mm (Fig. 55) is required to be investigated situated around the point M, since the abscissa $AP = p$ and the applied line $PM = q$ shall be found, and on the axis MR the minimum abscissa may be considered $Mq = t$ and the applied line $qm = u$; and there becomes $x = p + t$ and $y = q + u$; with which values substituted in the equation it may arrive at this equation



$$0 = At + Bu + Ctt + Dtu + Euu + Ft^3 + Gttu + \text{ etc.},$$

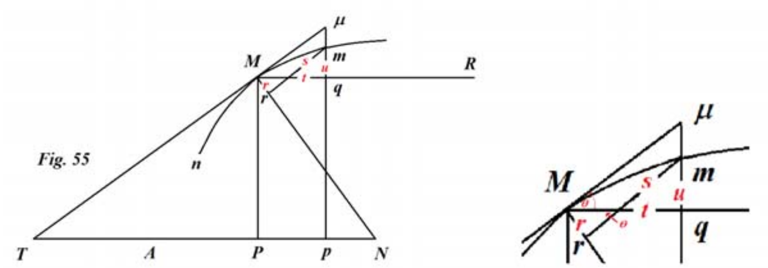
which expresses the nature of the curve related to the same axis MR. But because we have put in place these new coordinates t and u, the following terms will be as if infinitely smaller than the preceding ones and thus they can be rejected without error.

306. Therefore unless both the first coefficients A et B may vanish, with all the following terms rejected the equation $0 = At + Bu$ shows the straight line $M\mu$, which touches the curve at the point M and has a common direction with the curve at this place. Therefore there will be $Mq : q\mu = B : -A$; from which, on account of the known quantities A and B,

with the position of the tangent $M\mu$ becoming known, which since it may touch the curve at the point *M* only, we may see by how much the curve *Mm* may differ by a small interval at least from the right line $M\mu$. Towards this end we may take the normal *MN* for the axis, on which from *m* the applied line *mr* may be drawn to the perpendicular, and calling $Mr = r, rm = s$; there will be

$$t = \frac{-Ar + Bs}{\sqrt{(AA+BB)}} \quad \text{and} \quad u = \frac{-As - Br}{\sqrt{(AA+BB)}}$$

$$r = \frac{-At - Bu}{\sqrt{(AA+BB)}} \quad \text{and} \quad s = \frac{Bt - Au}{\sqrt{(AA+BB)}}.$$



Whereby , since there shall be

$$-At - Bu = Ctt + Dtu + Euu + Ft^3 + Gttu + \text{etc.},$$

r will be a quantity infinitely smaller than *t* and *u,* and therefore also *r* will be a quantity infinitely less than *s;* for *s* is determined by *t* and *u,* but *r* is determined by the squares or higher powers of *t* and *u*.

[In the supplementary diagram above, we have

$$t = \frac{r}{\sin\theta} + \left(s - \frac{r}{\tan\theta}\right)\cos\theta = r\sin\theta + s\cos\theta = \frac{-Ar}{\sqrt{A^2+B^2}} + \frac{Bs}{\sqrt{A^2+B^2}};$$

$$\text{as} \quad \frac{-A}{B} = \tan\theta ; \sin\theta = \frac{-A}{\sqrt{A^2+B^2}} ; \cos\theta = \frac{B}{\sqrt{A^2+B^2}} ; \text{etc.}]$$

307. Therefore we will have a much closer understanding of the curve *Mm*, if we may lead the terms $Ctt + Dtu + Euu$ into the computation also and we may neglect the following terms only ; and thus we will have the this equation between *t* and *u*

$$-At - Bu = Ctt + Dtu + Euu ,$$

in which, if we may substitute the above values in place of *t* and *u*, we will have

$$r\sqrt{(AA+BB)}=\frac{(AAC+ABD+BBE)rr}{AA+BB}+\frac{(AAD-BBD-2ABC+2ABE)rs}{AA+BB}$$
$$+\frac{(AAE-ABD+BBC)ss}{AA+BB}.$$

But, because *r* is infinitely less than *s*, the terms *rr* and *rs* vanish beside the term *ss*, and the equation becomes

$$ss=\frac{(AA+BB)r\sqrt{(AA+BB)}}{AAE-ABD+BBC},$$

which equation expresses the nature of the osculating curve at *M*.

308. Therefore the minimal arc of the curve *Mm* will agree with the vertex of the parabola described on the axis *MN*, whose latus rectum or parameter is

$$\frac{(AA+BB)\sqrt{(AA+BB)}}{AAE-ABD+BBC};$$

from which the curvature of this parabola at the vertex is just as great as the curvature of the proposed curve at the point *M*. But since the curvature of no curve can be known more clearly than the curvature of a circle, because its curvature is the same everywhere, and for that to be present, as the radius becomes smaller, it will be more convenient to define the curvature by the curvature of an equal circle, which is accustomed to be called the *osculating circle*. Hence on this account it will be necessary to define a circle whose curvature may agree with the curvature of the proposed parabola at its vertex, so that then it may be allowed to be substituted in place of the osculating parabola.

309. Towards effecting this we consider the curvature of the circle as unknown and we may express that in the same manner set out through the curvature of the parabola, for thus in turn the circle will be able to be substituted for the osculating parabola. Therefore with the proposed curve *Mm,* a circle with the radius $=a$ shall be described, the nature of which is expressed by the equation $yy=2ax-xx$. Therefore on taking $AP=p$ and $PM=q$ there will be $qq=2ap-pp$. Now there may be put

$$x=p+t \text{ and } y=q+u,$$

and this equation may arise

$$qq+2qu+uu=2ap+2at-pp-2pt-tt,$$

which on account of $qq = 2ap - pp$ is reduced to this form

$$0 = 2at - 2pt - 2qu - tt - uu\,,$$

which compared with the above form gives

$$A = 2a - 2p,\quad B = -2q,\quad C = -1,\quad D = 0 \quad \text{and} \quad E = -1\,,$$

from which the equation becomes :

$$AA{+}BB = 4(aa - 2ap + pp + qq) = 4aa\,,$$

and

$$\left(AA + BB\right)\sqrt{(AA + BB)} = 8a^3\,;$$

and also

$$AAE - ABD + BBC = -AA - BB = -4aa\,.$$

From which a circle, the radius of which $= a$, may osculate at whatever point on the curve with the vertex of a parabola, the nature of which is expressed by the equation $ss = 2ar$; and thus in turn as the vertex of the parabola $ss = br$ osculates with the curve, so the same circle will osculate, the radius of which is $= \tfrac{1}{2}b$.

310. Therefore since above we have found the osculating parabolic curve *Mm*, of which the equation shall be

$$ss = \frac{(AA + BB)\sqrt{(AA + BB)}}{AAE - ABD + BBC}\,r\,,$$

it is evident that the curvature of the curve at M agrees with the curvature of the circle, the radius of which shall be

$$= \frac{(AA + BB)\sqrt{(AA + BB)}}{2(AAE - ABD + BBC)}.$$

Therefore this expression gives the radius of the osculating circle and this radius is accustomed to be called the *radius of osculation ;* often also it may be called the *radius of curvature* or *curvature*. Therefore from the equation between *t* and *u*, as we have elicited from the proposed equation between *x* and *y*, at once can define the radius of osculation of the curve at the point *M* or the radius of the osculating circle of the curve at the point

M. For in the equation between t and u the terms may be re-entered, in which t and u retain more than two dimensions, and from the equation, which will be of this form

$$0 = At + Bu + Ctt + Dtu + Euu,$$

the radius of osculation may be found

$$= \frac{(AA+BB)\sqrt{(AA+BB)}}{2(AAE-ABD+BBC)}.$$

311. Truly because the sign has involved an ambiguity with the root $\sqrt{(AA+BB)}$, it is uncertain, whether that expression shall be positive or negative, evidently whether the point N of the curve may be considered concave or convex. Towards removing the doubt it must be sought, whether the point m shall be placed within the tangent $M\mu$ towards the axis AN or whether indeed it may fall outside the tangent. In the former case the curve will be concave towards N and the centre of the osculating falls on a part of the right line MN extending towards the axis ; truly in the latter case it falls on a part of the right line NM produced beyond M . Therefore all doubt will vanish, if it may be inquired, whether qm shall be less or greater than $q\mu$, for in the former case the curve will be concave towards N, in the latter truly convex.

312. Truly there is $q\mu = -\frac{At}{B}$ and $qm = u$, whereby it is required to be seen, whether $-\frac{At}{B}$ shall be greater or less than u. Therefore because $m\mu$ is as the smallest line, we may put $m\mu = w$ and there will be $u = -\frac{At}{B} - w$; from which, with the substitution made, the equation becomes

$$0 = -Bw + Ctt - \frac{ADtt}{B} - Dtw + \frac{AAEtt}{BB} + \frac{2AEtw}{B} + Eww;$$

where on account of w being the smallest term besides t , both the terms tw and ww vanish. Hence there becomes

$$w = \frac{(BBC-ABD+AAE)tt}{B^3}.$$

But if therefore w were a positive quantity, which comes about, if

$$\frac{BBC-ABD+AAE}{B^3} \text{ or } \frac{AAE-ABD+BBC}{B}$$

were a positive quantity, then the curve will be concave towards N ; but if it were a negative quantity, the convexity of the curve with regard to the point N.

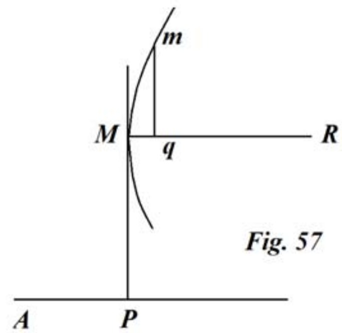


Fig. 57

313. So that these may be rendered clearer, the different cases which are able to occur (Fig. 57) are required to be set out separately. Therefore initially there shall be $B = 0$, in which case the applied line itself PM will be the tangent to the curve Mm and the radius of osculation will be $= \frac{A}{2E}$. But whether the curve shall be concave towards R, as the figure shows, or convex, is understood from the equation $0 = At + Ctt + Dtu + Euu$. For since there shall be $Mq = t$ and $qm = u$, on account of t being infinitely less than u the terms tt and tu vanish before uu and there will be $At + Euu = 0$; from which equation it is understood, if the coefficients A and E may have opposite signs or if $\frac{E}{A}$ were a negative quantity, then the curve becomes concave towards R. But if the coefficients A and E may have equal signs and $\frac{E}{A}$ were a positive quantity, then the will be placed on the other side of the tangent ; for the abscissa Mq must be put in place negative, so that it may correspond to a real applied line qm.
[Recall the general equation $-At - Bu = Ctt + Dtu + Euu + Ft^3 + Gttu + \text{etc.}$, which now becomes $-At = Euu$.]

314. Now the tangent $M\mu$ shall be inclined to the axis AP or parallel to that, thus so that the angle $RM\mu$ shall be acute and the normal MN may cut the axis at N beyond P (Fig. 55), in which case the positive applied lines u will correspond to the abscissas t ; so that the coefficients A and B have different signs and the fraction $\frac{A}{B}$ will be negative. Now in this case as we have seen before the curve becomes concave towards N, if

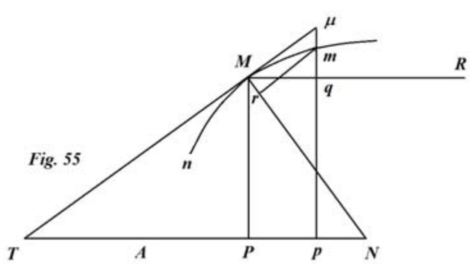


Fig. 55

$$\frac{AAE - ABD + BBC}{B}$$

were a positive quantity or, since $\frac{B}{A}$ shall be a negative quantity, if

$$\frac{AAE - ABD + BBC}{A}$$

were a negative quantity. For if

$$\frac{AAE - ABD + BBC}{B}$$

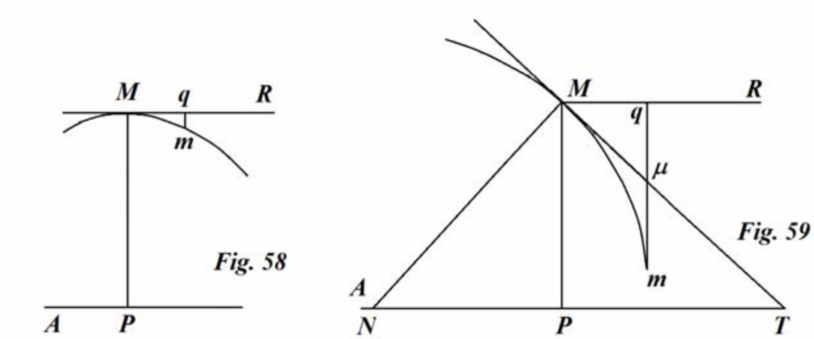
were a negative quantity, or

$$\frac{AAE - ABD + BBC}{A}$$

were positive, then the curve may direct the convexity towards *N*. Truly in each case the radius of osculation will be

$$= \frac{(AA + BB)\sqrt{(AA + BB)}}{2(AAE - ABD + BBC)}.$$

315. Now let $A = 0$, in which case (Fig. 58) *MR* will be the tangent to the curve likewise parallel to the axis and *u* infinitely less than *t*, from which there will be $0 = Bu + Ctt$.



Whereby, if *B* and *C* may have equal signs or if BC were a positive quantity, then *u* must have a negative value ; and thus the curve will be concave towards the point *P*, on which *N* falls, as the above rule with *A* made $= 0$ has shown; truly the radius of osculation will be $= \frac{B}{2C}$. But this same rule, which has been give above, prevails if (Fig. 59) the tangent *MT* may cross the axis beyond *P*; for then equally the curve will be either concave or convex towards *N*, according as this expression

$$\frac{AAE - ABD + BBC}{B}$$

were either positive or negative, and the radius of osculation as before will be

$$= \frac{(AA + BB)\sqrt{(AA + BB)}}{2(AAE - ABD + BBC)}.$$

316. An ellipse shall be proposed, or rather the quadrant (Fig. 60) *DMC,* the centre of is *A,* the one transverse semiaxis $AD = a$, the other conjugate semiaxis $AC = b$. Therefore with the abscissa *x* taken on the axis *AD* from the centre *A,* this equation will be had for the ellipse

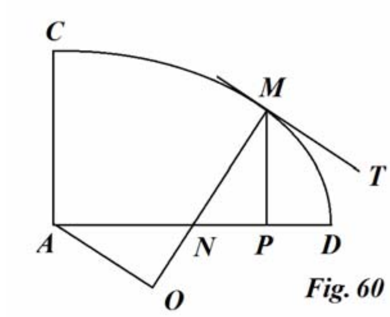


Fig. 60

$$aayy + bbxx = aabb.$$

Now with some abscissa taken $AP = p$ and on putting the applied line $PM = q$ the equation becomes

$$aaqq + bbpp = aabb.$$

Now there may be put $x = p + t$ and $y = q + u$, the equation becomes

$$aaqq + 2aaqu + aauu + bbpp + 2bbpt + bbtt = aabb$$

or

$$2bbpt + 2aaqu + bbtt + aauu = 0.$$

Therefore in the first place, on account of the positive coefficients of *t* and *u*, the normal *MN* will concur with the axes nearer than *P*, and there will be

$$PM : PN = B : A = aaq : bbp \text{ and } PN = \frac{bbp}{aa}$$

on account of $A = 2bbp$ and $B = 2aaq$. Therefore truly on account of $C = bb,\ D = 0$ and $E = aa$, there will be

$$\frac{AAE - ABD + BBC}{B} = \frac{4aabb(aaqq + bbpp)}{2aaq} = \frac{4a^4b^4}{2aaq}$$

and thus a positive quantity, from which it is shown that the curve is concave towards *N*.

317. Towards finding now the radius of osculation, there is

$$AA + BB = 4(a^4qq + b^4pp) \text{ and } AAE - ABD + BBC = 4a^4b^4;$$

from which the radius of osculation will be $\dfrac{(a^4qq + b^4pp)^{\frac{3}{2}}}{a^4b^4}$. But there is

$$MN = \sqrt{(qq + \frac{b^4 pp}{a^4})}$$

from which

$$\sqrt{(a^4 qq + b^4 pp)} = aa \cdot MN$$

and thus the radius of osculation

$$= \frac{aa \cdot MN^3}{b^4}$$

If from the centre A the perpendicular AO may be drawn to the normal MN produced, on account of $AN = p - \frac{bbp}{aa}$ and the similar triangles MNP and ANO, it will be

$$NO = \frac{aabbpp - b^4 pp}{a^4 \cdot MN}$$

and

$$MO = NO + MN = \frac{aaqq + bbpp}{aa \cdot MN} = \frac{bb}{MN},$$

from which $MN = \frac{bb}{MO}$, and hence the radius of osculation $= \frac{aabb}{MO^3}$, which expression is applied equally to each axis AD and AC.

318. Moreover for any curve with the radius of osculation found for any place on the curve, the nature of the curve is seen clearly enough. For if a part of the curve may be divided into many minimal parts, each small part can be regarded as the arc of a circle, of which the radius will be the radius of oscillation at that place. Hence also truly the description of the curve will be resolved more accurately by may points. For after many points were observed, through which the curve may pass, if for these individual points first the tangents and hence again the normals and then the radii of osculation are sought, the minute parts of the curve situated between the points will be able to be described with the aid of compasses. And in this manner truly the figure of the curve will be expressed more accurately, as the points will be closer together than the first considered.

319. Therefore because the small part of the curve (Fig. 55) at M agrees with the small arc of the circle described by the radius of osculation, not only the element Mm, but also the preceding element Mn will be granted the same curvature. For since the nature of the minimal part of the curve Mm may be expressed by an equation of this kind $ss = \alpha r$ between the coordinates $Mr = r$ and $rm = s$, any of the minimal abscissas

$Mr = r$ will correspond to a twofold equation for the applied line *s*, the one positive and the other negative ; and thus the curve will be continued towards *n* and equally towards *m*. Therefore wherever the radius of osculation, which is $\frac{1}{2}\alpha$, has a finite magnitude, there at any rate the curvature will be uniform on both sides through the minimal interval. Therefore in these cases neither will the curve from *M* change suddenly, with a cusp formed, nor is it reflected nor will the part *Mn* be able to change the curvature to become convex directed towards *N*, as long as the other part *Mm* is concave towards *N* ; a change of curvature of this kind is accustomed to be called an *inflection* or an *opposite turning point* : whereby, where the radius of osculation is finite, there neither cusps nor contrary turning points need to be considered.

320. Therefore since from the equation between *t* and *u*,

$$0 = At + Bu + Ctt + Dtu + Euu + Ft^3 + Gttu + Htuu + \text{etc.},$$

the radius of osculation may be found

$$= \frac{(AA+BB)\sqrt{(AA+BB)}}{2(AAE - ABD + BBC)},$$

it is evident, if $AAE - ABD + BBC = 0$, then the radius of osculation becomes infinitely great and therefore the circle of osculation changes into a right line. Therefore where this comes about, there the curved line has no curvature and two elements of the curve are placed as if in a direction. Therefore so that in these cases the nature of the curve may be examined more carefully, the substitution

$$t = \frac{-Ar + Bs}{\sqrt{(AA+BB)}} \quad \text{and} \quad u = \frac{-As - Br}{\sqrt{(AA+BB)}}$$

is required to be put in place also, in the terms $Ft^3 + Gttu + Htuu + Iu^3$. But since besides the first term $r\sqrt{(AA+BB)}$, all the following terms which contain *r* may vanish, with these terms removed and with the substitution made through the whole equation, an equation of this kind will be obtained

$$r\sqrt{(AA+BB)} = \alpha ss + \beta s^3 + \gamma s^4 + \delta s^5 + \text{etc}.$$

[Thus, only the first power of *r* is retained in this situation.]

321. From this equation it may be deduced at once, as above, that the radius of osculation will be

$$\frac{\sqrt{(AA+BB)}}{2\alpha};$$

but if there shall be $\alpha = 0$, in which case the radius of oscillation becomes infinite, towards knowing the nature of the curve more precisely, the following βs^3 term must be taken, thus so that

$$r\sqrt{(AA+BB)} = \beta s^3;$$

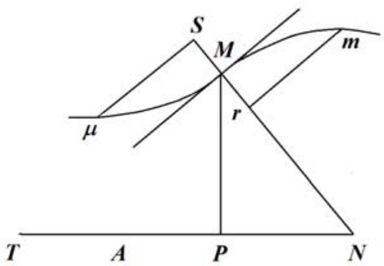


Fig. 61

unless indeed there shall be $\beta = 0$, all the following terms γs^4, δs^5 etc. vanish before this term. Therefore in this case the curve at M (Fig. 61) will osculate with another curve expressed by this equation

$$r\sqrt{(AA+BB)} = \beta s^3,$$

from which likewise the figure of the curve will become known about the point *M*. Therefore since the negative value of the applied line *s* may correspond to the abscissa *r* taken negatively, the curve about *M* will have the snake-like figure $mM\mu$ and thus at *M* there will be a point of opposite curvature.

322. But if besides α, there becomes also $\beta = 0$, then (Fig. 62) the nature of the curve around *M* will be expressed by this equation

$$r\sqrt{(AA+BB)} = \gamma s^4,$$

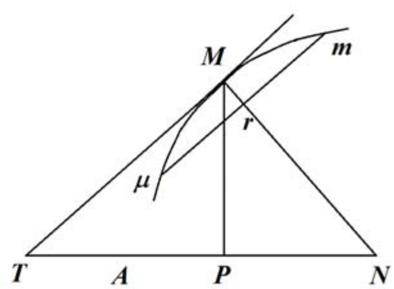


Fig. 62

from which since to whatever single abscissa *r* a two-fold applied line *s* may correspond, the one positive and the other negative, the abscissa *r* may not be taken on both sides, each part of the curve *Mm* and $M\mu$ will be positive on the same side of the tangent. But if, on account of α, β and γ vanishing, the nature of the curve about *M* may be expressed by the equation

$$r\sqrt{(AA+BB)} = \delta s^5,$$

then the curve at *M* again will have a point of opposite curvature, as in figure 61.

But if there were also $\delta = 0$, so that the equation becomes

$$r\sqrt{(AA+BB)} = \varepsilon s^6 \ ,$$

then the curve again will have a point of free from curvature as in figure 62. And generally, if the exponent of *s* were an odd number, the curve at *M* will have a point of opposite curvature, but if the exponent of *s* were an even number, the curve will be free from a point of opposite curvature, as in figure 62.

323. These therefore are the phenomena of curves, if the point *M* were simple or if in the equation

$$0 = At + Bu + Ctt + Dtu + Euu + Ft^3 + \text{etc}.$$

and each coefficient *A* and *B* does not vanish at the same time. But if there were both $A = 0$ and $B = 0$, and the curve may have two or more branches (Fig. 56) themselves intersecting at the point *M*, the curvature and nature of one or other of the branches will be investigated at *M* separately as before. For the equation for the tangent of any branch shall be $mt + nu = 0$ and the equation may be sought for this branch between the coordinates *r* and *s*, of which that one *r* (Fig. 55) may be taken on the normal *MN*, so that *r* shall be infinitely less than *s*. Therefore it will be necessary to put

$$t = \frac{-mr+ns}{\sqrt{(mm+nn)}} \quad \text{and} \quad u = \frac{-ms-nr}{\sqrt{(mm+nn)}};$$

with which done and with the terms vanishing before the rest on account of being infinitely small, the equation will produce, if *M* were a double point, an equation of this kind

$$rs = \alpha s^3 + \beta s^4 + \gamma s^5 + \delta s^6 + \text{ etc.},$$

but if *M* were a triple point, from such :

$$rss = \alpha s^4 + \beta s^5 + \gamma s^6 + \text{ etc.}$$

and thus again ; which equations are all reduced to this form :

$$r = \alpha ss + \beta s^3 + \gamma s^4 + \delta s^5 + \text{ etc.}$$

324. From this equation of the branch of this same curve, which we have considered, at *M* the radius of osculation $= \frac{1}{2}\alpha$, which if $\alpha = 0$, becomes $= \infty$. Therefore in this case the nature of the curve will be expressed either from this equation $r = \beta s^3$, $r = \gamma s^4$, or $r = \delta s^5$ etc.; from which as before, the branch of the curve at *M* is

gathered to have either a point of contrary turning, or to be free from such. Evidently the former arises, if the exponent of *s* itself were an odd number, and the latter if it were an even number. Therefore it will be required to judge separately in this manner concerning whichever branch will be passing through *M*, since if the tangent of this were found, and its tangent may disagree with the tangents of the rest of the curves themselves, at the same point of intersection *M* .

325. Moreover another indication will need to be advanced, if the tangents of two or more branches should meet at the point M (Fig. 55). Indeed with *A* and *B* vanishing in the equation

$$0 = Ctt + Dtu + Euu + Ft^3 + Gttu + \text{ etc }.$$

the first members $Ctt + Dtu + Euu$ shall both be simple equal factors or both branches crossing may themselves have a common tangent at the point *M*. Therefore there shall be

$$Ctt + Dtu + Euu = (mt + nu)^2 ,$$

and the equation for the coordinates $Mr = r$ and $rm = s$ transferred on putting

$$t = \frac{-mr + ns}{\sqrt{(mm + nn)}} \text{ and } u = \frac{-ms - nr}{\sqrt{(mm + nn)}}$$

will produce an equation of this kind :

$$rr = \alpha rss + \beta s^3 + \gamma rs^3 + \delta s^4 + \varepsilon rs^4 + \zeta s^5 + \text{etc.};$$

for the terms, in which *r* has two or more dimensions, vanish before the first term *rr*.

326. Here the first to be looked at is the term βs^3, which if it shall be present, all the rest will vanish before that, as *r* is infinitely less than *s*. Therefore unless there were $\beta = 0$, the nature of the curve around *M* will be expressed by this equation $rr = \beta s^3$; from which, since there shall be $r = s\sqrt{\beta s} = ss\sqrt{\frac{\beta}{s}}$, it is understood that the radius of osculation at *M* to be $= \frac{1}{2}\sqrt{\frac{s}{\beta}}$ or, on account of *s* vanishing at *M*, the radius of osculation also becomes $= 0$. Therefore the radius of osculation at *M* will be infinitely great or an element of the curve at *M* will be an infinitely small part. Because again the applied line *s* maintains the same value, whether the abscissa *r* may take a positive or negative value, it is apparent that the curve at *M* (Fig. 63) has a cusp and to be

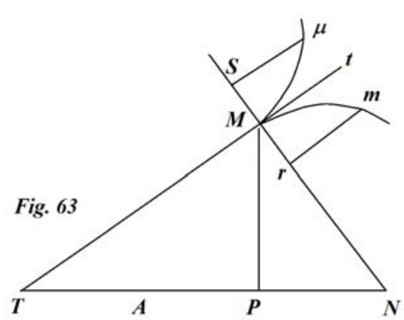


Fig. 63

stretched apart into two branches *Mm*, $M\mu$ mutually touching each other at *M* and turning the convexity towards the *Mt*.

327. Let $\beta = 0$, but the term δs^4 shall be present, before which γrs^3 vanishes ; and the nature of the curve about *M* is expressed by the equation $rr = \alpha rss + \delta s^4$,which, if $\alpha\alpha$ were less than -4δ on account of the imaginary factors indicate the point *p* to be a conjugate at M ; but if $\alpha\alpha$ were greater than -4δ then it is separated into two equations of this kind $r = fss$ and $r = gss$.

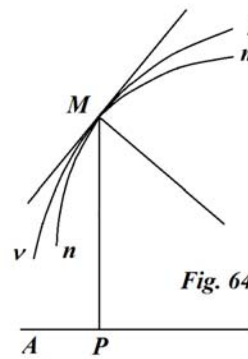


Fig. 64

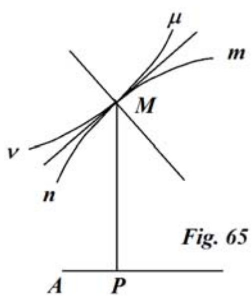


Fig. 65

Whereby at *M* the two branches of the curve mutually touch each other, of which one at M has the radius of osculation $= \frac{1}{2f}$, the other $= \frac{1}{2g}$. Therefore if these two branches (Fig. 64) turn the concavity in the same direction, the figure will be of two circular arcs touching within themselves, but if (Fig. 65) the concavities may be directed in opposite directions, the figure will be of two circular arcs touching each other outside.

328. But if δ also should vanish, then the equation will be resolvable into two equations or otherwise, in the first case two branches themselves arise at the tangent point *M*, the nature of each of which may be expressed by an equation of this kind $r = \alpha s^m$; therefore just as many different figures will be produced, as the number of combinations of the two branches are given, which establish a simple point at *M*, which we will call *branches of the first order,* which all are retained in the equation $r = \alpha s^m$. But in the second case, in which the equation cannot be seen to resolve itself two different forms, the nature of the curve will be expressed either by the equation $rr = \alpha s^5$, $rr = \alpha s^7$, $rr = \alpha s^9$, etc.; which branches with that, which we have found above $rr = \alpha s^3$, we will call *branches of the second order*, because they will hold in turn the place of two branches of the first order of the tangents at M . But these branches of the second order (Fig. 63) all will have a cusp at *M*, as the equation $rr = \alpha s^3$ provided ; yet with this distinction that, since the radius of osculation shall be infinitely small at *M* for the equation $rr = \alpha s^3$, the same may be produced with an infinite size for the remaining equations. Since indeed from the equation $rr = \alpha s^5$ there shall be $r = ss\sqrt{\alpha s}$, the radius of osculation at $M = \frac{1}{2\sqrt{\alpha s}}$, which is infinite on account of $s = 0$.

329. If three tangents of branches crossing themselves at *M* may be incident on each other in turn, then either three branches of the first order touch each other at the same point *M,*

or at M there will be the contact of one branch of the second order with a single branch of the first order, or a single *branch of the third order* will pass through *M.* But the nature of branches of the third order is expressed by equations of this kind : $r^3 = \alpha s^4$, $r^3 = \alpha s^5$, $r^3 = \alpha s^7$,$r^3 = \alpha s^8$ etc., or by these generally : $r^3 = \alpha s^m$, for some whole number *n* present greater than three nor divisible by three. Moreover the figure thus will be prepared of these branches, so that at *M* there shall be a point of opposite flexion, if *n* shall be an odd number ; truly a non-opposite or continuous point shall be present (as in figure 62), if *n* were an even number. Furthermore the radius of osculation at *M* will be infinitely small in these curves, if *n* were less than 6, but infinitely great, if *n* were greater than 6.

330. In a similar manner if four tangents of branches crossing each other at *M* may agree, then either four branches of the first order, or two of the first with one of the second, or two branches of the second order, or one of the first and one of the third meet each other at the same point *M* , or finally *one branch of the fourth order* will pass through *M*. But the nature of the branches of fourth order will be contained in this general equation of $r^4 = \alpha s^m$, with the whole number *n* odd and greater than 4. Moreover all these equations bear a cusp as branches of the second order (Fig. 63). But at *M* the radius of osculation will be infinitely small, if *n* were less than 8, and infinitely great, if *n* were greater than 8.

331. In the same way the nature of the branches of the *fifth* or superior order may be established ; but in the account of the figure of the branches of the fifth, seventh, ninth and of all the odd orders agreeing with the branches of the first order, the figure of which is two − fold either with or without an opposite point of flexion [*i.e.* an inflection point]. But the branches of the sixth, eighth and of all the even orders agree with the account of the figure with branches of the second and fourth orders, clearly all will have a cusp at *M,* as figure 63 shows. But according to the radius of osculation the curve may retain, because the nature of these arcs is expressed by this equation $r^m = \alpha s^n$, with the number *n* present greater than *m,* it is clear, if *n* were less than *2m,* the radius of osculation becomes infinitely small ; truly on the other hand, if *n* were greater than *2m,* infinitely great.

332. Therefore the phenomena, which in all curves offer themselves to be viewed, can be reduced to three kinds. Clearly in the first place the curve is progressing with *continued curvature* nor at any point does it have an inflection point, or a cusp or point of reflection. This first case comes about, if the radius of osculation everywhere were of finite magnitude, then truly also cases are given, in which the magnitude of the continued trace of the radius of osculation undisturbed becomes either infinitely great or infinitely small, which arises in use, if the nature of the curve around the point *M* is expressed by the equation $\alpha r^m = s^n$ with the odd number *m* present but with the even number *n* greater than *m.* The second phenomenon is the *point of opposite flexion,* which cannot be considered, unless the radius of osculation were either infinitely great or infinitely small ; moreover this is indicated by the equation $\alpha r^m = s^n$, if each exponent *m* and *n* were an odd number, with *n* being always greater than *m.* Indeed the radius of osculation will be

infinitely great, if *n* were greater than *2m,* but infinitely small, if *n* were less than *2m.* The third phenomenon is the *point of reflection* or *cusp,* where as if two branches turn towards each other to be convex coming together at a point, touching each other and terminated there ; the equation $\alpha r^m = s^n$ shows such a point*,* if *m* were an even number and *n* odd. Therefore at a cusp the radius of osculation always is either infinitely small or infinitely great.

333. Therefore because all the varieties may be contained in these three kinds of curves, on account of the continued trace : it may be understood that the first branch of the continued curve thus at no point may give a point of inflection, such as the finite angle *ACB* may establish at *C* (Fig. 66). Then, since at a point of reflection both branches turn themselves convexly, the point of reflection *ACB* at *C* does not give a point of reflection of this kind (Fig. 67), where the branches *AC* and *BC* may have a certain common tangent at *C*, but the one may be seen here to be concave and the other convex towards the other ; and as many times as a reflection of this kind may be seen to be present, so it shows how often the curve is not complete ; and, if the curve may be completed according to the norm of the equation and all the following parts may be expressed, a figure will arise, such as is shown in 64. Indeed the ways of describing the curves are given, in which cusps of the kind *ACB* arise, which therefore are called by L'Hôspital *cusps of the second kind*. Truly it is required to be noted that a mechanical description does not always produce the whole curve, which may be contained in a certain equation, but on many occasions only shows a certain part ; by which remark alone the dispute ends, which has arisen about cusps of the second kind.

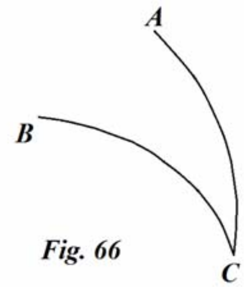


Fig. 66

Fig. 67

[Labey points out in his French translation that there are enumerable algebraic curves that present cusps of this kind, without having to consider cusps of the second kind. He considers for example the equation
$y^4 - 2y2x - 4yxx - x^3 + x^4 = 0,$ giving $y = \sqrt{x} \pm \sqrt[4]{x^3} = \sqrt{x} \pm \sqrt{x\sqrt{x}}$; here the first term must be positive, otherwise the second term becomes imaginary if the first term is made negative.]

334. If two branches (Fig. 64), which have a common tangent at *M* and thus surely are represented by the four arcs *Mm*, $M\mu$, *Mn*, *Mv* departing from *M* may be expressed by different equations, there is no doubt, which of these arcs shall be continued ; clearly these, which are contained in the same equation; and the arc *Mm* by continuation will become the arc *Mn*, and $M\mu$ from the arc *vM* continued. But truly if both these two branches may be expressed by the same equation, then on account of the first reason ceasing, the arc *Mm* equally is able to had for continuing the arcs *vM* and *nM*. But since each arc *Mn* and *Mv* shall be equally able to be had for the continuation of the arc *Mm*, the other also can be had for the other's continuation. Hence the arcs *mM* and $M\mu$ are assessed to constitute the continued curve, and equally for any two arcs whatever, and thus in this case two cusps of the second kind $mM\mu$ and *nNv* may themselves be considered at *M*.

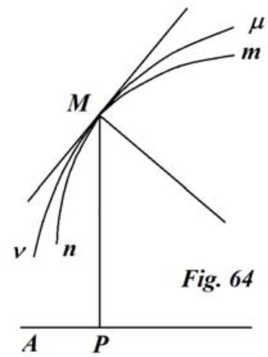


Fig. 64

335. Nor truly does it prevail only with two branches, which without inflection and without a cusp are mutual tangents at *M*, and which are expressed by the same equation, but also the same account will be continued, of whatever kind both these branches should be mutually tangent at *M* , as long as they may be expressed by a common equation. This arises, as many times as an equation of this kind is arising between *r* and *s* for then each branch will be expressed by the same equation $\alpha\alpha r^{2m} - 2\alpha\beta r^m s^n + \beta\beta s^{2n} = 0$; for then each branch is expressed by the same equation $\alpha r^m = \beta s^n$. Therefore in this case any two of the four arcs leaving the point *M* can be regarded as a single line, and hence innumerable cusps of the second kind will arise. But this account of the continuation is the reason why certain descriptions and mechanical constructions sometimes produce cusps of the second kind ; yet this cannot happen, except when the description does not contain the whole curve, but shows only one or some number of branches of this curve.

CAPUT XIV

DE CURVATURA LINEARUM CURVARUM

304. Quemadmodum in superiori capite lineas rectas indagavimus, quae in quovis puncto lineae curvae ipsius directionem indicabant, ita hic lineas curvas simpliciores investigabimus, quae in quovis loco cum curva proposita tam exacte congruant, ut saltem per minimum spatium quasi confundantur. Sic enim cognita indole curvae simplicioris simul curvae propositae natura inde colligetur. Simili methodo scilicet hic utemur, qua supra ad naturam ramorum in infinitum extensorum scrutandam sumus usi; primo videlicet investigando lineam rectam, quae curvam tangat, deinde vero lineam curvam simpliciorem, quae cum curva proposita multo magis conveniat eamque non solum tangat, sed quasi osculetur. Vocari autem eiusmodi linearum curvarum arctissimus contactus solet *osculatio.*

305. Sit igitur proposita aequatio quaecunque inter coordinatas orthogonales x et y, atque ad naturam minimae curvae portionis Mm (Fig. 55) circa punctum M versantis indagandam, cum inventa sit abscissa $AP = p$ et applicata $PM = q$, ponatur in axe MR abscissa minima $Mq = t$ et applicata $qm = u$; eritque $x = p + t$ et $y = q + u$; quibus valoribus in aequatione substitutis perveniatur ad hanc aequationem

$$0 = At + Bu + Ctt + Dtu + Euu + Ft^3 + Gttu + \text{ etc.},$$

quae exprimet naturam curvae eiusdem ad axem MR relatae. Quoniam autem has novas coordinatas t et u minimas statuimus, sequentes termini quasi infinities erunt minores quam antecedentes ideoque prae his sine errore reiici poterunt.

306. Nisi ergo ambo coefficientes primi A et B evanescant, reiectis sequentibus terminis omnibus aequatio $0 = At + Bu$ ostendet lineam rectam $M\mu$, quae curvam in puncto M tanget hocque loco cum curva communem habet directionem. Erit ergo $Mq : q\mu = B : -A$; unde, ob cognitas quantitates A et B, positio tangentis $M\mu$ innotescit, quae cum curvam in puncto tantum M contingat, videamus, quantum curva Mm porro a recta $M\mu$ saltem per minimum spatium aberret. In hunc finem assumamus normalem MN pro axe, in quem ex m applicata orthogonalis mr ducatur, ac vocetur $Mr = r, rm = s$; erit

$$t=\frac{-Ar+Bs}{\sqrt{(AA+BB)}}\quad\text{et}\quad u=\frac{-As-Br}{\sqrt{(AA+BB)}}$$

$$r=\frac{-At-Bu}{\sqrt{(AA+BB)}}\quad\text{et}\quad s=\frac{Bt-Au}{\sqrt{(AA+BB)}}.$$

Quare, cum sit

$$-At-Bu=Ctt+Dtu+Euu+Ft^3+Gttu+\text{etc.},$$

erit r quantitas infinities minor quam t et u, ac propterea erit quoque r quantitas infinities minor quam s; nam s per t et u, at r per ipsarum t et u quadrata vel potestates superiores determinatur.

307. Naturam ergo curvae *Mm* multo propius cognoscemus, si terminos quoque $Ctt+Dtu+Euu$ in computum ducamus atque sequentes tantum negligamus; sicque habebimus inter t et u hanc aequationem

$$-At-Bu=Ctt+Dtu+Euu,$$

in qua si loco t et u valores superiores substituamus, habebimus

$$r\sqrt{(AA+BB)}=\frac{(AAC+ABD+BBE)rr}{AA+BB}+\frac{(AAD-BBD-2ABC+2ABE)rs}{AA+BB}+\frac{(AAE-ABD+BBC)ss}{AA+BB}.$$

At, quia r infinities minor est quam s, termini rr et rs prae termino ss evanescent, fietque

$$ss=\frac{(AA+BB)r\sqrt{(AA+BB)}}{AAE-ABD+BBC},$$

quae aequatio exprimit naturam curvae curvam propositam in M osculantis.

308. Curvae ergo arcus minimus *Mm* congruet cum vertice parabolae super axe *MN* descriptae, cuius latus rectum seu parameter est

$$\frac{(AA+BB)\sqrt{(AA+BB)}}{AAE-ABD+BBC};$$

unde qualis est curvatura huius parabolae in vertice, talis erit curvae propositae curvatura in puncto *M*. Cum autem nullius curvae curvatura distinctius cognoscatur quam circuli, quoniam ipsius curvatura ubique est eadem eoque maior existit, quo minor fuerit radius, commodius erit curvaturam curvarum definire per circulum aequalis curvaturae, qui *circulus osculator* vocari solet. Hanc ob rem oportebit circulum definire, cuius curvatura conveniat cum curvatura propositae parabolae in ipsius vertice, quo tum circulum istum in locum parabolae osculantis substituere liceat.

309. Ad hoc efficiendum contemplemur curvaturam circuli tanquam incognitam eamque modo exposito per curvaturam parabolae exprimamus, sic enim vicissim pro parabola osculante circulus osculator substitui poterit. Sit igitur curva *Mm* proposita circulus radio $= a$ descriptus, cuius natura exprimetur aequatione $yy = 2ax - xx$. Sumta ergo $AP = p$ et $PM = q$ erit $qq = 2ap - pp$. Iam ponatur

$$x = p + t \text{ et } y = q + u,$$

atque orietur haec aequatio

$$qq + 2qu + uu = 2ap + 2at - pp - 2pt - tt,$$

quae ob $qq = 2ap - pp$ reducitur ad hanc formam

$$0 = 2at - 2pt - 2qu - tt - uu,$$

quae cum superiori forma comparata dat

$$A = 2a - 2p, \quad B = -2q, \quad C = -1, \quad D = 0 \quad \text{et} \quad E = -1,$$

unde fit

$$AA + BB = 4(aa - 2ap + pp + qq) = 4aa$$

et

$$\left(AA + BB\right)\sqrt{(AA + BB)} = 8a^3$$

atque

$$AAE - ABD + BBC = -AA - BB = -4aa.$$

Unde circulum, cuius radius $= a$, in quovis puncto osculatur parabolae vertex, cuius natura exprimitur aequatione $ss = 2ar$; ideoque vicissim quam curvam osculatur vertex parabolae $ss = br$, eandem osculabitur circulus, cuius radius est $= \tfrac{1}{2}b$.

310. Cum igitur supra invenerimus curvam *Mm* osculari parabolam, cuius aequatio sit

$$ss = \frac{(AA+BB)\sqrt{(AA+BB)}}{AAE-ABD+BBC}r,$$

manifestum est eiusdem curvae curvaturam in *M* convenire cum curvatura circuli, cuius radius sit

$$= \frac{(AA+BB)\sqrt{(AA+BB)}}{2(AAE-ABD+BBC)}.$$

Haec ergo expressio dat radium circuli osculatoris atque iste radius quoque vocari solet *radius osculi;* saepe etiam *radius curvedinis* seu *curvaturae* appellatur. Ex aequatione ergo inter *t* et *u*, quam ex aequatione inter *x* et *y* proposita elicuimus, statim definiri potest radius osculi curvae in puncto *M* seu radius circuli osculantis curvae in *M*. In aequatione enim inter *t* et *u* reiiciantur termini, in quibus *t* et *u* plures duabus dimensiones obtinent, atque ex aequatione, quae erit huius formae

$$0 = At + Bu + Ctt + Dtu + Euu,$$

invenietur radius osculi

$$= \frac{(AA+BB)\sqrt{(AA+BB)}}{2(AAE-ABD+BBC)}.$$

311. Quoniam vero signum radicale $\sqrt{(AA+BB)}$ ambiguitatem signi involvit, incertum est, utrum ista expressio sit affirmativa an negativa, scilicet utrum concavitas curvae punctum *N* respiciat an convexitas. Ad hoc dubium tollendum quaeri debet, utrum curvae punctum *m* intra tangentem $M\mu$ versus axem *AN* sit positum an vero extra tangentem cadat. Priori casu curva versus *N* erit concava atque centrum circuli osculantis in rectae *MN* portionem versus axem protensam incidet; posteriori casu vero in portionem rectae *NM* ultra *M* productam. Omnis ergo dubitatio evanescet, si inquiratur, utrum *qm* sit minor quam $q\mu$ an maior, priori enim casu curva versus *N* erit concava, posteriori vero convexa.

312. Est vero $q\mu = -\frac{At}{B}$ et $qm = u$, quare videndum est, utrum sit $-\frac{At}{B}$ maior minorve quam *u*. Quia igitur $m\mu$ est lineola quam minima, ponatur $m\mu = w$ eritque $u = -\frac{At}{B} - w$; unde, facta substitutione, fit

$$0 = -Bw + Ctt - \frac{ADtt}{B} - Dtw + \frac{AAEtt}{BB} + \frac{2AEtw}{B} + Eww;$$

ubi ob *w* prae *t* minimum termini *tw* et *ww* evanescunt. Hinc fit

$$w = \frac{(BBC - ABD + AAE)tt}{B^3}.$$

Quodsi ergo *w* fuerit quantitas affirmativa, quod evenit, si

$$\frac{BBC - ABD + AAE}{B^3} \text{ seu } \frac{AAE - ABD + BBC}{B}$$

fuerit quantitas affirmativa, tum curva erit concava versus *N*, sin autem fuerit quantitas negativa, curvae convexitas punctum *N* respiciet.

313. Quo haec clariora reddantur, diversi casus (Fig. 57), qui occurrere possunt, seorsim sunt evolvendi. Sit igitur primum $B = 0$, quo casu ipsa applicata *PM* erit tangens curvae *Mm* et radius osculi erit $= \frac{A}{2E}$. Utrum autem curva sit concava versus *R*, uti figura praesentat, an convexa, ex aequatione $0 = At + Ctt + Dtu + Euu$ intelligitur. Cum enim sit $Mq = t$ et $qm = u$, ob *t* infinities minus quam *u* termini *tt* et *tu* prae *uu* evanescent eritque $At + Euu = 0$; ex qua aequatione intelligitur, si coefficientes *A* et *E* habeant contraria signa seu si $\frac{E}{A}$ fuerit quantitas negativa, tum

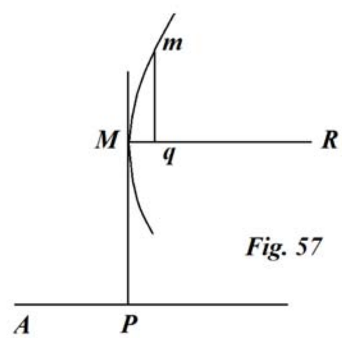


Fig. 57

curvam fore concavam versus *R*. At si coefficientes *A* et *E* habeant paria signa et $\frac{E}{A}$ fuerit quantitas affirmativa, tum curva ad alteram tangentis partem erit sita; abscissa enim *Mq* statui debet negativa, quo applicata *qm* respondeat realis.

314. Sit nunc tangens $M\mu$ inclinata ad axem *AP* seu ipsi parallelam, ita ut angulus $RM\mu$ sit acutus et normalis *MN* axem in *N* ultra *P* (Fig. 55) secet, quo casu abscissis *t* respondebunt applicatae *u* affirmativae; unde coefficientes *A* et *B* signa habebunt disparia et fractio $\frac{A}{B}$ erit negativa. De hoc casu iam ante vidimus curvam fore concavam versus *N*, si fuerit

$$\frac{AAE - ABD + BBC}{B}$$

quantitas affirmativa vel, cum $\frac{B}{A}$ sit quantitas negativa, si fuerit

$$\frac{AAE-ABD+BBC}{A}$$

quantitas negativa. Sin autem fuerit

$$\frac{AAE-ABD+BBC}{B}$$

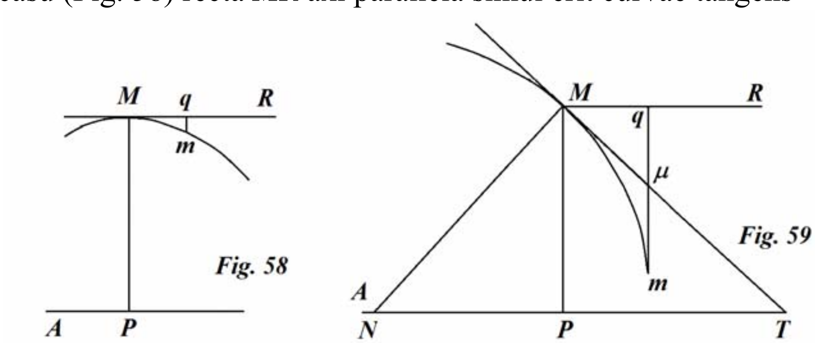
quantitas negativa seu

$$\frac{AAE-ABD+BBC}{A}$$

quantitas affirmativa, tum curva versus *N* convexitatem obvertet. Utroque vero casu radius osculi erit

$$=\frac{(AA+BB)\sqrt{(AA+BB)}}{2(AAE-ABD+BBC)}.$$

315. Sit nunc $A=0$, quo casu (Fig. 58) recta *MR* axi parallela simul erit curvae tangens et *u* infinities minor quam *t,* unde erit $0=Bu+Ctt$. Quare, si *B* et *C* habeant aequalia signa seu si *BC* fuerit quantitas affirmativa, tum *u* habere debet valorem negativum; ideoque curva erit concava versus punctum *P,* in quod *N* incidit, quod ipsum regula superior facto $A=0$ ostendit; radius osculi vero erit $=\frac{B}{2C}$ Haec autem eadem regula, quae supra est data, valet, si (Fig. 59) tangens *MT* ultra *P* cum axe concurrat; tum enim pariter curva versus *N* erit vel concava vel convexa, prout haec expressio



$$\frac{AAE-ABD+BBC}{B}$$

fuerit vel affirmativa vel negativa, eritque radius osculi ut ante

$$=\frac{(AA+BB)\sqrt{(AA+BB)}}{2(AAE-ABD+BBC)}\ .$$

316. Sit proposita ellipsis, seu saltem eius quadrans (Fig. 60) *DMC*, cuius centrum *A*, alter semiaxis transversus $AD=a$, alter semiaxis coniugatus $AC=b$. Sumtis ergo abscissis *x* in axe *AD* a centro *A*, habebitur haec aequatio pro ellipsi

$$aayy+bbxx=aabb.$$

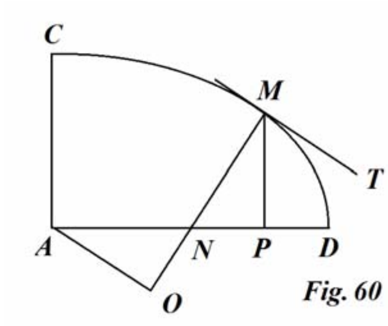


Fig. 60

Sumta iam quapiam abscissa $AP=p$ et posita applicata $PM=q$ erit

$$aaqq+bbpp=aabb.$$

Ponatur iam $x=p+t$ et $y=q+u$, erit

$$aaqq+2aaqu+aauu+bbpp+2bbpt+bbtt=aabb$$

seu

$$2bbpt+2aaqu+bbtt+aauu=0.$$

Primum ergo, ob coefficientes ipsarum *t* et *u*, normalis *MN* citra *P* cum axe concurrit, eritque

$$PM:PN=B:A=aaq:bbp \text{ et } PN=\frac{bbp}{aa}$$

ob $A=2bbp$ et $B=2aaq$. Praeterea vero ob $C=bb,\ D=0$ et $E=aa$, erit

$$\frac{AAE-ABD+BBC}{B}=\frac{4aabb(aaqq+bbpp)}{2aaq}=\frac{4a^4b^4}{2aaq}$$

ideoque quantitas affirmativa, qua indicatur curvam versus *N* esse concavam.

317. Ad ipsum iam radium osculi inveniendum, est

$$AA+BB=4(a^4qq+b^4pp) \text{ et } AAE-ABD+BBC=4a^4b^4;$$

unde radius osculi erit $\frac{(a^4qq+b^4pp)^{\frac{3}{2}}}{a^4b^4}$. At est

$$MN=\sqrt{(qq+\frac{b^4pp}{a^4})}$$

unde

$$\sqrt{(a^4qq+b^4pp)}=aa\cdot MN$$

ideoque radius osculi

$$=\frac{aa\cdot MN^3}{b^4}$$

Si in normalem *MN* productam ex centro *A* ducatur perpendiculum *AO,* erit ob $AN=p-\frac{bbp}{aa}$ et triangula *MNP* et *ANO* similla,

$$NO=\frac{aabbpp-b^4pp}{a^4\cdot MN}$$

et

$$MO=NO+MN=\frac{aaqq+bbpp}{aa\cdot MN}=\frac{bb}{MN},$$

und $MN=\frac{bb}{MO}$, hincque radius osculi $=\frac{aabb}{MO^3}$, quae expressio ad utrumque axem *AD* et *AC* aeque est accommodata.

318. Invento autem pro quovis curvae loco radio osculi natura curvae satis clare perspicitur. Si enim portio curvae in partes plurimas quam minimas dividatur, unaquaeque particula haberi potest pro arculo circuli, cuius radius erit ipse radius osculi in eo loco. Hinc vero etiam descriptio curvae per plurima puncta multo accuratius absolvetur. Postquam enim plura notata fuerint puncta, per quae curva transeat, si pro his singulis punctis primo quaerantur tangentes hincque porro normales atque tum radii osculi, portiunculae curvae intra puncta inventa sitae ope circini poterunt describi. Hocque modo eo accuratius vera curvae figura exprimetur, quo propiora fuerint puncta primum notata.

319. Quoniam igitur (Fig. 55) portiuncula curvae ad *M* cum arculo circuli radio osculi descripti congruit, non solum elementum *Mm,* sed etiam praecedens *Mn* eadem curvatura erit praeditum. Cum enim natura minimae curvae portionis *Mm* exprimatur huiusmodi aequatione $ss=\alpha r$ inter coordinatas $Mr=r$ et $rm=s$, unicuique abscissae minimae $Mr=r$ ex aequatione duplex respondebit applicata *s,* altera affirmativa, altera negativa; ideoque curva versus *n* aeque ac versus *m* continuabitur. Ubicunque ergo radius osculi,

qui est $\frac{1}{2}\alpha$, finitam habet magnitudinem, ibi curvatura utrinque saltem per minimum spatiolum erit uniformis. Neque ergo his casibus curva ex *M* subito, formata cuspide, reflectetur neque mutata curvatura portio *Mn* convexitatem versus *N* obvertere poterit, dum altera *Mm* est concava versus *N;* cuiusmodi curvaturae immutatio vocari solet *inflexio* vel *punctum flexus contrarii,* quare, ubi radius osculi est finitus, ibi neque cuspis neque punctum flexus contrarii locum habere potest.

320. Cum igitur ex aequatione inter *t* et *u*

$$0 = At + Bu + Ctt + Dtu + Euu + Ft^3 + Gttu + Htuu + \text{etc.}$$

inventus sit radius osculi

$$= \frac{(AA+BB)\sqrt{(AA+BB)}}{2(AAE-ABD+BBC)},$$

manifestum est, si fuerit $AAE - ABD + BBC = 0$, tum radium osculi fieri infinite magnum ideoque circulum osculantem in lineani rectam abire. Ubi ergo hoc evenit, ibi linea curva curvatura destituitur atque duo curvae elementa quasi in directum erunt sita. Quo igitur his casibus natura curvae penitius perspiciatur, substitutio

$$t = \frac{-Ar+Bs}{\sqrt{(AA+BB)}} \quad \text{et} \quad u = \frac{-As-Br}{\sqrt{(AA+BB)}}$$

etiam in terminis $Ft^3 + Gttu + Htuu + Iu^3$ est instituenda. Cum autem prae termino primo $r\sqrt{(AA+BB)}$ omnes termini sequentes, qui *r* continent, evanescant, his terminis reiectis atque substitutione per totam aequationem facta obtinebitur eiusmodi aequatio

$$r\sqrt{(AA+BB)} = \alpha ss + \beta s^3 + \gamma s^4 + \delta s^5 + \text{etc}.$$

321. Ex hac aequatione iam statim colligitur, ut supra, radius osculi

$$\frac{\sqrt{(AA+BB)}}{2\alpha};$$

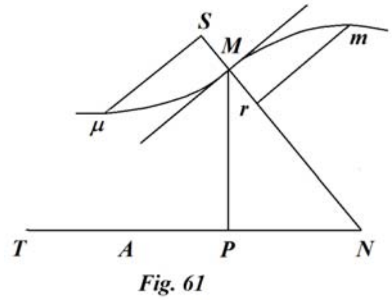


Fig. 61

sin autem sit $\alpha = 0$, quo casu radius osculi fit infinitus, ad curvae naturam exactius cognoscendam sumi debet terminus sequens βs^3, ita ut sit

$$r\sqrt{(AA+BB)} = \beta s^3;$$

nisi enim sit $\beta = 0$, termini sequentes γs^4, δs^5 etc. omnes prae hoc evanescunt. Curvam ergo hoc casu (Fig. 61) in *M* osculabitur curva hac aequatione

$$r\sqrt{(AA+BB)} = \beta s^3$$

expressa, ex qua simul figura curvae circa punctum *M* cognoscetur. Cum igitur abscissae *r* negative sumtae negativus valor applicatae *s* respondeat, curva circa *M* figuram habebit anguineam $mM\mu$ ideoque in *M* habebit punctum flexus contrarii.

322. Quodsi praeter α etiam fiat $\beta = 0$, tum (Fig. 62) natura curvae circa *M* exprimetur hac aequatione

$$r\sqrt{(AA+BB)} = \gamma s^4,$$

ex qua cum unicuique abscissae *r* duplex applicata *s* respondeat, altera affirmativa, altera negativa, neque abscissa *r* utrinque sumi queat, utraque curvae portio *Mm* et $M\mu$, ad eandem tangentis partem erit posita. At si, ob α, β et γ evanescentes, natura curvae circa *M* exprimatur aequatione

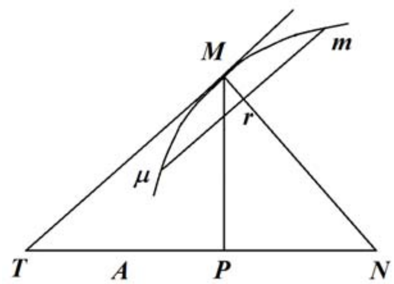


Fig. 62

$$r\sqrt{(AA+BB)} = \delta s^5,$$

tum curva ad *M* iterum habebit punctum flexus contrarii, uti in figura 61.
Sin autem fuerit etiam $\delta = 0$, ut fiat

$$r\sqrt{(AA+BB)} = \varepsilon s^6,$$

tum curva iterum puncto flexus contrarii destituetur uti figura 62. Atque generaliter, si exponens ipsius *s* fuerit numerus impar, curva in *M* habebit punctum flexus contrarii, sin autem exponens ipsius *s* fuerit numerus par, curva carebit puncto flexus contrarii, uti in figura 62.

323. Haec igitur sunt curvarum phaenomena, si punctum *M* fuerit simplex seu si in aequatione

$$0 = At + Bu + Ctt + Dtu + Euu + Ft^3 + \text{etc}.$$

non uterque coefficiens *A* et *B* simul evanescat. Quodsi autem fuerit et $A = 0$ et $B = 0$, curvaque habuerit (Fig. 56) duos pluresve ramos se in puncto *M* intersecantes, uniuscuiusque rami curvatura et indoles in *M* investigabitur seorsim ut ante. Sit enim pro tangente cuiusvis rami $mt + nu = 0$ et quaeratur aequatio pro hoc ramo inter coordinatas *r* et *s*, quarum illa *r* (Fig. 55) in normali *MN* capiatur, ut sit *r* infinities minor quam *s*. Poni ergo debebit

$$t = \frac{-mr+ns}{\sqrt{(mm+nn)}} \text{ et } u = \frac{-ms-nr}{\sqrt{(mm+nn)}};$$

quo facto et neglectis terminis ob infinitam parvitatem prae reliquis evanescentibus prodibit, si *M* fuerit punctum duplex, huiusmodi aequatio

$$rs = \alpha s^3 + \beta s^4 + \gamma s^5 + \delta s^6 + \text{ etc.},$$

sin autem *M* fuerit punctum triplex, talis

$$rss = \alpha s^4 + \beta s^5 + \gamma s^6 + \text{ etc.}$$

et ita porro; quae aequationes omnes reducuntur ad hanc formam

$$r = \alpha ss + \beta s^3 + \gamma s^4 + \delta s^5 + \text{ etc.}$$

324. Ex hac aequatione intelligitur istius curvae rami, quem consideramus, in *M* esse radium osculi $= \frac{1}{2}\alpha$, qui, si $\alpha = 0$, fiet $= \infty$. Hoc ergo casu natura curvae exprimetur vel hac aequatione $r = \beta s^3$ vel $r = \gamma s^4$ vel $r = \delta s^5$ etc.; ex quibus, ut ante, colligetur curvae ramum in *M* vel punctum flexus contrarii habere vel tali carere. Prius scilicet evenit, si exponens ipsius *s* fuerit numerus impar, posterius, si sit numerus par. Hoc ergo modo iudicandum erit de quovis ramo per punctum *M* transeunte seorsim, cum reperta fuerit eius tangens, eiusque tangens discrepet a tangentibus reliquorum ramorum sese in eodem puncto *M* intersecantium.

325. Aliud autem indicium erit ferendum, si duorum pluriumve ramorum tangentes (Fig. 55) in puncto *M* coincidant. Sint enim evanescentibus *A* et *B* in aequatione

$$0 = Ctt + Dtu + Euu + Ft^3 + Gttu + \text{ etc .}$$

primi membri $Ctt + Dtu + Euu$ ambo factores simplices aequales seu ambo rami se in puncto *M* decussantes communem habeant tangentem. Sit ergo

$$Ctt + Dtu + Euu = (mt+nu)^2,$$

atque aequatione ad coordinatas $Mr = r$ et $rm = s$ translata ponendo

$$t = \frac{-mr+ns}{\sqrt{(mm+nn)}} \text{ et } u = \frac{-ms-nr}{\sqrt{(mm+nn)}}$$

huiusmodi prodibit aequatio

$$rr = \alpha rss + \beta s^3 + \gamma rs^3 + \delta s^4 + \varepsilon rs^4 + \zeta s^5 + \text{etc.};$$

termini enim, in quibus *r* habet duas pluresve dimensiones, prae primo *rr* evanescunt.

326. Hic primum spectandus est terminus βs^3, qui si adfuerit, prae eo reliqui omnes evanescunt, propterea quod *r* infinities minus est quam *s*. Nisi ergo fuerit $\beta = 0$, natura curvae circa *M* exprimetur hac aequatione $rr = \beta s^3$; ex qua, cum sit $r = s\sqrt{\beta s} = ss\sqrt{\frac{\beta}{s}}$, intelligitur radium osculi in *M* esse $= \frac{1}{2}\sqrt{\frac{s}{\beta}}$ seu, ob *s* evanescens in *M*, radium osculi quoque fieri $= 0$. Erit ergo curvatura in *M* infinite magna seu elementum curvae in *M* erit portio circuli infinite parvi. Quoniam porro applicata *s* eundem obtinet valorem, sive abscissa *r* sumatur affirmativa sive negativa, patet (Fig. 63) curvam in *M* habere cuspidem atque in duos ramos *Mm*, $M\mu$ divaricari se mutuo in *M* contingentes atque tangenti *Mt* convexitatem obvertentes.

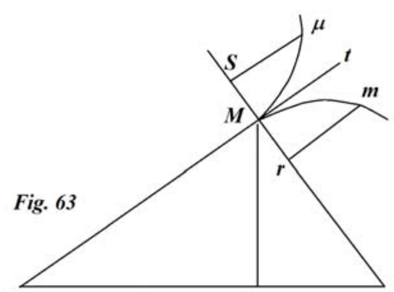


Fig. 63

327. Sit $\beta = 0$, adsit autem terminus δs^4, prae quo γrs^3 evanescit; atque natura curvae circa *M* exprimetur aequatione $rr = \alpha rss + \delta s^4$, quae, si fuerit $\alpha\alpha$ minor quam -4δ ob factores imaginarios punctum coniugatum *p* in *M* indicat; sin autem $\alpha\alpha$ maior quam -4δ dum in duas aequationes huiusmodi $r = fss$ et $r = gss$ dispescitur. Quare in *M* duo curvae rami se mutuo contingent, quorum alterius in *M* radius osculi est $= \frac{1}{2f}$, alterius $= \frac{1}{2g}$. Si ergo (Fig. 64) hi duo rami concavitatem in eandem plagam vertant, figura erit duorum arcuum circularium se intus tangentium, sin autem (Fig. 65) concavitates in plagas oppositas dirigantur, figura erit duorum arcuum circularium se extus tangentium.

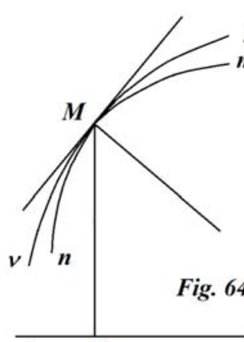


Fig. 64

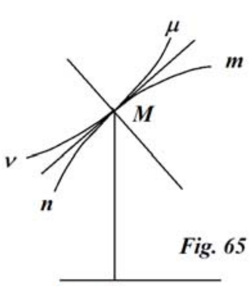


Fig. 65

328. Sin etiam δ evanescat, tum aequatio vel in duas aequationes erit resolubilis vel secus, priori casu duo oriuntur rami se in puncto *M* tangentes, quorum utriusque natura

exprimetur huiusmodi aequatione $r=\alpha s^m$; prodibunt ergo tot diversae figurae, quot dantur combinationes binorum ramorum, qui in *M* punctum simplex constituunt, quos vocemus *ramos primi ordinis,* qui omnes in aequatione $r=\alpha s^m$ continentur. Posteriori autem casu, quo aequatio in duas alias se resolvi non patitur, natura curvae exprimetur aequatione vel $rr=\alpha s^5$ vel $rr=\alpha s^7$ vel $rr=\alpha s^9$ etc.; quos ramos cum eo, quem supra invenimus $rr=\alpha s^3$, *ramos secundi ordinis* appellabimus, quia vicem tenent duorum ramorum primi ordinis se in *M* tangentium. Hi autem (Fig. 63) rami secundi ordinis omnes in *M* habebunt cuspidem, uti praebuit aequatio $rr=\alpha s^3$; hoc tamen discrimine, quod, cum radius osculi in *M* pro aequatione $rr=\alpha s^3$ esset infinite parvus, idem pro reliquis aequationibus prodeat infinite magnus. Cum enim ex aequatione $rr=\alpha s^5$ sit $r=ss\sqrt{\alpha s}$, erit radius osculi in $M=\dfrac{1}{2\sqrt{\alpha s}}$, hoc est ob $s=0$ infinitus.

329. Si tres tangentes ramorum se in *M* decussantium in se invicem incidant, tum vel tres rami primi ordinis se in eodem puncto *M* contingent vel in M erit contactus unius rami secundi ordinis cum uno ramo primi ordinis vel unicus per *M* transibit *ramus tertii ordinis.* Ramorum autem tertii ordinis natura exprimetur huiusmodi aequationibus $r^3=\alpha s^4$, $r^3=\alpha s^5$, $r^3=\alpha s^7$, $r^3=\alpha s^8$ etc., seu hac generali $r^3=\alpha s^m$, existente *n* numero quocunque integro ternario maiore neque per ternarium divisibili. Horum ramorum autem figura ita erit comparata, ut in *M* sit punctum flexus contrarii, si *n* fuerit numerus impar; flexus vero non contrarius seu continuus (ut in figura 62) adsit, si *n* fuerit numerus par. Ceterum in his curvis radius osculi in *M* erit infinite parvus, si *n* minor quam 6, at infinite magnus, si *n* maior quam 6.

330. Simili modo si quatuor tangentes ramorum se in *M* decussantium congruant, tum vel quatuor rami primi ordinis vel duo primi et unus secundi vel duo rami secundi ordinis vel unus primi et unus tertii ordinis se in eodem puncto *M* contingent vel denique unicus *ramus quarti ordinis* per *M* transibit. Ramorum autem quarti ordinis natura continetur hac aequatione generali $r^4=\alpha s^m$, existente *n* numero integro impari maiore quam 4. Hae autem aequationes omnes praebent cuspidem uti rami (Fig. 63) secundi ordinis.
At in *M* erit radius osculi infinite parvus, si *n* minor quam 8, infinite magnus autem, si *n* maior quam 8.

331. Eodem modo ramorum *quinti* superiorumve ordinum natura evolvetur; ratione figurae autem rami quinti, septimi, noni omniumque imparium ordinum conveniunt cum ramis primi ordinis, quorum duplex est figura vel cum puncto flexus contrarii vel sine eo. Rami autem sexti, octavi et omnium parium ordinum conveniunt ratione figurae cum ramis secundi et quarti ordinis, omnes scilicet habebunt cuspidem in *M,* uti figura 63 exhibet. Quod autem ad radium osculi attinet, quoniam horum arcuum natura exprimitur hac aequatione $r^m=\alpha s^n$ existente *n* numero maiore quam *m,* perspicuum est, si fuerit *n* minor quam *2m,* radium osculi fore infinite parvum; contra vero, si *n* maior quam *2m,* infinite magnum.

332. Phaenomena ergo, quae in omni curva conspectui se offerunt, ad tria genera reducuntur. Primo scilicet curva *continua curvatura* progreditur neque usquam punctum flexus contrarii habet neque cuspidem seu punctum reflexionis. Evenit hoc primum, si radius osculi ubique fuerit finitae magnitudinis, tum vero etiam dantur casus, quibus radii osculi magnitudo sive infinite magna sive infinite parva continuum tractum non perturbat, quod usu venit, si natura curvae circa punctum *M* exprimitur aequatione $\alpha r^m = s^n$ existente *m* numero impari at *n* numero pari maiori quam *m.* Secundum phaenomenon est *punctum flexus contrarii,* quod locum habere nequit, nisi radius osculi fuerit vel infinite magnus vel infinite parvus; indicatur autem aequatione $\alpha r^m = s^n$, si uterque exponens *m* et *n* fuerit numerus impar, existente semper *n* maiore quam *m.* Erit enim radius osculi infinite magnus, si *n* maior quam *2m,* at infinite parvus, si *n* minor quam *2m.* Tertium phaenomenon est *punctum reflexionis* seu *cuspis,* ubi duo quasi rami versus se invicem convexi in puncto coeuntes se tangunt atque terminantur; tale punctum monstrat aequatio $\alpha r^m = s^n$, si *m* fuerit numerus par et *n* impar. In cuspide ergo radius osculi semper est vel infinite parvus vel infinite magnus.

333. Quoniam igitur in his tribus generibus omnes curvarum, ratione tractus continui, varietates continentur, primum intelligitur curvae continuae ramum nunquam ita inflexum dari, ut (Fig. 66) in *C* angulum finitum *A 0 B* constituat. Deinde, cum in puncto reflexionis ambo rami sibi convexitatem obvertant, eiusmodi (Fig. 67) punctum reflexionis *ACB* in *C* non datur, ubi rami *AC* et *BC* in *C* quidem communem tangentem habeant, at alterius concavitas alterius convexitatem respiciat ; et quoties huiusmodi reflexio adesse videatur, toties curva non est completa; et, si curva ad normam aequationis compleatur ac secundum omnes partes exprimatur, orietur figura, qualis in figura 64 exhibetur. Dantur quidem curvarum describendarum modi, quibus eiusmodi cuspis *ACB* oritur, quae propterea ab Hospitalio *cuspis secundae speciei* vocatur. Verum notandum est descriptiones mechanicas non semper totam curvam, quae quidem aequatione contineatur, producere, sed saepenumero certam tantum partem exhibere, qua sola notatione lis, quae circa hanc cuspidem secundae speciei est mota, dirimitur.

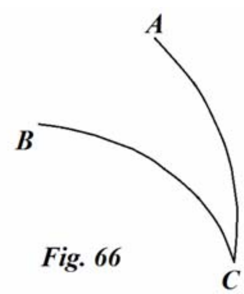


Fig. 66

Fig. 67

334. Si (Fig. 64) duo rami, qui in *M* communem habent tangentem ideoque quatuor arcus ex *M* exeuntes repraesentant nempe *Mm,* $M\mu$, *Mn, Mv,* diversis aequationibus exprimantur, dubium est nullum, quinam horum arcuum sint continui; ii scilicet, qui sub eadem aequatione continentur; eritque arcus *Mm* continuatio arcus *Mn* et $M\mu$, , continuatio arcus *vM.* Quodsi vero ambo rami illi eadem aequatione exprimantur, tum ob cessantem rationem priorem arcus *Mm* aeque haberi potest pro continuatione arcus *vM* atque arcus *nM.* Cum autem uterque arcus *Mn* et *Mv* aeque haberi possit pro continuatione arcus *Mm,* etiam alter pro alterius continuatione haberi poterit. Hinc arcus *mM* et $M\mu$, curvam continuam constituere censendi sunt, aeque ac bini arcus

quicunque alii, sicque hoc casu in *M* se respicient duae cuspides secundae speciei $mM\mu$ et *nNv*.

335. Neque vero solum valet de duobus ramis, qui sine flexu contrario ac sine cuspide se mutuo in *M* tangunt atque eadem aequatione exprimuntur, sed etiam eadem erit continuitatis ratio, cuiuscunque generis fuerint ambo illi rami se mutuo in *M* tangentes, dummodo communi aequatione exprimantur. Evenit hoc, quoties inter *r* et *s* ad huiusmodi pervenitur aequationem tum enim uterque ramus eadem aequatione $\alpha\alpha r^{2m} - 2\alpha\beta r^m s^n + \beta\beta s^{2n} = 0$; tum enim uterque ramus eadem aequatione $\alpha r^m = \beta s^n$ exprimetur. Hoc igitur casu quatuor arcuum ex puncto *M* exeuntium duo quicunque pro una linea continua haberi possunt, hincque nascentur innumerabiles cuspides secundae speciei. Haec autem ipsa continuitatis ratio in causa est, quod quaedam descriptiones ac constructiones mechanicae nonnumquam cuspides secundae speciei producant; hoc tamen evenire non potest, nisi quando descriptio non totam curvam in aequatione contentam, sed eius tantum ramum unum vel aliquot exhibet.