CHAPTER XIII

TOWARDS AN UNDERSTANDING OF CURVED LINES

285. Just as above we have described the nature of the infinite extensions of the branches thus, so that we have assigned a right line or a simpler curved line, which will merge with that curved line at infinity, thus in this chapter we have established some part of a curve present in a finite interval to be subjected to examination and to investigate some right line or simpler curve, which perhaps may agree with that part of the curve through the smallest interval. And certainly in the first place it is apparent every right line, which touches the curve, in that place where it touches, agrees with the line of the curve drawn or to have two points in common as a minimum. Then truly also other curved lines can be shown, which may agree more accurately with a part of the line and that as if osculating at that point. [The Latin translates as 'kissing'.] Moreover with these known, the state of a curved line and its properties at some point are most clearly evident.

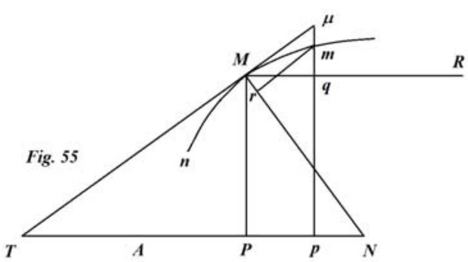

Fig. 55

286. Therefore let there be some equation proposed between the coordinates *x* and *y* for a certain curve. The value of some abscissa *x* may be granted, (Fig. 55), $AP = p$ and the values of the applied line *y* may be sought corresponding to this abscissa, which if there several, for argument's sake one may be taken $PM = q$, and *M* will be a point on the curve, or a point through which the curve will pass. Then truly, if in the proposed equation between *x* and *y* there may be written *p* in place of *x* and *q* in place of *y*, all the terms of the equation mutually cancel each other, so that nothing may be left. Now investigating the nature of the part of this curve, which passes through the point *M*, the right line *Mq* is drawn from the point *M* parallel to the axis *AP*, which now may be taken for the axis, and here the new abscissa may be called $Mq = t$, the applied line $qm = u$. Therefore because the point *m* is placed equally on the curve, if *mq* may be produced as far as to the former axis at *p* and $Ap = p + t$ may be substituted in place of *x* and $pm = q + u$ in place of *y*, an identical equation must be produced equally.

287. But with this substitution made in the equation proposed between *x* and *y*, all the terms, in which neither *t* nor *u* is absent, mutually cancel each other and these terms, which contain the new coordinates *t* and *u*, will be present only. Hence an equation of this kind will be produced

$$0 = At + Bu + Ctt + Dtu + Euu + Ft^3 + Fttu + Htuu + \text{etc.},$$

where *A*, *B*, *C*, *D* etc. are constant quantities composed from the constants of the first equations and from *p* and *q* themselves, which we have now for constants. Therefore the nature of the same curve is expressed in this new equation, truly referred to the axis *Mq*, and in which a point of the curve itself *M* is taken for the start of the abscissas.

288. And indeed in the first place it is apparent, if there may be put $Mq = t = 0$, then also there is $qm = u = 0$, because the point *m* falls on *M*. Then, because we wish to investigate only a minimum part of the curve moving around *M*, we will obtain this, if we assume values for *t* as minimal ; in which case also $qm = u$ will have a minimum value ; for we wish only as if to find the nature of the vanishing arc *Mm*. Because if truly for *t* and *u* values may be taken as minimal, the terms *tt*, *tu* and *uu* at this stage will be much smaller and the following $t^3,\ ttu,\ tuu,\ u^3$ etc. also will be much smaller than these and so one thus; on account of which, since the smallest terms may be able to be omitted besides these as if infinitely greater ones, this equation $0 = At + Bu$ will remain, which is the equation for the right line $M\mu$ passing through the point *M* and it will show that line, if the point *m* approaches *M* closely, to agree with the curve.

289. Therefore this right line $M\mu$ will be the tangent of the curve at the place *M*, and thus hence at some point of the curve *M* the tangent μMT can be drawn. Clearly, since from the equation $At + Bu = 0$ there shall be

$$\frac{u}{t} = -\frac{A}{B} = \frac{q\mu}{Mq},$$

there will be

$$q\mu : Mq = MP : PT = -A : B.$$

Therefore, since there shall be $PM = q$, there becomes $PT = -\frac{Bq}{A}$; moreover this part of the axis *PT* is accustomed to be called the *subtangent.* Therefore from these this is deduced :

RULE FOR FINDING THE SUBTANGENT

In the equation for a curve, upon finding $x = p$ *of the abscissas to satisfy the applied line* $y = q$, *there may be put* $x = p + t$ *and* $y = q + u$ *; but from the boundaries, which arise from the substitution, only these may be retained, in which t and u maintain a single dimension, with all the rest ignored. And thus the equation* $At + Bu = 0$ *will come to two terms only ; from which with A and B known the subtangent will be* $PT = -\frac{Bq}{A}$.

EXAMPLE I

The proposed curve shall be a parabola, the nature of which is expressed by this equation $yy = 2ax$*, with the principal axis being AP and the vertex A.*

There may be taken $AP = p$; and if on calling $PM = q$, the equation becomes $qq = 2ap$, or $q = \sqrt{2ap}$. Now there may be put $x = p + t$ and $y = q + u$, and the equation becomes

$$qq + 2qu + uu = 2ap + 2at;$$

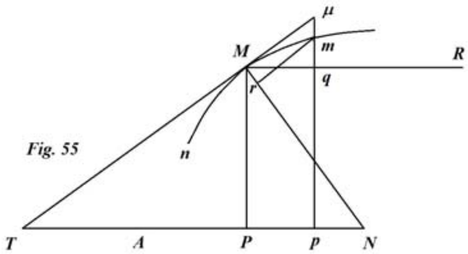

Fig. 55

from which, by the rule, only these terms $2qu = 2at$ may be retained, which give

$$at - qu = 0,\ \frac{u}{t} = \frac{a}{q} = -\frac{A}{B}.$$

Therefore the subtangent will be

$PT = \frac{qq}{a} = 2p$ on account of $qq = 2ap$. Hence the subtangent *PT* will be twice the abscissa *AP*.

EXAMPLE II

The curve shall be an ellipse described with centre A, the equation of which is

$$yy = \frac{bb}{aa}(aa - xx) \text{ or } aayy + bbxx = aabb.$$

Therefore with $AP = p$ and on putting $PM = q$ there will be $aaqq + bbpp = aabb$. Now there may be put $x = p + t$ and $y = q + u$; and, because these terms only must be retained, in which *t* and *u* have a single dimension, the rest can be ignored at once and the equation becomes

$$2aaqu + 2bbpt = 0$$

from which

$$\frac{u}{t} = -\frac{bbp}{aaq} = -\frac{A}{B}.$$

Therefore the subtangent will be

$$PT = -\frac{B}{A}q = -\frac{aaqq}{bbp} = \frac{-aa + pp}{p};$$

which expression, since it shall be negative, indicates the point *T* falls on the opposite side of the ellipse. Moreover this expression agrees uncommonly well with the tangent of the ellipse treated above.

EXAMPLE III

Let the proposed line be of the seventh kind of the third order

$$yyx = axx + bx + c .$$

Therefore on taking $AP = p$ and putting $PM = q$ there will be $pqq = app + bp + c$. Now there may be put $x = p + t$ and $y = q + u$ and the equation becomes

$$(p+t)(qq+2qu+uu) = a(pp+2pt+tt) + b(p+t) + c .$$

With all the superfluous terms rejected the equation becomes $2pqu + qqt = 2apt + bt$, from which there becomes

$$\frac{u}{t} = \frac{2ap+b-qq}{2pq} = -\frac{A}{B}$$

and thus the subtangent

$$PT = -\frac{B}{A} = \frac{2pqq}{2ap+b-qq} = \frac{2app+2bp+2c}{2ap+b-qq} = \frac{2ap^3+2bpp+2cp}{app-c},$$

or

$$PT = \frac{2ppqq}{app-c}$$

290. Therefore with the tangent to the curve known, likewise the direction is understood, which the curve follows at the point *M*. Indeed a curved line can be considered most appropriately as the way, which a point continually moving forwards with a continuous variation in the direction of the motion. And thus the point, which the curve $M\mu$ will describe in its motion at *M*, will be moving forwards along the direction of the tangent $M\mu$; which direction if it were conserved, will describe the right line $M\mu$, but it changes the direction of the motion from the vestigial direction, if indeed it will describe a curved line ; from which for knowing the course of the curved line it is necessary to define the position of the tangent, that which happens easily by the method treated here ; for nor indeed is any difficulty encountered, as long as the equation for the proposed curve should be rational and free from fractions. Moreover all equations are able always to be reduced to such a form. But if the equation were either irrational or involved fractions nor will it be free to be reduced to a rational and whole form, then indeed the

same method will be able to be used but with a certain moderation, which moderation itself leads on to the *differential calculus* ; on account of which we will reserve the method for finding tangents, if the equation for the proposed curve were neither rational nor whole, to the differential calculus.

291. Hence therefore with the inclination of the tangent $M\mu$ to the axis *AP* or of its parallel *Mq* known. For since there shall be $q\mu : Mq = -A : B$, if the coordinates were orthogonal and thus the angle $Mq\mu$ right, $-\frac{A}{B}$ will be the tangent of the angle $qM\mu$; but if the coordinates were oblique, then the angle $qM\mu$ may be found by trigonometry from the given angle $Mq\mu$, and in the ratio of the sides $Mq,\ q\mu$. Moreover it is apparent, if $A = 0$ in the resultant equation $At + Bu = 0$, then the angle $qM\mu$ vanishes and thus the tangent $M\mu$ becomes parallel to the axis *AP* . But if $B = 0$, then the tangent $M\mu$ of the applied line *PM* will be parallel to the applied line *PM* itself, touching the curve at the point *M* .

292. With the tangent *MT* found, if to that at the point of contact *M* the normal *MN* may be drawn, this will be likewise the normal to the curve itself ; therefore its position will be found easily in any case. It may be expressed most conveniently, if the coordinates *AP* and *PM* were orthogonal; for then there will be the similar triangles $Mq\mu$ and *MPN* and thus

$$Mq : q\mu = MP : PN \text{ or } -B : A = q : PN,$$

from which there becomes

$$PN = -\frac{Aq}{B}.$$

Moreover this part of the axis *PN*, between the applied line and the normal *MN* is accustomed to be called the *subnormal*. Therefore this subnormal, if the coordinates were orthogonal, is defined most easily from the subtangent *PT*, for there will be

$$PT : PM = PM : PN \text{ or } PN = \frac{PM^2}{PT}.$$

Truly in addition, if the angle *APM* were right, the tangent itself will be

$$MT = \sqrt{(PT^2 + PM^2)}$$

and the normal itself

$$MN = \sqrt{(PM^2 + PN^2)}$$

or, since there shall be $PT : TM = PM : MN$, it will be given by :

$$MN = \frac{PM:TM}{PT} = \frac{PM}{PT}\sqrt{(PT^2 + PM^2)}\,.$$

293. Because we have seen, if in equation $At + Bu = 0$ either $A = 0$ or $B = 0$, then the tangent becomes parallel either to the axis or to the applied line, the case remains requiring to be considered, in which each coefficient *A* and *B* shall both become $= 0$. Therefore since this comes about, the following [*i.e*. higher order] terms found in the above equation (in §286), in which *t* and *u* have two dimensions, are no longer able to be ignored with respect to these $At + Bu$ (which themselves vanish). Hence on this account it is required to consider this equation $0 = Ctt + Dtu + Euu$, with the higher order terms ignored, certainly which vanish before these, if *t* and *u* may be placed infinitely small. From this equation therefore, as from the general, it is evident, if there may be put $t = 0$, to be also $u = 0$ and thus *M* is a point on the curve, which indeed is in agreement with the hypothesis.

294. Therefore since this equation $0 = Ctt + Dtu + Euu$ may show the point *M* to be situated properly on the curve, it is evident, if *DD* were less than *4CE,* then the equation becomes imaginary, unless *t* and *u* shall be $= 0$. Therefore in this case the point *M* certainly will relate to the curve, truly it will be separated from the rest of the curve and thus it will be a conjoined oval in a vanishing point, a case of the kind we have observed in the preceding chapter. Therefore here indeed the idea of a tangent cannot be considered, because, if the tangent is a right line having two nearby points in common with the curve, a point from a right line cannot touch [the curve] in this way. And thus with this understood a point of conjunction, if which may be given on a certain curve, may be recognised and distinguished from the remaining points of the curve.

295. But if moreover *DD* were greater than *4CE,* the equation $0 = Ctt + Dtu + Euu$ will be resolvable into two equations of this form $\alpha t + \beta u = 0$ (Fig. 56), each of which meets the nature of the curve equally [*i.e.* adjusts itself equally to the nature of the curve at the point]. Therefore since the position of the tangent or the direction of the curve may be shown at the point *M*, it is necessary, that the two branches of the curve may cross each other at the point *M* and may put in place there a double point. Clearly on taking $Mq = t$, $q\mu$ and qv shall be both values of *u*, which that equation may provide, and the right lines $M\mu$ and *Mv* both will be tangents to the curve at the point *M.* Therefore the intersection of two branches of the curve will be at *M*, of which one is directed along $M\mu$ and the other along *Mv*. Therefore since equally a point of conjugation shall be required to be had for a double point, this equation $0 = Ctt + Dtu + Euu$ will always indicate a double point, just as the equation $At + Bu = 0$, as often as it may be considered, declares only a simple point of the curve.

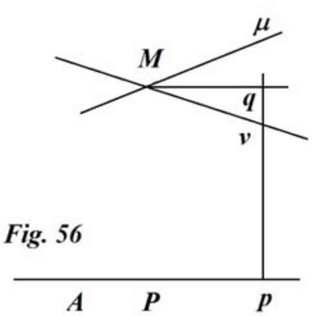

Fig. 56

296. But if there were $DD = 4CE$, then both these tangents $M\mu$ and *Mv* coincide and the angle μMv vanishes ; from which it is understood that the two branches of the curve not only meet at *M*, but also have the same direction and thus are tangents to each other in turn ; in which case the point *M* nevertheless will be double, because a right line drawn through the point must be considered to cut the curve in two points. Therefore when in the equation, as we have obtained in §286, both the first coefficients *A* and *B* vanish, then it is required to be concluded that the curve has a double point at *M*, of which there are three different kind given ; either an oval vanishing in a point (or a conjugal point), or the two branches of the curve mutually intersecting each other (or a node), or the touching of two branches of the curve; which different kinds of double points the three-fold constitution of the equation $0 = Ctt + Dtu + Euu$ defines.

297. If these three coefficients *C, D* and *E* also may vanish besides the coefficients *A* and *B*, then the following terms will have to be taken, in which *t* and *u* maintain three dimensions, and there will be $Ft^3 + Gttu + Htuu + Iu^3 = 0$. Which equation, if it may have a single real factor, here may show a single branch of the curve passing through the point *M* and likewise the direction or tangent ; truly the two remaining imaginary factors will prove to be an oval vanishing in the point *M* itself. But if all the roots of this equation were real, hence the three branches cross over each other at the same point *M* or touch, just as these roots were unequal or equal. Whichever of these will eventuate, the curve always will have a triple point at *M*, and likewise it is to be considered to be cut in three points *M* always.

298. But if in addition all these preceding four coefficients *F, G, H* and *I* may vanish, then it will be necessary to consider the following terms of the equation to understand the nature of the point *M* of the curve, in which *t* and *u* may have four dimensions; from which the point *M* will be declared a four-fold point. For at that point either two conjugate ovals coalesce, which arises, if all the roots of the equation of the fourth order were imaginary. Or at *M* there will be the intersection or contact of two branches of the curve with a conjugate point, which eventuates, if two roots were real, and the remaining two were imaginary. Or finally at *M* there will be the intersection of four branches of the curve, if all the roots were real; moreover the intersection either of two, three, or of all four will become a point of contact, if two, three, or all four become equal. But in a similar manner, with all these terms vanishing also, where *t* and *u* maintain four dimensions, there will be a progression in the reasoning to five terms or of a higher dimension.

299. From these careful assessments the general equation will be found easily for all curves, which not only may pass through the point *M*, but also in *M* the curves may have either a simple, double, triple, or any multiple point wished. For on putting $AP = p,\ PM = q$ and with *P, Q, R, S* etc. denoting some functions of the coordinates *x* and *y*, it is evident this equation

$$P(x-p)+Q(y-q)=0$$

expresses the curve passing through the point *M*; for if there may be put $x=AP=p$, it becomes $y=PM=q$, as long as neither *P* were divisible by $y-q$ nor *Q* by $x-p$, or provided these factors $x-p$ and $y-q$, on which the passage of the curve through the point *M* depends, may not be eliminated from the equation by division. Moreover it is evident all the curves, which indeed may pass through the point *M*, are contained in that equation $P(x-p)+Q(y-q)=0$; truly *M* will be a simple point, if this equation were not of this form, such as we shall show soon for multiple points.

300. If *M* has to be a double point, the equation for the curve will be contained in this general form

$$P(x-p)^2+Q(x-p)(y-q)+R(y-q)^2=0,$$

provided this form may not be ruined by division. Hence it is seen that a double point cannot fall on lines of the second order ; for since that equation shall be of the second order only, it is necessary, that *P, Q* and *R* shall be constant quantities ; but then the equation will not be for a curved line, but indeed two right lines. But if *P, Q, R* were functions of the first order such as $\alpha x+\beta y+\gamma$, then lines of the third order may be had having a double point at *M*. But truly a line of the third order, unless it may depend on three right lines, cannot have more than one double point. For we may consider two double points to be given and through these a right line may be drawn ; this right line may cut the curve in four points, which is contrary to the nature of lines of the third order. A line of the fourth order will have two double points only ; a line of the fifth order cannot have more than three, and thus so on.
[The maximum number of double points $\frac{(n-1)(n-2)}{2}$, for an irreducible curve of order *n*, established by Maclaurin in geometria organica 1720, page 137. Noted by A.S. in the *O.O*. edition.]

301. Let *M* be a triple point of the curve and the nature of the curve may be expressed by this equation

$$P(x-p)^3+Q(x-p)^2(y-q)+R(x-p)(y-q)^2+S(y-q)^3=0.$$

Therefore this equation, if it may define a curved line, will exceed the third order, for if *P, Q, R* and *S* shall be constants, so that the nature of lines of the third order is removed, then the equation may have three factors of the form $\alpha(x-p)+\beta(y-q)$ and thus becomes that for three right lines. Therefore a triple point cannot be present in a simpler curve than the fourth order ; nor can lines of the fifth order have more than one triple point, for otherwise a right line may be given cutting the line of the fifth order in six points. But nothing prevents a line of the sixth order from having two triple points.

302. If the equation may be contained in this form :

$$P(x-p)^4+Q(x-p)^3(y-q)+R(x-p)^2(y-q)^2+S(x-p)(y-q)^3+T(y-q)^4=0,$$

then the curve will have a quadruple point at *M*. Therefore the simplest curved line, which a point of the fourth order may enjoy, will depend on the fifth order of lines. Truly two fourfold points cannot arise unless on lines of the eighth or of a higher order. In a similar manner the general equations can be shown for lines, which may have a fifth order point at *M* , or for some multiple it pleases.

303. But if moreover *M* were either a double or triple point , or some multiple whatever, then either just as many branches of the curve mutually cut each other or are tangents at *M*, or if the number of branches intersecting each other shall be less, then one or more conjugal points will gather together at the same point *M*, which state of the curve may be known from these, which have been treated before. Clearly, in the functions *P*, *Q*, *R*, *S* etc. and everywhere *p* and *q* must be written in place of *x* and *y*, and also *t* and *u* in place of the factors $x-p$ and $y-q$; then indeed equations of the same kind will be produced, from which the nature of the curve, and the intersecting tangents of their branches, can be defined at *M*.

CAPUT XIII

DE AFFECTIONIBUS LINEARUM CURVARUM

285. Quemadmodum supra ramorum in infinitum extensorum indolem ita descripsimus, ut lineam rectam vel curvam simpliciorem assignaverimus, quae cum illa curva in infinito confunderetur, ita in hoc capite constituimus quamvis curvae portionem in spatio finito existentem examini subiicere atque rectam vel curvam simpliciorem investigare, quae cum illa curvae portione saltem per minimum spatium congruat. Ac primo quidem patet omnem lineam rectam, quae curvam tangit, in eo loco, ubi tangit, cum tractu lineae curvae congruere seu cum linea curva duo ad minimum puncta communia habere. Tum vero etiam aliae lineae curvae exhiberi possunt, quae cum data curvae portione accuratius congruant eamque quasi osculentur. His autem cognitis status lineae curvae in quovis loco eiusque affectiones clarissime erunt perspectae.

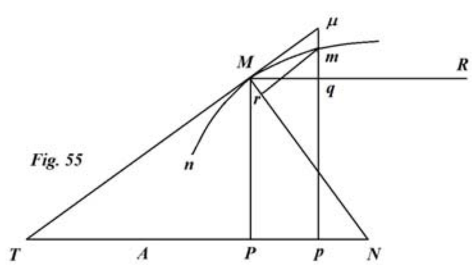

Fig. 55

286. Sit igitur proposita aequatio quaecunque inter coordinatas x et y pro curva quapiam. Tribuatur (Fig. 55) abscissae x ,valor quispiam $AP = p$ et quaerantur valores applicatae y huic abscissae respondentes, qui si plures fuerint, sumatur pro lubitu unus $PM = q$, eritque M punctum in curva seu punctum, per quod curva transibit. Tum vero, si in aequatione inter x et y proposita loco x scribatur p et q loco y, omnes aequationis termini se mutuo tollent, ita ut nihil remaneat. Iam ad naturam illius curvae portionis, quae per punctum M transit, indagandam ex M ducatur recta Mq axi AP parallela, quae nunc pro axe accipiatur, et vocetur hic nova abscissa $Mq = t$, applicata $qm = u$. Quia igitur punctum m pariter in curva est positum, si mq usque ad priorem axem in p producatur atque $Ap = p + t$ in locum ipsius x et $pm = q + u$ in locum ipsius y substituatur, aequatio pariter identica prodire debet.

287. Facta autem hac substitutione in aequatione inter x et y proposita, omnes termini, in quibus neque t nec u inest, se mutuo sponte destruent illique termini, qui novas coordinatas t et u continent, soli supererunt. Hinc ergo eiusmodi prodibit aequatio

$$0 = At + Bu + Ctt + Dtu + Euu + Ft^3 + Fttu + Htuu + \text{etc.},$$

ubi A, B, C, D etc. sunt quantitates constantes ex constantibus primae aequationis et ipsis p et q, quas nunc pro constantibus habemus, compositae. Ista igitur nova aequatione natura eiusdem curvae, exprimitur, verum ad axem Mq refertur, et in quo ipsum curvae punctum M pro initio abscissarum assumitur.

288. Ac primo quidem patet, si ponatur $Mq = t = 0$, tum quoque fore $qm = u = 0$, quia punctum min *M* incidit. Deinde, quia tantum minimam curvae portionem circa *M* versantem indagare volumus, hoc impetrabimus, si pro *t* valores quam minimos assumamus; quo casu quoque $qm = u$ valorem habebit minimum; naturam enim arcus *Mm* quasi evanescentis tantum desideramus. Quodsi vero prot et *u* sumantur valores quam minimi, termini *tt*, *tu* et *uu* multo adhuc erunt minores atque sequentes $t^3,\ ttu,\ tuu,\ u^3$ etc. multo quoque erunt minores quam illi et ita porro; quam ob causam, cum termini minimi prae allis quasi infinite maioribus omitti queant, remanebit ista aequatio $0 = At + Bu$, quae est aequatio pro linea recta $M\mu$ per punctum *M* transeunte atque indicat hanc rectam, si punctum *m* ad *M* proxime accedat, cum curva congruere.

289. Erit ergo haec recta $M\mu$ tangens curvae in loco *M*, ideoque hinc ad quodvis punctum curvae *M* tangens μMT duci potest. Scilicet, cum ex aequatione $At + Bu = 0$ sit

$$\frac{u}{t} = -\frac{A}{B} = \frac{q\mu}{Mq},$$

erit

$$q\mu : Mq = MP : PT = -A : B.$$

Ergo, cum sit $PM = q$, fiet $PT = -\frac{Bq}{A}$; vocari autem haec axis portio *PT* solet *subtangens.* Ex his ergo haec deducitur

REGULA PRO INVENIENDA SUBTANGENTE

In aequatione pro curva, postquam abscissae $x = p$ inventa fuerit satisfacere applicata $y = q$, ponatur $x = p + t$ et $y = q + u$; ex terminis autem, qui per substitutionem oriuntur, ii tantum retineantur, in quibus t et u unicam dimensionem tenent, reliquis omnibus neglectis. Sicque ad duos tantum terminos $At + Bu = 0$ pervenietur; unde cognitis A et B erit subtangens $PT = -\frac{Bq}{A}$.

EXEMPLUM I

Sit proposita curva parabola, cuius natura hac exprimitur aequatione $yy = 2ax$, existente AP axe principali et A vertice.

Sumatur $AP = p$; et, si vocetur $PM = q$, erit $qq = 2ap$, seu $q = \sqrt{2ap}$. Iam ponatur $x = p + t$ et $y = q + u$, eritque

$$qq + 2qu + uu = 2ap + 2at\,;$$

unde, per regulam, hi tantum termini $2qu = 2at$ retineantur, qui dant

$$at - qu = 0,\ \frac{u}{t} = \frac{a}{q} = -\frac{A}{B}.$$

Erit ergo subtangens $PT = \frac{qq}{a} = 2p$ ob $qq = 2ap$. Hinc subtangens *PT* erit dupla abscissae *AP*.

EXEMPLUM II

Sit curva ellipsis centro A descripta, cuius aequatio est

$$yy = \frac{bb}{aa}\left(aa - xx\right) \text{ seu } aayy + bbxx = aabb.$$

Sumta ergo $AP = p$ et posita $PM = q$ erit $aaqq + bbpp = aabb$.
Iam ponatur $x = p + t$ et $y = q + u$; et, quoniam ii tantum termini retineri debent, in quibus *t* et *u* unicam habent dimensionem, reliqui statim omitti possunt fietque

$$2aaqu + 2bbpt = 0$$

unde

$$\frac{u}{t} = -\frac{bbp}{aaq} = -\frac{A}{B}.$$

Erit ergo subtangens

$$PT = -\frac{B}{A}q = -\frac{aaqq}{bbp} = \frac{-aa + pp}{p}\,;$$

quae expressio, cum sit negativa, indicat punctum *T* in partem contrariam cadere. Ceterum haec expressio egregie convenit cum determinatione tangentium ellipsis supra tradita.

EXEMPLUM III

Sit proposita linea tertii ordinis speciei septimae

$$yyx = axx + bx + c\,.$$

Sumto ergo $AP = p$ et posita $PM = q$ erit $pqq = app + bp + c$.
Iam statuatur $x = p + t$ et $y = q + u$ eritque

$$(p+t)(qq+2qu+uu)=a(pp+2pt+tt)+b(p+t)+c\ .$$

Reiectis omnibus terminis superfluis erit $2pqu+qqt=2apt+bt$, unde fit

$$\frac{u}{t}=\frac{2ap+b-qq}{2pq}=-\frac{A}{B}$$

ideoque subtangens

$$PT=-\frac{B}{A}=\frac{2pqq}{2ap+b-qq}=\frac{2app+2bp+2c}{2ap+b-qq}=\frac{2ap^3+2bpp+2cp}{app-c},$$

vel

$$PT=\frac{2ppqq}{app-c}$$

290. Cognita ergo hoc modo tangente curvae, simul cognoscitur directio, quam curva sequitur in puncto *M*. Linea enim curva aptissime considerari potest tanquam via, quam describit punctum continuo promotum cum variata continuo motus directione. Ideoque punctum, quod curvam $M\mu$ motu suo describit in *M*, promovebitur secundum directionem tangentis $M\mu$; quam directionem si conservaret, describeret rectam $M\mu$, at e vestigio directionem motus inflectit, siquidem lineam curvam describit; unde ad tractum lineae curvae cognoscendum in singulis punctis positionem tangentis definire oportet, id quod facile fit methodo hic tradita; neque enim ulla offenditur difficultas, dummodo aequatio pro curva proposita fuerit rationalis atque a fractionibus libera. Ad talem autem formam aequationes omnes semper reduci possunt. Sin autem aequatio fuerit vel irrationalis vel fractionibus implicata neque eam ad formam rationalem et integram reducere vacaverit, tum eadem quidem methodus, at cum moderatione quadam, adhiberi potest, quae ipsa moderatio *calculum differentialem* produxit; quamobrem methodum inveniendi tangentes, si aequatio pro curva proposita non fuerit rationalis et integra, in calculum differentialem reservabimus.

291. Hinc ergo innotescit inclinatio tangentis $M\mu$ ad axem *AP* seu eius parallelam *Mq*. Cum enim sit $q\mu : Mq=-A:B$, si coordinatae fuerint orthogonales ideoque angulus $Mq\mu$ rectus, erit $-\frac{A}{B}$ tangens anguli $qM\mu$; sin autem coordinatae fuerint obliquangulae, tum ex angulo $Mq\mu$ dato et ratione laterum Mq, $q\mu$ per trigonometriam reperietur angulus $qM\mu$. Patet autem, si in aequatione resultante $At+Bu=0$ fuerit $A=0$, tum angulum $qM\mu$ evanescere ideoque tangentem $M\mu$ fore axi *AP* parallelam. Sin autem fuerit $B=0$, tum tangens $M\mu$ applicatis *PM* erit parallela seu ipsa applicata *PM* curvam in puncto *M* tanget.

292. Inventa tangente *MT*, si ad eam in puncto contactus *M* ducatur normalis *MN*, erit haec ad ipsam curvam simul normalis; cuius propterea positio quovis casu facile reperitur. Commodissime autem exprimitur, si coordinatae *AP* et *PM* fuerint orthogonales; tum enim erunt triangula $Mq\mu$ et *MPN* similia ideoque

$$Mq : q\mu = MP : PN \text{ seu } -B : A = q : PN,$$

unde fit

$$PN = -\frac{Aq}{B}.$$

Vocari autem haec axis portio *PN*, inter applicatam et normalem *MN* intercepta, solet *subnormalis.* Haec igitur subnormalis, si coordinatae fuerint orthogonales, ex inventa subtangente *PT* facillime definitur, erit enim

$$PT : PM = PM : PN \text{ seu } PN = \frac{PM^2}{PT}.$$

Praeterea vero, si angulus *AP M* fuerit rectus, erit ipsa tangens

$$MT = \sqrt{(PT^2 + PM^2)}$$

et ipsa normalis

$$MN = \sqrt{(PM^2 + PN^2)}$$

seu, cum sit $PT : TM = PM : MN$, erit

$$MN = \frac{PM : TM}{PT} = \frac{PM}{PT}\sqrt{(PT^2 + PM^2)}.$$

293. Quoniam vidimus, si in aequatione $At + Bu = 0$ fuerit vel $A = 0$ vel $B = 0$, tum tangentem fore vel axi vel applicatis parallelam, superest casus, quo uterque coefficiens *A* et *B* simul fit $= 0$, considerandus. Hoc ergo cum evenit, in aequatione supra (paragrapho 286) inventa sequentes termini, in quibus *t* et *u* duas obtinent dimensiones, non amplius prae his $At + Bu$ (qui ipsi evanescunt) negligi poterunt. Hanc ob rem consideranda veniet haec aequatio $0 = Ctt + Dtu + Euu$, neglectis sequentibus terminis, quippe qui prae his, si *t* et *u* statuantur infinite parva, evanescunt. Ex hac igitur aequatione, uti ex generali, manifestum est, si ponatur $t = 0$, fore et $u = 0$ ideoque *M* esse punctum in curva, quod quidem hypothesi est consentaneum.

294. Cum igitur haec aequatio $0 = Ctt + Dtu + Euu$ statum curvae prope punctum *M* declaret, manifestum est, si fuerit *DD* minor quam *4CE*, tum aequationem fore imaginariam, nisi sint *t* et $u = 0$. Hoc igitur casu punctum *M* quidem ad curvam pertinebit, verum erit seiunctum a reliqua curva eritque ideo ovalis coniugata in punctum evanescens, cuiusmodi casum in capite praecedente notavimus. Hic igitur ne idea quidem tangentis locum habet, quia, si tangens est recta duo puncta proxima cum curva habens communia, punctum a recta tangi hoc modo non potest. Hoc itaque pacto punctum coniugatum, si quod datur in curva quapiam, agnoscetur atque a reliquis curvae punctis discernetur.

295. Quodsi autem fuerit *DD* maior quam *4CE*, aequatio $0 = Ctt + Dtu + Euu$ resolubilis erit in duas aequationes huius formae $\alpha t + \beta u = 0$ (Fig. 56), quarum utraque in curvae naturam aeque competit. Cum igitur utraque positionem tangentis seu directionem curvae in puncto *M* exhibeat, necesse est, ut duo curvae rami se in puncto *M* decussent ibique punctum duplex constituant. Sumta scilicet $Mq = t$, sint $q\mu$ et qv ambo valores ipsius *u*, quos illa aequatio praebet, atque rectae $M\mu$ et *Mv* erunt ambae tangentes curvae in puncto *M*. In *M* ergo erit intersectio duorum curvae ramorum, quorum alter secundum $M\mu$ et alter secundum *M v* dirigitur. Cum igitur punctum coniugatum pariter pro puncto duplici sit habendum, haec aequatio $0 = Ctt + Dtu + Euu$ semper punctum duplex indicabit, quemadmodum aequatio $At + Bu = 0$, quoties locum habet, punctum curvae tantum simplex declarat.

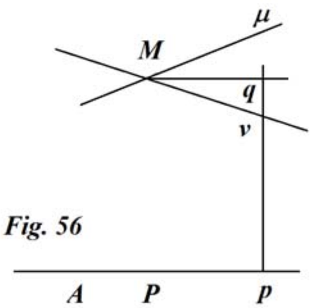

Fig. 56

296. Sin autem fuerit $DD = 4CE$, tum ambae istae tangentes $M\mu$ et *Mv* coincident et angulus μMv evanescet; ex quo intelligitur duos curvae ramos in *M* non solum concurrere, sed etiam eandem directionem habere ideoque se invicem tangere; quo casu punctum *M* nihilominus erit duplex, quia recta per hoc punctum ducta curvam hoc loco in duobus punctis secare est censenda. Quando ergo in aequatione, quam paragrapho 286 obtinuimus, ambo coefficientes primi *A* et *B* evanescunt, tum concludenda est curva in *M* punctum duplex habere, cuius tres dantur species diversae; vel ovalis in punctum evanescens seu punctum coniugatum vel duorum curvae ramorum intersectio mutua seu nodus vel duorum curvae ramorum contactus, quas diversas puncti duplicis species triplex aequationis $0 = Ctt + Dtu + Euu$ constitutio definit.

297. Si praeter coefficientes *A* et *B* etiam hi tres *C, D* et *E* omnes evanescant, tum sequentes sumi debebunt termini, in quibus *t* et *u* tres obtinent dimensiones, eritque $Ft^3 + Gttu + Htuu + Iu^3 = 0$. Quae aequatio si unicum habeat factorem simplicem realem, hic ostendet unum curvae ramum per punctum *M* transeuntem eiusque simul directionem seu tangentem; bini vero reliqui factores imaginarii in ipso puncto *M* ovalem evanescentem arguent. Sin autem omnes radices illius aequationis fuerint reales, hinc cognoscetur tres curvae ramos se in eodem puncto *M* vel decussare vel tangere,

prout illae radices fuerint vel inaequales vel aequales. Quicquid horum evenerit, curva in *M* semper habebit punctum triplex atque recta per *M* ducta curvam simul in tribus punctis secare putanda est.

298. Quodsi praeter omnes coefficientes praecedentes etiam hi quatuor *F, G, H* et *I* evanescant, tum ad naturam puncti curvae *M* cognoscendam contemplari oportebit terminos aequationis sequentes, in quibus *t* et *u* quatuor habeant dimensiones; unde punctum *M* quadruplex erit iudicandum. In eo enim vel duae ovales coniugatae coalescunt, quod evenit, si aequationis quarti gradus omnes radices fuerint imaginariae. Vel in *M* erit intersectio seu contactus duorum curvae ramorum cum puncto coniugato, quod evenit, si duae radices fuerint reales, duae reliquae vero imaginariae. At in *M* denique erit intersectio quatuor curvae ramorum, si omnes radices aequationis fuerint reales; intersectio autem vel duorum vel trium vel omnium quatuor abibit in contactum, si duae, tres vel omnes quatuor radices fiant aequales. Simili autem modo in iudicio erit progrediendum, si etiam his terminis, ubi *t* et *u* quatuor obtinent dimensiones, evanescentibus procedendum erit ad terminos quinque ulteriorumve dimensionum.

299. His perpensis facile erit aequationem generalem pro omnibus curvis invenire, quae non solum per punctum *M* transeant, sed etiam in *M* habeant punctum vel simplex vel duplex vel triplex vel totuplex, prout quis voluerit. Positis enim $AP = p,\ PM = q$ ac denotantibus *P*, *Q*, *R*, *S* etc. functiones quascunque coordinatarum *x* et *y*, manifestum est hanc aequationem

$$P(x-p)+Q(y-q)=0$$

exprimere curvam per punctum *M* transeuntem; si enim ponatur $x = AP = p$, fiet $y = PM = q$, dummodo neque *P* per $y-q$ nec *Q* per $x-p$ fuerit divisibile, vel dummodo hi factores $x-p$ et $y-q$, a quibus transitus curvae per punctum *M* pendet, ex aequatione per divisionem non eliminentur. Perspicuum autem est omnes curvas, quae quidem per punctum *M* transeant, in ista aequatione $P(x-p)+Q(y-q)=0$ contineri; erit vero *M* punctum simplex, si haec aequatio non fuerit eius formae, qualem pro punctis multiplicibus mox exhibebimus.

300. Si *M* debeat esse punctum duplex, aequatio pro curva in hac forma generali continebitur

$$P(x-p)^2+Q(x-p)(y-q)+R(y-q)^2=0,$$

dummodo haec forma per divisionem non pereat. Perspicitur hinc in lineas secundi ordinis punctum duplex cadere non posse; quo enim illa aequatio secundi tantum sit, necesse est, ut *P*, *Q* et *R* sint quantitates constantes; tum autem aequatio non erit pro linea curva, sed pro duabus rectis. Sin autem *P*, *Q*, *R* sint functiones primi ordinis ut $\alpha x+\beta y+\gamma$, tum lineae habebuntur tertii ordinis in *M* punctum duplex habentes. At vero linea tertii ordinis, nisi ex tribus rectis constet, plus uno puncto duplici habere nequit. Ponamus enim dari duo puncta duplicia atque per ea lineam rectam duci; haec linea recta curvam in quatuor punctis secaret, quod naturae linearum tertii ordinis adversatur. Linea quarti ordinis duo tantum habebit puncta duplicia; linea quinti ordinis plura tribus habere non poterit et ita porro.

301. Sit *M* punctum curvae triplex atque natura linea curvae hac exprimetur aequatione

$$P(x-p)^3+Q(x-p)^2\left(y-q\right)+R(x-p)(y-q)^2+S(y-q)^3=0\,.$$

Haec aequatio igitur, si lineam curvam definiat, tertium ordinem superabit, namque si *P, Q, R* et *S* essent constantes, quod linearum tertii ordinis natura exigit, tum aequatio tres haberet factores formae $\alpha\left(x-p\right)+\beta(y-q)$ ideoque foret pro tribus rectis. In curvas ergo quarto ordine simpliciores punctum triplex non cadit; neque lineae quinti ordinis plus uno puncto triplici habere possunt, alioquin enim daretur recta lineam quinti ordinis in sex punctis secans. Nihil autem impedit, quominus linea sexti ordinis duo habeat puncta triplicia.

302. Si aequatio in hac forma contineatur:

$$P(x-p)^4+Q(x-p)^3\left(y-q\right)+R(x-p)^2(y-q)^2+S(x-p)(y-q)^3+T(y-q)^4=0\,,$$

tum curva in *M* habebit punctum quadruplex. Linea ergo curva simplicissima, quae puncto quadruplici gaudeat, ad linearum ordinem quintum pertinebit. Duo vero puncta quadruplicia non cadunt nisi in lineas aut octavi aut altioris gradus. Simili modo aequationes generales exhiberi possunt pro lineis, quae in *M* habeant punctum quintuplex vel pro lubitu multiplex.

303. Quodsi autem *M* fuerit vel punctum duplex vel triplex vel utcunque multiplex, tum vel totidem curvae rami se mutuo in puncto *M* secabunt sive tangent, vel, si numerus ramorum se intersecantium sit minor, tum unum plurave puncta coniugata in eodem puncto *M* concrescent, qui curvae status cognoscetur ex iis, quae ante sunt tradita. Scilicet, in functionibus *P, Q, R, S* etc. ubique loco *x* et *y* scribi debent *p* et *q,* et *t* et *u* loco factorum $x-p$ et $y-q$; tum enim prodibunt eiusmodi aequationes, ex quibus constitutio curvae et ramorum se in *M* intersecantium tangentes definiri poterunt.