CHAPTER XII

CONCERNING THE INVESTIGATION OF FIGURES OF CURVED LINES

272. The matters which have been set out in these chapters, help in the understanding of the infinite extensions of figures of curved lines. Certainly a curved figure shall have a certain figure in a finite space, most difficult on many occasions to understand from the equation. For it requires on this account to elicit from the equation, whatever the finite abscissas values, the corresponding individual applied line values and to distinguish between these which are real and those which are imaginary ; which undertaking, if the equation shall be of a higher order, generally exceeds the known analytical strengths. For if some known value may be attributed to the abscissa, in turn it will sustain the unknown applied lines in an equation. And hence the number of dimensions of the applied line will depend on the resolution of the equation. Moreover this labour can be greatly lightened by this reduction of the equation, as long as both the axis and the inclination of the axis are assumed most appropriate and convenient ; then also, in the same way, whichever of the coordinates may be taken for the abscissas, the labour will be maximally diminished, if these may be taken of the coordinates, of which the least dimensions of the applied lines occur in the equation.

273. Thus, if we wish to investigate the figures of lines of the third order, which pertain to the first kind, we will assume the most simple equation for this kind shown from §258 and from the coordinates *t* and *u* the first *t* for the applied line, the second truly *u* for the abscissa, because *t* has only two dimensions.

Therefore we will have a form of the equation of this kind :

$$yy = \frac{2by + axx + cx + d - nnx^3}{x}$$

which resolved gives

$$y = \frac{b \pm \sqrt{(bb + dx + cxx + ax^3 - nnx^4)}}{x}$$

with neither *b* nor $n = 0$.

274. Therefore the values of *x* of the function $bb + dx + cxx + ax^3 - nnx^4$ which lead on to a positive value, a two-fold applied line will correspond to these ; in the cases in which this function indeed will vanish, a single applied line will agree from the same abscissa *x*, or the two applied lines *y* will be equal to each other. But if that function may maintain a

negative value, then no applied line will correspond to the abscissa. But values of this function, if they should be positive, shall be unable to change into negative values, unless first they shall be made equal, or the function should disappear. Therefore the cases are required to be considered mainly, in which $bb+dx+cxx+ax^3-nnx^4$ shall become $=0$; which indeed arises with certainty in two cases, because, if x should pass over a certain limit, either positive or negative, its value will become negative. Hence the whole curve will correspond to the separation of the abscissas, beyond which all the applied lines become negative.

275. We may consider the expression $bb+dx+cxx+ax^3-nnx^4$ to have only two real factors or to be able to vanish in two cases only ; which comes about, if (Fig. 49) the abscissa may be determined at the points P and S , where only a single applied line may be found. Therefore through the whole interval PS the applied lines will be twins and real, truly outside the interval PS all the applied lines will be imaginary and thus the whole curve will lie between the applied lines Kk and Nn. Truly the applied lines will be the asymptotes of the curve at the beginning of the abscissas A, which besides will cut the curve at some point ; for if x may be put [close to] $x=0$, the equation becomes [approximately]

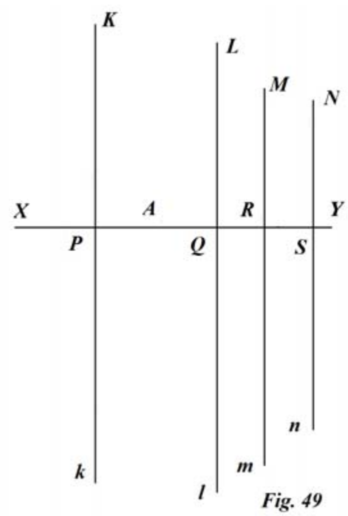

Fig. 49

$$\sqrt{(bb+dx+cxx+ax^3-nnx^4)}=b+\frac{dx}{2b},$$

from which there will be :

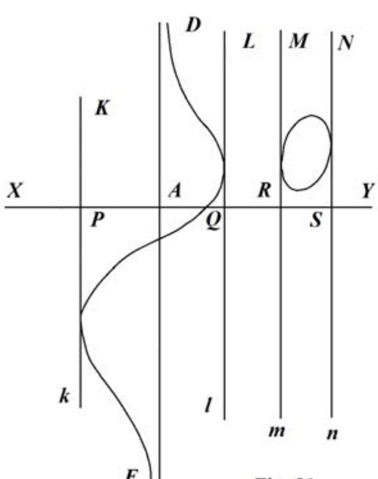

Fig. 51

$$y=\frac{b\pm\left(b+\frac{dx}{2b}\right)}{x},$$

that is, either there will be

$$y=\infty \text{ or } y=-\frac{d}{2b}.$$

Therefore in this case the curve will have a form of this kind, such as figure 50 shows.

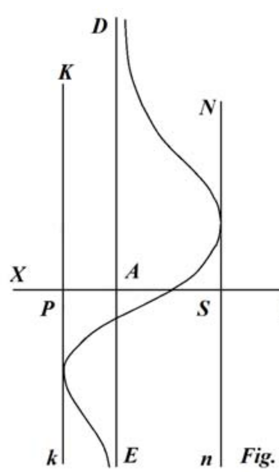

Fig. 50

276. We may consider the expression $bb + dx + cxx + ax^3 - nnx^4$ to have four simple unequal real factors and thus to vanish in four cases. Therefore the applied lines restrict the curve to a singe point in just as many places *P, Q, R* and *S*. Therefore since the applied lines through the axis of the interval *XP* should be imaginary, now they will be imaginary again through the interval *QR*, and real anew through the interval *RS*. Beyond *S* towards *Y* they must become imaginary again. Hence the curve will be consistent with two parts separated from each other in turn, the one part of which lies between the right lines *Kk* and *Ll*, the other is contained between the right lines *Mm* and *Nn*. Truly since the applied lines shall be real at the start of the abscissas *A*, it is necessary, that is shall be placed on the axis either in the interval *PQ* or *RS*. Therefore in this case the curve will have a figure, such as figure 51 showed, evidently it will be consistent with an oval standing apart related to the rest of the curve according to the asymptote *DE*, which is called the *conjoined oval*.

277. If two roots become equal to each other, either the points *P* and *Q* , *Q* and *R,* or *R* and *S* may meet. Truly, if the first may come about, because *A* lies between *P* and *Q*, each root must be *x* [*i.e*. zero], which cannot happen, because *b* cannot be absent. But if the points *R* and *S* may come together, the conjoined oval becomes infinitely small and will become a *conjoined point.* But if the points *Q* and *R* may come together, the oval will be joined together with the remaining curve, so that the curve with the node of figure 52 may emerge [literally, with a knot in it]. But if truly three roots may agree or the points *Q, R* and *S* meet, then the node vanishes into the sharpest point [i.e. *cusp*], such as figure 53 shows. Therefore thus five different varieties are present in the first kind, out of which Newton put in place just as many kinds.

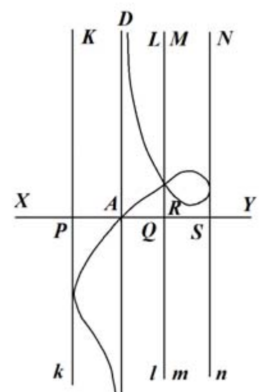

Fig.52

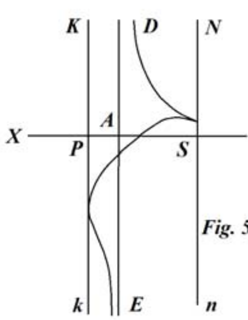

Fig. 53

278. In a similar manner the subdivisions of the remaining Newtonian kinds have been made, because all the equations have been prepared thus, so that may not have more than two dimensions. Truly when the other coordinate has a single dimension, the form of the curve will be easily recognised. For the equation will be of this kind $y = P$ with *P* being some rational function of the abscissa *x;* therefore whatever the value of *x* attributed, the applied line also will maintain always a single value and thus the curve will accompany each axis to infinity on being continually extended. If the function *P* shall be a fraction, it can happen, that the applied line is made infinite at one or more points and thus the asymptote of the curve will be shown, which arises, when the denominator of the function *P* vanishes.

279. Therefore there may be put $y=\frac{P}{Q}$,and all the real roots of the equation $Q=0$ will show these infinite applied lines ; indeed whatever root of this equation may be declared, for example $x=f$,if the abscissa $x=f$ may be taken, the applied line y becomes infinite, because Q becomes $=0$. Then truly it is apparent, if the applied lines y were positive, then x must be greater than f, with the same x made less than f to become negative, and thus the applied line will be an asymptote of the kind $u=\frac{A}{t}$; and this is required to be understood with all the factors unequal. But if the denominator Q should have two equal factors, for example $(x-f)^2$, then, if the applied lines shall be positive with f taken greater than x, they will remain positive, if x were made less than f, and there will be an asymptote y of the kind $uu=\frac{A}{t}$ on making $x=f$. But if the denominator Q shall have three equal factors, surely $(x-f)^3$, then the applied lines before and after that asymptote, which shall be infinite, shall have the same sign as in the first case.

280. After these the equations which may be contained in this form $yy=\frac{2Py-R}{Q}$ are easily treated, with the functions P, Q and R present, whatever the entirety of the abscissas x. Therefore for each of the abscissas x either two or none of the applied lines will meet ; clearly two applied lines may be produced, if PP were greater than QR, and zero, if PP were less than QR; therefore within some limit, which divided the real applied lines from the imaginary ones or the zero ones, there will be $PP=QR$ and thus there becomes $y=\frac{P}{Q}$, or this applied line restricts the curve into a single point or tangent. Therefore towards understanding the form of the curve, the equation $PP-QR=0$ will be considered, of which the individual real roots will give the places, where the applied lines restrict the curve to a single point. These points may be noted on the axis, and, if all the roots were unequal, the parts of the axis contained between these points alternately will have twin real applied lines and imaginary applied lines ; and thus the whole curve will depend on parts separated from each other, as often as the alternative kinds are taken to be present, from which the conjugate ovals arise.

281. If the two roots of the equation $PP-QR=0$ become equal, then two of the noted points come together on the axis and hence on the axis a part having either real or imaginary applied lines vanishes. In the first case the curve will produce a node as in figure 52, in the second case the conjugated oval vanishes into a conjugated point [*i.e. a shared or common point*]. But if moreover that equation had three equal roots, the node becomes infinitely small and will change into a cusp as in figure 53; if four roots of the equation were present, either the two separate ovals condense into a point, or in that cusp a node will be given, or two cusps placed opposite to the peak. But if five roots were

present equal, new forms barely appear ; for a cusp arises, in which not one as before but two ovals merge together into a point ; nor also will a greater multitude of equal roots produce a new distinction in the resulting figures.

282. A node of the intersection of two branches of a curve is also accustomed to be called a *double point,* because therefore a right line cutting the curve at that point is agreed to cut the in two points. And if through the node the branch of another curve may cross, then in that intersection a *triple point* will arise; truly a *quadruple point* will arise, if two double points meet, from which the genesis and nature of points of any multiplicity is seen. Therefore also the vanishing oval or the conjugation point will be a double point, and equally with cusps, which arise from a conjugated point reunited with the rest of the curve.

283. If the equation, by which the applied line y is expressed by the abscissa x, shall be cubic or of a higher order, thus so that y will be a multiform function of z , then whichever one of the abscissas or so many applied lines come together , so y will have and equation of just as many dimensions, or the number of these may be diminished by two, four, six, etc. Therefore two applied lines always begin to be imaginary at the same time and before this, as they avoided being imaginary, they were equal to each other. Hence out of the transition from imaginaries to reals more varieties arise, which moreover either come together or are composed from these, which we have just explained. But if moreover all the values of the applied lines are sought both for the positive as well as for the negative abscissas, then a curve will be readily delineated passing through these points found and the figure of which will be known.

284. We may show this by an example, because, whatever shall arise from an equation of higher degree, yet the applied line y may be expressed by square roots only. There shall be without doubt

$$2y = \pm\sqrt{(6x-xx)} \pm \sqrt{(6x+xx)} \pm \sqrt{(36-xx)}\,,$$

from which equation an eighth-fold applied line corresponds to each abscissa. But it is evident, if the abscissa x may be placed negative, then the applied line becomes imaginary ; which likewise comes about, if x may be taken greater than 6; from which the whole curve will be contained between the limits $x = 0$ and $x = 6$. Therefore the successive values 0, 1, 2, 3, 4, 5, 6 are put for x and there will be :

If	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
$\sqrt{(6x-xx)}$	0,000	2,236	2,828	3,000	2,828	2,236	0,000
$\sqrt{(6x+xx)}$	0,000	2,646	4,000	5,196	6,325	7,416	8,485
$\sqrt{(36-xx)}$	6,000	5,916	5,657	5,196	4,472	3,317	0,000
Hence the sum :	6,000	10,798	12,485	13,392	13,625	12,969	8,485
From where y, if							
$+\ +\ +$	3,000	5,399	6,242	6,696	6,812	6,484	4,242
$-\ +\ +$	3,000	3,163	3,414	3,696	3,984	4,248	4,242
$+\ -\ +$	3,000	2,753	2,242	1,500	0,487	0,932	-4,242
$+\ +\ -$	-3,000	-0,517	0,586	1,500	2,341	3,167	4,242

The remaining four permutations of the signs differ only on account of the signs. Hence an eightfold applied line corresponds to any abscissa, which if they may be shown in the figure [54], will produce the constant two inter-folding curved lines *AFBEcagbcDA* and *afbEOAGBODa*, having two cusps at *A* and *a* and four double point or intersections of branches at *D*, *E*, *C* and *c*.

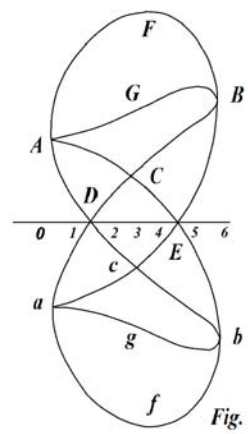

Fig. 54

CAPUT XII

DE INVESTIGATIONE FIGURAE LINEARUM CURVARUM

272. Quae in his capitibus sunt exposita, inserviunt figurae linearum curvarum in infinitum extensarum cognoscendae. Cuiusmodi vero figuram habeat quaepiam linea curva in spatio finita, saepenumero difficillimum est ex aequatione cognoscere. Oportet enim ad hoc pro quavis abscissa finita valores applicatae respondentes singulos ex aequatione eruere atque reales ab imaginariis discernere; quod negotium, si aequatio sit altioris gradus, plerumque vires analyseos cognitae superat. Quodsi enim abscissae valor quicunque cognitus tribuatur, applicata in aequatione incognitae vicem sustinebit. Hincque a numero dimensionum, quem applicata obtinet, pendebit aequationis resolutio. Negotium autem hoc per reductionem aequationis ad formam simpliciorem, dum et axis commodissimus et inclinatio coordinatarum aptissima assumitur, valde sublevari potest; tum etiam, quia perinde est, utra coordinatarum pro abscissa accipiatur, labor maxime diminuetur, si ea coordinatarum, cuius paucissimae dimensiones in aequatione occurrunt, pro applicata assumatur.

273. Sic, si figuras linearum tertii ordinis, quae ad speciem primam pertinent, investigare velimus, assumemus aequationem pro hac specie simplicissimam paragrapho 258 exhibitam et ex coordinatis *t* et *u* priorem *t* pro applicata, alteram vero *u* pro abscissa, quia *t* duas tantum dimensiones habet.
Huiusmodi ergo aequationis formam habebimus

$$yy = \frac{2by + axx + cx + d - nnx^3}{x}$$

quae resoluta dat

$$y = \frac{b \pm \sqrt{(bb + dx + cxx + ax^3 - nnx^4)}}{x}$$

existente neque *b* neque $n = 0$.

274. Qui ergo valores ipsius *x* functioni $bb + dx + cxx + ax^3 - nnx^4$ valorem affirmativum induunt, iis duplex applicata respondet; quibus casibus vero haec functio evanescit, iisdem unica applicata *y* abscissae *x* convenit seu binae applicatae inter se fiunt aequales. At si functio illa valorem negativum obtinet, tum abscissae nulla prorsus applicata respondet. Sed valores istius functionis, si fuerint affirmativi, in negativos abire

nequeunt, nisi prius facti sint aequales, seu functio evanuerit. Casus igitur potissimum erunt considerandi, quibus functio $bb+dx+cxx+ax^3-nnx^4$ fit $=0$; quod quidem certo duobus evenit casibus, quoniam, si x certum limitem sive affirmative sive negative transgrediatur, eius valor fit negativus. Hinc tota curva determinato abscissae spatio respondebit, ultra quod omnes applicatae fiant imaginariae.

275. Ponamus expressionem $bb+dx+cxx+ax^3-nnx^4$ duos tantum habere factores reales seu duobus tantum casibus evanescere posse; quod eveniat, si (Fig. 49) abscissa determinetur in punctis P et S , ubi unica tantum applicata reperiatur. Per totum ergo spatium PS applicatae erunt geminae et reales, extra spatium vero PS omnes applicatae erunt imaginariae ideoque tota curva intra applicatas Kk et Nn iacebit. Applicata vero in initio abscissarum A erit asymptota curvae, quae preaterea curvam in puncto quopiam secabit; si enim ponatur $x=0$, fiet

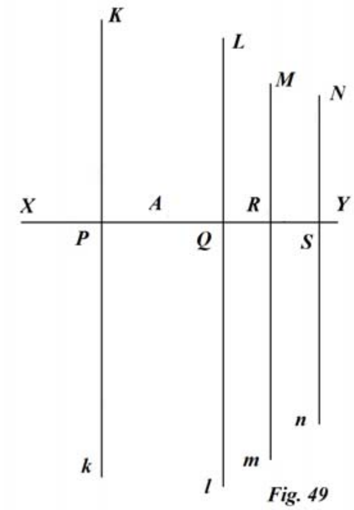

Fig. 49

$$\sqrt{(bb+dx+cxx+ax^3-nnx^4)}=b+\frac{dx}{2b},$$

unde erit

$$y=\frac{b\pm\left(b+\frac{dx}{2b}\right)}{x},$$

hoc est, erit vel $y=\infty$ vel $y=-\frac{d}{2b}$. Curva ergo hoc casu eiusmodi habet formam, qualem figura 50 repraesentat.

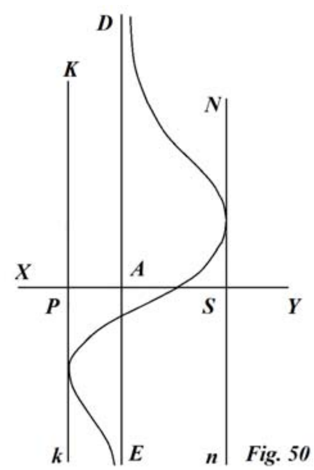

Fig. 50

276. Ponamus expressionem $bb+dx+cxx+ax^3-nnx^4$ quatuor habere factores simplices reales inaequales ideoque quatuor casibus evanscere. In totidem ergo locis P, Q, R et S applicatae curvam in unico puncto stringent. Cum igitur applicatae per axis spatium XP fuissent imaginariae, nunc per spatium QR erunt iterum imaginariae ac per RS rursus reales. Extra S versus Y denou fient imaginariae. Hinc curva constabit duabus partibus a se invicem separatis, quarum altera intra rectas Kk et Ll, altera intra rectas Mm and Nn continetur. Cum vero in abscissarum initio A applicatae sint reales, necesse est, ut id vel in axis intervallo PQ vel RS sit situm. Hoc ergo casu curva figuram habebit, qualem figura 51 ostendit, scilicet constabit ovali a reliqua curva ad asymptotam DE relata distante, quae vocatur *ovalis coniugata*.

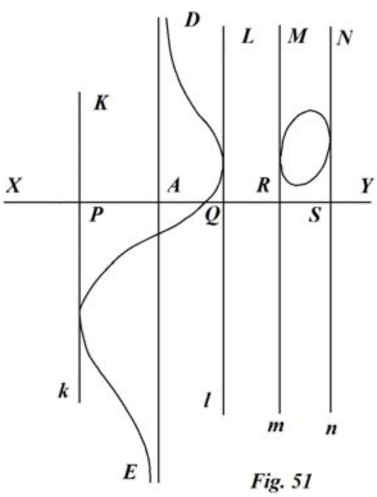

Fig. 51

277. Si duae radices fiant inter se aequales, vel puncta *P* et *Q* vel *Q* et *R* vel *R* et *S* convenient. Verum, si prius eveniat, quia *A* intra *P* et *Q* iacet, utraque radix deberet esse *x*, quod, quia *b* deesse nequit, fieri non potest. Sin autem puncta *R* et *S* conveniant, ovalis coniugata fiet infinite parva et abibit in *punctum coniugatum.* At si puncta *Q* et *R* conveniant, ovalis cum reliqua curva ita coniungetur, ut prodeat curva *nodata* figurae 52. Quodsi vero tres radices congruant seu puncta *Q, R* et *S* conveniant, tum nodus in *cuspidem* acutissimam evanescet, qualem figura 53 repraesentat. Sic igitur quinque diversae varietates in specie prima locum habent, ex quibus NEWTONUS totidem constituit species.

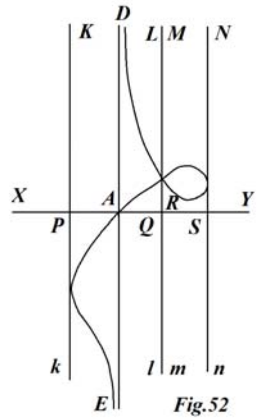

Fig.52

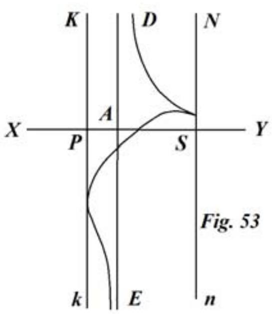

Fig. 53

278. Simili modo subdivisiones reliquarum specierum a NEWTONO sunt factae, quoniam omnes aequationes ita sunt comparatae, ut altera coordinata plures duabus non habeat dimensiones. Quando vero altera coordinata unicam habet dimensionem, forma curvae facillime cognoscetur. Aequatio enim erit huiusmodi $y = P$ existente P functione quapiam rationali abscissae *x;* quicunque ergo ipsi *x* valor tribuatur, applicata quoque semper unum obtinet valorem ideoque curva continuo tractu axem utrinque in infinitum comitabitur. Si functio *P* sit fracta, fieri potest, ut applicata in uno pluribusve locis fiat infinita ideoque curvae asymptotam exhibeat, quod evenit, ubi denominator functionis *P* evanescit.

279. Ponatur ergo $y = \dfrac{P}{Q}$, atque istas applicatas infinitas ostendent omnes radices reales aequationis $Q = 0$; quaelibet enim radix huius aequationis, puta $x = f$, declarat, si sumatur abscissa $x = f$, fore applicatam *y* infinitam, quia fit $Q = 0$. Tum vero patet, si fuerint applicatae *y* affirmativae, dum esset *x* maior quam *f*, easdem facto *x* minore quam *f* futuras esse negativas, ideoque applicata erit asymptota speciei $u = \dfrac{A}{t}$; hocque de omnibus factoribus inaequalibus est tenendum. Sin autem denominator *Q* duos habuerit factores aequales, puta $\left(x - f\right)^2$, tum, si applicatae sint affirmativae sumto *f* maiore quam *x*, manebunt affirmativae, si ponatur *x* minor *f*, eritque applicata *y* facto $x = f$ asymptota speciei $uu = \dfrac{A}{t}$. At si denominator *Q* tres habuerit factores aequales, nempe $\left(x - f\right)^3$, tum applicatae ante et post illam, quae fit infinita, diversa habebunt signa uti casu primo.

280. Post has aequationes facillime tractantur, quae in hac forma continentur $yy = \frac{2Py - R}{Q}$, existentibus *P, Q* et *R* functionibus quibuscunque integris abscissae *x.* Cuique igitur abscissae *x* vel geminae convenient applicatae vel nulla; duae scilicet prodeunt applicatae, si fuerit *PP* maior quam *QR,* et nulla, si *PP* minor quam *QR;* in quolibet ergo limite, qui applicatas reales ab imaginariis seu nullis dirimit, erit $PP = QR$ ideoque fit $y = \frac{P}{Q}$, seu haec applicata curvam in unico puncto stringet vel tanget. Ad curvae ergo formam cognoscendam consideretur aequatio $PP - QR = 0$, cuius singulae radices reales dabunt loca, ubi applicatae curvam in unico puncto stringunt. Notentur haec puncta in axe, atque, si omnes radices fuerint inaequales, axis partes inter haec puncta contentae alternatim habebunt applicatas geminas reales et imaginarias; sicque curva tot constabit partibus a se invicem seiunctis, quot huiusmodi alternationes adesse deprehenduntur, unde ovales coniugatae originem ducunt.

281. Si aequationis $PP - QR = 0$ duae radices fiant aequales, tum illorum in axe notatorum punctorum duo convenient hincque in axe portio vel imaginarias habens applicatas vel reales evanescet. Priori casu curva prodibit nodata uti in figura 52, posteriori ovalis coniugata in punctum coniugatum evanescet. Quodsi autem ilia aequatio tres habuerit radices aequales, nodus fiet infinite parvus atque in cuspidem abibit ut in figura 53; si quatuor affuerint radices aequationis aequales, vel duae ovales separatae concrescent in punctum vel in ipsa cuspide dabitur nodus seu duae cuspides ad verticem oppositae. Sin quinque radices aequales affuerint, novae fere formae non proveniunt; cuspis enim oritur, in qua non una ut ante sed duae ovales in punctum coalescunt; neque etiam maior radicum aequalium multitudo novum discrimen in figuris resultantibus producit.

282. Nodus seu intersectio duorum curvae ramorum vocari etiam solet *punctum duplex,* propterea quod linea recta curvam in eo puncto secans eam in duobus punctis secare censenda est. Atque si per nodum alius curvae ramus transiret, tum in hac intersectione nascetur punctum curvae *triplex;* punctum vero *quadruplex* orietur, si duo puncta duplicia conveniunt, ex quo genesis et natura punctorum quorumvis *multiplicium* perspicitur. Erit ergo etiam ovalis evanescens seu punctum coniugatum punctum duplex, pariter ac cuspis, quae oritur a puncto coniugato cum reliqua curva connexo.

283. Si aequatio, qua applicata *y* per abscissam *x* exprimitur, sit cubica vel altioris gradus, ita ut *y* aequetur functioni multiformi ipsius *z* , tum unicuique abscissae convenient vel tot applicatae, quot *y* in aequatione habet dimensiones, vel earum numerus minuetur binario vel quaternario vel senario etc. Perpetuo ergo binae applicatae simul imaginariae esse incipiunt atque prius, quam imaginariae evadunt, inter se fiunt aequales. Hinc ex transitione ab imaginariis ad reales plures nascuntur varietates, quae autem cum his,

quas modo explicavimus, vel conveniunt vel ex iis ipsis sunt compositae. Quodsi autem pro plurimis abscissis tam affirmativis quam negativis quaerantur omnes applicatae valores, tum per haec puncta inventa curva facile delineabitur eiusque figura cognoscetur.

284. Illustremus haec exemplo, quod, quamvis ortum sit ex aequatione altioris gradus, tamen applicata y per solas radices quadratas exprimatur. Sit nimirum

$$2y = \pm\sqrt{(6x-xx)} \pm \sqrt{(6x+xx)} \pm \sqrt{(36-xx)}\,,$$

ex qua aequatione cuivis abscissae octuplex applicata respondet. Perspicuum autem est, si abscissa x statuatur negativa, tum applicatam fore imaginariam; quod idem evenit, si abscissa x sumatur maior quam 6; ex quo tota curva intra limites $x=0$ et $x=6$ continebitur. Ponantur ergo pro x successive valores 0, 1, 2, 3, 4, 5, 6 eritque

si	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
$\sqrt{(6x-xx)}$	0,000	2,236	2,828	3,000	2,828	2,236	0,000
$\sqrt{(6x+xx)}$	0,000	2,646	4,000	5,196	6,325	7,416	8,485
$\sqrt{(36-xx)}$	6,000	5,916	5,657	5,196	4,472	3,317	0,000
summa	6,000	10,798	12,485	13,392	13,625	12,969	8,485
hinc y, si							
+ + +	3,000	5,399	6,242	6,696	6,812	6,484	4,242
- + +	3,000	3,163	3,414	3,696	3,984	4,248	4,242
+ - +	3,000	2,753	2,242	1,500	0,487	0,932	-4,242
+ + -	-3,000	-0,517	0,586	1,500	2,341	3,167	4,242

Reliquae quatuor signorum permutationes ab his tantum ratione signorum differunt. Hinc cuilibet abscissae octuplex applicata respondet, quae si in figura [54] exhibeantur, prodibit lineae curva duplici plexu *AFBEcagbcDA* et *afbEOAGBODa* constans, duas habens cuspides in *A* et *a* et puncta duplicia seu ramorum intersectiones quatuor in *D, E, C* et *c*.

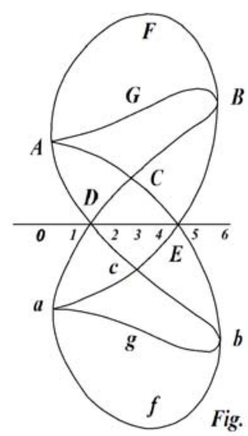

Fig. 54