CHAPTER XI

CONCERNING LINES OF THE FOURTH ORDER

260. The general equation for lines of the fourth order is

$$\alpha y^4 + \beta y^3 x + \gamma yyxx + \delta yx^3 + \varepsilon x^4 + \zeta y^3 + \eta yyx + \theta yxx + \iota x^3$$
$$+ \chi yy + \lambda yx + \mu xx + \nu y + \xi x + o = 0\ ;$$

but which can be reduced (as well as by variations of the inclination of the coordinates, by the position of the axis, and the start of the abscissas) in many ways to simpler forms for the diverse cases. Therefore so that the following method treats all the *species* or rather *kinds* of lines, which are contained in this order, it will be necessary that they be enumerated with respect to the greatest member, from which the following different cases are generated :

I.
If all four simple factors of the greatest member are imaginary.

II.

If two factors only are real and unequal to each other.

III.

If two factors only are real and equal.

IV.

If all four factors are real and unequal.

V.

If two factors are equal, with the remaining two present unequal to each other.

VI.

If besides two equal factors also the remaining two shall be equal to each other.

VII.

If three simple factors were equal to each other.

VIII.
If all four factors were equal to each other.

CASE I

261. If all factors of the greatest member were imaginary, generally there the curve will be devoid of branches extending to infinity ; because from the diversity of infinite branches we seek therefore a distinction of the kinds, this case provides a single genus. Therefore there becomes

GENUS I

with the general lack of branches of curves extending to infinity, the nature of which may be expressed by this simplest equation

$$(yy+mmxx)(yy-2pxy+qqxx)+ayyx+byxx+cyy+dyx+exx+fy+gx+h=0$$

with *pp* being less than *qq*. For because the terms y^4 and x^4 by necessity are present in the greatest member [in the main equation in §260], in effect the coordinates *x* and *y* can be either increased or deceased by a given amount, so that they may be greater than the terms y^3 and x^3 from the second member [hence enabling the equation to have no real roots].

CASE II

262. If only two factors of the greatest member shall be real and unequal, by the obliquity of the coordinates and the change of the axis it is possible to effect, that the one factor shall be *y* and the other truly *x* ; therefore the equation thus will be had :

$$yx(yy-2myx+nnxx)+ayyx+byxx+cyy+dyx+exx+fy+gx+h=0$$

with *mm* being present less than *nn* [to assure imaginary roots for this factor].
Because indeed the terms y^3x and yx^3 are present in the greatest member by necessity above, the terms y^3 and x^3 in the second member can be omitted. Therefore the curve will have two right asymptotes, the one expressed from the equation $y=0$, the other from the equation $x=0$. Therefore the nature of the first will be expressed by this equation

$$nnyx^3+exx+gx+h=0,$$

and of the latter by

$$xy^3 + cyy + fy + h = 0.$$

Hence there will be formed

GENUS II

with two right asymptotes, each of the kind $u = \frac{A}{t}$, with the provision that neither c nor e shall be a vanishing quantity.
[Note as in previous chapters that the coefficients at the end of a calculation are not necessarily the same as those at the start, as these are redefined *en passant* by Euler.]

GENUS III

has two right asymptotes, the one of the nature $u = \frac{A}{t}$, the other of the nature $u = \frac{A}{tt}$, and expressed from the equation

$$yx(yy - 2myx + nnxx) + ayyx + byxx + cyy + dyx + fy + gx + h = 0$$

with neither $c = 0$ nor $g = 0$.

GENUS IV

has two right asymptotes, the one of the kind $u = \frac{A}{t}$, the other of the kind $u = \frac{A}{t^3}$, and expressed by the equation

$$yx(yy - 2myx + nnxx) + ayyx + byxx + cyy + dyx + fy + h = 0$$

with c not $= 0$.

GENUS V

has two right asymptotes, both of the kind $u = \frac{A}{tt}$, and contained by the equation

$$yx(yy - 2myx + nnxx) + ayyx + byxx + dyx + fy + gx + h = 0$$

provided neither $f = 0$ nor $g = 0$.

GENUS VI

has two right asymptotes, the one of the kind $u = \frac{A}{tt}$ and the other of the kind $u = \frac{A}{t^3}$, moreover contained by this equation

$$yx(yy - 2myx + nnxx) + ayyx + byxx + dyx + fy + h = 0$$

with f present not $= 0$.

GENUS VII

has two right asymptotes, both of the kind $u = \frac{A}{t^3}$ and contained by this equation

$$yx(yy - 2myx + nnxx) + ayyx + byxx + dyx + h = 0$$

with *nn* everywhere greater than *mm.*

CASE III

263. Both these factors of the greatest member, which are accustomed to be real, shall be equal to each other, and the equation will be of this kind :

$$yy(yy - 2myx + nnxx) + ayyx + bx^3 + cyy + dyx + exx + fy + gx + h = 0$$

again with *nn* greater than *mm,* which equation, unless there shall be $b = 0$, gives

GENUS VIII

having one parabolic asymptote of the kind $uu = At$.
But if *b* shall be $= 0$, on putting $x = \infty$ there becomes

$$yy + \frac{ay}{nn} + \frac{e}{nn} + \frac{g}{nnx} + \frac{h}{nnxx} = 0.$$

Hence, if *aa* were less than *4nne,* there will be produced

GENUS IX

having no branches extending to infinity.
If there were $b = 0$ and *aa* greater than *4nne* and g not $= 0$, there will be produced

GENUS X

having two asymptotes of the kind $u = \frac{A}{t}$ parallel to each other.

GENUS X[a] [missing in the original]

having two asymptotes, one of the kind $u = \frac{A}{t}$ and one of the kind $u = \frac{A}{tt}$.
If there were both $b = 0$ and $g = 0$ and *aa* were greater than *4nne*, there is produced

GENUS XI

having two asymptotes parallel to each other of the kind $u = \frac{A}{tt}$.
If there were $b = 0$ and $aa = 4nne$ nor truly $g = 0$, there will be produced

GENUS XII

having an asymptote of the kind $uu = \frac{A}{t}$.
If there were $b = 0,\ g = 0$ and $aa = 4nne$ and h were a negative quantity, there will be produced

GENUS XIII

having a hyperbolic asymptote of the kind $uu = \frac{A}{tt}$.
But if $b = 0,\ g = 0,\ aa = 4nne$ and h were a positive quantity, there will be produced

GENUS XIV

evidently having no branches extending to infinity.

CASE IV

264. All four factors of the greatest member shall be real, simple, and unequal, and will have an equation of this form

$$yx(y-mx)(y-nx) \ + ayyx + byxx + cyy + dyx + exx + fy + gx + h = 0.$$

Therefore the curve will have four right asymptotes either of the kind $u = \frac{A}{t}$, or $u = \frac{A}{tt}$, or $u = \frac{A}{t^3}$. Hence, according to the precept given in paragraph 251, the following kinds will arise.

GENUS XV

having four hyperbolic asymptotes all of the kind $u = \frac{A}{t}$.

GENUS XVI

having four hyperbolic asymptotes, three of the kind $u = \frac{A}{t}$ and one of the kind $u = \frac{A}{tt}$.

GENUS XVII

having four hyperbolic asymptotes, three of the kind $u = \frac{A}{t}$ and one of the kind $u = \frac{A}{t^3}$.

GENUS XVIII

having four hyperbolic asymptotes, two of the kind $u = \frac{A}{t}$ and two of the kind $u = \frac{A}{tt}$.

GENUS XIX

having four hyperbolic asymptotes, two of the kind $u = \frac{A}{t}$, one of the kind $u = \frac{A}{tt}$ and one of the kind $u = \frac{A}{t^3}$.

GENUS XX

having four hyperbolic asymptotes, two of the kind $u = \frac{A}{t}$ and two of the kind $u = \frac{A}{t^3}$.

GENUS XXI

having four hyperbolic asymptotes, all of the kind $u = \frac{A}{tt}$.

GENUS XXII

having four hyperbolic asymptotes, three of the kind $u=\frac{A}{tt}$ and one of the kind $u=\frac{A}{t^3}$.

GENUS XXIII

having four hyperbolic asymptotes, two of the kind $u=\frac{A}{tt}$ and two of the kind $u=\frac{A}{t^3}$.

GENUS XXIV

having four hyperbolic asymptotes, all of the kind $u=\frac{A}{t^3}$.

CASE V

265. Two factors of the greatest member shall be equal to each other, with the remaining being unequal, and the equation will be of this kind

$$yyx(y+nx)+ayxx+bx^3+cyy+dyx+exx+fy+gx+h=0.$$

Hence in the first place, by reason of the equal factors, all genera arise, which is as in *case III*, and each one occurs with just as many varieties in total, as the number of unequal factors provides, that is, as many genera as the second case contains. Therefore six times seven, that is forty two genera arise from this case. But two genera are impossible to be produced, clearly if both the parallel asymptotes were of the kind $u=\frac{A}{tt}$ and with one of the remaining $u=\frac{A}{t}$, and with the other present either $u=\frac{A}{tt}$ or $u=\frac{A}{t^3}$. Whereby this case provides forty genera, which prepared with the preceding number makes sixty four kinds, which would be exceedingly long to describe here individually ; nor also because the individual genera to be set out are empty, as all are allowed to be real. But if anyone has a need, following the precepts given, they may wish to undertake this business by themselves, to restrain and amend the number of genera [i.e. kinds].

CASE VI

266. This case, in which two pairs of equal factors are present, will be contained in this equation

$$yyxx + ay^3 + bx^3 + cyy + dyx + exx + fy + gx + h = 0 .$$

Moreover each pair of equal factors considered by themselves gives seven, from which both pairs will produce forty nine genera. Truly because *h* is unable likewise to be positive and negative, two genera become impossible, and thus from this case forty seven kinds will arise in total, which number also is greater, than so that the individual members are able to be counted here. Therefore up to the present we have one hundred and eleven kinds arising. [According to the *O.O.* edition, this number is ninety in total.]

CASE VII

267. If three factors were equal to each other, the equation will be of this kind

$$y^3x + ayxx + bx^3 + cyy + dyx + exx + fy + gx + h = 0 .$$

Here the factor *x* provides an asymptote of the kind $u = \frac{A}{t}$, if c were not $= 0$; but if $c = 0$, nor truly were $f = 0$, it gives an asymptote of the kind $u = \frac{A}{tt}$; but if $c = 0$ and $f = 0$, it gives an asymptote of the kind $u = \frac{A}{t^3}$. Accordingly the factor y^3, unless there were $b = 0$, gives a parabolic asymptote of the kind $u^3 = Att$; but if $b = 0$, on making *x* infinite, the equation becomes

$$y^3 + ayx + dy + ex + g + \frac{cyy + fy + h}{x} = 0.$$

This, if e shall not be $= 0$, will become $y^3 + ayx + ex = 0$; from which, if a shall not be $= 0$, the asymptotes will be both $yy + ax = 0$ and $ay + e = 0$; therefore a place will be had likewise for parabolic asymptotes of the kind $uu = At$ and hyperbolic asymptotes expressed by this equation [the last term is incomplete]

$$(ay + e)x - \frac{e^3}{a^3} - \frac{de}{a} + g + \frac{cee - afe + aah}{aax} = 0.$$

Therefore unless there shall be $e^3+aade-a^3g=0$, this asymptote is of the kind $u=\frac{A}{t}$; truly on the other hand of the kind $u=\frac{A}{tt}$. But if $a=0$ with e not $=0$, it will be $y^3+ex=0$; which gives a parabolic asymptote of the kind $u^3=At$. But if $e=0$ and $a=0$, there becomes $y^3+dy+g=0$, which equation gives rise to either a single asymptote of the kind $u=\frac{A}{t}$, three of the same kind, or a single of the kind $u=\frac{A}{t}$ and one of the kind $uu=\frac{A}{t}$ or one of the kind $u^3=\frac{A}{t}$. Therefore altogether eight varieties occur, which multiplied by three arising from the factor *x*, will give twenty four. Therefore all the cases treated up to this stage give *one hundred and thirty five* kinds.

CASE VIII

268. If all the factors shall be equal to each other, this equation will be accommodated :

$$y^4+ayyx+byxx+kx^3+cyy+dyx+exx+fy+gx+h=0.$$

This, if k were not $=0$, will produce

GENUS CXXXVI

having a single parabolic asymptote of the kind $u^4=At^3$.
Let $k=0$, truly b not $=0$, there will be $y^4+byxx+exx=0$ and hence $y^3+bxx=0$ and $by+e=0$; from which, for the right asymptote $by+e=0$, there will be

$$(by+e)xx+\frac{e^4}{b^4}+\frac{aeex}{bb}+\frac{cee}{bb}-\frac{dex}{b}-\frac{ef}{b}+gx+h=0;$$

therefore, unless there shall be $aee-bde+bbg=0$, the asymptote will be of the kind $u=\frac{A}{t}$, truly otherwise of the kind $u=\frac{A}{tt}$; from which the

GENUS CXXXVII

will be produced having a parabolic asymptote of the kind $u^3=Att$ and one hyperbolic asymptote of the kind $u=\frac{A}{t}$, and

GENUS CXXXVIII

having a parabolic asymptote of the kind $u^3 = Att$ and one hyperbolic asymptote of the kind $u = \frac{A}{tt}$.

269. If now both $k = 0$ and $b = 0$, so that there shall be

$$y^4 + ayyx + cyy + dyx + exx + fy + gx + h = 0;$$

if e shall not be $= 0$, there becomes $y^4 + ayyx + exx = 0$, which equation, if *aa* were less than *4e*, is impossible, but if *aa* were greater than *4e*, two parabolic asymptotes related to the same axis, of the kind $uu = At$, will be produced ; but if $aa = 4e$, these two parabolas merge into one, in which case the genera CXXXIX, CXL and CXLI are put in place.

But, if $e = 0$, so that this equation may be had :

$$y^4 + ayyx + cyy + dyx + fy + gx + h = 0,$$

if a shall not be $= 0$, there will be $y^4 + ayyx + cyy + dyx + gx = 0$, and therefore both $yy + ax = 0$ and $y =$ a constant, there will be $ayy + dy + g = 0$, from which *y* either has two different values, either equal or nothing real. In the first case the curve in addition to the one parabolic asymptote will have two parallel asymptotes of the kind $u = \frac{A}{t}$, in the second case one of the kind $uu = \frac{A}{t}$, and in the third case nothing, from which again the three genera are put in place, surely CXLII, CXLIII and CXLIV.

270. Now also there shall be $a = 0$, so that there shall be

$$y^4 + cyy + dyx + fy + gx + h = 0 .$$

Here, if d shall not be $= 0$, the curve will have a parabolic asymptote of the kind $u^3 = At$ and containing one right line equation $dy + g = 0$, and one of the kind $u = \frac{A}{t}$.
Finally if also $d = 0$, the curve will have one parabolic asymptote of the kind $u^4 = At$ *;* and thus altogether they have established *one hundred and forty six* genera of lines of the fourth order ; but which include within themselves generally several more individual kinds.

271. Now from these it is clearly seen, how greatly the number of genera may be multiplied in lines of the fifth and higher order, so that by counting, how many we have

made for the third order, higher orders cannot evidently be put in place, unless it were wished to devote a whole volume to which work.
But what may concern the properties of lines of the fourth and higher orders, these will be derived in a similar manner from a general equation, just as we have used above with lines of the third order, and thus I will not tarry in the analysis of these.

CAPUT XI

DE LINEIS QUARTI ORDINIS

260. Aequatio generalis pro lineis quarti ordinis est

$$\alpha y^4+\beta y^3x+\gamma yyxx+\delta yx^3+\varepsilon x^4+\zeta y^3+\eta yyx+\theta yxx+\iota x^3$$
$$+\chi yy+\lambda yx+\mu xx+\nu y+\xi x+o=0\ ;$$

quae autem (variatis tum coordinatarum inclinatione, tum axis positione, tum abscissarum initio) multis modis pro diversis casibus ad simpliciorem formam reduci potest. Quo igitur secundum methodum traditam omnes *species* vel potius *genera* linearum, quae in hoc ordine continentur, enumerentur, ad membrum supremum respici oportet, unde sequentes casus nascuntur diversi:

I.
Si supremi membri omnes quatuor factores simplices sunt imaginarii.

II.

Si duo factores tantum sunt reales et inaequales inter se.

III.

Si duo factores tantum sunt reales et aequales.

IV.

Si omnes quatuor factores sunt reales et inaequales.

V.

Si duo factores inter se sunt aequales, reliquis binis inter se existentibus inaequalibus.

VI.

Si praeter duos factores aequales etiam reliqui duo sint inter se aequales.

VII.

Si tres factores simplices fuerint inter se aequales.

VIII.

Si omnes quatuor factores inter se aequales fuerint.

CASUS I

261. Si omnes factores membri supremi fuerint imaginarii, curva ramis in infinitum excurrentibus omnino erit destituta; quoniam igitur ex diversitate ramorum infinitorum discrimen generum petimus, iste casus unicum praebebit genus. Erit ergo

GENUS I

curvarum ramis in infinitum extensis omnino carentium, quarum natura hac aequatione simplicissima exprimetur

$$(yy+mmxx)(yy-2pxy+qqxx)+ayyx+byxx+cyy+dyx+exx+fy+gx+h=0$$

existente *pp* minore quam *qq*. Quoniam enim in supremo membro termini y^4 et x^4 necessario adsunt, coordinatis *x* et *y* quantitate data sive augendis sive minuendis effici potest, ut termini y^3 et x^3 ex secundo membro excedant.

CASUS II

262. Si duo factores membri supremi tantum sint reales et inaequales, per obliquitatem coordinatarum et axis mutationem effici potest, ut alter sit *y* alter vero *x;* aequatio ergo ita se habebit

$$yx(yy-2myx+nnxx)+ayyx+byxx+cyy+dyx+exx+fy+gx+h=0$$

existente *mm* minore quam *nn*.
Quia enim in supremo membro termini y^3x et yx^3 necessario adsunt, in secundo membro termini y^3 et x^3 omitti possunt. Habebit ergo curva duas asymptotas rectas, alteram aequatione $y=0$, alteram aequatione $x=0$ expressam. Prioris ergo indoles exponetur hac aequatione

$$nnyx^3+exx+gx+h=0,$$

posterioris hac

$$xy^3 + cyy + fy + h = 0.$$

Hinc formabitur

GENUS II

duabus asymptotis rectis, utraque indolis $u = \frac{A}{t}$, praeditum, si neque c neque e sit quantitas evanescens.

GENUS III

duas habet asymptotas rectas, alteram indolis $u = \frac{A}{t}$, alteram indolis $u = \frac{A}{tt}$, et exprimitur aequatione

$$yx(yy - 2myx + nnxx) + ayyx + byxx + cyy + dyx + fy + gx + h = 0$$

non existente $c = 0$ neque $g = 0$.

GENUS IV

duas habet asymptotas rectas, alteram indolis $u = \frac{A}{t}$, alteram indolis $u = \frac{A}{t^3}$, et exprimitur aequatione

$$yx(yy - 2myx + nnxx) + ayyx + byxx + cyy + dyx + fy + h = 0$$

non existente $c = 0$.

GENUS V

duas habet asymptotas rectas, ambas generis $u = \frac{A}{tt}$, et continetur aequatione

$$yx(yy - 2myx + nnxx) + ayyx + byxx + dyx + fy + gx + h = 0$$

non existente $f = 0$ neque $g = 0$.

GENUS VI

duas habet asymptotas rectas, alteram indolis $u = \frac{A}{tt}$ et alteram indolis $u = \frac{A}{t^3}$, continetur autem hac aequatione

$$yx(yy-2myx+nnxx)+ayyx+byxx+dyx+fy+h=0$$

non existente $f=0$.

GENUS VII

duas habet asymptotas rectas, ambas indolis $u=\frac{A}{t^3}$ et continetur hac aequatione

$$yx(yy-2myx+nnxx)+ayyx+byxx+dyx+h=0$$

existente ubique *nn* maiore quam *mm*.

CASUS III

263. Sint ambo illi factores supremi membri, qui soli sunt reales, inter se aequales, atque aequatio erit huiusmodi:

$$yy(yy-2myx+nnxx)+ayyx+bx^3+cyy+dyx+exx+fy+gx+h=0$$

existente iterum *nn* maiore quam *mm*, quae aequatio, nisi sit $b=0$, dat

GENUS VIII

habens unam asymptotam parabolicam speciei $uu=At$.
Si autem *b* sit $=0$, posito $x=\infty$ fiet

$$yy+\frac{ay}{nn}+\frac{e}{nn}+\frac{g}{nnx}+\frac{h}{nnxx}=0.$$

Hinc, si fuerit *aa* minor quam *4nne*, prodit

GENUS IX

nullum habens ramum in infinitum extensum.
Si fuerit $b=0$ et *aa* maior quam *4nne* neque sit $g=0$, prodit

GENUS X

duas habens asymptotas inter se parallelas speciei $u=\frac{A}{t}$.

GENUS X[a]

duas habens asymptotas, unam speciei $u=\frac{A}{t}$ et unam speciei $u=\frac{A}{tt}$.
Si fuerit et $b=0$ et $g=0$ et aa maior quam $4nne$, prodit

GENUS XI

duas habens asymptotas inter se parallelas speciei $u=\frac{A}{tt}$.
Si fuerit $b=0$ et $aa=4nne$ nec vero $g=0$, prodit

GENUS XII

asymptotam habens speciei $uu=\frac{A}{t}$.
Si fuerit $b=0,\ g=0$ et $aa=4nne$ atque h quantitas negativa, prodit

GENUS XIII

asymptotam habens hyperbolicam speciei $uu=\frac{A}{tt}$.
At si $b=0,\ g=0,\ aa=4nne$ et h quantitas affirmativa, prodit

GENUS XIV

nullos prorsus habens ramos in infinitum extensos.

CASUS IV

264. Sint membri supremi omnes quatuor factores simplices reales et inaequales, atque aequatio huiusmodi formam habebit

$$yx(y-mx)(y-nx)\ +ayyx+byxx+cyy+dyx+exx+\ fy+gx+h=0.$$

Curva igitur quatuor habebit asymptotas rectas speciei vel $u=\frac{A}{t}$ vel $u=\frac{A}{tt}$ vel $u=\frac{A}{t^3}$. Hinc, ad praeceptum paragrapho 251 datum, sequentia orientur genera.

GENUS XV

habens quatuor asymptotas hyperbolicas omnes speciei $u = \frac{A}{t}$.

GENUS XVI

habens quatuor asymptotas hyperbolicas, tres speciei $u = \frac{A}{t}$ et unam speciei $u = \frac{A}{tt}$.

GENUS XVII

habens quatuor asymptotas hyperbolicas, tres speciei $u = \frac{A}{t}$ et unam speciei $u = \frac{A}{t^3}$.

GENUS XVIII

habens quatuor asymptotas hyperbolicas, duas speciei $u = \frac{A}{t}$ et duas speciei $u = \frac{A}{tt}$.

GENUS XIX

habens quatuor asymptotas hyperbolicas, duas speciei $u = \frac{A}{t}$, unam speciei $u = \frac{A}{tt}$ et unam speciei $u = \frac{A}{t^3}$.

GENUS XX

habens quatuor asymptotas hyperbolicas, duas speciei $u = \frac{A}{t}$ et duas speciei $u = \frac{A}{t^3}$.

GENUS XXI

habens quatuor asymptotas hyperbolicas, omnes speciei $u = \frac{A}{tt}$.

GENUS XXII

habens quatuor asymptotas hyperbolicas, tres speciei $u = \frac{A}{tt}$ et unam speciei $u = \frac{A}{t^3}$.

GENUS XXIII

habens quatuor asymptotas hyperbolicas, duas speciei $u = \frac{A}{tt}$ et duas speciei $u = \frac{A}{t^3}$

GENUS XXIV

habens quatuor asymptotas hyperbolicas, omnes speciei $u = \frac{A}{t^3}$.

CASUS V

265. Sint duo factores membri supremi inter se aequales, reliquis existentibus inaequalibus, aequatio erit huiusmodi

$$yyx(y+nx)+ayxx+bx^3+cyy+dyx+exx+fy+gx+h=0.$$

Hinc primo, ratione factorum aequalium, omnia oriuntur genera, quae in *casu III,* et unumquodque cum tot varietatibus occurrit, quot factores inaequales suggerunt, hoc est, quot casus secundus continet genera. Omnino ergo

sexies septem, hoc est quadraginta-duo general) ex hoc casu nascuntur. Duo autem hinc prodeunt genera impossibilia, nempe si ambae asymptotae parallelae speciei fuerint $u = \frac{A}{tt}$ et reliquarum una $u = \frac{A}{t}$, altera existenta vel $u = \frac{A}{tt}$ vel $u = \frac{A}{t^3}$. Quare hic casus quadraginta genera praebet, quae cum antecedentibus numerum generum *sexaginta-quatuor* conficiunt, quae singula hic describere nimis foret longum. Neque etiam, quia singula haec genera evolvere non vacavit, firmiter affirmare licet omnia esse realia. Qui autem secundum praecepta data hoc negotium in se suscipere voluerit, numerum generum, si opus fuerit, restringet atque emendabit.

CASUS VI

266. Hic casus, quo duo factorum aequalium paria adsunt, ista aequatione continebitur

$$yyxx+ay^3+bx^3+cyy+dyx+exx+fy+gx+h=0 .$$

Utrumque autem factorum aequalium par in se spectatum varietates dat septem, unde ambo paria praebebunt genera quadraginta novem. Quia vero *h* simul affirmativum et negativum esse nequit, duo genera fiunt impossibilia, ideo que ex hoc casu omnino nascuntur genera quadraginta-septem, qui numerus etiam maior est, quam ut singula hic recenseri queant. Hactenus ergo nacti sumus genera *centum et undecim.*

CASUS VII

267. Si tres factores inter se fuerint aequales, aequatio erit eiusmodi

$$y^3x + ayxx + bx^3 + cyy + dyx + exx + fy + gx + h = 0 .$$

Hic factor x praebet asymptotam speciei $u = \frac{A}{t}$, si non fuerit $c = 0$; at si $c = 0$, nec vero $f = 0$, asymptotam dat speciei $u = \frac{A}{tt}$; at si $c = 0$ et $f = 0$, asymptotam dat speciei $u = \frac{A}{t^3}$. Deinde factor y^3, nisi fuerit $b = 0$, dat asymptotam parabolicam speciei $u^3 = Att$; sin autem $b = 0$, posito x infinito, fit

$$y^3 + ayx + dy + ex + g + \frac{cyy + fy + h}{x} = 0.$$

Hic, si non sit $e = 0$, erit $y^3 + ayx + ex = 0$; unde, si nec $a = 0$, erit et $yy + ax = 0$ et $ay + e = 0$; simul ergo locum habet asymptota parabolica speciei $uu = At$ et hyperbolica hac aequatione expressa

$$(ay + e)x - \frac{e^3}{a^3} - \frac{de}{a} + g + \frac{cee - afe + aah}{aax} = 0.$$

Nisi ergo sit $e^3 + aade - a^3g = 0$, haec asymptota est speciei $u = \frac{A}{t}$; contra vero speciei $u = \frac{A}{tt}$. At si $a = 0$ non existente $e = 0$, erit $y^3 + ex = 0$; quae dat asymptotam parabolicam speciei $u^3 = At$. Sin autem sit $e = 0$ et $a = 0$, fiet $y^3 + dy + g = 0$, quae aequatio vel unicam praebet asymptotam speciei $u = \frac{A}{t}$ vel tres eiusdem speciei vel unam speciei $u = \frac{A}{t}$ et unam speciei $uu = \frac{A}{t}$ vel unam speciei $u^3 = \frac{A}{t}$. Omnino ergo octo varietates occurrunt, quae, per tres ex factore x ortas multiplicatae, dabunt genera viginti-quatuor. Ergo omnes casus hactenus tractati dant genera *centum triginta quinque* .

CASUS VIII

268. Si omnes factores sint inter se aequales, haec aequatio locum habebit:

$$y^4 + ayyx + byxx + kx^3 + cyy + dyx + exx + fy + gx + h = 0 .$$

Hic, si non fuerit $k = 0$, prodit

GENUS CXXXVI

unicam habens asymptotam parabolicam speciei $u^4 = At^3$.
Sit $k = 0$, non vero $b = 0$, erit $y^4 + byxx + exx = 0$ hincque $y^3 + bxx = 0$ et $by + e = 0$; unde, pro asymptota recta $by + e = 0$, erit

$$(by + e)xx + \frac{e^4}{b^4} + \frac{aeex}{bb} + \frac{cee}{bb} - \frac{dex}{b} - \frac{ef}{b} + gx + h = 0 ;$$

ergo, nisi sit $aee - bde + bbg = 0$, asymptota erit speciei $u = \frac{A}{t}$, contra vero speciei $u = \frac{A}{tt}$; unde prodeunt

GENUS CXXXVII

unam habens asymptotam parabolicam speciei $u^3 = Att$ et unam hyperbolicam speciei $u = \frac{A}{t}$, et

GENUS CXXXVIII

unam habens asymptotam parabolicam speciei $u^3 = Att$ et unam hyperbolam speciei. $u = \frac{A}{tt}$.

269. Sit iam $k = 0$ et $b = 0$, ut sit

$$y^4 + ayyx + cyy + dyx + exx + fy + gx + h = 0 ; ;$$

si non sit $e = 0$, erit $y^4 + ayyx + exx = 0$, quae aequatio, si fuerit *aa* minor quam *4e*, est impossibilis, sin *aa* maior quam *4e*, duas praebet asymptotas parabolicas ad eundem axem relatas, speciei $uu = At$*;* sin $aa = 4e$, hae duae parabolae in unam coeunts, quibus genera CXXXIX, CXL et CXLI constituuntur.

At, si $e = 0$, ut habeatur haec aequatio

$$y^4 + ayyx + cyy + dyx + fy + gx + h = 0,$$

si non sit $a=0$, erit $y^4+ayyx+cyy+dyx+gx=0$, ergo et $yy+ax=0$ et $y=$constanti, erit $ayy+dy+g=0$, unde y vel duos habet valores diversos vel aequales vel nullum realem. Casu primo curva praeter unam asymptotam parabolicam habebit duas asymptotas parallelas speciei $u=\frac{A}{t}$, secundo unam speciei $uu=\frac{A}{t}$, tertio nullam, unde iterum tria genera constituuntur nempe CXLII, CXLIII et CXLIV.

270. Sit nunc etiam $a=0$, ut sit

$$y^4+cyy+dyx+fy+gx+h=0\ .$$

Hic, si non sit $d=0$, curva habebit asymptotam parabolicam speciei $u^3=At$ et unam rectam aequatione $dy+g=0$ contentam, speciei $u=\frac{A}{t}$. Denique si et $d=0$, curva unam habebit asymptotam parabolicam speciei $u^4=At$*;* sicque omnino linearum quarti ordinis constituta sunt genera *centum quadragintasex* ; quae autem singula plerumque plures species notabiliter differentes sub se complectuntur.

271. Ex his iam clare perspicitur, quantopere generum numerus in lineis quinti altiorisve ordinis multiplicetur, ut recensio, qualem pro ordine tertio fecimus, institui prorsus nequeat, nisi quis integrum volumen huic operi destinare velit. Quod autem ad primarias proprietates linearum quarti altiorisve ordinis attinet, eae ex aequatione generali simili modo derivabuntur, quo supra in lineis tertii ordinis sumus usi, neque idcirco earum explicationi immorabimur.