CHAPTER X

CONCERNING THE PRINCIPAL PROPERTIES OF LINES OF THE THIRD ORDER

239. Just as above we have deduced the principal properties of second order lines from the general equation, thus also the principal properties of lines of the third order will be understood from the general equation, and likewise also it will be allowed to infer the general properties of lines of the fourth and of higher orders from the equation of that order. On account of which we will consider the most general equation for lines of the third order, which is

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0,$$

which expresses the nature of a line of the third order of any kind between the coordinates *x* and *y*, at whatever angle of inclination, and with any right line taken for the axis.

240. Therefore unless α shall be = 0, each and every abscissa *x* will correspond to one or three real applied lines. We may consider three real applied lines to be given ; and it is evident that the relation of these can be defined by an equation. And thus on putting $\alpha = 1$, the equation will be of this kind :

$$y^3 + (\beta x + \varepsilon)yy + (\gamma xx + \zeta x + \theta)y + \delta x^3 + \eta xx + \iota x + \chi = 0$$

and the sum of these three applied lines corresponding to the same abscissa will be $= -\beta x - \varepsilon$*;* the sum of the three rectangles formed from two applied lines will be $= \gamma xx + \zeta x + \theta$*;* and finally the product of all, or the parallelepiped formed from these will be $= -\delta x^3 - \eta xx - \iota x - \chi$. If two applied lines were imaginary, indeed these will prevail the same, but they will not be able to accommodate the figure of the lines, because from that neither the sum nor the rectangle of the two imaginary applied lines will be able to be understood.

241. Therefore let some line of the third order (Fig. 44) be related to the axis *AZ*, to which below from a given angle the applied lines shall be the ordinates *LMN, lmn* cutting the curve in three points. Therefore on putting the abscissa $AP = x$, the triple applied line *y* will have the value *PL, PM* and −*PN*; from which there will be

$$PL + PM - PN = -\beta x - \varepsilon.$$

Whereby, if there may be taken

$$PO = z = \frac{PL + PM - PN}{3},$$

the point O thus will be in the middle position, so that there shall be $LO = MO + NO$. Therefore since there shall be $z = \frac{-\beta x + \varepsilon}{3}$, this point O shall be placed on the right line OZ, which right line therefore will cut all the ordinates lmn parallel to LMN itself in o, so that there shall be $lo + mo = no$; which property is analogous to the property of diameters, which lines of the second order have provided. But if therefore the two parallel ordinates and cutting the curve in three points thus will be cut in the points O et o, so that the two applied lines adjacent on one side taken likewise shall be equal to the third placed on the other side, the right line drawn through these points O and o will cut the remaining ordinates similarly to these, and the line will be as if a diameter of the third order.

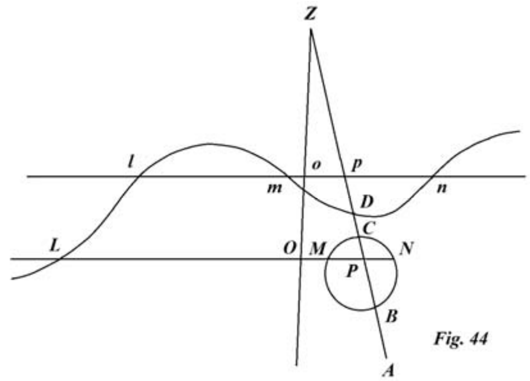

Fig. 44

242. Because for lines of the second order all the diameters mutually intersect each other at the same point, we may consider in what manner several diameters of lines of this kind of the third order may be compared among themselves. Therefore we may conceive the applied lines under some other angle to the same axis AP, and the [new] abscissa shall be $= t$ and the [new] applied line $= u$; there will be $y = nu$ and $x = t - mu$, which values substituted into the general equation [where n and m can be viewed as scaling factors]

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0$$

will give this equation :

$$\left.\begin{array}{l} + n^3u^3 \quad + \beta nnuut \quad + \gamma nutt + \delta t^3 + \varepsilon nnuu \ + \zeta nut + \eta tt + \theta nu + \iota t + \chi \\ -\beta mnnu^3 \ - 2\gamma mnuut - 3\delta mutt \quad - \zeta mnuu - 2\eta mut \quad - \iota mu \\ + \gamma mmnu^3 + \ 3\delta mmuut \qquad\qquad + \eta mmuu \\ -\delta m^3u^3 \end{array}\right\} = 0.$$

Hence for that right line of the diameter sustained in turn, if its applied line drawn under the same angle to the abscissa t may be called $= v$, there will be

$$3v = \frac{-\beta nnt + 2\gamma mnt - 3\delta mmt - \varepsilon nn + \zeta mn - \zeta mm}{n^3 - \beta mnn + \gamma mmn - \delta m^3}.$$

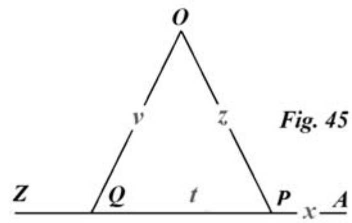

Fig. 45

243. Now O shall be the intersection (Fig. 45) of these two diameters, so that in the first place OP may be drawn parallel to the former applied lines for the axis AZ, then truly OQ will be parallel to the latter, and there will be $AP = x,\ PO = z,\ AQ = t$ and $OQ = v$.
Then truly there will be

$$z = nv \text{ and } x = t - mv$$

and thus

$$v = \frac{z}{n} \text{ and } t = x + \frac{m}{n}z.$$

And thus in the first place $3z = \beta x - \varepsilon$, and again

$$3v = -\frac{\beta x}{n} - \frac{\varepsilon}{n} \text{ and } t = x - \frac{\beta mx}{3n} - \frac{\varepsilon m}{3n}.$$

These values may be substituted into the equation found before, and it will produce :

$$\left.\begin{array}{llll}
-\beta nnx & +\beta\beta mnx & -\beta\gamma mnx & +\dfrac{\beta\delta m^3 x}{n} \\
-\varepsilon nn & +\beta\varepsilon mn & -\gamma\varepsilon mm & +\dfrac{\delta\varepsilon m^3}{n} \\
+\beta nnx & -\dfrac{\beta\beta mnx}{3} & -\dfrac{\beta\varepsilon mn}{3} & +\varepsilon nn \\
-2\gamma mnx & +\dfrac{2\beta\gamma mmx}{3} & +\dfrac{2\gamma\varepsilon mm}{3} & -\zeta mn \\
+3\delta mmx & -\dfrac{\beta\delta m^3 x}{n} & -\dfrac{\delta\varepsilon m^3}{n} & +\eta mm
\end{array}\right\} = 0.$$

or

$$\left.\begin{array}{llll}
\frac{2}{3}\beta\beta mnx & -\frac{1}{3}\beta\gamma mmx & -2\gamma mnx & +3\delta mmx \\
+\frac{2}{3}\beta\varepsilon mn & -\frac{1}{3}\gamma\varepsilon mm & -\zeta mn & +\eta mm
\end{array}\right\} = 0.$$

244. Therefore the intersection of the diameters *O* will depend everywhere on the inclination of the applied lines to the axis, which will be contained by the letters *m* and *n* ; nor on that account (if the intersection of the diameters can be called the *centre*), do all the lines of the third order have a centre. Yet meanwhile cases can be shown, in which the mutual intersection of the diameters fall on the same fixed point. Clearly this comes about, if the terms including *mn* and *mm* may separately be made equal to zero and the values of *x* thence arising to be put equal. But from these two equalities there becomes :

$$x=\frac{3\zeta-2\beta\varepsilon}{2\beta\beta-6\gamma}=\frac{3\eta-\gamma\varepsilon}{\beta\gamma-9\delta};$$

so that which two values may agree, it is necessary that there shall be

$$6\beta\beta\eta-2\beta\beta\gamma\varepsilon-18\gamma\eta+6\gamma\gamma\varepsilon=3\beta\gamma\zeta-2\beta\beta\gamma\varepsilon-27\delta\zeta+18\beta\delta\varepsilon$$

or

$$\beta\gamma\zeta-2\beta\beta\eta-9\delta\zeta+6\gamma\eta+6\beta\delta\varepsilon-2\gamma\gamma\varepsilon=0,$$

from which there becomes

$$\eta=\frac{\beta\gamma\zeta-9\delta\zeta+6\beta\delta\varepsilon-2\gamma\gamma\varepsilon}{2\beta\beta-6\gamma}.$$

Therefore as many times as a value η of this kind may be had, so all the diameters mutually intersect each other in one and the same point ; and thus these lines of the third order do have a centre, which will be found by taking on the axis

$$AP=\frac{3\zeta-2\beta\varepsilon}{2\beta\beta-6\gamma}\ \text{ and }\ PO=\frac{-\beta\zeta+2\gamma\varepsilon}{2\beta\beta-6\gamma}.$$

245. This same determination of the centre can be accommodated, if which may be given if for the first coefficient α may not be put as one. For if the most general equation were proposed for lines of the third order :

$$\alpha y^3+\beta yyx+\gamma yxx+\delta x^3+\varepsilon yy+\zeta yx+\eta xx+\theta y+\iota x+\chi=0\ ,$$

these curves will be provided with a centre, if there were

$$\eta=\frac{\beta\gamma\zeta-9\alpha\delta\zeta+6\beta\delta\varepsilon-2\gamma\gamma\varepsilon}{2\beta\beta-6\alpha\gamma}.$$

Then truly the centre will be at *O* with

$$AP = \frac{3\alpha\zeta - 2\beta\varepsilon}{2\beta\beta - 6\alpha\gamma} \text{ and } PO = \frac{2\gamma\varepsilon - \beta\zeta}{2\beta\beta - 6\alpha\gamma} \text{ present.}$$

Whereby, if a single ordinate thus may be divided cutting the curve in three points, so that the two applied lines placed on the one side will be equal to the third adjacent on the other side, then a right line drawn through the centre and this point of division will cut all the other ordinates parallel to that.

246. If these may be adapted to equations of the kinds enumerated above, it will be apparent that the first, second, third, fourth and fifth kinds have a centre, but only if there shall be $\alpha = 0$, and in this case the centre is put at the beginning of the abscissas itself. The sixth and seventh kinds are completely without a centre, because the coefficient α cannot be absent. Truly the eighth, ninth, tenth, eleventh, twelfth and thirteenth have a centre, always in the position of the start of the abscissas. In the fourteenth, fifteenth, and sixteenth kinds the centre is infinitely distant, and thus all these lines divided into three parts among themselves are parallel.

247. From these remarks concerning the sum of each of three applied lines of noted values, we may consider the product of the same, because nothing noteworthy is found from the sum of the rectangles [*i.e.* the intermediate terms in the general equation]. Therefore from the general equation of paragraph 239 there will be

$$-PM \cdot PL \cdot PN = -\delta x^3 - \eta xx - \iota x - \chi\,;$$

we will attend to this expression requiring to be explained, so that, if there may be put $y = 0$, there becomes $\delta x^3 + \eta xx + \iota x + \chi = 0$, the roots of which equation will give the intersections of the curve with the axis *AZ*. Which if they shall be at the points *B, C* and *D,* will be

$$\delta x^3 + \eta xx + \iota x + \chi = \delta(x - AB)(x - AC)(x - AD)\;;$$

on account of which there will be [see Fig. 44]

$$PL \cdot PM \cdot PN = \delta \cdot PB \cdot PC \cdot PD\,;$$

and thus some other ordinates *lmn* taken parallel to the first will be

$$PL \cdot PM \cdot PN : \; PB \cdot PC \cdot PD = pl \cdot pm \cdot pn : \; pB \cdot pC \cdot pD\,;$$

which property generally is similar to that, as we have found above for lines of the second order in the ratio of rectangles ; and similar properties will happen for lines of the fourth, fifth, and higher orders.

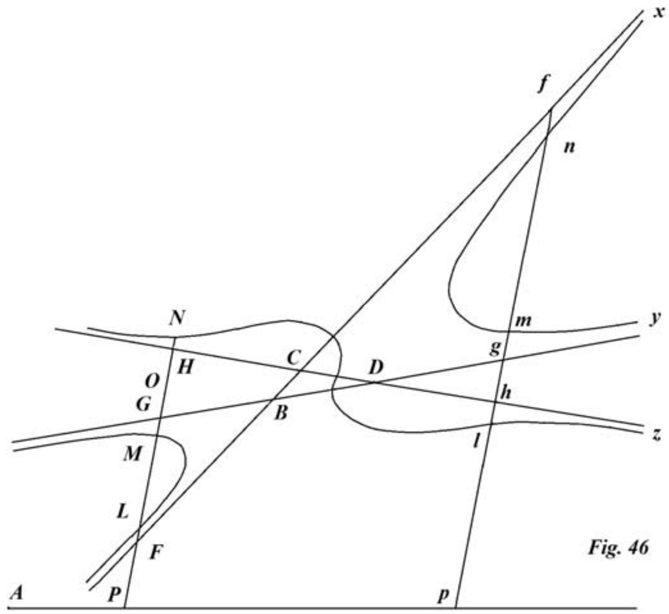

Fig. 46

248. Now a line of the third order may also have the three right asymptotes *FBf, GDg, HCh* (Fig. 46). Because a line of the third order will change into these three asymptotes, if the equation for the curve becomes resolvable into three simple factors of the form $py+qx+r$, so that an equation will be able to show the asymptotes in-folding the nature of the lines, the greatest member of which will agree with the greatest member for the curve. Then truly, because the position of the asymptotes is determined from the second member of the equation, the equation for the asymptotes and the equation for the second curve also will have a common member also. Whereby, if this were the equation for the curve related to the axis *AP* between the abscissa $AP = x$ and the applied line $PM = y$

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0,$$

for the equation relating the asymptotes to the same axis *AP*, the following equation will be had between the abscissa $AP = x$ and the applied line $PG = z$:

$$z^3 + (\beta x + \varepsilon)zz + (\gamma xx + \zeta x + B)z + \delta x^3 + \eta xx + Cx + D = 0,$$

in which the coefficients *B, C, D* thus have been prepared, so that the equation may emerge resolvable into three simple factors.

249. But if therefore some applied line *PN* may be drawn, since cutting the curve in the three points *L, M, N,* then also the asymptotes cutting in the three points *F, G, H,* from the equation for the curve there will be

$$PL + PM + PN = -\beta x - \varepsilon .$$

But from the equation for the asymptotes in the same manner there will be

$$PF+PG+PH=-\beta x-\varepsilon .$$

Hence on this account there will be :

$$PL+PM+PN=PF+PG+PH \text{ or } FL-GM+HN=0 .$$

And, if some other applied line *pf* may be drawn, in the same manner there will be :

$$fn-gm+hl=0 .$$

Therefore if some right line cuts both the curve as well as the three asymptotes in three points, the two parts of the line contained between the asymptotes and the curve, which lie in the same region, will be equal to the part lying on the opposite side.

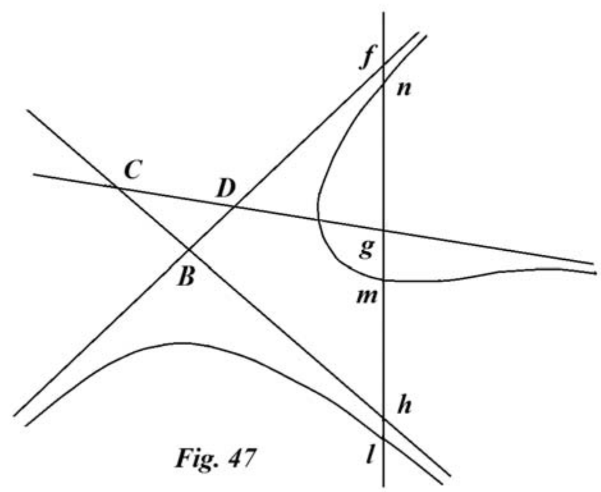

Fig. 47

250. Therefore in lines of the third order, which have three right asymptotes, the three legs to these asymptotes do not all converge to the same sides of the asymptotes : they are able to be set out; but, if two lie in the same region, the third by necessity extends to the opposite region. On this account lines of this kind of the third order, whatever figure they may represent (Fig.47), is impossible, because the right line cutting the asymptote in the points *f, g, h,* truly the curve in *l, m , n,* presents the parts *fn, gm, hl* situated on the same side, the sum of which cannot be equal to zero [*i.e.* essentially they are all quantities of the same sign, each branch lying below its asymptote]. For parts inclining in the same sense retain the same sign, for example + , those which tend in the opposite direction , the sign − ; from which it is apparent the sum of the three parts cannot vanish, unless they shall be given a difference of signs.

251. Hence now the reason is clearly evident, why two right asymptotes of the kind $u=\frac{A}{tt}$, cannot be given in lines of the third order, while the third asymptote shall be of the kind $u=\frac{A}{t}$, as therefore those hyperbolic legs converge to an infinitely greater extent than the hyperbolic leg of the kind $u=\frac{A}{t}$ to its asymptote. For we may put the right line *fl* to be removed to infinity and the intervals *fn, gm, hl* become infinitely small. But, if

the two branches *nx, my* may be placed of the kind $u = \frac{A}{tt}$, truly the third branch *lz* of the kind $u = \frac{A}{t}$, while the intervals fn and gm become infinitely smaller than the interval hl and thus there cannot be $gm = fn + hl$.

252. Therefore with lines of the greater orders, which have just as many asymptotes as dimensions, a single asymptote of the kind $u = \frac{A}{t}$ cannot be present, while the rest shall be of higher kinds $u = \frac{A}{tt}, u = \frac{A}{t^3}$, etc.; but, if one of the kind $u = \frac{A}{t}$ shall be present, by necessity another also must be present. By the same account, if no asymptote of the kind $u = \frac{A}{t}$ shall be present, it cannot happen, that only one of the kind $u = \frac{A}{tt}$ shall be present, but at least two must be present. For hyperbolic legs of the kind $u = \frac{A}{t^3}$, $u = \frac{A}{t^4}$ etc. converge at an infinitely greater extent to their asymptotes, than the kind $u = \frac{A}{tt}$. Hence therefore in the enumeration of kinds, in which a certain order of superiority will be contained, the impossible cases will be excluded easily and by this made conspicuous, bothersome calculations will be able to be avoided.

253. Moreover we may put a line of the third order to be cut by some right line only at two points, and from all the other right lines parallel to this either to be cut in the same two points or not be cut at all. Therefore if on some axis, the applied lines y may be established parallel to this right line, an equation will be prepared thus :

$$yy + \frac{(\gamma xx + \zeta x + \theta)y}{\beta x + \varepsilon} + \frac{\delta x^3 + \eta xx + \iota x + \chi}{\beta x + \varepsilon} = 0.$$

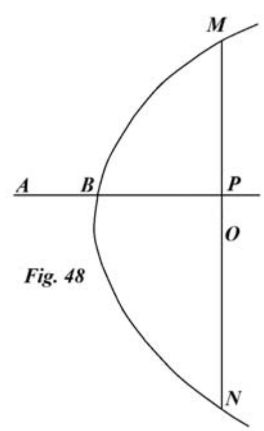

Fig. 48

Evidently, (Fig. 48) if the abscissa AP may be said $= x$, two applied lines will be had y, to wit PM and $-PN$; but from the nature of the equation there will be,

$$PM - PN = \frac{-\gamma xx - \zeta x - \theta}{\beta x + \varepsilon}.$$

The ordinate MN will be bisected at the point O, and there will be

$$PO = \frac{1}{2}\frac{\gamma xx + \zeta x + \theta}{\beta x + \varepsilon};$$

hence, if there is put $PO = z$, there will be

$$z(\beta x + \varepsilon) = \tfrac{1}{2}(\gamma xx + \zeta x + \theta) ;$$

from which it is apparent all the points *O* which bisect the ordinates parallel to *MN* are situated on a hyperbola [with the asymptote $z = \frac{\gamma}{2\beta}x$], unless $\gamma xx + \zeta x + \theta$ were divisible $\beta x + \varepsilon$, in which case the point *O* will be placed on a right line.

254. But therefore if $\gamma xx + \zeta x + \theta$ were divisible by $\beta x + \varepsilon$, then the given curve will be a diameter, or a right line bisecting all the parallel ordinates MN ; which property agrees with all lines of second order. Truly, if $\gamma xx + \zeta x + \theta$ shall be divisible by $\beta x + \varepsilon$ it must vanish, if there may be put $x = \frac{-\varepsilon}{\beta}$; whereby, if there were $\gamma\varepsilon\varepsilon - \beta\varepsilon\zeta + \beta\beta\theta = 0$, then the line arising will be a diameter of the third order line.

255. Hence therefore we may be able to determine all the most general cases, in which the diameters of lines of the third order are present. For let the proposed general equation be

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0,$$

of which the applied lines *y*, because they have either a triple or single value, are unable to receive the diameter property. Therefore under some other angle to the same axis other applied lines *u* are drawn, thus so that there shall be $y = nu$ and $x = t - mu$, and with the substitution made :

$$\left.\begin{array}{l} +\alpha n^3u^3 \quad +\beta nnuut \quad +\gamma nutt + \delta t^3 + \varepsilon nnuu + \zeta nut + \eta tt + \theta nu + \iota t + \chi \\ -\beta mnnu^3 - 2\gamma mnuut - 3\delta mutt \quad - \zeta mnuu - 2\eta mut \quad - \iota mu \\ +\gamma mmnu^3 + 3\delta mmuut \qquad\qquad + \eta mmuu \\ -\delta m^3u^3 \end{array}\right\} = 0.$$

At first therefore, so that these new applied lines are returned adapted for receiving the diameter property, it is necessary, that they may adopt a double value only, and on that account there will be :

$$\alpha n^3 - \beta mnn + \gamma mmn - \delta m^3 = 0 .$$

256. In addition therefore it is required, that the quantity by which u is multiplied, surely

$$(\gamma n-3\delta m)tt+(\zeta n-2\eta m)t+\theta n-\iota m,$$

shall be divisible by that, which multiplies uu, which is

$$(\beta nn-2\gamma mn+3\delta mm)t+\varepsilon nn-\zeta mn+\eta mm;$$

or so that the first equation may be made equal to zero, if there is put

$$t=\frac{-\varepsilon nn+\zeta mn-\eta mm}{\beta nn-2\gamma mn+3\delta mm}.$$

Hence therefore there becomes [from the main equation in §255, and what has been established previously in §254] :

$$\iota=\frac{\theta n}{m}-\frac{(\zeta n-2\eta m)(\varepsilon nn-\zeta mn+\eta mm)}{(\beta nn-\ 2\gamma mn+3\delta mm)m}+\frac{(\gamma n-3\delta m)(\varepsilon nn-\zeta mn+\eta mm)^2}{(\beta nn-2\gamma mn+3\delta mm)^2 m}.$$

257. If we apply this to the kinds enumerated above, it will be apparent at once that no diameters can be had in the first kind. But in the second kind in which the abscissas x are taken parallel to a diameter, the ordinates to the axis will be bisected. The third kind at once allows no diameters. The fourth kind always have one diameter bisecting the ordinates of one asymptote. Truly the fifth kind will have three diameters, which will bisect the ordinates of the individual asymptotes. The sixth kind can have no asymptotes. The seventh has one diameter always for the parallel ordinates of the asymptote arising from the factor $x-my$. The eighth has one diameter for the ordinates parallel to the axis. The ninth kind has two diameters ; the one for the ordinates parallel to the axis, the other for the ordinates parallel to the asymptotes. The tenth is as the eighth and the eleventh is comparable to the ninth. The twelfth on account of the diameters is equal to the eighth, and the thirteenth to the ninth. The fourteenth has one diameter for the ordinates parallel to the axis. The fifteen and the sixteenth kinds generally do not allow ordinates which cut the curve in two points, and thus cannot have diameters. Moreover these properties of diameters have been observed properly by Newton, as a consideration of these on that account will help the work given here to be put in place.

258. Although in the equations which we have given above for the individual kinds of lines of the third order, we have put the x and y coordinates normal to each other, yet the nature of the kind is not changed, even if these may be inclined in some manner to each other. For the equation, with the coordinates placed orthogonal, providing some number of legs extending to infinity, will bear just as many also running off to infinity, if the applied lines may be inclined to the axis in some manner. Nor truly also will the nature of the legs departing to infinity be changed, with the inclination of the coordinates changed ;

indeed legs which shall be parabolic, the same will remain parabolic, and hyperbolic legs will retain the same nature. So that neither the kind of parabolic nor of hyperbolic legs will be changed. Whereby all curves, as the equation for the first kind shown provides, whether rectangular or oblique angled coordinates be put in place, will be referring always to the first kind of curve, and the reasoning of all the remaining kinds has been prepared in a like manner.

259. Therefore any obliquity of the coordinates put in place, will not restrict the equations given above, if in place of y there may be put vu, and $t-\mu u$ in place of x with $\mu\mu+vv=1$ present. But with the angle of obliquity taken as it pleases, the above will be able to be returned simpler. Hence for the individual kinds the following most simple equations will be formed between the oblique angled coordinates t and u :

First Kind

$$u(tt+nnuu)+auu+bt+cu+d=0$$

with neither $n=0$ nor $b=0$.

Second Kind

$$u(tt+nnuu)+auu+cu+d=0$$

with n not $=0$.

Third Kind

$$u\left(tt-nnuu\right)+auu+bt+cu+d=0$$

with neither $n=0$ nor $b=0$ nor $\pm nb+c+\frac{4aa}{nn}=0.$

Fourth Kind

$$u\left(tt-nnuu\right)+auu+cu+d=0$$

with neither $n=0$ nor $c+\frac{aa}{4nn}=0.$

Fifth Kind

$$u(tt-nnuu)+auu-\frac{aau}{4nn}+d=0$$

with n not $=0$.

Sixth Kind

$$tuu+att+bt+cu+d=0$$

with neither $a=0$ nor $c=0$.

Seventh Kind

$$tuu + att + bt + d = 0$$

with a not $= 0$.

Eighth Kind

$$tuu + bbt + cu + d = 0$$

with neither $b = 0$ nor $c = 0$.

Ninth Kind

$$tuu + bbt + d = 0$$

with b not $= 0$.

Tenth Kind

$$tuu - bbt + cu + d = 0$$

with neither $b = 0$ nor $c = 0$.

Eleventh Kind

$$tuu - bbt + d = 0$$

with b not $= 0$.

Twelfth Kind

$$tuu + cu + d = 0$$

with c not $= 0$.

Thirteenth Kind

$$tuu + d = 0.$$

Fourteenth Kind

$$u^3 + att + cu + d = 0.$$

Fifteenth Kind

$$u^3 + atu + bt + d = 0$$

with a not $= 0$.

Sixteenth Kind

$$u^3 + at = 0.$$

CAPUT X

DE PRAECIPUIS LINEARUM TERTII ORDINIS PROPRIETATIBUS

239. Quemadmodum supra linearum secundi ordinis proprietates praecipuas ex aequatione generali deduximus, ita etiam linearum tertii ordinis praecipuae proprietates ex aequatione generali cognosci poterunt similique modo licebit linearum quarti altiorisve gradus proprietates ex aequatione concludere. Quamobrem consideremus aequationem generalissimam pro lineis tertii ordinis, quae est

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0,$$

quae exprimet naturam lineae tertii ordinis cuiusvis inter coordinatas x et y ad quemvis angulum inclinatas et recta quacunque pro axe assumta.

240. Nisi igitur α sit = 0, unicuique abscissae x vel una respondebit applicata realis vel tres. Ponamus dari tres applicatas reales; atque manifestum est earum relationem per aequationem definiri posse. Posita itaque $\alpha = 1$, istiusmodi erit aequatio

$$y^3 + (\beta x + \varepsilon)yy + (\gamma xx + \zeta x + \theta)y + \delta x^3 + \eta xx + \iota x + \chi = 0$$

atque summa istarum trium applicatarum eidem abscissae x respondentium erit $= -\beta x - \varepsilon$; summa trium rectangulorum ex binis applicatis formatorum erit $= \gamma xx + \zeta x + \theta$; ac denique productum omnium seu parallelepipedum ex illis formatum erit $= -\delta x^3 - \eta xx - \iota x - \chi$. Si duae applicatae essent imaginariae, haec quidem eadem valerent, at ad linearum figuram accommodari non possent, quia ex ea neque summa neque rectangulum duarum applicatarum imaginariarum intelligi potest.

241. Sit igitur (Fig. 44) linea quaecunque tertii ordinis ad axem AZ relata, ad quem sub dato angulo applicatae sint ordinatae LMN, lmn curvam secantes in tribus punctis. Posita ergo abscissa $AP = x$, applicata y triplicem habebit valorem PL, PM et $-PN$; unde erit

$$PL + PM - PN = -\beta x - \varepsilon.$$

Quare, si capiatur

$$PO = z = \frac{PL + PM - PN}{3},$$

punctum *O* ita erit in medio situm, ut sit $LO = MO + NO$. Cum igitur sit $z = \frac{-\beta x + \varepsilon}{3}$, hoc punctum *O* situm erit in linea recta *OZ*, quae recta propterea omnes ordinatas *lmn* ipsi *LMN* parallelas ita secabit in *o*, ut sit $lo + mo = no$; quae proprietas analoga est proprietati diametrorum, qua lineae secundi ordinis sunt praeditae. Quodsi ergo duae ordinatae parallelae et curvam in tribus punctis secantes ita secentur in punctis *O* et *o*, ut binae applicatae ad unam partem iacentes simul sumtae aequales sint tertiae ad partem alteram sitae, recta per haec puncta *O* et *o* ducta omnes reliquas ordinatas illis parallelas similiter secabit eritque quasi diameter lineae tertii ordinis.

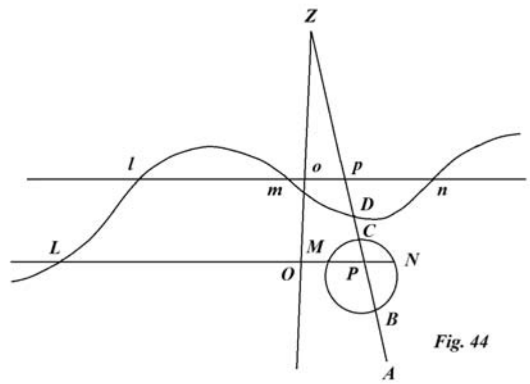

Fig. 44

242. Quoniam in lineis secundi ordinis omnes diametri se mutuo in eodem puncto intersecant, videamus, quomodo plures huiusmodi diametri linearum tertii ordinis inter se sint comparatae. Concipiamus ergo ad eundem axem *AP* sub alio quovis angulo applicatas sitque abscissa $= t$ et applicata $= u$; erit $y = nu$ et $x = t - mu$, qui valores in aequatione generali

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0$$

substituti hanc dabunt aequationem

$$\left.\begin{array}{llllll} +\,n^3u^3 & +\beta nnuut & +\gamma nutt + \delta t^3 & +\varepsilon nnuu & +\zeta nut + \eta tt + \theta nu + \iota t + \chi \\ -\beta mnnu^3 & -2\gamma mnuut - 3\delta mutt & & -\zeta mnuu - 2\eta mut & -\iota mu \\ +\,\gamma mmnu^3 & +\,3\delta mmuut & & +\eta mmuu & \\ -\delta m^3u^3 & & & & \end{array}\right\} = 0.$$

Hinc pro linea illa recta diametri vicem sustinente, si eius applicata sub eodem angulo ad abscissam *t* ducta vocetur $= v$, erit

$$3v = \frac{-\beta nnt + 2\gamma mnt - 3\delta mmt - \varepsilon nn + \zeta mn - \zeta mm}{n^3 - \beta mnn + \gamma mmn - \delta m^3}.$$

243. Sit iam (Fig. 45) *O* intersectio, harum duarum diametrorum, unde ad axem *AZ* primo prioribus applicatis parallela ducatur *OP*, tum vero posterioribus parallela *OQ*, eritque $AP = x, PO = z, AQ = t$ et $OQ = v$.

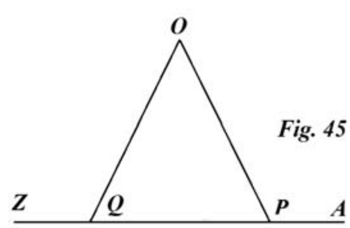

Fig. 45

Tum vero erit

$$z = nv \text{ et } x = t - mv$$

ideoque

$$v=\frac{z}{n} \text{ et } t=x+\frac{m}{n}z.$$

Primo itaque habetur $3z=\beta x-\varepsilon$, porroque

$$3v=-\frac{\beta x}{n}-\frac{\varepsilon}{n} \text{ et } t=x-\frac{\beta mx}{3n}-\frac{\varepsilon m}{3n}.$$

Substituantur hi valores in aequatione ante inventa et prodibit

$$\left.\begin{array}{llll}
-\beta nnx & +\beta\beta mnx & -\beta\gamma mnx & +\dfrac{\beta\delta m^3x}{n} \\
-\varepsilon nn & +\beta\varepsilon mn & -\gamma\varepsilon mm & +\dfrac{\delta\varepsilon m^3}{n} \\
+\beta nnx & -\dfrac{\beta\beta mnx}{3} & -\dfrac{\beta\varepsilon mn}{3} & +\varepsilon nn \\
-2\gamma mnx & +\dfrac{2\beta\gamma mmx}{3} & +\dfrac{2\gamma\varepsilon mm}{3} & -\zeta mn \\
+3\delta mmx & -\dfrac{\beta\delta m^3x}{n} & -\dfrac{\delta\varepsilon m^3}{n} & +\eta mm
\end{array}\right\} = 0.$$

seu

$$\left.\begin{array}{llll}
\dfrac{2}{3}\beta\beta mnx & -\dfrac{1}{3}\beta\gamma mmx & -2\gamma mnx & +3\delta mmx \\
\dfrac{2}{3}\beta\varepsilon mn & -\dfrac{1}{3}\gamma\varepsilon mm & -\zeta mn & +\eta mm
\end{array}\right\}=0.$$

244. Pendet ergo utique intersectio diametrorum O ab inclinatione applicatarum ad axem, quae litteris m et n continetur; neque idcirco (si intersectionem diametrorum *centrum* vocare lubeat), lineae tertii ordinis omnes centro gaudent. Interim tamen casus exhiberi possunt, quibus diametrorum intersectio mutua in idem punctum fixum incidat. Fiet scilicet hoc, si termini per mn et mm affecti seorsim nihilo aequales ponantur ac valores ipsius x inde orituri aequales statuantur. Fiet autem ex his duabus aequalitatibus

$$x=\frac{3\zeta-2\beta\varepsilon}{2\beta\beta-6\gamma}=\frac{3\eta-\gamma\varepsilon}{\beta\gamma-9\delta};$$

qui duo valores ut congruant, necesse est, ut sit

$$6\beta\beta\eta-2\beta\beta\gamma\varepsilon-18\gamma\eta+6\gamma\gamma\varepsilon=3\beta\gamma\zeta-2\beta\beta\gamma\varepsilon-27\delta\zeta+18\beta\delta\varepsilon$$

seu

$$\beta\gamma\zeta - 2\beta\beta\eta - 9\delta\zeta + 6\gamma\eta + 6\beta\delta\varepsilon - 2\gamma\gamma\varepsilon = 0,$$

unde fit

$$\eta = \frac{\beta\gamma\zeta - 9\delta\zeta + 6\beta\delta\varepsilon - 2\gamma\gamma\varepsilon}{2\beta\beta - 6\gamma}.$$

Quoties ergo η huiusmodi habuerit valorem, toties omnes diametri se mutuo in uno eodemque puncto intersecant; ideoque hae lineae tertii ordinis centro gaudebunt, quod reperietur sumendo in axe

$$AP = \frac{3\zeta - 2\beta\varepsilon}{2\beta\beta - 6\gamma} \text{ et } PO = \frac{-\beta\zeta + 2\gamma\varepsilon}{2\beta\beta - 6\gamma}.$$

245. Haec eadem centri determinatio, si quod datur, locum habet, si pro primo coefficiente α non ponatur unitas. Si enim proposita fuerit aequatio generalissima pro lineis tertii ordinis

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0 ,$$

hae curvae centro erunt praeditae, si fuerit

$$\eta = \frac{\beta\gamma\zeta - 9\alpha\delta\zeta + 6\beta\delta\varepsilon - 2\gamma\gamma\varepsilon}{2\beta\beta - 6\alpha\gamma}.$$

Tum vero centrum erit in *O* existente

$$AP = \frac{3\alpha\zeta - 2\beta\varepsilon}{2\beta\beta - 6\alpha\gamma} \text{ et } PO = \frac{2\gamma\varepsilon - \beta\zeta}{2\beta\beta - 6\alpha\gamma}.$$

Quare, si unica ordinata curvam in tribus punctis secans ita dividatur, ut binae applicatae ad unam partem sitae aequentur tertiae ad alteram partem iacenti, tum recta per centrum et hoc divisionis punctum ducta omnes alias ordinatas illa parallelas similiter secabit.

246. Si haec ad aequationes specierum supra enumeratarum accommodentur, patebit species primam, secundam, tertiam, quartaen et quintam centro gaudere, si modo sit $\alpha = 0$, hocque casu centrum in ipso abscissarum, initio esse positum. Species sexta et septima, centro prorsus carent, quia, coefficiens *a* abesse nequit. Species vero octava, nona, decima, undecima, duodecima et decima-tertia centrum habent, semper in abscissarum initio positum. In speciebus decima-quarta, decima-quinta et decima-sexta centrum infinite distat ideoque omnes illae lineae triametri inter se erunt parallelae.

247. His de summa trium cuiusque applicatae valorum notatis contemplemur eorundem productum, quoniam de rectangulorum aggregato nihil admodum notatu dignum reperitur. Erit ergo ex aequatione generali paragraphi 239

$$-PM \cdot PL \cdot PN = -\delta x^3 - \eta xx - \iota x - \chi \,;$$

ad quam expressionem explicandam ad hoc attendamus, quod, si ponatur $y = 0$, fiat $\delta x^3 + \eta xx + \iota x + \chi = 0$, cuius propterea aequationis radices dabunt axis *AZ* et curvae intersectiones. Quae si sint in punctis *B, C* et *D,* erit

$$\delta x^3 + \eta xx + \iota x + \chi = \delta(x - AB)(x - AC)(x - AD) \;;$$

quapropter erit

$$PL \cdot PM \cdot PN = \delta \cdot PB \cdot PC \cdot PD\,;$$

ideoque sumta alia quacunque ordinata *lmn* priori parallela erit

$$PL \cdot PM \cdot PN : \; PB \cdot PC \cdot PD = pl \cdot pm \cdot pn : \; pB \cdot pC \cdot pD\,;$$

quae proprietas omnino similis est illi, quam supra pro lineis secundi ordinis ratione rectangulorum invenimus; atque similis proprietas in lineas quarti, quinti et superiorum ordinum competet.

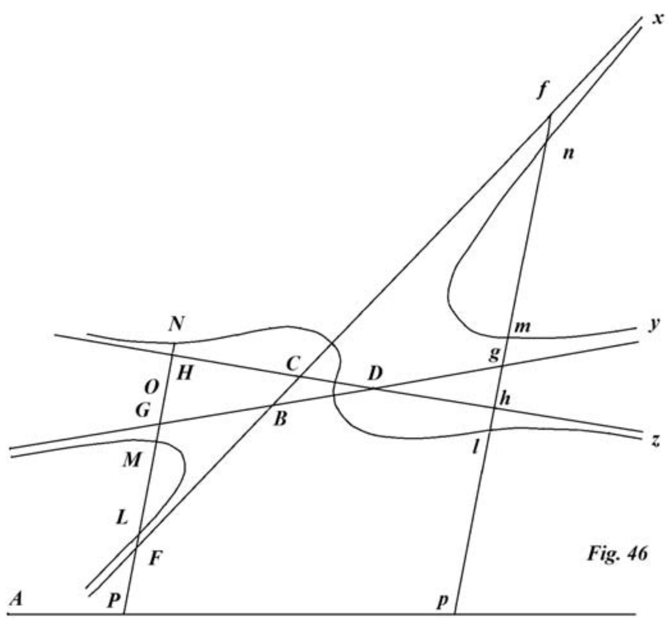

Fig. 46

248. Habeat nunc linea tertii ordinis tres quoque asymptotas rectas *FBf, GDg, HCh* (Fig. 46). Quoniam ipsa linea tertii ordinis in has tres asymptotas abit, si aequatio pro curva resolubilis fiat in tres factores simplices formae $py + qx + r$, pro asymptotis tanquam linea complexa peculiaris aequatio exhiberi poterit, cuius supremum membrum conveniet cum supremo membro pro curva. Deinde vero, quia asymptotarum positio ex secundo aequationis membro determinatur, aequatio pro asymptotis et aequatio pro curva secundum quoque membrum commune habebunt. Quare, si pro curva ad axem *AP* relata haec fuerit aequatio inter abscissam $AP = x$ et applicatam $PM = y$

$$\alpha y^3 + \beta yyx + \gamma yxx + \delta x^3 + \varepsilon yy + \zeta yx + \eta xx + \theta y + \iota x + \chi = 0,$$

pro asymptotis ad eundem axem AP relatis sequens habebitur aequatio inter abscissam $AP = x$ et applicatam $PG = z$:

$$z^3 + (\beta x + \varepsilon)zz + (\gamma xx + \zeta x + B)z + \delta x^3 + \eta xx + Cx + D = 0,$$

in qua coefficientes B, C, D ita sunt comparati, ut aequatio in tres factores simplices resolubilis evadat.

249. Quodsi ergo ducatur applicata quaecunque PN, cum curvam secans in tribus punctis $L, M, N,$ tum etiam asymptotas in tribus punctis F, G, H secans, erit ex aequatione pro curva

$$PL + PM + PN = -\beta x - \varepsilon.$$

At ex aequatione pro asymptotis erit pari modo

$$PF + PG + PH = -\beta x - \varepsilon.$$

Hanc ob rem erit

$$PL + PM + PN = PF + PG + PH \quad \text{seu} \quad FL - GM + HN = 0.$$

Atque, si alia quaecunque applicata pf ducatur, erit eodem modo

$$fn - gm + hl = 0.$$

Si igitur recta quaecunque cum curvam tum tres asymptotas secet in tribus punctis, binae partes lineae inter asymptotas et curvam contentae, quae ad eandem regionem vergunt, aequales erunt parti in regionem oppositam vergenti.

250. In linea igitur tertii ordinis, quae tres habet asymptotas rectas, tria crura ad has asymptotas convergencia non omnia ad easdem asymptotarum partes: possunt esse disposita; sed, si duo ad eandem partem vergant, tertium necessaria ad oppositas tendet. Hanc ob rem huiusmodi linea tertii ordinis, qualem figura (Fig.47) repraesentat, est impossibilis, quoniam recta secans asymptotas in punctis $f, g, h,$ curvam vero in $l, m, n.,$ praebet partes fn, gm, hl in eandem plagam vergentes, quaram

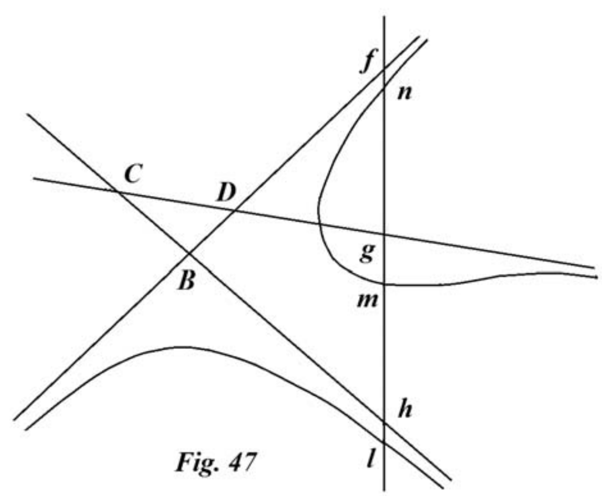

Fig. 47

summa nihilo aequalis esse nequit. Partes enim in eandem plagam vergentes obtinent idem signum, puta + , quae vero in contrariam plagam tendunt, signum − ; unde patet summam trium harum partium evanescere non posse, nisi signis diversis sint praeditae.

251. Hinc iam clare perspicitur ratio, cur in linea tertii ordinis dari nequeant duae asymptotae rectae speciei $u = \frac{A}{tt}$, dum tertia asymptota sit speciei $u = \frac{A}{t}$, propterea quod illa crura hyperbolica infinities magis ad suam asymptotam convergant, quam crus hyperbolicum speciei $u = \frac{A}{t}$. Ponamus enim rectam *fl* in infinitum removeri fientque intervalla *fn, gm, hl* infinite parva. At, si rami duo *nx, my* ponantur speciei $u = \frac{A}{tt}$, tertius vero ramus *lz* speciei $u = \frac{A}{t}$, tum intervalla *fn* et *gm* infinities erunt minora quam intervallum *hl* ideoque esse nequit $gm = fn + hl$.

252. In lineis ergo superiorum ordinum, quae tot habent asymptotas quot dimensiones, unica asymptota speciei $u = \frac{A}{t}$ adesse nequit, dum reliquae sint specierum superiorum $u = \frac{A}{tt}$, $u = \frac{A}{t^3}$, etc.; sed, si una adsit speciaie $u = \frac{A}{t}$, necessario et altera adesse debet. Ob eandem rationem, si asymptota speciei $u = \frac{A}{t}$ nulla adsit, fieri non potest, ut una tantum speciei $u = \frac{A}{tt}$ adsit, sed ad minimum duae adesse debebunt. Crura enim hyperbolica speciei $u = \frac{A}{t^3}$, $u = \frac{A}{t^4}$ etc. infinities magis as suas asymptotas convergunt, quam species $u = \frac{A}{t^4}$. Hinc igitur in enumeratione specierum, quae in ordine quopiam superiori continentur, casus impossibiles facile excludi hocque insignes calculi molestiae evitari poterunt.

253. Ponamus autem lineam tertii ordidis a recta quapiam induobus tantum punctis secari, atque ab omnibus aliis rectis huic parallelis vel in duobus etiam punctis vel nusquam secabitur. Si igitur in axe quocunque statuantur applicatae *y* huic rectae parallelae, aequatio ita erit comparata

$$yy + \frac{(\gamma xx + \zeta x + \theta)y}{\beta x + \varepsilon} + \frac{\delta x^3 + \eta xx + \iota x + \chi}{\beta x + \varepsilon} = 0.$$

Scilicet, si (Fig. 48) abscissa *AP* dicatur $=x$, duae habebuntur applicatae *y*,nempe *PM* et $-$ *PN;* erit autem, ex natura aequationum,

$$PM-PN=\frac{-\gamma xx-\zeta x-\theta}{\beta x+\varepsilon}.$$

Bisecetur ordinata *MN* in puncto *O*, erit

$$PO=\frac{1}{2}\frac{\gamma xx+\zeta x+\theta}{\beta x+\varepsilon};$$

hinc, si ponatur $PO=z$, erit

$$z(\beta x+\varepsilon)=\tfrac{1}{2}(\gamma xx+\zeta x+\theta)\ ;$$

unde patet omnia puncta *O* ordinatas parallelas *MN* bisectantia sita esse in hyperbola, nisi fuerit $\gamma xx+\zeta x+\theta$ divisibile per $\beta x+\varepsilon$, quo casu punctum *O* positum erit in linea recta.

254. Quodsiergo $\gamma xx+\zeta x+\theta$ divisibile fuerit per $\beta x+\varepsilon$, tum curva praedita erit diametro seu recta omnes ordinatas parallelas *MN* bisecante; quae proprietas in omnes lineas secundi ordinis competit. Verum, si $\gamma xx+\zeta x+\theta$ divisibile sit per $\beta x+\varepsilon$ evanescere debebit, si ponatur $x=\frac{-\varepsilon}{\beta}$; quare, si fuerit $\gamma\varepsilon\varepsilon-\beta\varepsilon\zeta+\beta\beta\theta=0$, tum linea tertii ordinis diametro erit praedita.

255. Hinc igitur generalissime omnes casus determinare poterimus, quibus lineae tertii ordinis diametris sunt praeditae. Sit enim proposita aequatio generalis

$$\alpha y^3+\beta yyx+\gamma yxx+\delta x^3+\varepsilon yy+\zeta yx+\eta xx+\theta y+\iota x+\chi=0,$$

cuius applicatae *y,* quia triplicem valorem vel unicum habent, diametri proprietatem recipere nequeunt. Ducantur ergo sub alia quocunque angulo ad eundem axem aliae applicatae *u,* ita ut sit $y=nu$ et $x=t-mu$, ac fiat substitutio

$$\left.\begin{array}{llllllll}
+\alpha n^3u^3 & +\beta nnuut & +\gamma nutt+\delta t^3 & +\varepsilon nnuu & +\zeta nut & +\eta tt & +\theta nu+\iota t+\chi \\
-\beta mnnu^3 & -2\gamma mnuut & -3\delta mutt & -\zeta mnuu & -2\eta mut & & -\iota mu \\
+\gamma mmnu^3 & +3\delta mmuut & & +\eta mmuu & & & \\
-\delta m^3u^3 & & & & & &
\end{array}\right\}=0.$$

Primum ergo, quo hae novae applicatae ad diametrum recipiendam aptae reddantur, necesse est, ut duplicem tantum valorem induere possint, eritque idcirco

$$\alpha n^3 - \beta mnn + \gamma mmn - \delta m^3 = 0 .$$

256. Praeterea vero requiritur, ut quantitas, per quam u est multiplicata, nempe

$$(\gamma n - 3\delta m)tt + (\zeta n - 2\eta m)t + \theta n - \iota m ,$$

divisibilis sit per eam, quae uu multiplicat, quae est

$$(\beta nn - 2\gamma mn + 3\delta mm)t + \varepsilon nn - \zeta mn + \eta mm ;$$

sive illa nihilo fieri debet aequalis, si ponatur

$$t = \frac{-\varepsilon nn + \zeta mn - \eta mm}{\beta nn - 2\gamma mn + 3\delta mm}.$$

Hinc ergo fiet

$$\iota = \frac{\theta n}{m} - \frac{(\zeta n - 2\eta m)(\varepsilon nn - \zeta mn + \eta mm)}{(\beta nn - 2\gamma mn + 3\delta mm)m} + \frac{(\gamma n - 3\delta m)(\varepsilon nn - \zeta mn + \eta mm)^2}{(\beta nn - 2\gamma mn + 3\delta mm)^2 m}.$$

257. Si haec ad species supra enumeratas applicemus, apparebit in specie prima nullam prorsus diametrum locum habere posse. In specie autem secunda ordinatae axi, in quo abscissae x capiuntur parallelae diametro, bisecabuntur. Species tertia nullam prorsus diametrum admittit. Species quarta semper unam habet diametrum ordinatas uni asymptotae parallelas bisecantem. Quinta vero species tres habebit diametros, quae ordinatas singulis asymptotis parallelas bisecabunt. Species sexta nullam prorsus habere potest diametrum. Septima unam diametrum semper habet pro ordinatis asymptotae ex factore $x - my$ ortae parallelis. Octava unam diametrum habet pro ordinatis axi parallelis. Nona species duas habet diametros; alteram pro ordinatis axi parallelis, alteram pro ordinatis alteri asymptotae parallelis. Decima uti octava et undecima uti nona est comparata. Duodecima ratione diametrorum par est octavae, et decima-tertia nonae. Decima-quarta unam habet diametrum pro ordinatis axi parallelis. Species decima-quinta et sexta omnino ordinatas, quae in duobus punctis curvam secent, non admittunt ideoque diametro gaudere nequeunt. Hae autem diametrorum proprietates a NEWTONO probe sunt notatae, quam ob causam earum commemorationem hic data opera attulisse iuvabit.

258. Quanquam in aequationibus, quas supra pro singulis speciebus linearum tertii ordinis dedimus, coordinatas x et y inter se normales posuimus, tamen speciei natura non mutatur, etiamsi eae quomodocunque ad se invicem sint inclinatae. Quot enim aequatio,

positis coordinatis orthogonalibus, praebet crura in infinitum extensa, totidem quoque praebebit eadem aequatio, si applicatae ad axem utcunque inclinentur. Neque vero etiam natura crurum in infinitum excurrentium mutatur, mutata coordinatarum inclinatione; quae enim crura sunt parabolica, eadem manebunt parabolica, et quae sunt hyperbolica, eandem naturam retinebunt. Quin etiam species crurum tam parabolicorum quam hyperbolicorum non alterabitur. Quare omnis curva, quam aequatio pro prima specie exhibita praebet, sive coordinatae statuantur rectangulae sive obliquangulae, semper ad eandem speciem primam erit referenda, similique modo reliquarum specierum omnium ratio est comparata.

259. Admissa ergo eoordinatarum obliquitate quacunque, aequationes supra datae non restringentur, si loco *y* ponatur vu et $t-\mu u$ loco *x* existente $\mu\mu+vv=1$. Sumto autem angulo obliquitatis pro lubitu, aequationes supra datae simpliciores reddi poterunt. Hinc pro singulis speciebus sequentes simplicissimae aequationes inter coordinatas obliquangulas *t* et *u* formabuntur:

SPECIES PRIMA

$$u(tt+nnuu)+auu+bt+cu+d=0$$

existente nec $n=0$ nec $b=0$.

SPECIES SECUNDA

$$u(tt+nnuu)+auu+cu+d=0$$

non existente $n=0$.

SPECIES TERTIA

$$u\left(tt-nnuu\right)+auu+bt+cu+d=0$$

existente nec $n=0$ nec $b=0$ nec $\pm nb+c+\frac{4aa}{nn}=0.$

SPECIES QUARTA

$$u\left(tt-nnuu\right)+auu+cu+d=0$$

existente nec $n=0$ nec $c+\frac{aa}{4nn}=0.$

SPECIES QUINTA

$$u(tt-nnuu)+auu-\frac{aau}{4nn}+d=0$$

non existente $n=0$.

SPECIES SEXTA

$$tuu+att+bt+cu+d=0$$

existente nec $a=0$ nec $c=0$.

SPECIES SEPTIMA

$$tuu+att+bt+d=0$$

non existente $a=0$.

SPECIES OCTAVA

$$tuu+bbt+cu+d=0$$

existente nec $b=0$ nec $c=0$.

SPECIES NONA

$$tuu+bbt+d=0$$

non existente $b=0$.

SPECIES DECIMA

$$tuu-bbt+cu+d=0$$

existente nec $b=0$ nec $c=0$.

SPECIES UNDECIMA

$$tuu-bbt+d=0$$

non existente $b=0$.

SPECIES DUODECIMA

$$tuu+cu+d=0$$

non existente $c=0$.

SPECIES DECIMA-TERTIA

$$tuu + d = 0.$$

SPECIES DECIMA-QUARTA

$$u^3 + att + cu + d = 0.$$

SPECIES DECIMA-QUINTA

$$u^3 + atu + bt + d = 0$$

non existente $a = 0$.

SPECIES DECIMA-SEXTA

$$u^3 + at = 0.$$