CHAPTER VI

THE INTERSECTION OF TWO SURFACES

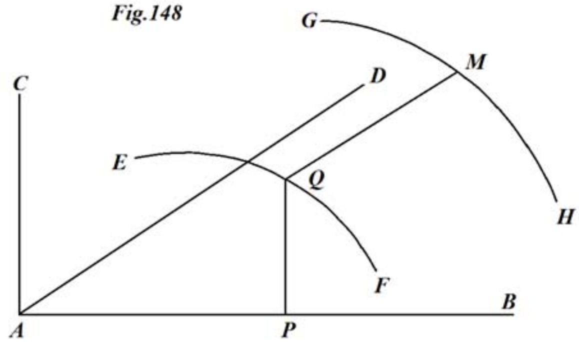
131. Now the method of investigating the nature of a section has been set out above, which arises if some surface may be cut by a plane. For since the curved line which the section may form, shall be put wholly in the same plane by which the section has been made, and we have assumed two coordinates in the same plane, from the relation of which the nature of curved lines of this kind is accustomed to be expressed, so that with this agreed upon the recognition of the section may be reduced to the usual method. But if the cutting surface should not be a plane, because then the section will not be placed in the same plane, its nature cannot be understood by two coordinates ; by which account another way will have to be used for including sections of this kind, by which the position of any point truly will be indicated.

132. But the locations of points not situated in the same plane can be defined, if three planes normal to each other may be added and with the help of which any point may be assigned by these three distances, by which that point is separated from any plane desired. Hence three variables are required to express the nature of the curved line not set up in the same plane; thus so that, if one variable may be defined as it pleases, from that the two remaining determined values may be obtained. Therefore one equation between these three coordinates does not suffice towards putting this in place, certainly which will indicate the nature of a certain surface ; concerning which there will be a need for two equations, with the help of which, if to one a value may be attributed, likewise the two remaining values may be determined.

133. Therefore the nature of any curved line, which cannot be agreed to be set up in the same plane, is expressed most conveniently by two equations between the three variables, *e.g.*, $x, y, z,$ which will represent just as many coordinates between the normals themselves. Therefore with the aid of two equations of this kind both the variables will be able to be determined from the third ; clearly both y and z will be equal to a certain function of x. Also by choice one of the variables will be able to be eliminated, from which three equations involving only two variables will be formed, one between x and y, the second between x and z and the third between y and z. Truly of these three equations any one [equation] you please may be determined at once from the two remaining, thus so that, if equations may be had between x and y, and between x and z, from these the third equation now may be found be the elimination of x.

134. Therefore let some curved line be proposed (Fig. 148) not placed in the same plane, once certain point of which shall be *M*. By choice three normal axes may be taken in turn *AB*, *AC* and *AD*, by which in turn the three normal planes may be determined in turn *BAC*, *BAD* and *CAD* may be determined. From some point *M* of the curve the perpendicular *MQ* may be sent to the plane *BAC*, and from the point *Q*, *QP* may be drawn normal to *AD*, these three coordinates will be *AP*, *PQ* and *QM*, between which if two equations may be given, the nature of the curve may be determined. Therefore they may be called $AP = x$, $PQ = y$ and $QM = z$; and from the two equations between *x*, *y* and *z* proposed by eliminating *z* an equation will be formed containing only the two variables *x* and *y*, which will determine the position of the point *Q* in the plane *BAC* ; and the individual points *Q* arising from *M* will provide the curved line *EQF*, the nature of which will be expressed by that equation found between *x* and *y*.

Fig.148



135. Therefore in this manner from two equations between the three coordinates proposed the nature of the curve *EQF* is recognised easily, which may be formed by sending perpendiculars *MQ* from the individual points *M* of the curve requiring to be found in the plane *BAC*. But this curve *EQF* may be called the *projection* of the curve *GMH* into the plane *BAC*. But in whatever manner the projection made into the plane *BAC* it is found by eliminating the variable *z*, thus the projection of the same curve into the plane *BAD* or into the plane *CAD* will be found, if either the variable *y* or *x* may be eliminated. But one projection *EQF* will not suffice towards recognizing the curve *GMH*, but if the perpendiculars $QM = z$ were known for the individual points *Q*, from the projection *EQF* that curve *GMH* would be constructed easily. Therefore for this there is a need, that besides the equation between *x* and *y*, by which the nature of the projection is expressed, the equation may be had between *z* and *x* or between *z* and *y* or also between the three *z*, *x*, *y*, from which the length of the perpendicular $QM = z$ for any point *Q* becomes known.

136. But since the equation between *z* and *x* may express the projection of the curve *GMH* made into the plane *BAD*, moreover the equation between *z* and *y* expresses the projection into the plane *CAD*, and the equation between the three variables *z*, *y* and *x* may show the surface, in which the curve *GMH* may be found ; in the first place it is evident from the two projections of the same curve *GMH* made into the two planes that the curve *GMH* itself becomes known. Then truly it is evident, if the surface may be given, in which the curved line *GMH* may be contained, and in addition its projection into a certain plane, that curve equally becomes known. For from the individual

projected points the right normals *QM* may be erected, the intersection of which with the surface will define the curve *GMH* sought.

137. From these premises, which pertain to understanding the nature of any curve not to be established in the same plane, it will not be difficult to define the intersection of any two surfaces. For just as the intersection of two planes will be a right line, thus the intersection of some two surfaces will be a line, either right or curved, and this will either be placed in the same plane or otherwise. But in whatever manner it may be prepared, the individual points of that will relate to each surface and thus will be contained in the equation of each surface. But if therefore both surfaces may be expressed between three coordinates, which may refer to the same three principal planes normal to each other or to the same three axes normal between themselves *AB, AC* and *AD*, then both these equations jointly express the nature of the intersection.

138. Therefore for two surfaces mutually cutting each other, the nature of each must be expressed by an equation between the three coordinates, which may be referred to the same principal axes, and the nature of which must be expressed by an equation between the three coordinates, and thus two equations will be had made between the three coordinates x, y and z, from which if one may be eliminated, an equation between the two remaining will present the projection of the intersection in the plane, which is constituted by these two coordinates. Therefore in this manner the intersection of each surface made by a plane can be investigated also ; for since the general equation for a plane shall be $\alpha z+\beta y+\gamma x=f$, if in the equation of the surface in place of z the value of that arising from that equation may be substituted, surely $z=\frac{f-\beta y-\gamma x}{\alpha}$, an equation will be produced for the projection of the intersection made in the plane of the coordinates x and y. Likewise truly the equation $z=\frac{f-\beta y-\gamma x}{\alpha}$ for any point Q of the projection will provide the magnitude of the perpendicular *QM* relating to the same intersection.

139. But if it may eventuate, that the equation for the projection becomes impossible, as if it may be found that $xx+yy+aa=0$, then this will be an indication that both surfaces never intersect each other. But if the equation of the projection may lead to a single point or if the projection may vanish in a single point, then that intersection also will be a point and thus both surfaces will touch each other mutually in a point ; which contact point therefore will be able to be known from an equation. But in addition a contact line may be given, when two surfaces touch at an infinitude of points, and the contact line will be either right or curved. Clearly it will be a right line, if the plane may touch a cylinder or a cone, but a right cone may be touched by a sphere within along the periphery of a circle. Which contacts will be recognised from the equation, if an equation may be produced from a projection of this kind, which may have two equal roots, because otherwise therefore the contact is zero, unless there shall be a meeting of the two intersections.

140. Towards explaining this more clearly we may consider a sphere to be cut by some plane. We may take the equation applied to the centre of the sphere

$$zz + yy + xx = aa ,$$

but for some plane put in place this equation will be had :

$$\alpha z + \beta y + \gamma x = f ,$$

from which, since there shall be $z = \frac{f - \beta y - \gamma x}{\alpha}$, the following equation will arise between *x* and *y* for the projection :

$$0 = ff - \alpha\alpha aa - 2\beta fy - 2\gamma fx + (\alpha\alpha + \beta\beta)yy + 2\beta\gamma xy + (\alpha\alpha + \gamma\gamma)xx ,$$

which appears to be an ellipse, if indeed the equation were real; but if it were imaginary, the sphere in now way may be touched by the plane ; but if the ellipse may vanish into a point, the plane and the sphere touch each other in turn. So that which case may be elicited, there may be sought :

$$y = \frac{\beta f - \beta\gamma x \pm \alpha\sqrt{(aa(\alpha\alpha + \beta\beta) - ff + 2\gamma fx - (\alpha\alpha + \beta\beta + \gamma\gamma)xx)}}{\alpha\alpha + \beta\beta},$$

where, if f had a value of this kind, so that under no circumstance could the value of the root become real, there will be no contact nor intersection.

141. We may put $f = a\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}$

and there will be

$$y = \frac{\beta f - \beta\gamma x \pm \alpha x\sqrt{-(\alpha\alpha + \beta\beta + \gamma\gamma)} \mp \alpha\gamma a\sqrt{-1}}{\alpha\alpha + \beta\beta},$$

for which equation it is unable to be satisfied by real values, unless there shall be

$$x = \frac{\gamma a}{\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}} \quad \text{and} \quad y = \frac{\beta a}{\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}}.$$

Whereby, if the plane were $f = a\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}$, which is expressed by the equation

$\alpha z+\beta y+\gamma x=f$, it will touch the sphere ; and the point of contact will be had, if there may be taken

$$x=\frac{\gamma a}{\sqrt{(\alpha\alpha+\beta\beta+\gamma\gamma)}},\ y=\frac{\beta a}{\sqrt{(\alpha\alpha+\beta\beta+\gamma\gamma)}} \text{ and } z=\frac{\alpha a}{\sqrt{(\alpha\alpha+\beta\beta+\gamma\gamma)}},$$

the truth of which value can be proved from elementary geometry, where the contact of a sphere by a plane is shown.

142. Hence therefore the general rule is deduced, with the help of which it can become known, whether any surface may be touched by a plane or not. For with one variable eliminated from both equations it is required to be seen, whether or not the resulting equation can be resolved into simple factors or not. For if it may have two simple imaginary factors, the contact will be given at a point, which may become known by putting each factor $=0$. But if it may have two simple real factors and these equal to each other, then the surfaces mutually touch each other along a right line. But if truly that equation may have two non simple equal factors or if it were divisible by a quadratic, then the root of that put equal to zero will show the projection of that line which arises from the contact. Hence also it is evident, if that same equation were to have four imaginary factors, then the surfaces mutually touch in two points.

143. So that this may be explained more fully, we may investigate the contact of a cone and a sphere, the centre of which shall be placed on the axis of the cone. The equation for the sphere is $zz+yy+xx=aa$, but for the cone it is $(f-z)^2=mxx+nyy$, because the vertex of the cone shall be removed from the centre of the sphere by the interval f put in place. Hence we may eliminate the variable *y*, and there will be

$$(f-z)^2=naa-nzz+(m-n)xx$$

for the projection of the intersection in the plane of the coordinates *x* and *z*. In the first place the cone shall be right or $m=n$ and here will be

$$z=\frac{f\pm\sqrt{(n(1+n)aa-nff)}}{1+n}.$$

Whereby, if there were $f=a\sqrt{(1+n)}$, there will be the double point $z=\dfrac{a}{\sqrt{(1+n)}}$ and thus the contact shall be of a line, evidently by a circle, the projection of which is a right line in a plane passing through the axis is a right line normal to the axis.

144. But for a scalene cone, where m and n are unequal, the equation found always may be seen to give an intersection, since yet more often none exist. For always, if indeed m were greater than n, a real equation will be produced for the projection of the intersection ; but truly it is to be observed that a real projection does not always indicate a real intersection. So indeed that the intersection may be real does not suffice that the projection is real, but in addition it is necessary that the perpendiculars drawn from the projection to the intersection are real. Therefore it can be concluded, although all real curves may have whatever real projections, yet the reality of the curve which is sought does not follow in turn from the reality of the projection. And this caution is to be used properly always, lest we misuse the reality of the equations, which we have found from the projections.

145. We will avoid this inconvenience, if we may look of the projection in the plane of the ordinates x and y, because indeed in this plane no point may be given, to which no point in the surface of the cone may correspond ; if the projection in this plane were real, the intersection itself will also be real. Therefore since there shall be $z=\sqrt{(aa-xx-yy)}$, from the other equation it shall become

$$f-\sqrt{(aa-xx-yy)}=\sqrt{(mxx+nyy)}$$

or

$$aa+ff-(1+m)xx-(1+n)yy=2f\sqrt{(aa-xx-yy)}$$

and thence

$$\begin{aligned}&(aa-ff)^2-\big(2(aa-ff)+2(aa+ff)m\big)x^2-\big(2(aa-ff)+2(aa+ff)n\big)y^2\\&+(1+m)^2x^4+2\big(1+m\big)(1+n)xxyy+(1+n)^2y^4=0,\end{aligned}$$

from which there becomes

$$\left.\begin{aligned}&\frac{aa-ff+n(aa-ff)-(1+m)(1+n)xx}{(1+n)^2}\\&\pm\frac{2f}{(1+n)^2}\sqrt{(n(1+n)aa-nff+(m-n)(1+n)xx)}\end{aligned}\right\}=yy$$

and

$$\left.\begin{array}{l}\dfrac{aa-ff+m(aa+ff)-(1+m)(1+n)yy}{(1+n)^2}\\ \pm\dfrac{2f}{(1+n)^2}\sqrt{(m(1+m)aa-mff+(n-m)(1+m)yy)}\end{array}\right\}=xx.$$

146. Therefore in order that the equation found may have real factors, it is required that either $ff=(1+n)aa$ or $ff=(1+m)aa$. In the first case the equation becomes

$$yy=\frac{naa-(1+m)xx}{(1+n)}\pm\frac{2fx\sqrt{(m-n)}}{(1+n)\sqrt{(1+n)}},$$

from which, if *m* shall be less than *n*, it is necessary that there shall be

$$x=0\,,\ y=\pm a\sqrt{\frac{n}{1+n}}\ \text{ and }\ z=\frac{a}{\sqrt{(1+n)}}.$$

Therefore two contact points are given equally distant on either side from the axis of the cone. But if *m* were greater than *n*, the other equation must be taken

$$xx=\frac{maa-(1+n)yy}{1+m}\pm\frac{2fy\sqrt{(n-m)}}{(1+m)\sqrt{(1+m)}},$$

which cannot be real, unless there shall be $y=0$; in which case the coordinates become

$$x=\pm a\sqrt{\frac{m}{1+m}}\ \text{ and }\ z=\frac{a}{\sqrt{(1+m)}}.$$

And therefore in this case two other points of contact will be given; indeed the contact is present in each side of the cone, where it is nearest. And thus in a similar manner the contact will be able to be judged in individual cases.

147. But the manner of determining tangent planes of whatever surfaces can be deduced far more easily by the method of finding the tangents of curved lines treated above.
Let the nature of the surface (Fig. 149), of which we seek the tangent planes, be expressed by an equation between the three coordinates

$$AP = x, \ PQ = y \ \text{ and } \ QM = z,$$

from which it is required to define the position of the tangent plane at the point *M*. Therefore in the first place we may consider, if the surface may be cut by some plane passing through the point *M*, from which the tangent of the section arising at the point *M* is to be placed in the plane of the tangent. Whereby, if we have found the tangents of two sections of this kind at the point *M*, the plane, which is defined by these two right line tangents, must together be the tangent to the surface at the point *M*.

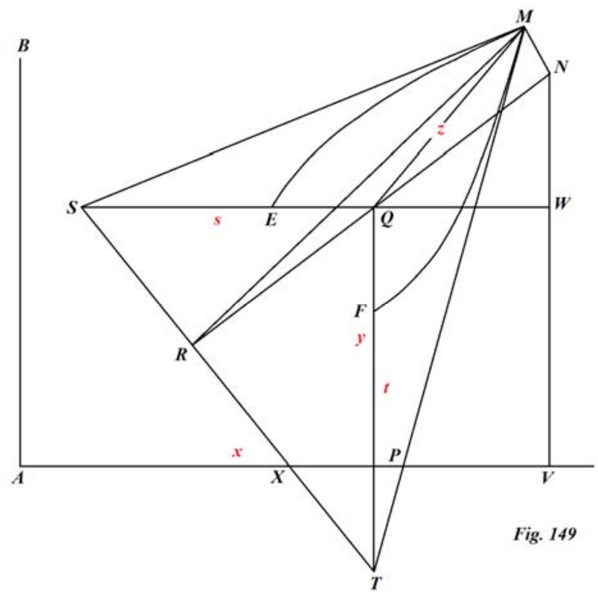


Fig. 149

148. Therefore in the first place the surface may be cut by a plane normal to the plane *APQ* along the right line *QS* parallel to the axis *AP*. Then in a similar manner a cut is made through the point *M* equally normal to the plane *APQ*, but along the right line *QP* normal to the axis *AP* ; or the first section shall be normal to the axis *AB*, truly the latter to the axis *AP*. Let *EM* be the curve of the first section, of which the right line of the tangent *MS* may be sought crossing *QS* at the point *S*, thus so that *QS* shall be the subtangent. The latter section shall be the curved line *FM*, the tangent of which shall be the right line *MT* and the subtangent *QT*. With which found the plane *SMT* will touch the surface at the point *M*. Therefore *ST* drawn will give the intersection of the tangent plane with the plane *APQ* ; and, if from *Q* to *ST* the normal *QR* may be drawn, then *QR* will be to *QM* as the whole sine to the tangent of the angle *MRQ*, by which the tangent plane is inclined to the plane *APQ*.

149. By the method of the tangents treated above we may put the subtangents found to be $QS = s$ and $QT = t$; there will be

$$PT = t - y \ \text{ and } \ PX = s - \frac{sy}{t};$$

from which there shall be

$$AX = x + \frac{sy}{t} - s.$$

Therefore the point X will be known, in which the right line ST cuts the axis AP, and because the angle $AXS = TSQ$, of which the tangent of the angle will be $= \frac{t}{s}$, from which the position of the intersection of the tangent plane with the plane APQ is known. Then on account of $ST = \sqrt{(ss+tt)}$ there will be

$$QR = \frac{st}{\sqrt{(ss+tt)}},$$

through which if it may be divvied by QM, the tangent of the angle of inclination MRQ will be produced

$$= \frac{z\sqrt{(ss+tt)}}{st}.$$

If again the normal MN may be drawn to MR, this will be both normal to the tangent plane as well as to the surface itself at the point M. Therefore the position of this may be deduced from

$$QN = \frac{zz\sqrt{(ss+tt)}}{st}.$$

The perpendicular NV may be sent from N to the axis AP, on account of the $QNV = QST$ there will be

$$PV = \frac{zz}{s} = QW \quad \text{and} \quad NW = \frac{zz}{t}.$$

Whereby, if in this manner the position of the point N may be defined in the plane APQ, the right line NM will be normal to the surface.

150. Just as the intersection of two surfaces by projections must be investigated, as now it has been shown above. We may inquire moreover, of what order the projection for an order shall become, to which the surface may be referred. And indeed in the first place two surfaces of the first order or planes for the projection of each give a line of the first order. Then also we have seen that this projection cannot rise beyond the second order, if either surface were of the first or second order. In a similar manner it is evident, if one surface were of the third order and the other of the first, the projection cannot pass beyond the third order and thus henceforth. But if two surfaces of the second order may cut each other mutually, the projection of the intersection will be either of the fourth order or less ; and generally, if the one surface shall be of order m, the other of order n, the projection of the intersection cannot be referred to a higher order than which is indicated by the number mn.

151. When neither of the surfaces cutting each other mutually is a plane, generally the mutual section of these is a curved line not established in the same plane. Yet this does not become an obstacle, as the whole section may be put in the same plane ; that shall come about, if both the equations of the surfaces taken jointly are included in an equation of this kind $\alpha z + \beta + \gamma x = f$. Whether that may arise from the two proposed equations, the two variables *z* and *y* may be defined by the third *x* and $z = P$ and $y = Q$, with *P* and *Q* being functions of *x*. Then it may be examined, whether a number *n* of this kind may be given, so that all the powers of *x* in $P + nQ$ mutually cancel out besides the lowest *x* and constant terms. Because if it comes about and there will be $P + nQ = mx + k$, the section will be in the same plane and this plane will be indicated by the equation $z + ny = mx + k$.

152. Let there be of example's sake, the following two surfaces proposed of the second order, the one for the right cone $zz = xx + yy$, the other for the elliptic-hyperboloid of the second order

$$zz = xx + 2yy - 2ax - aa.$$

From which since there shall be

$$xx + 2yy - 2ax - aa = xx + yy\ ,$$

there will become

$$y = \sqrt{(2ax + aa)} \text{ and } z = x + a\ ,$$

which last equation now indicates the whole section to be situated in the same plane, the position of which may be determined from the equation $z = x + a.$ Therefore by this method several questions relating to the nature of surfaces are able to be resolved. But those which pass beyond the method presented here require the analysis of infinite quantities, towards an understanding which these, which have been examined in these books, prepare the way.

THE END.

CAPUT VI

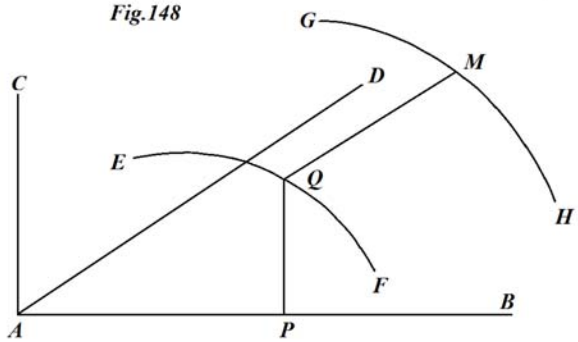
DE INTERSECTIONE DUARUM SUPERFICIERUM

131. Supra iam exposita est methodus investigandi naturam sectionis, quae oritur, si superficies quaecunque a plano secatur. Cum enim linea curva, quam sectio format, tota posita sit in eodem plano, quo sectio est facta, binas coordinatas, quarum relatione natura huiusmodi linearum curvarum exprimi solet, in eodem plano assumsimus, ut hoc pacto cognitio ad receptam rationem reduceretur. At si superficies secans non fuerit plana, quoniam tum sectio non in eodem plano iacebit, eius natura duabus coordinatis comprehendi nequit; quapropter alio modo erit utendum ad huiusmodi sectiones aequationibus includendas, quibus cuiusque puncti positio vera indicetur.

132. Punctorum autem non in eodem plano sitorum loca definiri possunt, si tria plana inter se normalia in subsidium adhibeantur atque pro quovis puncto ternae illae distantiae assignentur, quibus id a quolibet plano distat. Hinc tres variabiles requirentur ad naturam lineae curvae non in eodem plano constitutae exprimendam; ita ut, si una pro libitu definiatur, ex ea binae reliquae valores determinatos obtineant. Una igitur aequatio inter tres illas coordinatas non sufficit ad hoc praestandum, quippe quae indolem universae superficiei cuiusdam indicaret; quocirca duabus opus erit aequationibus, quarum ope, si uni variabili datus valor tribuatur, simul binarum reliquarum valores determinentur.

133. Natura igitur cuiusque lineae curvae, quam in eodem plano constitutam esse non constat, commodissime exprimitur duabus aequationibus inter tres variabiles, puta x, y, z, quae totidem coordinatas inter se normales repraesentabunt. Ope duarum ergo huiusmodi aequationum binae variabiles ex tertia determinari poterunt; aequabitur scilicet tam y quam z functioni cuipiam ipsius x. Poterit etiam pro arbitrio una variabilium eliminari, unde tres aequationes duas tantum variabiles involventes formabuntur, una inter x et y, altera inter x et z et tertia inter y et z. Harum trium vero aequationum quaevis per binas reliquas sponte determinatur, ita ut, si habeantur aequationes inter x et y et inter x et z, ex his tertia iam per eliminationem ipsius x inveniatur.

Fig.148



134. Sit ergo proposita (Fig. 148) linea quaecunque curva non in eodem plano posita, cuius unum quoddam punctum sit M. Sumantur pro arbitrio tres axes invicem normales AB, AC et AD, quibus tria plana invicem normalia BAC, BAD et CAD determinantur. Ex puncto curvae M in planum BAC demittatur perpendiculum MQ et ex puncto Q ad axem

AD ducatur normalis *QP*, erunt *AP*, *PQ* et *QM* tres illae coordinatae, inter quas si duae dentur aequationes, natura curvae determinatur. Vocentur ergo $AP = x,\ PQ = y$ et $QM = z$; et ex duabus aequationibus inter *x*, *y* et *z* propositis eliminando *z* formetur aequatio duas tantum variabiles *x* et *y* continens, quae determinabit positionem puncti *Q* in plano *BAC* ; atque singula puncta *Q* ex singulis *M* orta praebebunt lineam curvam *EQF*, cuius natura aequatione illa inter *x* et *y* inventa exprimetur.

135. Hoc igitur modo ex duabus aequationibus inter tres coordinatas propositis facile cognoscitur natura curvae *EQF*, quae formatur demittendis ex singulis curvae indagandae punctis *M* perpendiculis *MQ* in planum *BAC*. Curva autem haec *EQF* vocatur *proiectio* curvae *GMH* in planum *BAC*. Quemadmodum autem proiectio in plano *BAC* facta invenitur eliminando variabilem *z*, ita eiusdem curvae proiectio in plano *BAD* vel in plano *CAD* obtinebitur, si vel variabilis *y* eliminetur vel *x*. Una autem proiectio *EQF* non sufficit ad curvam *GMH* cognoscendam, sin autem pro singulis punctis *Q* cognita fuerint perpendicula $QM = z$, ex proiectione *EQF* ipsa curva *GMH* facile construetur. Ad hoc igitur opus est, ut praeter aequationem inter *x* et *y*, qua natura proiectionis exprimitur, habeatur aequatio inter *z* et *x* vel inter *z* et *y* vel etiam inter tres *z*, *x*, *y*, ex qua longitudo perpendiculi $QM = z$ pro quovis puncto *Q* innotescat.

136. Cum autem aequatio inter *z* et *x* exprimat proiectionem curvae *GMH* in plano *BAD* factam, aequatio autem inter *z* et *y* proiectionem in plano *CAD* atque aequatio inter tres variabiles *z*, *y* et *x* exhibeat superficiem, in qua curva *GMH* versetur, manifestum est primum ex duabus proiectionibus eiusdem curvae *GMH* in duobus planis factis ipsam curvam *GMH* cognosci. Tum vero perspicuum est, si detur superficies, in qua linea curva *GMH* contineatur, atque praeterea eius proiectio in quodam plano, pariter curvam illam fore cognitam. Erigantur enim ex singulis proiectionis punctis rectae normales *QM*, quarum intersectio cum superficie definiet curvam *GMH* quaesitam.

137. His praemissis, quae ad indolem cuiusque curvae non in eodem plano constitutae cognoscendam pertinent, non difficile erit intersectionem duarum quarumvis superficierum definire. Quemadmodum enim intersectio duorum planorum est linea recta, ita intersectio duarum superficierum quarumvis erit linea, sive recta sive curva, haecque vel in eodem plano posita vel secus. Utcunque autem fuerit comparata, singula eius puncta ad utramque superficiem pertinebunt ideoque in aequatione utriusque superficiei continebuntur. Quodsi ergo ambae superficies exprimantur aequationibus inter ternas coordinatas, quae ad eadem tria plana principalia inter se normalia seu ad eosdem tres axes inter se normales *AB*, *AC* et *AD* referantur, tum ambae istae aequationes coniunctae naturam intersectionis expriment.

138. Propositis ergo duabus superficiebus se mutuo secantibus, utriusque natura exprimi debet aequatione inter tres coordinatas, quae ad eosdem axes principales referantur, sicque habebuntur duae aequationes inter tres coordinatas *x*, *y* et *z*, ex quibus si una eliminetur, aequatio inter binas reliquas praebebit proiectionem intersectionis in plano, quod his duabus coordinatis constituitur, factae. Hoc igitur modo quoque intersectio

cuiusque superficiei a plano factae investigari poterit; cum enim aequatio generalis pro plano sit $\alpha z+\beta y+\gamma x=f$, si in aequatione superficiei loco z substituatur eius valor ex illa aequatione oriundus, nempe $z=\frac{f-\beta y-\gamma x}{\alpha}$, prodibit aequatio pro proiectione intersectionis in plano coordinatarum x et y facta. Simul vero aequatio $z=\frac{f-\beta y-\gamma x}{\alpha}$ pro quovis puncto Q proiectionis praebebit quantitatem perpendiculi QM ad ipsam intersectionem pertingentis.

139. Quodsi eveniat, ut aequatio pro proiectione fiat impossibilis, uti si inveniretur $xx+yy+aa=0$, tum hoc erit indicium ambas superficies se mutuo nusquam intersecare. Sin autem aequatio proiectionis in unicum punctum deducat seu si proiectio in punctum evanescat, tum ipsa quoque intersectio erit punctum ideoque ambae superficies se mutuo in puncto contingent; qui contactus itaque ex aequatione cognosci poterit. Datur autem praeterea contactus linearis, quando duae superficies se in infinitis punctis contingunt, lineaque contactus vel erit recta vel curva. Recta scilicet erit, si planum tangat cylindrum vel conum, conus rectus autem a globo intus tangetur per peripheriam circuli. Qui contactus ex aequatione cognoscentur, si pro proiectione eiusmodi prodierit aequatio, quae duas habeat radices aequales, propterea quod contactus nil aliud est, nisi concursus duarum intersectionum.

140. Ad haec clarius explicanda ponamus globum secari a plano quocunque. Sumamus aequationem ad centrum globi accommodatam

$$zz+yy+xx=aa,$$

pro plano autem utcunque posito haec habebitur aequatio

$$\alpha z+\beta y+\gamma x=f,$$

unde, cum sit $z=\frac{f-\beta y-\gamma x}{\alpha}$, sequens orietur aequatio inter x et y pro proiectione

$$0=ff-\alpha\alpha aa-2\beta fy-2\gamma fx+(\alpha\alpha+\beta\beta)yy+2\beta\gamma xy+(\alpha\alpha+\gamma\gamma)xx,$$

quam patet esse ellipsin, siquidem aequatio fuerit realis; sin autem fuerit imaginaria, globus a plano nusquam tangetur; at si ellipsis in punctum evanescat, planum et globus se mutuo tangent. Qui casus ut eruatur, quaeratur

$$y=\frac{\beta f-\beta\gamma x\pm\alpha\sqrt{(aa(\alpha\alpha+\beta\beta)-ff+2\gamma fx-(\alpha\alpha+\beta\beta+\gamma\gamma)xx)}}{\alpha\alpha+\beta\beta},$$

ubi, si f eiusmodi habuerit valorem, ut quantitas radicalis nunquam fieri possit realis, nullus dabitur contactus neque intersectio.

141. Ponamus esse $f = a\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}$

eritque

$$y = \frac{\beta f - \beta\gamma x \pm \alpha x\sqrt{-(\alpha\alpha + \beta\beta + \gamma\gamma)} \mp \alpha\gamma a\sqrt{-1}}{\alpha\alpha + \beta\beta},$$

cui aequationi realiter satisfieri nequit, nisi sit

$$x = \frac{\gamma a}{\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}} \quad \text{et} \quad y = \frac{\beta a}{\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}}.$$

Quare, si fuerit $f = a\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}$ planum, quod exprimitur aequatione $\alpha z + \beta y + \gamma x = f$, globum tanget; punctumque contactus habebitur, si capiatur

$$x = \frac{\gamma a}{\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}},\ y = \frac{\beta a}{\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}} \quad \text{et} \quad z = \frac{\alpha a}{\sqrt{(\alpha\alpha + \beta\beta + \gamma\gamma)}},$$

quorum va1orum veritas per geometriam elementarem, ubi contactus sphaerae a plano docetur, comprobari potest.

142. Hinc igitur generalis regula deducitur, cuius ope cognosci potest, utrum superficies quaecunque a plano aliave superficie tangatur an non. Eliminata enim ex ambabus aequationibus una variabili videndum est, an aequatio resultans resolvi possit in factores simplices an minus. Si enim habeat duos factores simplices imaginarios, dabitur contactus in puncto, quod innotescet ponendo utrumque factorem $= 0$. Sin autem habeat duos factores simplices reales eosque inter se aequales, superficies se mutuo secundum lineam rectam tangent. Quodsi vero illa aequatio habeat duos factores non simplices aequales seu si fuerit per quadratum divisibilis, tum eius radix nihilo aequalis posita exhibebit proiectionem illius lineae, quae ex contactu oritur. Hinc quoque patet, si eadem illa aequatio quatuor habuerit factores imaginarios, tum superficies se mutuo in duobus punctis contingere.

143. Quo haec plenius explicentur, investigemus contactum coni et globi, cuius centrum in axe coni sit positum. Aequatio pro globo est $zz + yy + xx = aa$, pro cono

autem $(f-z)^2 = mxx+nyy$, posito quod vertex coni intervallo *f* a centro globi sit remotus. Eliminemus hinc variabilem *y* eritque

$$(f-z)^2 = naa - nzz + (m-n)xx$$

pro proiectione intersectionis in plano coordinatarum *x* et *z*. Sit primum conus rectus seu $m = n$ eritque

$$z = \frac{f \pm \sqrt{(n(1+n)aa - nff)}}{1+n}.$$

Quare, si fuerit $f = a\sqrt{(1+n)}$, erit dupliciter $z = \frac{a}{\sqrt{(1+n)}}$ ideoque contactus erit linearis, scilicet per circulum, cuius proiectio in plano per axem transeunte est linea recta ad axem normalis.

144. Pro cono autem scaleno, ubi *m* et *n* sunt inaequales, aequatio inventa videtur semper dare intersectionem, cum tamen saepius nulla existat. Semper enim, siquidem *m* superet *n*, prodibit aequatio realis pro proiectione intersectionis; at vero notandum est realitatem proiectionis non semper indicare intersectionem realem. Ut enim ipsa intersectio sit realis, non sufficit proiectionem esse realem, sed insuper perpendicula a proiectione ad intersectionem ducta realia esse oportet. Quamvis igitur omnis curva realis habeat quasvis proiectiones reales, tamen non vicissim ex realitate proiectionis realitas ipsius curvae, quae quaeritur, concludi potest. Haecque cautela perpetuo probe est adhibenda, ne realitate aequationum, quas pro proiectionibus invenimus, abutamur.

145. Hoc incommodum evitabimus, si proiectionem in plano ordinatarum *x* et *y* quaeramus, quia enim in hoc plano nullum datur punctum, cui non punctum in conica superficie respondeat; si proiectio in hoc plano fuerit realis, ipsa quoque intersectio erit realis. Cum igitur sit $z = \sqrt{(aa - xx - yy)}$, fiet ex altera aequatione

$$f - \sqrt{(aa-xx-yy)} = \sqrt{(mxx+nyy)}$$

seu

$$aa + ff - (1+m)xx - (1+n)yy = 2f\sqrt{(aa-xx-yy)}$$

porroque

$$(aa-ff)^2-\left(2(aa-ff)+2(aa+ff)m\right)x^2-\left(2(aa-ff)+2(aa+ff)n\right)y^2$$
$$+(1+m)^2x^4+2\left(1+m\right)(1+n)xxyy+(1+n)^2y^4=0,$$

unde fit

$$\left.\begin{array}{l}\dfrac{aa-ff+n(aa-ff)-(1+m)(1+n)xx}{(1+n)^2}\\ \pm\dfrac{2f}{(1+n)^2}\sqrt{(n(1+n)aa-nff+(m-n)(1+n)xx)}\end{array}\right\}=yy$$

et

$$\left.\begin{array}{l}\dfrac{aa-ff+m(aa+ff)-(1+m)(1+n)yy}{(1+n)^2}\\ \pm\dfrac{2f}{(1+n)^2}\sqrt{(m(1+m)aa-mff+(n-m)(1+m)yy)}\end{array}\right\}=xx.$$

146. Ut igitur aequatio inventa habeat factores, oportet esse vel

$ff=(1+n)aa$ vel $ff=(1+m)aa$. Priori casu fit

$$yy=\frac{naa-(1+m)xx}{(1+n)}\pm\frac{2fx\sqrt{(m-n)}}{(1+n)\sqrt{(1+n)}},$$

unde, si sit *m* minor quam *n*, necesse est, ut sit

$$x=0\ \text{ et }\ y=\pm a\sqrt{\frac{n}{1+n}}\ \text{ et }\ z=\frac{a}{\sqrt{(1+n)}}.$$

Dantur ergo duo puncta contactus ab axe coni utrinque aequaliter distantia. Sin autem fuerit *m* maior quam *n*, sumi debet altera aequatio

$$xx=\frac{maa-(1+n)yy}{1+m}\pm\frac{2fy\sqrt{(n-m)}}{(1+m)\sqrt{(1+m)}},$$

quae realis esse nequit, nisi sit $y=0$; quo casu fit

$$x = \pm a\sqrt{\frac{m}{1+m}} \text{ et } z = \frac{a}{\sqrt{(1+m)}}.$$

Hocque ergo casu dabuntur duo alia contactus puncta; contactus enim existet in ea coni parte, ubi est arctissimus. Simili itaque modo in singulis casibus contactus iudicari debebit.

147. Modus autem longe facilior determinandi plana tangentia quarumcunque superficierum deduci potest ex methodo inveniendi tangentes linearum curvarum supra tradita. Sit (Fig. 149) natura superficiei, cuius plana tangentia quaerimus, expressa aequatione inter tres coordinatas

$$AP = x,\ PQ = y \text{ et } QM = z,$$

ex qua definiri oportet positionem plani superficiem in puncto *M* tangentis. Primum igitur consideramus, si superficies secetur plano quocunque per punctum *M* transeunte, sectionis inde ortae tangentem in puncto *M* sitam fore in plano tangente. Quare, si duarum huiusmodi sectionum tangentes in puncto *M* invenerimus, planum, quod his duabus rectis tangentibus definitur, ipsam superficiem in puncto *M* contingere debere.

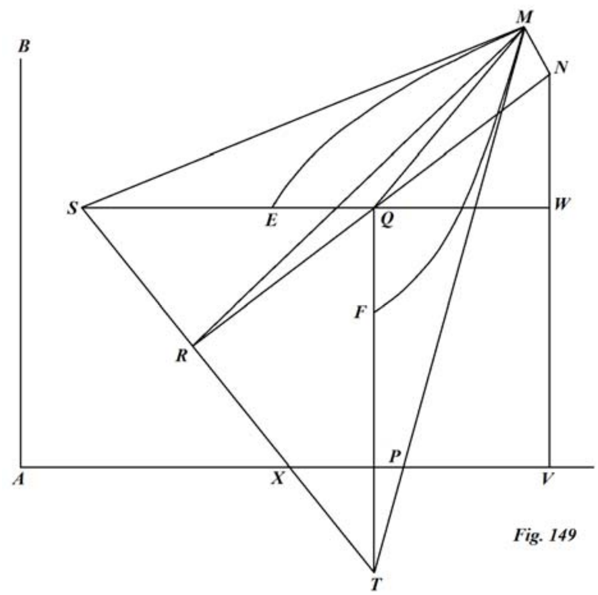


Fig. 149

148. Secetur ergo primum superficies plano ad planum *APQ* normali secundum rectam *QS* parallelam axi *AP.* Tum simili modo fiat sectio per punctum *M* pariter normalis ad planum *APQ,* sed secundum rectam *QP axi AP* normalem; seu prior sectio sit normalis ad axem *AB,* posterior vero ad axem *AP.* Sit curva *EM* prior sectio, cuius quaeratur tangens *MS* rectae *QS* in puncto *S* occurrens, ita ut sit *QS* subtangens. Sectio posterior sit linea curva *FM*, cuius tangens sit recta *MT* et subtangens *QT.* Quibus inventis planum *SMT* superficiem in puncto *M* tanget. Ducta ergo *ST* dabit intersectionem plani tangentis cum plano *APQ;* atque, si ex *Q* ad *ST* normalis ducatur *QR*, tum erit *QR* ad *QM* uti sinus totus ad tangentem anguli *MRQ*, quo planum tangens ad planum *APQ* inclinatur.

149. Ponamus per methodum tangentium supra traditam inventas esse subtangentes $QS = s$ et $QT = t$; erit

$$PT = t - y \text{ et } PX = s - \frac{sy}{t};$$

unde fit

$$AX = x + \frac{sy}{t} - s.$$

Innotescit ergo hinc punctum *X*, in quo recta *ST* axem *AP* traiicit, et, quia angulus $AXS = TSQ$, erit huius anguli tangens $= \frac{t}{s}$, ex quo positio intersectionis plani tangentis cum plano *APQ* cognoscitur. Deinde ob $ST = \sqrt{(ss+tt)}$
erit

$$QR = \frac{st}{\sqrt{(ss+tt)}},$$

per quam si dividatur *QM*, prodibit

$$\text{tangens anguli inclinationis } MRQ = \frac{z\sqrt{(ss+tt)}}{st}.$$

Si porro ad *MR* normalis ducatur *MN*, erit haec cum ad planum tangens tum ad ipsam superficiem in puncto *M* normalis. Eius ergo positio colligitur ex

$$QN = \frac{zz\sqrt{(ss+tt)}}{st}.$$

Demittatur ex *N* ad axem *AP* perpendicularis *NV*, ob angulum $QNV = QST$
erit

$$PV = \frac{zz}{s} = QW \text{ et } NW = \frac{zz}{t}.$$

.
Quare, si hoc modo definiatur positio puncti *N* in plano *APQ*, recta *NM* erit normalis in superficiem.

150. Quemadmodum intersectio duarum superficierum per proiectiones indagari debet, supra iam est ostensum. Inquiramus autem, cuius ordinis futura sit proiectio pro ordine, ad quem superficies referuntur. Ac primo quidem duae superficies primi ordinis seu planae pro intersectione eiusque proiectione dant lineam primi ordinis. Deinde quoque vidimus hanc proiectionem ultra secundum ordinem assurgere non posse, si altera superficies fuerit primi ordinis altera secundi. Simili modo manifestum est, si altera superficies fuerit tertii ordinis altera primi, proiectionem tertium gradum non esse transgressuram et ita porro. Sin autem duae lineae secundi ordinis se mutuo secent, proiectio intersectionis erit vel quarti ordinis vel inferioris; atque generaliter, si altera superficies sit ordinis *m*, altera ordinis *n*, intersectionis proiectio ad altiorem ordinem, quam qui numero *mn* indicatur, nunquam referetur.

151. Quando neutra superficierum se mutuo secantium est plana, plerumque sectio earum mutua est linea curva non in eodem plano constituta. Hoc tamen non obstante fieri potest, ut tota sectio in eodem plano sit posita; id quod eveniet, si ambae superficierum aequationes iunctim sumtae huiusmodi aequationem $\alpha z+\beta+\gamma x= f$ in se complectantur. Quod utrum eveniat, ex duabus aequationibus propositis definiantur binae variabiles z et y per tertiam x fiatque $z=P$ et $y=Q$, existentibus P et Q functionibus ipsius x. Tum dispiciatur, an eiusmodi numerus n detur, ut in $P+nQ$ omnes potestates ipsius x se mutuo tollant praeter infimam x et terminos constantes. Quod si eveniat fueritque $P+nQ=mx+k$, sectio erit in eodem plano hocque planum indicabitur aequatione $z+ny=mx+k$.

152. Sint, verbi gratia, propositae sequentes duae superficies secundi ordinis, altera pro cono recto $zz=xx+yy$, altera pro superficie secundi generis elliptico-hyperbolica

$$zz=xx+2yy-2ax-aa.$$

Ex quibus cum sit

$$xx+2yy-2ax-aa=xx+yy,$$

erit

$$y=\sqrt{(2ax+aa)} \text{ et } z=x+a,$$

quae ultima aequatio iam indicat totam sectionem in eodem plano esse sitam, cuius positio determinetur aequatione $z=x+a$. Hac igitur ratione plurimae quaestiones ad naturam superficierum pertinentes resolvi poterunt. Quae autem methodum hic expositam transgrediuntur, eae analysin infinitorum requirunt, ad quam scientiam haec, quae his libris tradita sunt, viam praeparant.

FINIS