CHAPTER IV

CONCERNING THE CHANGING OF COORDINATES

86. Just as the equations for curved lines situated in the same plane are able to be transformed into innumerable diverse forms with either the starting point of the abscissas being changed, or the position of the axis, or both ; thus in the present exercise a much greater variety will be considered. Indeed in the first place in the same plane, in which both the coordinates have been placed, these can be varied in an infinite number of ways. Then truly this plane itself, which contains the two coordinates, and thus in the former variations can be increased to be changed in an infinite number of ways. Clearly with an equation given between three coordinates normal to each other, another equation can be found constantly between any three other coordinates equally normal to each other, the position of which with respect to the first infinitudes can be varied much more, than if two coordinates only were present, as usually arises in the equations of curved lines.

87. In the first place we may put the starting point of the abscissas x on the axis to be changed only, thus so that the two remaining coordinates y and z may remain the same, and the new abscissa will differ by a constant quantity from x. Therefore the new abscissa shall be $=t$, there will be $x=t\pm a$, with which value substituted into the equation for the surface an equation will be produced between the three coordinates t, y et z, which, although different from the start, yet will be for the same surface. In a similar manner the remaining coordinates y and z can be increased or decreased by constant quantities and, if there may be put $x=t\pm a,\ y=u\pm b$ and $z=v\pm c$, an equation will arise between the three variables t, u and v for the same surface ; and thus these new coordinates will be parallel to the former. Meanwhile in this manner the equation for the surface, although it is more general, yet will not be changed much.

[At this stage, the origin has been translated ; the next section shows how to rotate the translated coordinates in the plane through an angle about a normal axis.]

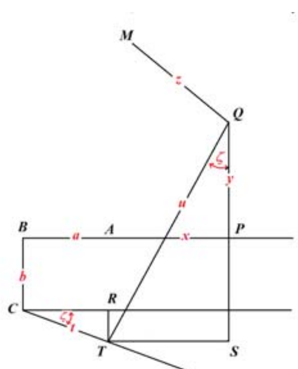

Fig. 140

88. Because the three orthogonal coordinates, an equation of which has expressed the nature of the surface, are referred to three planes normal to each other, we may consider one plane (Fig. 140) to remain unchanged, in which two of the coordinates x and y are taken, but on that some other line CT may be assumed for the axis besides AP [*i.e.* the planes normal to this plane are changed, or equivalently, a normal through the point of rotation parallel to MQ remains

unchanged]. Therefore since the first coordinates for the axis *AP* shall be $AP = x,\ PQ = y,\ QM = z,$ for the new axis *CQ* the coordinate $QM = z$ will remain the same, but the two remaining coordinates will become $CT = t,\ TQ = u$, with *QT* drawn normal to the new axis *CT*. Therefore for finding the equation between these new coordinates *t, u* and *z, CR* is drawn parallel to the first axis *AP*, then from *C* a perpendicular *CB* is drawn to that and calling $AB = a,\ BC = b$ and the angle $RCT = \zeta$. Finally *TR* may be drawn normal to *CR* and from *T* the perpendicular *TS* to *QP* produced.

89. With these made, in triangle *TCR* there will be $TR = t \cdot \sin.\zeta,\ CR = t \cdot \cos.\zeta$; but in triangle *QTS*, the angle of which equally at *Q* will be $= \zeta$, there becomes $TS = u \cdot \sin.\zeta$ and $QS = u \cdot \cos.\zeta$. Now there will be obtained from these :

$$AP = x = CR + TS - AB = t \cdot \cos.\zeta + u \cdot \sin.\zeta - a$$

and

$$QP = QS - TR - BC = y = u \cdot \cos.\zeta - t \cdot \sin.\zeta - b.$$

[*i.e.*, in matrix notation, not of course available to Euler, we have

$$: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\zeta & \sin\zeta & 0 \\ -\sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix} - \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} ;$$

note that the new coordinates are related to the old ones rather than vise-versa, and that we are concerned only with changes in the axes, and not in the orientation of the actual surface.]

But if therefore these values may be substituted in place of *x* and *y* in the proposed equation, an equation will result between the three new variables *t, u* and *z*, by which the nature of the same surface will be expressed. This new equation therefore itself brings forwards a more widely extended form, since in that three new arbitrary constants *a, b* and the angle ζ shall be introduced, which did not enter into the first equation. And this will be the general equation, when the same plane may be maintained, and in which the two coordinates *x* and *y* are changed in this way.

[The next section shows how to rotate the plane coordinates into another plane through an angle about a common axis.]

90. Now the plane also may be varied, in which both the initial coordinates *x* and *y* were assumed, and indeed in the first place thus (Fig. 141), so that the intersection of the new plane with the first plane *APQ* falls on the same right line

AP, which also may be considered as the axis for the new coordinates. Therefore *APT* shall be this new plane, the inclination of which to the first plane *APQ* will be the angle *QPT*, which may be put η. [Thus, the new plane is rotated by the angle η about the line *AP*, and so the normal *QM* rotates through the same angle to *TM*.] The normal *MT* may be drawn from *M* to *PT*, which likewise will be perpendicular in the new plane and in turn will be extended to the third coordinate. Therefore the three new coordinates may be put in place $AP = x,\ PT = u$ and $TM = v$ and with *TR* drawn to *PQ* and *TS* to the normal *QM* there will be :

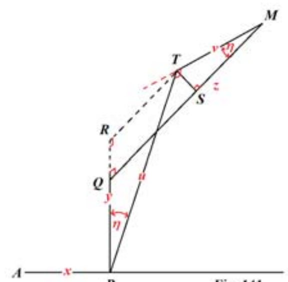

Fig. 141

$$TR = u\cdot\sin.\eta;\ PR = u\cdot\cos.\eta,\ TS = v\cdot\sin.\eta \text{ and } MS = v\cdot\cos.\eta.$$

Hence there will be :

$$PQ = y = u\cdot\cos.\eta - v\cdot\sin.\eta \text{ and } QM = z = v\cdot\cos.\eta + u\cdot\sin.\eta,$$

$$[i.e., \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & -\sin\eta \\ 0 & \sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix},]$$

which values for *y* and *z* substituted in the proposed equation will give the equation between the three new coordinates *x*, *u* and *v*, by which the nature of the same surface will be expressed.
[Note that the first transformation is anticlockwise about the normal to the point *B* (*a*,*b*) in the plane of the page, while the second is clockwise about the axis *TP*.]

[The next section shows how to combine these two transformations.]

91. Now the intersection of the new cutting plane with the plane *APQ* may fall (Fig. 140) on some line *CT* and η shall be the inclination between these planes ; and this right line *CT* may be taken for the [new *x*-] axis in this plane. In the first place the equation may be sought between the coordinates in and plane *APQ* related to the axis *GT*, which thus may be found from the preceding, so that on putting $AB = a,\ BC = b$, with the angle $TCR = \zeta$ and with the coordinates $CT = p,\ TQ = q$ and $QM = r$, so that there shall be

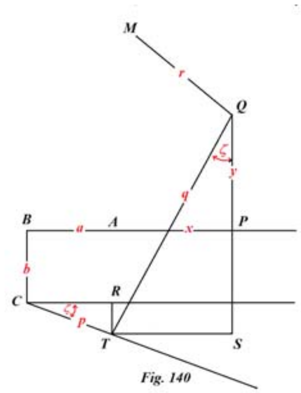

Fig. 140

$$x = p\cdot\cos.\zeta + q\cdot\sin.\zeta - a,\ y = q\cdot\cos.\zeta - p\cdot\sin.\zeta - b \text{ and } z = r.$$

[*i.e.*, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\zeta & \sin\zeta & 0 \\ -\sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$, ignoring the change of origin.]

Now truly from the preceding paragraph, with the new coordinates *t, u* and *v* put in place there becomes

$$p = t,\ q = u\cdot\cos.\eta - v\cdot\sin.\eta \text{ and } r = v\cdot\cos.\eta + u\cdot\sin.\eta.$$

[i.e., $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & -\sin\eta \\ 0 & \sin\eta & \cos\eta \end{bmatrix}\begin{bmatrix} t \\ u \\ v \end{bmatrix}$]

[Thus, a subsequent clockwise rotation takes place about the axis *cT*, as above, through the angle η, with the initial coordinates relabeled as *p, q*, and *r* and related as shown to the final coordinates *t, u*, and *v*.]

With these substituted the principal coordinates *x, y, z* thus will be determined from the new, so that there shall be

$$x = t\cdot\cos.\zeta + u\cdot\sin.\zeta\cdot\cos.\eta - v\cdot\sin.\zeta\cdot\sin.\eta - a,$$

$$y = -t\cdot\sin.\zeta + u\cdot\cos.\zeta\cdot\cos.\eta - v\cdot\cos.\zeta\cdot\sin.\eta - b,$$

and

$$z = u\cdot\sin.\eta + v\cdot\cos.\eta.$$

[*i.e.*,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\zeta & \sin\zeta & 0 \\ -\sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & -\sin\eta \\ 0 & \sin\eta & \cos\eta \end{bmatrix}\begin{bmatrix} t \\ u \\ v \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\zeta & \sin\zeta\cos\eta & -\sin\zeta\sin\eta \\ -\sin\zeta & \cos\zeta\cos\eta & -\cos\zeta\sin\eta \\ 0 & \sin\eta & \cos\eta \end{bmatrix}\begin{bmatrix} t \\ u \\ v \end{bmatrix},$$

on ignoring the change of origin for the rotation of the axis. Thus to date we have rotated about an axis normal to the plane of the page through an angle ζ, keeping *z* fixed; followed by a rotation about a horizontal axis through a point in the plane of the page through an angle η, keeping the new *x*-axis *t* fixed ; subsequently we will rotate through an angle θ , but vertical in the new plane of the page, keeping *v* fixed. It would appear that this is not the best series of rotations to use, as they are not independent from each other, as the final rotational matrix below shows; indeed a rotation about the new second axis would be needed to produce a set of three rotations that could be performed in any order.]

92. Now in this new plane some other line may be taken for the axis (Fig. 140), in which the coordinates *t* and *u* have been placed, and thus the most general equation will arise for the proposed surface. Towards this end *AP, PQ, QM* shall be the coordinates *t, u* et *v*, which we have now found, thus so that *AP* will represent the intersection of the plane mentioned with plane, in which the principal coordinates *x* and *y* are understood to be placed. And the right line *CT* shall be the new axis, to which the most general coordinates which we can find may be referred, which will be called $CT = p,\ TQ = q$ and $QM = r$. In addition, the lines *AB* and *BC* are constant, but the angle *CTR* may be put $= \theta$.
With these in place there will be, from § 89 :

$$t = p \cdot \cos.\theta + q \cdot \sin.\theta - AB ,$$

$$u = -p \cdot \sin.\theta + q \cdot \cos.\theta - BC ,$$

and

$$v = r.$$

[i.e. on setting

$$\begin{bmatrix} t \\ u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} .]$$

Which values, if they may be substituted into the expressions in the preceding paragraphs, there will be found :

$$x = p\left(\cos.\zeta \cdot \cos.\theta - \sin.\zeta \cdot \cos.\eta \cdot \sin.\theta\right) + q(\cos.\zeta \cdot \sin.\theta + \sin.\zeta \cdot \cos.\eta \cdot \cos.\theta) - r \cdot \sin.\zeta \cdot \sin.\eta + f ,$$

$$y = -p \cdot \left(\sin.\zeta \cdot \cos.\theta + \cos.\zeta \cdot \cos.\eta \cdot \sin.\theta\right) - q \cdot (\sin.\zeta \cdot \sin.\theta - \cos.\zeta \cdot \cos.\eta \cdot \cos.\theta) - r \cdot \cos.\zeta \cdot \sin.\eta + q,$$

and

$$z = -p \cdot \sin.\eta \cdot \sin.\theta + q \cdot \sin.\eta \cdot \cos.\theta + r \cdot \cos.\eta + k ,$$

where *f, g* and *k* are constant lines arising, which have been introduced in the composition of these.

[*i.e.*,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\zeta & \sin\zeta\cos\eta & -\sin\zeta\sin\eta \\ -\sin\zeta & \cos\zeta\cos\eta & -\cos\zeta\sin\eta \\ 0 & \sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\zeta\cos\theta - \sin\zeta\cos\eta\sin\theta & \cos\zeta\sin\theta + \sin\zeta\cos\eta\cos\theta & -\sin\zeta\sin\eta \\ -\sin\zeta\cos\theta - \cos\zeta\cos\eta\sin\theta & -\sin\zeta\sin\theta + \cos\zeta\cos\eta\cos\theta & -\cos\zeta\sin\eta \\ -\sin\eta\sin\theta & \sin\eta\cos\theta & \cos\eta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

.]

93. It is apparent therefore that the most general equation for some surface includes six arbitrary constants, which, however they may be determined, the equation will always express the nature of the same surface. But however simple or concise the equation should be for the surface between the coordinates *x, y, z,* if from that, the most general equation may be put together between *p, q* and *r,* on that account a great number of arbitrary constants by necessity becomes maximally complicated, especially if higher dimensions of *x, y* and *z* were present. Therefore scarcely ever will the case be given, in which it will be appropriate to rise to the most general equation. Yet nevertheless that general form thence may be understood to be useful, so that the equation may be reduced to its simplest with these constants defined in some suitable manner, yet on account of the prolixity of the calculation this labour will become overwhelming and most troublesome. Yet meanwhile in the following this method will not be without use of forming the most general equations, because thence unusual properties will be elicited and demonstrated.

94. But although the most general and complete equation shall be the most complicated, yet if we may examine the dimensions, which the coordinates taken together may constitute, the number of these is always equal to the number of dimensions, which the first coordinates *x, y* and *z* may give. Thus, since the equation for a sphere $xx + yy + zz = aa$ shall be of two dimensions, also the most general equation will not contain more than two dimensions of the coordinates *p, q* and *r.* Hence the number of dimensions, which the coordinates constitute in the equation of a certain surface, makes available for us the essential nature of this surface ; on account of which, however the position of the coordinates may be varied, yet always a number of the same dimensions emerges. Clearly here a similar account may be observed about surfaces, as we have understood above with curved lines ; from which we may divide these into certain orders. Therefore in the same way it will be agreed that surfaces be set out in orders following the dimensions of the coordinates and we will have surfaces of the first order, the equation of which contains the single order only ; we will refer to second order

surfaces, in the equation of which the coordinates rise to two dimensions; and thus again from the number of dimensions the following orders will be put in place.

95. If now these may be brought together, since from these which have been treated above concerning the discovery of sections planes of each surface, we may consider the order of the sections always to agree with the order, to which the surface belongs. Indeed let the equation for some surface proposed between the coordinates x, y and z pertaining to the order n, but the normal coordinates of each section shall be t and u. And above, in § 85, we have seen the equation between t and u to be found, if in the equation for the surface the following values may be substituted

$$x = f + t \cdot \cos.\theta - u \cdot \sin.\theta \cdot \cos.\varphi ,$$

$$y = \quad t \cdot \sin.\theta + u \cdot \cos.\theta \cdot \cos.\varphi ,$$

and

$$z = u \cdot \sin.\varphi.$$

Therefore it is evident that the equation for the section cannot be assigned more dimensions than the equation will have between x, y and z, but always just as many dimensions to be produced.

96. Therefore a surface of the first order cannot have other sections made by a plane besides lines of the first order, or right lines. Then from a section of the surface of the second order no other lines may arise except of the second order or conic sections ; for the surface of a cone is also that of the second order, since its equation shall be :

$$zz = \alpha xx + \beta yy .$$

In a similar manner from a surface of the third order lines of the third order will be produced by plane sections, and thus so forth. Yet it can happen and when it is possible, that an equation for a certain section may admit divisors, in which case the section will be composed from two or more lines of lesser order. Thus the section of a cone made by a vertical plane will consist of two right lines, which yet taken together will imitate a line of the second order, as we have recorded above.

97. Therefore we will be concerned with the construction of surfaces which belong to the first order, before being concerned with the rest. The equation expressing the nature of these will be therefore

$$\alpha x + \beta y + \gamma z = a ,$$

since all the sections of which made by a plane shall be right lines, it is evident these surfaces cannot be planes ; if indeed they have convexity or concavity, by necessity a curvilinear section will be given. For although in the remaining orders surfaces of this kind are given, certain sections of which are right lines (as we have seen to arise in the cylinder, the cone and with others used), yet in these curvilinear sections are not excluded. Evidently a similar account occurs here, such as we have observed with curved lines ; for just as a line, which cannot be cut by a right line in more than a single point, by necessity is a right line, thus a surface, which cut by a plane always gives a right line, by necessity is deduced to be itself a plane.

98. Moreover this nature can be shown most clearly from the most general equation. For the most general equation will be formed between the coordinates *p*, *q* and *r* from the equation $\alpha x+\beta y+\gamma z=a$ following § 92. And, because six new arbitrary constants are introduced, nothing stands in the way, why these may be determined thus, so that the coefficients of two of the coordinates *p* and *q* may vanish and an equation of this kind may remain $r=f$, expressing the nature of the same surface. But this equation $r=f$ shows the proposed surface to be a plane, in which the two coordinates *p* et *q* are present, and thus parallel to the same plane. It can be arranged also, that there is made $r=0$, and thus it is evident that the plane itself, in which *p* and *q* are assumed, to be the surface sought.

99. Therefore since it may be agreed that the surface expressed by the equation $\alpha x+\beta y+\gamma z=a$ shall be a plane, there is a need that we may define its position with respect to the plane in which the coordinates *x* and *y* are assumed. Therefore (Fig. 142) *M* shall be some point of this surface and let the three coordinates be $AP=x,\ PQ=y$ and $QM=z$. In the first place put $z=0$ and the equation will arise $\alpha x+\beta y=a$, which will express the intersection of the surface sought with the plane *APQ*, which is apparent to be the right line *BCR*, of which the position with respect to the axis *AP* will be such, so that the normal right line *AB* will be to the axis *AP* in the plane *APQ*

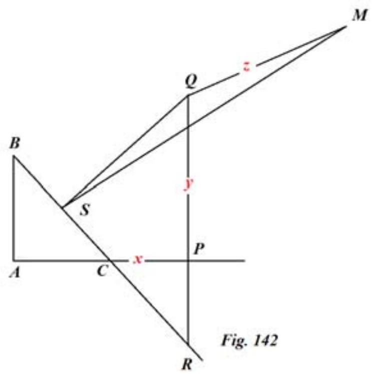

Fig. 142

$=\dfrac{\alpha}{\beta}$ and $AC=\dfrac{a}{\alpha}$, from which the angle of the tangent *ACB* will be $=\dfrac{\alpha}{\beta}$ and thus [recall that Euler always used a left-handed *x*-axis]

$$\text{the sine}=\frac{\alpha}{\sqrt{(aa+\beta\beta)}}\quad\text{and the cosine}=\frac{\beta}{\sqrt{(aa+\beta\beta)}}.$$

Then QP may be produced as far as the crossing of the right line BC at R and on account of $CP = x - \frac{a}{\alpha}$ there will be

$$CR = \frac{x\sqrt{(aa+\beta\beta)}}{\beta} - \frac{\alpha\sqrt{(aa+\beta\beta)}}{\alpha\beta} \text{ and } PR = \frac{\alpha x}{\beta} - \frac{\alpha}{\beta}.$$

100. The normal QS may be sent from Q to BC, and with MS joined it will be apparent the angle MSQ to measure the inclination of the proposed surface to the plane APQ.

Therefore since there shall be $PR = \frac{\alpha x - a}{\beta}$, there will be

$$QR = \frac{\alpha x + \beta y - a}{\beta} = -\frac{yz}{\beta}$$

and on account of the angle $RQS = ACB$ there will be

$$QS = \frac{\gamma z}{\sqrt{(aa+\beta\beta)}},$$

from which the tangent of the angle QSM shall be made $= \frac{-\sqrt{(aa+\beta\beta)}}{\gamma}$,

and therefore the cosine $= \frac{\gamma}{\sqrt{(aa+\beta\beta+\gamma\gamma)}}$.

Therefore the surface sought to the plane, in which x and y are involved, will be inclined by an angle, the tangent of which is $= -\frac{\sqrt{(aa+\beta\beta)}}{\gamma}$, truly in a similar manner the same surface will be inclined to the plane of the x and z coordinates at an angle, of which the tangent $= -\frac{\sqrt{(aa+\gamma\gamma)}}{\beta}$, and to the plane of the coordinates y and z at an angle, of which the tangent is $= -\frac{\sqrt{(\beta\beta+\gamma\gamma)}}{\alpha}$.

CAPUT IV

DE IMMUTATIONE COORDINATARUM

86. Quemadmodum aequationes pro lineis curvis in eodem plano sitis in innumerabiles formas diversas transformari possunt immutandis cum abscissarum initio tum axis positione tum utroque, ita in praesenti negotio multo adhuc maior varietas locum habet. Primum enim in eodem plano, in quo binae coordinatae sunt sitae, hae infinitis modis variari possunt. Deinde vero hoc ipsum planum, quod duas continet coordinatas, mutari sicque prior varietas in infinitum augeri poterit. Data scilicet aequatione inter tres coordinatas inter se normales, perpetuo inveniri potest alia aequatio inter tres quascunque alias coordinatas pariter inter se normales, quarum positio respectu priorum infinities magis variari potest, quam si duae tantum essent coordinatae, uti usu venit in aequationibus linearum curvarum.

87. Ponamus primum solum abscissarum x initium in axe mutari, ita ut binae reliquae coordinatae y et z maneant eaedem, atque nova abscissa quantitate constante ab x discrepabit. Sit igitur nova abscissa $= t$, erit $x = t \pm a$, quo valore in aequatione pro superficie substituto prodibit aequatio inter tres coordinatas t, y et z, quae, etsi a priori diversa, tamen pro eadem erit superficie. Simili modo reliquae coordinatae y et z quantitatibus constantibus augeri minuive poterunt atque, si ponatur $x = t \pm a,\ y = u \pm b$ et $z = v \pm c$, orietur aequatio inter tres variabiles t, u et v pro eadem superficie; atque adeo hae novae coordinatae prioribus erunt parallelae. Interim hoc modo aequatio pro superficie, etsi est magis generalis, tamen non multum variatur.

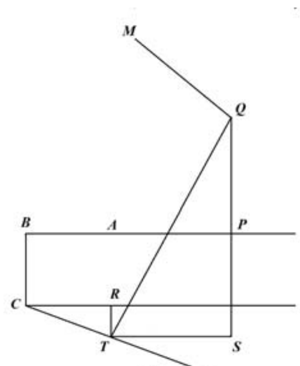

Fig. 140

88. Quoniam tres coordinatae orthogonales, quarum aequatio naturam superficiei exprimit, ad tria plana inter se normalia referuntur, ponamus (Fig. 140) planum unum, in quo binae coordinatarum x et y capiuntur, invariatum manere, in eo autem lineam quamcunque aliam CT praeter AP pro axe assumi. Cum igitur priores coordinatae pro axe AP essent $AP = x,\ P = y,\ QM = z,$ pro novo axe CQ manebit coordinata QM z eadem, at binae reliquae evadent $CT = t,\ TQ = u$, ducta QT ad novum axem CT normali. Ad aequationem igitur inter has novas coordinatas t, u et z inveniendam ducatur CR parallela priori axi AP, tum ex C ad eum perpendicularis ducatur CB ac vocetur $AB = a,\ BC = b$ et angulus $RCT = \zeta$. Denique ducatur TR normalis ad CR et ex T in QP productam perpendiculum TS.

89. His factis in triangulo *TCR* erit $TR = t \cdot \sin.\zeta$, $CR = t \cdot \cos.\zeta$; in triangulo autem *QTS,* cuius angulus ad *Q* pariter erit $= \zeta$, fiet $TS = u \cdot \sin.\zeta$ et $QS = u \cdot \cos.\zeta$. Ex his iam obtinebitur

$$AP = x = CR + TS - AB = t \cdot \cos.\zeta + u \cdot \sin.\zeta - a$$

et

$$QP = QS - TR - BC = y = u \cdot \cos.\zeta - t \cdot \sin.\zeta - b.$$

Quodsi ergo isti valores loco *x* et *y* in aequatione pro superficie proposita substituantur, resultabit aequatio inter ternas novas coordinatas *t, u* et *z,* qua eiusdem superficiei natura exprimetur. Haec igitur nova aequatio multo latius patentem speciem prae se feret, cum in eam ingrediantur tres novae constantes arbitrariae *a, b* et angulus ζ, quae in priori aequatione non inerant. Haecque erit aequatio generalis, quando quidem idem planum, in quo binae coordinatae *x* et *y* versantur, retineatur.

90. Varietur nunc quoque planum, in quo binae priores coordinatae *x* et *y* erant assumtae, ac primo quidem ita (Fig. 141), ut intersectio novi plani cum priori *APQ* incidat in ipsam rectam *AP*, quae etiam pro novis coordinatis tanquam axis spectetur. Sit igitur *APT* hoc novum planum, cuius ad prius *APQ* inclinatio erit angulus *QPT,* qui ponatur η. Ex *M* in *PT* ducatur normalis *MT*, quae simul in novum planum erit perpendicularis et vicem tertiae coordinatae tenebit. Ponantur ergo tres novae coordinatae $AP = x$, $PT = u$ et $TM = v$ et ducta *TR* ad *PQ* et *TS* ad *QM* normali erit

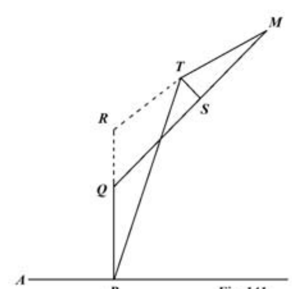

Fig. 141

$$TR = u \cdot \sin.\eta;\ PR = u \cdot \cos.\eta,\ TS = v \cdot \sin.\eta \text{ et } MS = v \cdot \cos.\eta.$$

Hinc erit

$$PQ = y = u \cdot \cos.\eta - v \cdot \sin.\eta \text{ et } QM = z = v \cdot \cos.\eta + u \cdot \sin.\eta,$$

qui valores in aequatione proposita pro *y* et *z* substituti dabunt aequationem inter tres novas coordinatas *x, u* et *v,* qua eiusdem superficiei natura exprimetur.

91. Cadat nunc (Fig. 140) intersectio novi plani secantis cum plano *APQ* in lineam quamcunque *CT* sitque η inclinatio istorum planorum; ac sumatur recta haec *CT* pro axe in hoc plano. Quaeratur primum aequatio inter coordinatas in

plano *APQ* ad axem *GT* relatas, quae ex praecedentibus ita reperietur, ut positis $AB = a,\ BC = b$, angulo $TCR = \zeta$ et coordinatis $CT = p,\ TQ = q$ et $QM = r$, ut sit

$$x = p\cdot\cos.\zeta + q\cdot\sin.\zeta - a,\ y = q\cdot\cos.\zeta - p\cdot\sin.\zeta - b \text{ et } z = r.$$

Nunc vero ex paragrapho praecedente positis novis coordinatis *t, u* et *v* fiet

$$p = t,\ q = u\cdot\cos.\eta - v\cdot\sin.\eta \text{ et } r = v\cdot\cos.\eta + u\cdot\sin.\eta .$$

His substitutis coordinatae principales *x, y, z* ex novis ita determinabuntur, ut sit

$$x = t\cdot\cos.\zeta + u\cdot\sin.\zeta\cdot\cos.\eta - v\cdot\sin.\zeta\cdot\sin.\eta - a$$

et

$$y = -t\cdot\sin.\zeta + u\cdot\cos.\zeta\cdot\cos.\eta - v\cdot\cos.\zeta\cdot\sin.\eta - b$$

atque

$$z = u\cdot\sin.\eta + v\cdot\cos.\eta .$$

92. Sumatur iam (Fig. 140) in plano isto novo, in quo coordinatae *t* et *u* sunt sitae, alia linea quaecunque pro axe sicque orietur aequatio generalissima pro superficie proposita. Sint in hunc finem *AP, PQ, QM* coordinatae *t, u* et *v,* quas modo invenimus, ita ut *AP* repraesentet intersectionem memorati plani cum plano, in quo principales coordinatae *x* et *y* positae concipiuntur. Sitque recta *CT* novus axis, ad quem novae generalissimae coordinatae, quas quaerimus, referantur, quae vocentur $CT = p,\ TQ = q$ et $QM = r$. Praeterea sunt *AB* et *BC* lineae constantes, angulus autem *CTR* ponatur $= \theta$.
His positis erit ex § 89.

$$t = p\cdot\cos.\theta + q\cdot\sin.\theta - AB$$

et

$$u = -p\cdot\sin.\theta + q\cdot\cos.\theta - BC$$

atque

$$v = r.$$

Qui valores si substituantur in expressionibus paragraphi praecedentis, reperietur

$$x = p\left(\cos.\zeta\cdot\cos.\theta - \sin.\zeta\cdot\cos.\eta\cdot\sin.\theta\right) + q(\cos.\zeta\cdot\sin.\theta + \sin.\zeta\cdot\cos.\eta\cdot\cos.\theta) - r\cdot\sin.\zeta\cdot\sin.\eta + f$$

et

$$y = -p \cdot \left(\sin.\zeta \cdot \cos.\theta + \cos.\zeta \cdot \cos.\eta \cdot \sin.\theta\right) - q \cdot (\sin.\zeta \cdot \sin.\theta - \cos.\zeta \cdot \cos.\eta \cdot \cos.\theta) - r \cdot \cos.\zeta \cdot \sin.\eta + q$$

atque

$$z = -p \cdot \sin.\eta \cdot \sin.\theta + q \cdot \sin.\eta \cdot \cos.\theta + r \cdot \cos.\eta + k,$$

ubi *f, g* et *k* sunt lineae constantes ex compositione earum, quae in calculum sunt introductae, ortae.

93. Patet ergo aequationem generalissimam pro quavis superficie sex constantes arbitrarias complecti, quae, utcunque determinentur, aequatio perpetuo eiusdem superficiei naturam exprimet. Quantumvis autem simplex et succincta fuerit aequatio pro superficie inter coordinatas *x, y, z*, si ex ea confletur aequatio generalissima inter *p, q* et *r*, ea ob ingentem constantium arbitrariarum numerum necessario fiet maxime intricata, praesertim si altiores dimensiones ipsarum *x, y* et *z* affuerint. Vix igitur dari poterit casus, in quo conveniret ad aequationem generalissimam assurgere. Quanquam enim ea utilitas inde percipi posset, ut idoneo modo constantibus illis definiendis aequatio simplicissima redderetur, tamen ob calculi prolixitatem hic labor plerumque fieret molestissimus. Interim tamen in sequentibus ista methodus aequationes generalissimas formandi usu non carebit, quoniam inde egregiae proprietates elicientur ac demonstrabuntur.

94. Quamquam autem aequatio generalissima plerumque sit maxime complicata, tamen, si ad dimensiones, quas coordinatae iunctim sumtae constituunt, spectemus, earum numerus perpetuo aequalis est numero dimensionum, quas primae coordinatae *x, y* et *z* confecerunt. Sic, cum aequatio pro sphaera $xx + yy + zz = aa$ sit duarum dimensionum, aequatio quoque generalissima non plures quoque quam duas continebit dimensiones coordinatarum *p, q* et *r*. Hinc numerus dimensionum, quas coordinatae in aequatione cuiuspiam superficiei constituunt, nobis suppeditat essentialem characterem naturae istius superficiei; propterea quod, utcunque positio coordinatarum varietur, perpetuo tamen idem dimensionum numerus emergit. Similis scilicet hic ratio circa superficies observatur, quam supra in lineis curvis deprehendimus; unde eas in certos ordines divisimus. Eodem ergo modo conveniet superficies secundum dimensiones coordinatarum in ordines disponere eritque nobis superficies ordinis primi, cuius aequatio unicam tantum dimensionem complectitur; ad ordinem secundum superficiem referemus, in cuius aequatione coordinatae ad duas dimensiones assurgunt; atque ita porro ex dimensionum numero sequentes ordines constituentur.

95. Si iam cum his conferantur ea, quae supra de inventione sectionum planarum cuiusque superficiei tradita sunt, ordinem sectionum perpetuo cum

ordine, ad quem superficies pertinet, congruere deprehendemus. Sit enim aequatio pro superficie quacunque proposita inter coordinatas x, y et z ad ordinem n pertinens, sectionis autem eius cuiusvis coordinatae normales sint t et u. Atque supra, § 85, vidimus aequationem inter t et u inveniri, si in aequatione pro superficie sequentes valores substituantur

$$x = f + t \cdot \cos.\theta - u \cdot \sin.\theta \cdot \cos.\varphi$$

et

$$y = \quad t \cdot \sin.\theta + u \cdot \cos.\theta \cdot \cos.\varphi$$

atque

$$z = u \cdot \sin.\varphi.$$

Manifestum igitur est aequationem pro sectione plures dimensiones assequi non posse, quam habebat aequatio inter x, y *et* z, sed perpetuo totidem prodituras esse dimensiones.

96. Superficies ergo primi ordinis alias sectiones a plano factas habere nequit praeter lineas primi ordinis seu rectas. Deinde ex sectione superficiei secundi ordinis aliae lineae non oriuntur nisi secundi ordinis seu sectiones conicae; est enim superficies coni ea quoque secundi ordinis, cum eius aequatio sit

$$zz = \alpha xx + \beta yy .$$

Simili modo ex superficie tertii ordinis per sectiones planas prodibunt lineae tertii ordinis, atque ita porro. Fieri tamen quandoque potest, ut aequatio pro sectione quapiam divisores admittat, quo casu sectio erit composita ex duabus pluribusve lineis inferiorum ordinum. Sic sectio coni per verticem facta constabit ex duabus lineis rectis, quae tamen coniunctim lineam secundi ordinis mentiuntur, uti supra annotavimus.

97. Constitutis igitur superficierum ordinibus investigemus prae reliquis eas superficies, quae ad ordinem primum pertinent. Aequatio ergo earum naturam exprimens erit

$$\alpha x + \beta y + \gamma z = a ,$$

cuius cum omnes sectiones plano factae sint lineae rectae, perspicuum est has superficies non planas esse non posse; si enim haberent convexitatem vel concavitatem, necessario daretur sectio curvilinea. Quanquam enim in reliquis ordinibus dantur eiusmodi superficies, quarum certae quaedam sectiones sunt lineae rectae (uti in cylindro, cono aliisque usu venire vidimus), tamen in iis sectiones curvilineae non excluduntur. Similis scilicet hic occurrit ratio, qualem in lineis observavimus; quemadmodum enim linea, quae a linea recta in pluribus uno punctis nullo modo secari potest, est necessario recta, ita superficies, quae a plano secta semper dat lineam rectam, necessario ipsa plana esse colligitur.

98. Ex aequatione autem generalissima ista indoles clarissime potest demonstrari. Formetur enim ex aequatione $\alpha x + \beta y + \gamma z = a$ aequatio generalissima inter coordinatas *p, q* et *r* secundum § 92. Et, quoniam sex novae constantes arbitrariae inducuntur, nil obstat, quominus eae ita determinentur, ut binarum coordinatarum *p* et *q* coefficientes evanescant atque huiusmodi aequatio $r = f$ remaneat, eiusdem superficiei naturam exprimens. Haec autem aequatio $r = f$ ostendet superficiem propositam esse plano, in quo binae coordinatae *p* et *q* existunt, parallelam ideoque ipsam planam. Effici quoque potest, ut fiat $r = 0$, sicque evidens erit ipsum planum, in quo *p* et *q* assumuntur, esse superficiem quaesitam.

99. Cum igitur constet superficiem aequatione $\alpha x + \beta y + \gamma z = a$ expressam esse planam, opus est, ut eius positionem respectu plani, in quo coordinatae *x* et *y* assumuntur, definiamus. Sit igitur (Fig. 142) *M* punctum quodcunque huius superficiei atque tres coordinatae $AP = x,\ PQ = y$ et $QM = z$. Ponatur primum $z = 0$ atque orietur aequatio $\alpha x + \beta y = a$, quae exprimet intersectionem superficiei quaesitae cum plano *APQ*, quam patet esse lineam rectam *BCR*, cuius positio respectu axis *AP* talis erit, ut sit recta *AB* ad axem *AP* in plano *APQ* normalis $= \frac{\alpha}{\beta}$ et $AC = \frac{\alpha}{\alpha}$, unde anguli *ACB* tangens erit $= \frac{\alpha}{\beta}$ ideoque

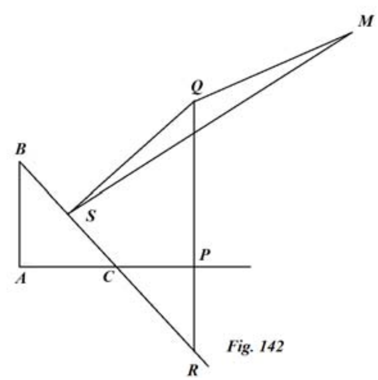

Fig. 142

$$\text{sinus} = \frac{\alpha}{\sqrt{(aa + \beta\beta)}} \quad \text{et cosinus} = \frac{\beta}{\sqrt{(aa + \beta\beta)}}.$$

Tum producatur *QP* usque ad occursum rectae *BC* in *R* atque ob $CP = x - \frac{a}{\alpha}$ erit

$$CR = \frac{x\sqrt{(aa + \beta\beta)}}{\beta} - \frac{\alpha\sqrt{(aa + \beta\beta)}}{\alpha\beta} \quad \text{et} \quad PR = \frac{\alpha x}{\beta} - \frac{\alpha}{\beta}.$$

100. Demittatur ex *Q* ad *BC* normalis *QS*, iunctaque *MS* patebit angulum *MSQ* metiri inclinationem superficiei propositae ad planum *APQ*.

Cum igitur sit $PR = \frac{\alpha x - a}{\beta}$, erit

$$QR = \frac{\alpha x + \beta y - a}{\beta} = -\frac{yz}{\beta}$$

et ob angulum $RQS = ACB$ erit

$$QS = \frac{\gamma z}{\sqrt{(aa + \beta\beta)}},$$

unde fit anguli QSM tangens $= \frac{-\sqrt{(aa+\beta\beta)}}{\gamma}$,

et propterea cosinus $= \frac{\gamma}{\sqrt{(aa+\beta\beta+\gamma\gamma)}}$.

Superficies ergo quaesita ad planum, in quo versantur x et y, inclinatur angulo, cuius tangens est $= -\frac{\sqrt{(aa+\beta\beta)}}{\gamma}$, pari vero modo eadem superficies ad planum coordinatarum x et z inclinabitur angulo, cuius tangens est $= -\frac{\sqrt{(aa+\gamma\gamma)}}{\beta}$, atque ad planum coordinatarum y et z angulo, cuius tangens est $= -\frac{\sqrt{(\beta\beta+\gamma\gamma)}}{\alpha}$.