CHAPTER III

CONCERNING THE SECTIONS OF THE CYLINDER, CONE, AND SPHERE

52. Because these bodies are accustomed to be considered in the elements of steriometry, here it will be convenient to investigate the sections of these as we may progress to less known solids. Therefore in the first place two kinds of cylinders occur in the elements, evidently of the *right* and of the *scalene* kinds. A cylinder may be called right, all the sections of which normal to the axis shall be circles equal to each other having the same centres arranged on the same line. But a *scalene* cylinder has circular sections not normal to the axis but inclined at some given angle to the axis ; which condition may be expressed more conveniently thus, as we may say an oblique or scalene cylinder to be one, of which all the sections shall be equal ellipses, the centres of which shall be placed on the same right line, which may be called the axis of the cylinder.

53. Therefore the cylinder (Fig. 130) shall be either right or scalene, of which the axis *CD* shall be placed perpendicularly on the plane of the table ; and its base

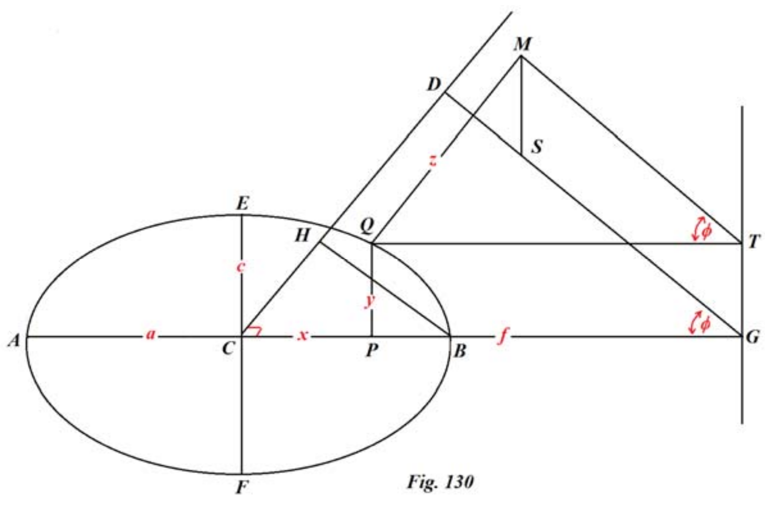

Fig. 130

AEBF or the section formed by the plane of the table shall be either a circle or an ellipse. Truly I will assume this base to be some ellipse having centre at *C* and conjugate axes *AB* and *EF*, because, what may be examined concerning the scalene cylinder, will be adapted easily to the right cylinder. Therefore the one semiaxis may be put $AC = BC = a$, truly the other $CE = CF = c$; now with the three coordinates $CP = x, PQ = y$ and $QM = z$ in place from the nature of the ellipse there will be $aacc = aayy + ccxx$; which same equation will express the nature of the cylinder, since the third variable *z* may not enter into the equation on account of all the sections equal to each other parallel to the plane *CPQ*.

54. Therefore all the sections of the cylinder parallel to the base shall be similar and equal to each other, evidently circles in the right cylinder and ellipses in the scalene cylinder. Then truly the sections, which are made along the planes normal to *APQ*, will be two right lines, parallel to each other which, and where the cylinder may be the tangent to a plane, they will merge into one ; and thus become imaginary, if the plane therefore is not crossed by the cylinder. This itself follows at once from the equation ; for if either x, y or $x \pm \alpha xy$ may be put constant at the denoted intersection of the plane and the base, then the equation will have two simple roots. And thus already we will have determined all the sections, which arise from a plane parallel to one of the three principal planes.

55. Towards investigating the nature of the remaining sections we may put the cutting plane to constitute the right line intersection *GT* with the plane of the base, which line in the first place shall be parallel either to the conjugate axis *EF* or normal to the other axis *AB* produced in *G*. With this put in place the distance CG shall be $= f$ and the inclination of the cutting plane *GTM* to the base may be measured by the angle $= \varphi$. The cutting plane *GTM* may cross to the axis of the cylinder at *D;* and with the right line *DG* drawn the angle $DGC = \varphi$, and therefore

$$DG = \frac{f}{\cos.\varphi} \text{ and } CD = \frac{f \cdot \sin.\phi}{\cos.\varphi}$$

From some point *M* of the section sought, *MT* may be drawn parallel to *DG* itself and on account of $TQ = f - x$ and the angle $QTM = \varphi$ there will be

$$TM = \frac{f - x}{\cos.\varphi} \text{ and } QM = \frac{(f - x)\cdot \sin.\phi}{\cos.\varphi} = z.$$

MS may be drawn parallel to *TG* and thus the normal to *DG*, will be

$$MS = TG = PQ = y \text{ and } DS = \frac{x}{\cos.\phi}.$$

56. Now the right lines *DS* and *SM* may be taken for the coordinates of the section sought and there shall be $DS = t$ and $SM = u$. Hence there will be $y = u$, $x = t\cdot\cos.\varphi$ and on account of

$$z = \frac{(f - x)\cdot \sin.\varphi}{\cos.\varphi} \text{ there will be } z = f\cdot \text{tang}.\varphi - t\cdot\sin.\varphi.$$

These values may be substituted into the equation for the cylinder $aacc = aayy + ccxx$ and this equation will result for the section sought

$$aacc = aauu + cctt\cos.\varphi^2,$$

which indicates the section to be an ellipse having the centre at the point *D*, of which the one principal axis falls on the line *DG*, the other truly will be normal to this. Truly the semi-axis falling on the line DG $\left(\text{by making } u = 0\right) = \frac{a}{\cos.\varphi}$. Or a right line may be drawn parallel to *GD* itself, the one semiaxis of the section sought will be $BH = \frac{a}{\cos.\varphi}$, truly the other conjugate axis will be $= c = CE$.

57. Therefore the section of the cylinder arising in this manner will be an ellipse, the conjugate semiaxes of which will be $\frac{a}{\cos.\varphi}$ and *c*. So that if therefore on the base *AEBF* the major semiaxis were $AC = a$, then on account of $\frac{a}{\cos.\varphi}$ being greater than *a* the sections will be ellipses more oblong than the base. But if *c* were greater than *a* of if the intersection *GT* were parallel to the major axis, then it can happen, that in the section both the axes may become equal to each other and thus the section of a circle may emerge. This will happen, if there should be $\frac{a}{\cos.\varphi} = c$ or $\cos.\varphi = \frac{a}{c}$. Therefore since in the triangle *BCH* with the right angle at *C*, the angle CBH shall be $= \varphi$, there will be

$$\cos.\varphi = \frac{BC}{BH} = \frac{a}{BH}.$$

Whereby, if there may be taken $BH = CE$, the sections will be circles, which since the right line BH may become $= CE$ in two ways, either by being put in place above or below [the plane containing] the right line $BH = CE$, two series of circular sections are present, which will be inclined obliquely to the axis *CD*; according to which cylinders of this kind may be called scalene.

58. Now the intersection of the cutting plane with the base shall be the right line *GT* (Fig. 131), obliquely placed in some manner, to which the perpendicular $GC = f$ may be sent from the centre of the base *C*, and putting the angle $BCG = \theta$; and the angle of inclination shall be $CGD = \varphi$, to which the angle *QTM* shall be equal, with *QT* drawn normal to *GT*. Therefore there will be

$$DG = \frac{f}{\cos.\varphi} \quad \text{and} \quad CD = \frac{f\cdot\sin.\varphi}{\cos.\varphi},$$

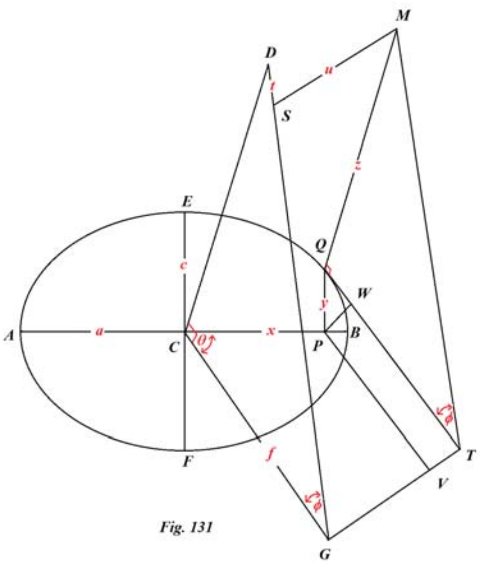

Fig. 131

M shall be a point on the section sought, from which the perpendicular *MQ* may be sent, and again to the axis *QP* thus so that, on calling $CP = x$, $PQ = y$ and $QM = z$, there shall be $aacc = aayy + ccxx$. Again the normals *PV* and *QT* may be drawn to the intersection *GT*; there will be

$$GV = x\cdot\sin.\theta, \quad PV = f - x\cdot\cos.\theta;$$

and on account of the angle $QPW = \theta$ there becomes

$$QW = y\cdot\sin.\theta, \text{ and } PW = VT = y\cdot\cos.\theta$$

et

$$QT = f - x\cdot\cos.\theta + y\cdot\sin.\theta.$$

Finally, with *MT* drawn, on account of the angle $MTQ = \varphi$ there will be

$$TM = \frac{z}{\sin.\varphi} \quad \text{and} \quad QT = \frac{z\cdot\cos.\varphi}{\sin.\varphi}.$$

59.The right-angled parallelogram *GSMT* may be drawn and on calling $DS = t$, $SM = GT = u$ there will be

$$u = GV + VT = x\cdot\sin.\theta + y\cdot\cos.\theta.$$

But on account of

$$QT = f - x\cdot\cos.\theta + y\cdot\sin.\theta$$

there will be

$$QT - CG = y \cdot \sin.\theta - x \cdot \cos.\theta,$$

from which there becomes

$$DS = TM - DG = \frac{y.\sin.\theta - x.\cos.\theta}{\cos.\varphi} = t.$$

Therefore since there shall be

$$x \cdot \sin.\theta + y \cdot \cos.\theta = u \text{ and } y \cdot \sin.\theta - x \cdot \cos.\theta = t \cdot \cos.\varphi,$$

there will be had

$$y = u \cdot \cos.\theta + t \cdot \sin.\theta \cdot cos\varphi \text{ and } x = u \cdot \sin.\theta - t \cdot \cos.\theta \cdot \cos.\varphi,$$

Which values substituted into the equation $aacc = aayy + ccxx$ in place of x and y will give

$$aacc = aauu\cos.\theta^2 + 2aaut \cdot \sin.\theta \cdot \cos.\theta \cdot \cos.\varphi + aatt\sin.\theta^2 \cdot \cos.\varphi^2$$
$$+ ccuu\sin.\theta^2 - 2ccut \cdot \sin.\theta \cdot \cos.\theta \cdot \cos.\varphi + cctt\cos.\theta^2 \cdot \cos.\varphi^2,$$

which appears to be the equation for an ellipse, the centre of which shall be D; but the coordinates DS and SM shall not be normal to the principal axes, unless there shall be $a = c$ or a right cylinder.

60. Towards inquiring more closely into this section (Fig.132), let $aMebf$ be the curve, the equation of which has been found between the coordinates $DS = t$ and $MS = u$, and for brevity's sake let this equation be

$$aacc = \alpha uu + 2\beta tu + \gamma tt,$$

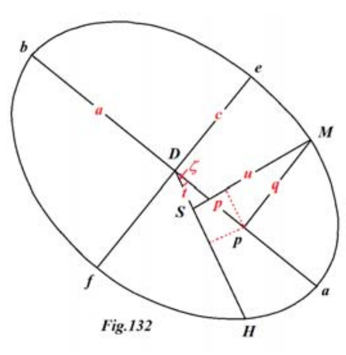

Fig.132

thus so that for the preceding case there may be had

$$\alpha = aa\cos.\theta^2 + cc\sin.\theta^2,$$

$$\beta = (aa - cc) \cdot \sin.\theta \cdot \cos.\theta \cdot \cos.\varphi,$$

and

$$\gamma = aa\sin.\theta^2\cos.\varphi^2 + cc\cos.\theta^2\cos.\varphi^2.$$

Now *ab* and *ef* shall be the conjugate principal axis of this section and with the applied line *Mp* drawn to either of these, calling $Dp = p$ and $Mp = q$ and the angle *aDH* may be put $= \zeta$; there will be

$$u = p \cdot \sin.\zeta + q \cdot \cos.\zeta \text{ and } t = p \cdot \cos.\zeta - q \cdot \sin.\zeta ,$$

and with the values substituted there becomes

$$\begin{aligned} aacc = &+\alpha \cdot \sin.\zeta^2 \cdot pp && +2\alpha \cdot \sin.\zeta \cdot \cos.\zeta \cdot pq && +\alpha \cdot \cos.\zeta^2 \cdot qq \\ &+2\beta \cdot \sin.\zeta \cdot \cos.\zeta \cdot pp && +2\beta \cdot (\cos.\zeta^2 - \sin.\zeta^2) \cdot pq && -2\beta \cdot \sin.\zeta \cdot \cos.\zeta \cdot qq \\ &+\gamma \cdot \cos.\zeta^2 \cdot pp && -2\gamma \cdot \sin.\zeta \cdot \cos.\zeta \cdot pq && +\gamma \cdot \sin.\zeta^2 \cdot qq \end{aligned}$$

61. Now since this equation may refer to an orthogonal diameter, the coefficient of pq must become $= 0$, from which on account of

$$2 \cdot \sin.\zeta \cdot \cos.\zeta = \sin.2\zeta \text{ and } \cos.\zeta^2 - \sin.\zeta^2 = \cos.2\zeta$$

the coefficient becomes

$$(\alpha - \gamma) \cdot \sin.2\zeta + 2\beta \cdot \cos.2\zeta = 0 \text{ and thus } \tang.2\zeta = \frac{2\beta}{\gamma - \alpha},$$

from which the angle *aDH*, and hence the position of the principal diameters is known. Hence in turn the semiaxes themselves are defined in this manner :

$$aD = \frac{ac}{\sqrt{(\alpha \cdot \sin.\zeta^2 + 2\beta \cdot \sin.\zeta \cdot \cos.\zeta + \gamma \cdot \cos.\zeta^2)}}$$

and

$$eD = \frac{ac}{\sqrt{(\alpha \cdot \cos.\zeta^2 - 2\beta \cdot \sin.\zeta \cdot \cos.\zeta + \gamma \cdot \sin.\zeta^2)}}.$$

62. Because

$$2\beta \text{ is equal to } \frac{2(\gamma - \alpha) \cdot \sin.\zeta \cdot \cos.\zeta}{\cos.\zeta^2 - \sin.\zeta^2},$$

with this value substituted, the expressions will be found

$$aD = \frac{ac\sqrt{(\cos.\zeta^2 - \sin.\zeta^2)}}{\sqrt{(\gamma\cdot\cos.\zeta^2 - \alpha\cdot\sin.\zeta^2)}} = \frac{ac\sqrt{2\cdot\cos.2\zeta}}{\sqrt{\left((\alpha+\gamma)\cdot\cos.2\zeta - \alpha + \gamma\right)}}$$

and

$$eD = \frac{ac\sqrt{(\cos.\zeta^2 - \sin.\zeta^2)}}{\sqrt{(\alpha\cdot\cos.\zeta^2 - \gamma\cdot\sin.\zeta^2)}} = \frac{ac\sqrt{2\cdot\cos.2\zeta}}{\sqrt{\left((\alpha+\gamma)\cdot\cos.2\zeta + \alpha - \gamma\right)}}$$

Therefore the product of these semi-axis will be

$$aD.eD = \frac{2aacc\cdot\cos.2\zeta}{\sqrt{\left(2\alpha\gamma(1+\cos.2\zeta^2) - (\alpha\alpha+\gamma\gamma)\cdot\sin.2\zeta^2\right)}}.$$

But since there shall be

$$(\gamma-\alpha)\cdot\sin.2\zeta = 2\beta\cdot\cos.2\zeta\ ,$$

this becomes [on squaring]

$$(\alpha\alpha+\gamma\gamma)\sin.2\zeta^2 = 4\beta\beta\cdot\cos.2\zeta^2 + 2\alpha\gamma\sin.2\zeta^2$$

and

$$aD.eD = \frac{2aacc\cdot\cos.2\zeta}{\sqrt{\left(4\alpha\gamma\cos.2\zeta^2 - 4\beta\beta\cos.2\zeta^2\right)}} = \frac{aacc}{\sqrt{(a\gamma-\beta\beta)}} = \frac{ac}{\cos.\varphi}.$$

63. In a similar manner, since the squares shall be

$$aD^2 = \frac{2aacc\cdot\cos.2\zeta}{(\alpha+\gamma)\cdot\cos.2\zeta - \alpha + \gamma}$$

and

$$eD^2 = \frac{2aacc\cdot\cos.2\zeta}{(\alpha+\gamma)\cdot\cos.2\zeta + \alpha - \gamma}.$$

there will be

$$aD^2 + eD^2 = \frac{4aacc\cdot(\alpha+\gamma)\cos.2\zeta^2}{4\alpha\gamma\cos.2\zeta^2 - 4\beta\beta\cos.2\zeta^2} = \frac{(\alpha+\gamma)aacc}{\alpha\gamma-\beta\beta}.$$

And hence there is elicited

$$aD + eD = \frac{ac(\alpha + \gamma + 2\sqrt{(\alpha\gamma - \beta\beta)})}{\sqrt{(\alpha\gamma - \beta\beta)}}$$

and

$$aD - eD = \frac{ac(\alpha + \gamma - 2\sqrt{(\alpha\gamma - \beta\beta)})}{\sqrt{(\alpha\gamma - \beta\beta)}}.$$

Therefore the semiaxis *aD* and *eD* will be the roots of this equation :

$$\left(\alpha\gamma - \beta\beta\right)x^4 - \left(\alpha + \gamma\right)aaccxx + a^4c^4 = 0,$$

but there is the equation

$$\sqrt{(a\gamma - \beta\beta)} = ac \cdot \cos.\varphi.$$

64. Since there shall be $aD.eD = \frac{a \cdot c}{\cos.\varphi}$ and φ shall be the angle, which the cutting plane makes with the plane of the base, hence we obtain the following elegant theorem :

THEOREM

If any cylinder may be cut by some plane, the rectangle of the axis of the section will be to the rectangle of the axes of the base of the cylinder as the secant of the angle, which the plane of the section establishes with the plane of the base, to the total sine.

Whereby, since all the parallelograms described about conjugate diameters shall be equal to the rectangles formed from the axes, also these parallelograms formed about the base and some section of the cylinder will maintain the same ratio to each other.

65. But the nature of oblique sections of this kind of a cylinder can be explained more conveniently in the following manner. If *AEBF* were the base of an elliptic cylinder (Fig. 133), the semiaxes of which $AC = BC = a,\ EC = CF = c,$ and the right line *CD* to the centre of the base *C* of the perpendicular axis of the cylinder; this cylinder will be cut by a plane, of which the intersection with the plane of the base shall be the right line *TH* produced to the axis *AB* placed obliquely, to which from *C* the perpendicular *CH* may be sent, and the angle shall be $GCH = \theta$, the cutting plane may intersect the axis of the cylinder through the point *D* ; with *DH* drawn the angle *CHD* will be the inclination of the cutting plane to the plane of the base, which angle may be called $= \varphi$. Therefore on putting $CG = f$ there will be

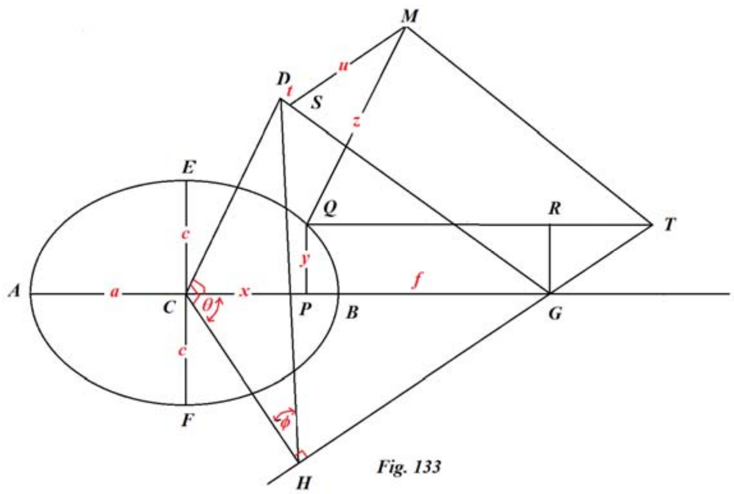

Fig. 133

$$GH = f\cdot\sin.\theta,\ CH = f\cdot\cos.\theta,\ DH = \frac{f\cdot\cos.\theta}{\cos.\varphi},\ \text{and } CD = \frac{f\cdot\cos.\theta\cdot\sin.\varphi}{\cos.\varphi}.$$

Hence on account of the triangle *DCG* right angled at *C*, there will be

$$DG = \frac{f\cdot\sqrt{(1-\sin.\theta^2\cdot\sin.\varphi^2)}}{\cos.\varphi} \text{ and the sine of the angle } DGH = \frac{\cos.\theta}{\sqrt{(1-\sin.\theta^2\cdot\sin.\varphi^2)}},$$

$$\text{the cosine} = \frac{\sin.\theta\cos.\varphi}{\sqrt{(1-\sin.\theta^2\cdot\sin.\varphi^2)}} \text{ and the tangent} = \frac{\cos.\theta}{\sin.\theta\cdot\cos.\varphi}.$$

[As $DG\cdot\sin DGH\cdot\cos\varphi = f\cdot\sqrt{(1-\sin.\theta^2\cdot\sin.\varphi^2)}\cdot\sin DGH = CH = f\cdot\cos\theta$, etc.]

66. Now from some point *M* of the section sought the perpendicular *MQ* may be sent to the base, and with the applied line drawn *QP* there shall be $GP = x,\ PQ = y$, there will be $aacc = aayy + ccxx$. *QT* may be drawn parallel to *CG* and to the same from the *G* of the normal *GR;* there will be $GR = y$ and $QR = f - x$. Therefore because the angle $TGR = GCH = \theta$, there will be

$$GT = \frac{y}{\cos.\theta} \quad \text{and} \quad TR = \frac{y \cdot \sin.\theta}{cos.\theta},$$

from which there shall become

$$QT = f - x + \frac{y \cdot \sin.\theta}{\cos.\theta}.$$

And thus on account of the similar triangles *CDG* and *QMT* there will be

$$CG : DG = QT : MT \quad \text{and} \quad CG : (CG - QT) = DG : DS,$$

with *MS* drawn parallel to *GT*. Hence there will be

$$DS = \frac{(x \cdot \cos.\theta - y \cdot \sin.\theta)\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}}{\cos.\theta \cdot \cos.\varphi}$$

Therefore on putting $DS = t,\ MS = u$, there will be

$$x \cdot \cos.\theta - y \cdot \sin.\theta = \frac{t \cdot \cos.\theta \cdot \cos.\varphi}{\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}}, \quad y = u \cdot \cos.\theta,$$

from which an equation between *t* and *u* will be found, which at this stage will be complicated enough.

67. But if now, in place of the principal axes of the base, a diameter *EF* may be drawn (Fig. 133') parallel to the intersection *TH* and conjugate to that diameter *AB*, which produced will cross with *TH* at *G*, then truly the same may remain, which we put in place before

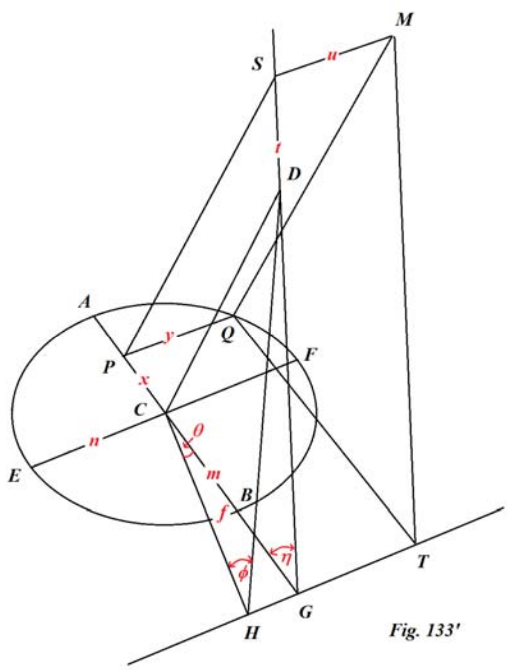

Fig. 133'

$$CG = f,\ GCH = \theta,\ CHD = \varphi,$$
$$CA = CB = m,\ CE = CF = n,$$

and it became with *QP* drawn parallel to the diameter *EF*, and on putting

$$CP = x,\ PQ = y,$$

so that there shall be $mmnn = mmyy + nnxx$, and there will be

$$GT = MS = y \text{ and } DS = \frac{DG \cdot x}{CG} = \frac{x\sqrt{(1-\sin.\theta^2 \cdot \sin.\varphi^2)}}{\cos.\varphi}.$$

Whereby, on putting $DS = t$ and $MS = u$, there becomes

$$x = \frac{t \cdot \cos.\varphi}{\sqrt{(1-\sin.\theta^2 \cdot \sin.\varphi^2)}} \text{ and } y = u,$$

truly $\frac{CG}{DG}$ will be the cosine of the angle CGD ; from which, if the angle CGD may be put $= \eta$, there will be $x = t \cdot \cos.\eta$; and thus there will be the equation for the section sought :

$$mmnn = mmuu + nntt \cdot \cos.\eta^2,$$

for the conjugate diameters, with the centre present at D ; and the semi-diameter in the direction $DS = \frac{m}{\cos.\eta}$ and the other $= n$. Truly the tangent of the angle *GSM*, to which these diameters in turn will be inclined, will be

$$= \frac{\cos.\theta}{\sin.\theta \cos.\varphi} \text{ and the cosine} = \frac{\sin.\theta \cdot \cos.\varphi}{\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}} = \sin.\theta \cos.\eta.$$

And with this agreed upon, the nature of the section will be understood easily.

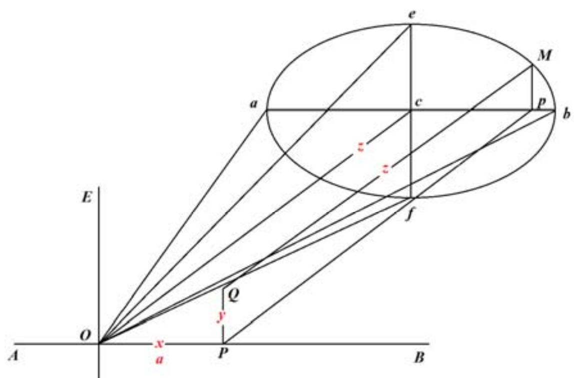

Fig. 134

68. Therefore with the sections of the cylinder established we may progress to the cone, either right or scalene ; truly in this I observe that the scalene cone only differs from the right cone, as in the scalene cone sections normal to the axis of the cone shall be ellipses having their centres on the axis of the cone, while in the right cone these sections are circles. Therefore (Fig. 134) *OaebfO* shall be some cone having the vertex at O and the axis Oc, which I put normal to the plane of the table, thus so that the table may represent a plane drawn through the vertex of the cone O and with the normal Oc to the axis of the cone. Through O in the plane of the table the right lines AB, EF may be drawn parallel to the axis ab and ef and of each normal section of the axis. So that therefore if from some point M of the section *aebf* the normal MQ may be sent to the plane of the table, and from Q to AB the perpendicular PQ, if there may be put $OP = x$, $PQ = y$, $QM = z$, also there will be the abscissa of the section $cp = x$, the applied line $pM = y$, from which, since the axes ab, ef shall maintain a constant ratio to $Oc = QM = z$, if there may be put $ac = bc = m \cdot z$ and $ec = fc = n \cdot z$, there will be

$$mmnnzz = mmyy + nnxx ,$$

which is the equation expressing the nature of the surface of a cone between the three variables *x, y* et *z.*

69. Therefore since all the sections normal to the axis *Oc* shall be ellipses, as may be apparent from the equation $mmnnzz = mmyy + nnxx$ (by attributing a constant value to *z*), in a similar manner the sections may be recognised, which will be normal either to the right line *AB* or *EF*. For if this cone may be cut by a plane normal to *AB* and by passing through the point *P*, by putting $OP = a$ the equation for this section will be had between the coordinates $Pp = z$ and $pM = y$, $mmnnzz = mmyy + nnaa$ which therefore appears to be a hyperbola having the centre at *P*, of which the transverse semi-axis will be $= \frac{a}{m}$ and the conjugate semi-axis $= \frac{na}{m}$. In a similar manner, if *y* may be put constant, the section of the normal right line *EF* may be understood to be a hyperbola having the centre on the right line *EF.*

70. If the plane (Fig. 135), by which the cone is cut, indeed shall be perpendicular to the plane *AEBF*, but truly normal to neither of the lines *AB*, *EF*, the section of the cone is defined easily. For this plane will cut the base *AEBF* along the right line *BE* and we may call $OB = a, OE = b$. Now from some point *M* of the section the normal *MQ* may be sent and from *Q* the applied line *QP*, so that there shall be $OP = x$, $PQ = y$ and $QM = z$ and from the nature of the cone $mmnnzz = mmyy + nnxx$. Therefore there will be

$$a : b = (a - x) : y \text{ or } y = b - \frac{bx}{a}.$$

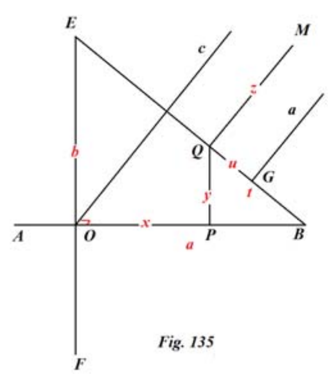

Fig. 135

The coordinates of the section may be put to be $BQ = t$ and $QM = z$, there will be

$$b : \sqrt{(aa + bb)} = y : t,$$

and thus

$$y = \frac{bt}{\sqrt{(aa + bb)}} \text{ and } a - x = \frac{at}{\sqrt{(aa + bb)}}.$$

Let $\sqrt{(aa + bb)} = c$; there will be

$$y=\frac{bt}{c},\quad x=a-\frac{at}{c}$$

and the following equation will be produced between t and z :

$$mmnncczz=mmbbtt+nnaacc-2nnaact+nnaatt.$$

Make $t-\frac{nnaac}{mmbb+nnaa}=GQ=u$, with $BG=\frac{nnaac}{mmbb+nnaa}$, and there will be

$$mmnncczz=(mmbb+nnaa)uu+\frac{mmnnaabbcc}{mmbb+nnaa}.$$

71. Therefore this section of the cone will be a hyperbola having the centre at the point G, the transverse semi-axis of which will be

$$Ga=\frac{ab}{\sqrt{(mmbb+nnaa)}}$$

and the conjugate semi-axis $=\frac{mnabc}{mmbb+nnaa}$. Truly the asymptotes of this hyperbola, which will cross the axis Ga at the centre G, will make an angle with Ga, the tangent of which is $=\frac{mnc}{\sqrt{(mmbb+nnaa)}}$. Therefore so that the hyperbola becomes equilateral, it is necessary that

$$mmnnaa+mmnnbb=mmbb+nnaa\ ,$$

or

$$\frac{b}{a}=\text{tang.}OBE=\frac{n\sqrt{(mm-1)}}{m\sqrt{(1-nn)}}.$$

Therefore unless $\frac{mm-1}{1-nn}$ shall be greater than zero, an equilateral hyperbola is unable to be formed in this way. Indeed in the right cone, where $m=n$, the tangent of the angle which the asymptotes make with the axis of the section will be $=m$ and the angle $=$ angle aOc.

72. Now let the section become oblique (Fig. 136), yet thus, so that its intersection *BT* with the plane *AEBF* shall be normal to the right line *AB*. Putting $OB = f$ and *M* the angle of inclination of the plane to the plane of the base or the angle $OBC = \varphi$, thus so that this cutting plane may cross the axis of the cone *OC* at the point *C* ; there becomes

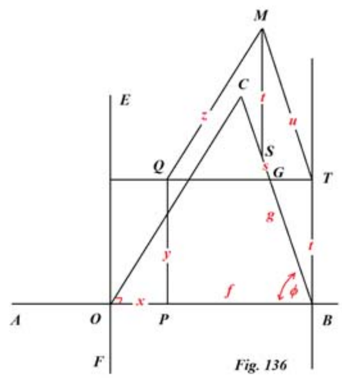

Fig. 136

$$BC = \frac{f}{\cos.\varphi} \text{ and } OC = \frac{f \cdot \sin.\varphi}{\cos.\varphi}.$$

From some point *M* of the section sought the perpendicular *MT* may be drawn to *BT* ; then truly to the plane of the base the perpendicular *MQ* and from *Q* to *OB* the normal *QP*, thus so that on putting $OP = x,\ PQ = y$ and $QM = z$ there may be had $mmnnzz = mmyy + nnxx.$ Putting $BT = t, TM = u$; for the coordinates of the section, there will be on account of the angle $QTM = \varphi,\ QM = z = u \cdot \sin.\varphi$:

$$TQ = u \cdot \cos.\varphi = f - x\ ;$$

from which there becomes :

$$y = t,\ z = u \cdot \sin.\varphi \text{ and } x = f - u \cdot \cos.\varphi$$

and thus

$$mmnnuu \cdot \sin.\varphi^2 = mmtt + nn(f - u \cdot \cos.\varphi)^2.$$

73. Putting $BC = \dfrac{f}{\cos.\varphi} = g$, so that there becomes $f = g \cdot \cos.\varphi$, there will be

$$x = (g - u) \cdot \cos.\varphi$$

and the equation will be for the section

$$mmnnuu \cdot \sin.\varphi^2 = mmtt + nngg \cdot \cos.\varphi^2 - 2nngu \cdot \cos.\varphi^2 + nnuu \cdot \cos.\varphi^2.$$

Putting

$$u - \frac{g.\cos.\varphi^2}{\cos.\varphi^2 - mm \cdot \sin.\varphi^2} = SG = s,$$

with *MS* drawn parallel to *BT*, and by taking

$$BG = \frac{g \cdot \cos.\varphi^2}{\cos.\varphi^2 - mm \cdot \sin.\varphi^2} = \frac{f \cdot \cos.\varphi}{\cos.\varphi^2 - mm \cdot \sin.\varphi^2} = \frac{f \cdot \cos.\varphi}{1-(1+mm) \cdot \sin.\varphi^2},$$

thus, so that the coordinates shall become $GS = s$ and $SM = t$, and this equation shall arise

$$mmtt + nn(\cos.\varphi^2 - mm \cdot \sin.\varphi^2)ss - \frac{mmnnff \cdot \sin.\varphi^2}{(\cos.\varphi^2 - mm \cdot \sin.\varphi^2)} = 0.$$

Therefore this equation will be the section of the cone having its centre at *G*. Therefore the equation will be for a parabola, if the centre *G* shall go to infinity, which shall happen, if $\text{tang}.\varphi = \frac{1}{m}$ or if the right line *BC* were parallel to the side of the cone *Oa* (Fig.134). Truly in this case there will be

$$mmtt + nnff - 2nnfu \cdot \cos.\varphi = 0;$$

the vertex of the parabola will be at *G* by taking $BG = \frac{f}{2\cos.\varphi}$, and the latus rectum will be (Fig. 136)

$$= \frac{2nnf \cdot \cos.\varphi}{mm}.$$

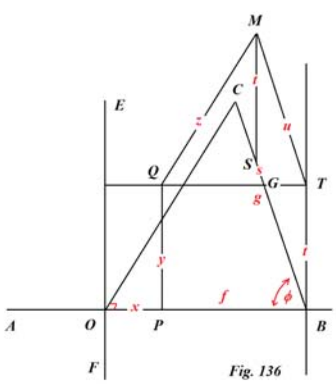

Fig. 136

74. Because the section is a parabola, if there were $\cos.\varphi^2 - mm \cdot \sin.\varphi^2 = 0$, it is evident that it becomes an ellipse, if $\cos.\varphi^2$ shall be greater than $mm \cdot \sin.\varphi^2$ or $\text{tang}.\varphi$ greater than $\frac{1}{m}$, in which case indeed the right line *BC* rises to meet the side of the cone opposite *Oa.* Therefore since there shall be

$$BG = \frac{g}{1 - mm \cdot \text{tang}.\varphi^2},$$

BG will be greater than *BC*, with *G* the centre of the section sought. Therefore the semi-axis of the section sought will be put in the direction *BC*

$$= \frac{mf \cdot \sin.\varphi}{\cos.\varphi^2 - mm \cdot \sin.\varphi^2},$$

truly the other conjugate semi-axis

$$= \frac{nf \cdot \sin.\varphi}{\sqrt{(\cos.\varphi^2 - mm \cdot \sin.\varphi^2)}},$$

and the semi-latus rectum

$$= \frac{nn}{m} f \cdot \sin.\varphi.$$

From which the section will be a circle, if there were

$$m = n\sqrt{(\cos.\varphi^2 - mm \cdot \sin.\varphi^2)} \text{ or } mm = nn - nn(1+mm) \cdot \sin.\varphi^2;$$

and hence this becomes

$$\sin.\varphi = \frac{\sqrt{(nn-mm)}}{\sqrt{(1+mm)}} = \sin.OBC \text{ and } \cos.\varphi = \frac{\sqrt{(1+nn)}}{\sqrt{(1+mm)}}.$$

Therefore unless *n* shall be greater than *m*, no section of this kind will be able to be a circle.

75. If $mm \cdot \sin.\varphi^2$ were greater than $\cos.\varphi^2$ or $\text{tang.}\,\varphi$ greater than $\frac{1}{m}$, thus so that the right line *BC* may diverge upwards from the side of the cone opposite *Oa*, then the section will be a hyperbola, of which the transverse half side will be

$$= \frac{mf \cdot \sin.\varphi}{-\cos.\varphi^2 + mm \cdot \sin.\varphi^2}$$

and the conjugate half side

$$= \frac{nf \cdot \sin.\varphi}{\sqrt{mm \cdot \sin.\varphi^2 - \cos.\varphi^2}}$$

and the semi-latus rectum

$$= \frac{nn}{m} f \cdot \sin.\varphi$$

and the tangent of the angle, within which the asymptotes cross the axis at the centre *G*, will be

$$= \frac{n}{m}\sqrt{mm \cdot \sin.\varphi^2 - \cos.\varphi^2}.$$

Whereby the hyperbola will be equilateral, if there were

$$mmnn \cdot \sin.\varphi^2 - nn \cdot \cos.\varphi^2 = mm = (mm+1)nn \cdot \sin.\varphi^2 - nn = mm,$$

or

$$\sin.\varphi = \frac{\sqrt{(mm+nn)}}{n\sqrt{(1+mm)}} \text{ and } \cos.\varphi = \frac{m\sqrt{(nn-1)}}{n\sqrt{(1+mm)}}.$$

Therefore it is necessary for this, that *n* shall be greater than one, otherwise an equiangular hyperbola cannot be produced by a section of this kind.

76. If the cone is right or $m = n$,then all the sections for these which we have established, can be referred to, because the position of the right line *AB* depends on our choice. But it remains for the scalene cone, that we may investigate the sections which may be formed by some oblique plane put in place according to the right line *AB*. Therefore (Fig.137) *BR* shall be the intersection of the cutting plane with the plane of the base *AEBF*. Putting $OB = f$, the angle $OBR = \theta$ and the angle of inclination of the cutting plane to the base $= \varphi$ there will be, by sending the perpendicular *OR* from *O* to *BR*, $OR = f \cdot \sin.\theta$ and $BR = f \cdot \cos.\theta$. Then with the right *RC* drawn in the cutting plane there will be on account of the angle $ORC = \varphi$

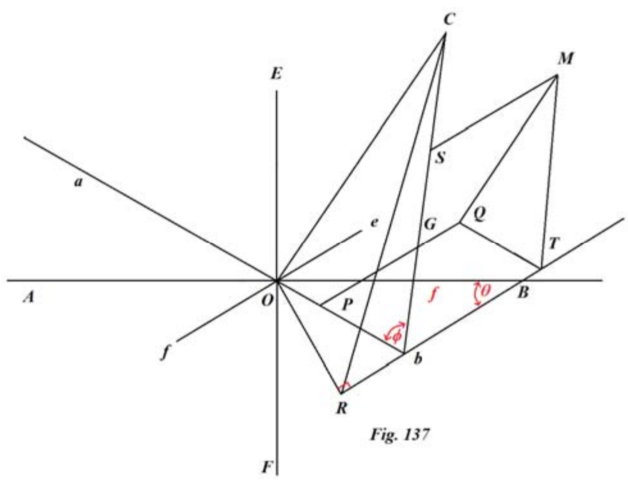

Fig. 137

$$RC = \frac{f \cdot \sin.\theta}{\cos.\varphi} \text{ and } OC = \frac{f \cdot \sin.\theta \cdot \sin.\varphi}{\cos.\varphi}.$$

Now if the section normal to the axis of the cone *OC* may be projected into the plane of the base, the principal axis of this will be along the right lines *AB* and *EF* the one will be as *m*, and the other as *n*.

77. In this projected section the diameter *ef* may be drawn parallel to *BR*: the angle *BOe* will be $= \theta$; and *aOb* shall be in the position of its conjugate diameter. The semi-diameter may be put $Oa = \mu,\ Oe = v$, there will be

$$\mu = \frac{\sqrt{m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2}}{\sqrt{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}}$$

and

$$v=\frac{mn}{\sqrt{mm\cdot\sin.\theta^2+nn\cdot\cos.\theta^2}},$$

and also

$$\text{tang}.BOb=\frac{nn\cdot\cos.\theta}{mm\cdot\sin.\theta},$$

of which therefore the sine of the angle will be

$$=\frac{nn\cdot\cos.\theta}{\sqrt{m^4\cdot\sin.\theta^2+n^4\cdot\cos.\theta^2}}$$

and the cosine

$$=\frac{mm\cdot\sin.\theta}{\sqrt{m^4\cdot\sin.\theta^2+n^4\cdot\cos.\theta^2}}.$$

Now the angle $ObR=\theta+BOb$, therefore

$$\sin.ObR=\frac{mm\cdot\sin.\theta^2+nn\cdot\cos.\theta^2}{\sqrt{m^4\cdot\sin.\theta^2+n^4\cdot\cos.\theta^2}}$$

and

$$\cos.ObR=\frac{(mm-nn)\cdot\sin.\theta\cdot\cos.\theta}{\sqrt{m^4\cdot\sin.\theta^2+n^4\cdot\cos.\theta^2}}$$

But there is:

$$\mu v=\frac{mn\cdot\sqrt{(m^4\cdot\sin.\theta^2+n^4\cdot\cos.\theta^2)}}{mm\cdot\sin.\theta^2+nn\cdot\cos.\theta^2}.$$

78. Therefore since there shall be $OR=f\cdot\sin.\theta$, there becomes

$$Ob=\frac{OR}{\sin.ObR}=\frac{f\cdot\sin.\theta\sqrt{(m^4\cdot\sin.\theta^2+n^4\cdot\cos.\theta^2)}}{mm\cdot\sin.\theta^2+nn\cdot\cos.\theta^2}$$

and

$$Rb=\frac{(mm-nn)f\cdot\sin.\theta\cdot\cos.\theta}{mm\cdot\sin.\theta^2+nn\cdot\cos.\theta^2}.$$

Hence from the triangle *RbC* with the right angle at *R* the tangent of the angle *CbR*

$$=\frac{mm\cdot\sin.\theta^2+nn\cdot\cos.\theta^2}{(mm-nn)\cdot\cos.\theta\cdot\cos.\varphi},$$

from which the angle *CbR* will be known. Now, from some point of the section *M* to the right line *RT*, *MT* may be drawn parallel to *Cb*, and from *M* to *Cb*, *MS* may be drawn parallel to *RT* itself, and they shall be called $bT=MS=t,\ bS=TM=u$, which may be seen as the oblique angled coordinates of the section sought, with the tangent of the angle present $bSM\ [=CbR]$

$$=\frac{mm\cdot\sin.\theta^2+nn\cdot\cos.\theta^2}{(mm-nn)\cdot\cos.\theta\cdot\cos.\varphi}.$$

Therefore it is apparent these coordinates become orthogonal in the right cone case, because therefore there will become $m=n$.

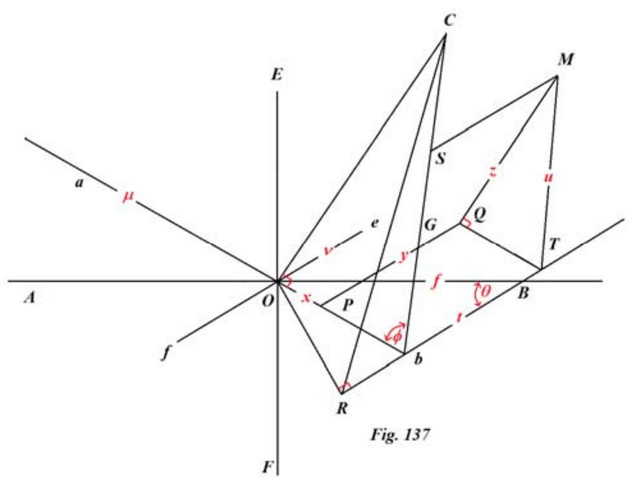

Fig. 137

79. From the point *M* of the section the perpendicular *MQ* may be sent to the plane *AEBF* and with *TQ* joined parallel to the diameter *ab;* then from *Q* the other ordinate *QP* may be drawn parallel to the other diameter *ef*. And on calling $OP=x,\ PQ=y$ and $QM=z$, from the nature of the conic there will be :

$$\mu\mu\nu\nu zz=\mu\mu yy+\nu\nu xx.$$

Now if from the point *M* a section of the cone may be considered parallel to the base, the semi-diameters *ab* and *ef* of that will be parallel to the right lines μz and νz [these lines are not shown in the diagram]. But since the sides *OC* and *Ob* of the right-angled triangle *COb* shall have been found, the hypotenuse will be

$$Cb=\frac{f\cdot\sin.\theta\sqrt{(m^4\cdot\sin.\theta^2+n^4\cdot\cos.\theta^2-(mm-nn)^2\cdot\sin.\theta^2\cdot\cos.\theta^2\cdot\sin.\varphi^2)}}{(mm\cdot\sin.\theta^2+nn\cdot\cos.\theta^2)\cdot\cos.\varphi}$$

and on account of the similar triangles *TMQ* and *bCO* there will be

$$TM\left(=u\right):TQ(=Ob-x):QM\left(=z\right)=bC:Ob:OC,$$

therefore $x = Ob - \frac{Ob \cdot u}{Cb}$, $z = \frac{OC \cdot u}{Cb}$ and $y = t$ and thus

$$\mu\mu\nu\nu \cdot OC^2 \cdot uu = \mu\mu \cdot Cb^2 \cdot tt + \nu\nu \cdot Ob^2 (Cb - u)^2.$$

80. This equation expanded out will give this :

$$0 = \mu\mu \cdot Cb^2 \cdot tt + \nu\nu(Ob^2 - \mu\mu \cdot OC^2)uu - 2\nu\nu \cdot Ob^2 \cdot Cb \cdot u + \nu\nu \cdot Ob^2 \cdot Cb^2,$$

in which if there may be put $u - \frac{Ob^2 \cdot Cb}{Ob^2 - \mu\mu \cdot OC^2} = s$, or on taking

$$bG = \frac{Ob^2 \cdot Cb}{Ob^2 - \mu\mu \cdot OC^2} = \frac{Cb}{1 - (mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2) \cdot \text{tang}.\varphi^2}$$

and on calling $GS = s$, G will be the centre of the conic section, the equation of which between the coordinates t and s will be

$$\mu\mu Cb^2 \cdot tt + \nu\nu(Ob^2 - \mu\mu \cdot OC^2)ss = \frac{\mu\mu \cdot \nu\nu \cdot Ob^2 \cdot OC^2 \cdot Cb^2}{Ob^2 - \mu\mu \cdot OC^2},$$

of which the transverse semi-diameter will be $= \frac{\mu \cdot Ob \cdot OC \cdot Cb}{Ob^2 - \mu\mu \cdot OC^2}$

and the conjugate semi-diameter $= \frac{\nu \cdot Ob \cdot OC}{\sqrt{(Ob^2 - \mu\mu \cdot OC^2)}}$

and the semi-latus rectum $= \frac{\nu\nu \cdot Ob \cdot OC}{\sqrt{(Ob^2 - \mu\mu \cdot OC^2)}}$.

But yet it is apparent, if tang. φ shall be less than $\frac{1}{\sqrt{(mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2)}}$

or tang. φ less than $\frac{\nu}{mn}$, the curve becomes an ellipse; if the equation shall be tang. $\varphi = \frac{\nu}{mn}$, a parabola, and if tang. φ shall be greater than $\frac{\nu}{mn}$, a hyperbola.

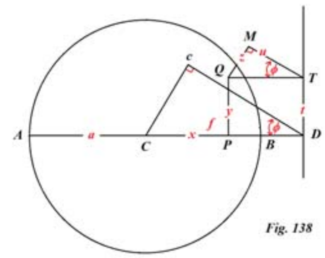

Fig. 138

81. The third body, the sections of which made by a plane we have set up to be examined here, of which indeed it is understood from elementary geometry that all the plane sections are circles. Yet meanwhile so that the method may be clearer, just as for whatever solid some sections of that must be elicited from a given equation, I may resolve the same matters here by analysis, which is accustomed to be treated commonly by synthesis. Therefore (Fig.138) *C* shall be the centre of a sphere, through which the plane of the table may be considered to pass through, thus so that the section made in this plane shall be a great circle, of which the radius $CA = CB$ may be put $= a$, which likewise will be the radius of the sphere. Again the right line *DP* shall be the intersection of the cutting plane with this plane of the table, to which the normal *CD* may be drawn from *C*, which shall be $= f$, moreover the angle of inclination shall be $= \varphi$.

82. Let *M* be some point of the section sought ; from which a perpendicular *MQ* may be sent to the plane of the table and from there the perpendicular *QP* to the right line *CD* assumed for the axis. So that if now the coordinates may be called $CP = x, PQ = y$ and $QM = z$, from the nature of the sphere there will be $xx + yy + zz = aa$. Equally from *M* the normal *MP* may be drawn to the right line *DT*, and with *QT* joined, on account of both *QT* and *MP* being normal to *DP*, the angle *MPQ* shall measure the inclination of the cutting plane to the plane of the base, which is $= \varphi$. Whereby, if *DT* and *MT* may be viewed as the coordinates of the section sought and they may be called $DT = t$, $TM = u$, there becomes $MQ = u \cdot \sin.\varphi$ and $PQ = u \cdot \cos.\varphi$.
Therefore there will be :

$$CP = x = f - u \cdot \cos.\varphi, \quad PQ = y = t \;\text{ et }\; QM = z = u \cdot \sin.\varphi .$$

With which values substituted, this equation will arise for the section of the sphere sought:

$$ff - 2fu \cdot \cos.\varphi + uu + tt = aa .$$

83. Now it is evident that the equation be for a circle. In as much as if there is put $u - f \cdot \cos.\varphi = s$, the equation will become

$$ff \cdot \sin.\varphi^2 + ss + tt = aa,$$

from which the radius of the section will be $= \sqrt{(aa - ff \cdot \sin.\varphi^2)}$. Whereby, if from D, Dc may be drawn parallel to the applied line TM, and to the same from the centre C the perpendicular Cc may be drawn, on account of $CD = f$ and the angle $CDc = \varphi$, there will be $Dc = f \cdot \cos.\varphi$ and $Cc = f \cdot \sin.\varphi$.
Hence, since the coordinates s and t may be referred to the centre, the point c will be the centre of the section and $\sqrt{(CB^2 - Cc^2)}$ the radius of this circle, as it is evident from the elements of geometry. Moreover in a similar manner, any sections made by planes of all other solids will be able to be investigated, provided the nature of these shall be expressed by an equation between the three variables.

84. Yet so that the whole operation may be seen better, some solid may be proposed (Fig. 139), the nature of which shall be expressed by an equation between the three coordinates $AP = x$, $PQ = y$ and $QM = z$, of which the first two shall be in the plane of the table, and the latter z shall be normal to the plane. Now this solid will be cut by some plane, the intersection of which with the plane of the table shall be the right line DT and the inclination of the angle $= \varphi$. Putting the right line $AD = f$, the angle $ADE = \theta$ and there will be with the perpendicular AE sent from A to DE :

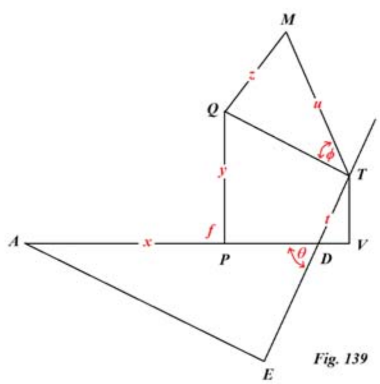

Fig. 139

$$AE = f \cdot \sin.\theta \text{ and } DE = f \cdot \cos.\theta.$$

Then from the point M of the section sought the perpendicular MP may be drawn to DT and with QT joined the angle MTQ will be equal to the given angle of inclination φ. Whereby if DP and PM may be taken for the coordinates of the section sought and they may be called $DT = t$, $TM = u$, there will be

$$QM = u \cdot \sin.\varphi \text{ and } PQ = u \cdot \cos.\varphi.$$

85. The perpendicular TV may be sent from T to the [x-] axis AD and on account of the angle $TDV = \theta$ there will be $TV = t \cdot \sin.\theta$ and $DV = t \cdot \cos.\theta$. Because again the angle TQP is $= s$, there will be

$$PV = u \cdot \sin.\theta \cdot \cos.\varphi \text{ and } PQ - TV = u \cdot \cos.\theta \cdot \cos.\varphi .$$

And thus from these the coordinates *x*, *y* and *z* may be defined in the following manner by *t* and *u*, so that there shall be

$$AP = x = f + t \cdot \cos.\theta - u \cdot \sin.\theta \cdot \cos.\varphi ;$$

and

$$PQ = y = t \cdot \sin.\theta - u \cdot \cos.\theta \cdot \cos.\varphi ;$$

and also

$$QM = z = u \cdot \sin.\varphi.$$

Whereby, if these values may be substituted into the equation between *x*, *y* and *z* for a given solid, the equation will be obtained between *t* and *u* or the coordinates of the section sought, the nature of which thus will become known. Moreover this method almost agrees with that we have used previously in § 50.

CAPUT III

DE SECTIONIBUS CYLINDRI CONI ET GLOBI

52. Quoniam haec corpora in elementis stereometriae considerari solent, eorum sectiones hic antea investigari conveniet, quam ad solida alia minus nota progrediamur. Primum igitur cylindrorum duae occurrunt species in elementis, *rectorum* scilicet ac *scalenorum.* Cylindrus *rectus* vocatur, cuius omnes sectiones ad axem normales sint circuli inter se aequales atque centra in eadem linea recta disposita habentes. Cylindrus autem *scalenus* sectiones ad axem non normales sed sub dato angulo inclinatas habet circulares; quae affectio commodius ita exprimetur, ut dicamus cylindrum obliquum seu scalenum esse, cuius omnes sectiones ad axem normales sint ellipses aequales, quarum centra in eadem linea recta, quae axis cylindri vocatur, sint posita.

53. Sit igitur (Fig. 130) cylindrus sive rectus sive scalenus, cuius axis *CD*

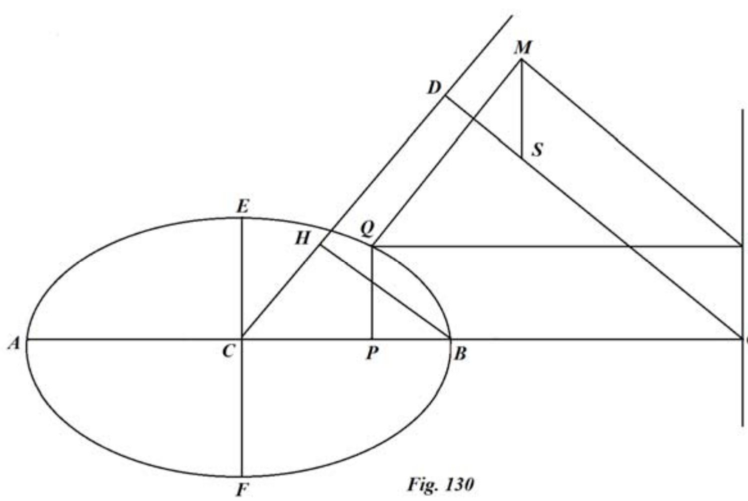

Fig. 130

perpendiculariter insistat plano tabulae; sitque eius basis *AEBF* seu sectio a plano tabulae formata vel circulus vel ellipsis. Assumam vero hanc basin esse ellipsin quamcunque centrum in *C* et axes coniugatos *AB* et *EF* habentem, quoniam, quae de cylindro scaleno tradentur, facillime ad rectum accommodabuntur. Ponatur ergo alter semiaxis $AC = BC = a$ alter vero $CE = CF = c$; positis nunc tribus coordinatis $CP = x, PQ = y$ et $QM = z$ erit ex natura ellipsis $aacc = aayy + ccxx$; quae eadem aequatio exprimet naturam cylindri, cum tertia variabilis z ob omnes sectiones plano CPQ parallelas inter se aequales in aequationem non ingrediatur.

54. Huius ergo cylindri omnes sectiones basi parallelae eidem erunt similes et aequales, scilicet circuli in cylindro recto et ellipses in scaleno. Tum vero sectiones, quae fiunt secundum plana ad APQ normalia, erunt lineae rectae, binae inter se parallelae, quae, ubi cylindrus tangetur a plano, in unum coalescent atque adeo imaginariae evadunt, si planum cylindro prorsus non occurrat. Hoc ipsum ex aequatione sponte sequitur; si enim vel x vel y vel $x \pm \alpha xy$ ponatur constans ad denotandam intersectionem plani secantis et basis, tum aequatio duas habebit radices simplices. Sicque determinavimus iam sectiones omnes, quae fiunt per plana uni trium planorum principalium parallela.

55. Ad naturam reliquarum sectionum indagandam ponamus planum secans cum plano basis intersectionem constituere rectam lineam GT, quae primo sit parallela alteri axi coniugato EF seu ad alterum AB productum in G normalis. Hoc posito sit distantia $CG = f$ et inclinatio plani secantis GTM ad basim mensuretur angulo $= \varphi$. Occurrat planum secans GTM axi cylindri in D; et ducta recta DG erit $DGC = \varphi$, ac propterea

$$DG = \frac{f}{\cos.\varphi} \text{ et } CD = \frac{f \cdot \sin.\phi}{\cos.\varphi}$$

Ex sectionis quaesitae puncto quovis M ducatur MT parallela ipsi DG atque ob $TQ = f - x$ et angulum $QTM = \varphi$ erit

$$TM = \frac{f-x}{\cos.\varphi} \text{ et } QM = \frac{(f-x)\cdot\sin.\phi}{\cos.\varphi} = z.$$

Ducatur MS parallela ipsi TG ideoque normalis in DG, erit

$$MS = TG = PQ = y \text{ et } DS = \frac{x}{\cos.\phi}.$$

56. Sumantur nunc rectae DS et SM pro coordinatis sectionis quaesitae sitque $DS = t$ et $SM = u$. Hinc erit $y = u,\ x = t\cdot\cos.\varphi$ et

$$\text{ob } z = \frac{(f-x)\cdot\sin.\varphi}{\cos.\varphi} \text{ erit } z = f\cdot\text{tang}.\varphi - t\cdot\sin.\varphi.$$

Substituantur isti valores in aequatione pro cylindro $aacc = aayy + ccxx$ atque resultabit pro sectione quaesita ista aequatio

$$aacc = aauu + cctt\cos.\varphi^2,$$

quae, indicat sectionem fore ellipsin centrum in puncto *D* habentem, cuius alter axis principalis in rectam *DG* cadat, alter vero ad hunc sit normalis. Erit vero semiaxis in rectam *DG* cadens $\left(\text{facto } u=0\right)=\dfrac{a}{\cos.\varphi}$. Vel ducatur recta parallela ipsi *GD*, erit $BH=\dfrac{a}{\cos.\varphi}$ alter semiaxis sectionis quaesitae, alter vero coniugatus erit $=c=CE$.

57. Erit ergo sectio cylindri hoc modo orta ellipsis, cuius semiaxes coniugati erunt $\dfrac{a}{\cos.\varphi}$ et *c*. Quodsi ergo in basi *AEBF* fuerit $AC=a$ semiaxis maior, tum ob $\dfrac{a}{\cos.\varphi}$ maiorem quam *a* sectiones erunt ellipses magis oblongae quam basis. Sin autem fuerit *c* maior quam *a* seu si intersectio *GT* fuerit axi maiori basis parallela, tum fieri potest, ut in sectione ambo axes fiant inter se aequales atque adeo sectio circulus evadat. Eveniet hoc, si fuerit $\dfrac{a}{\cos.\varphi}=c$ seu $\cos.\varphi=\dfrac{a}{c}$. Cum igitur sit in triangulo *BCH* ad *C* rectanagulo angulus $CBH=\varphi$, erit

$$\cos.\varphi=\frac{BC}{BH}=\frac{a}{BH}.$$

Quare, si sumatur *BH* = *CE*, sectiones erunt circuli, quod cum duplici modo fieri queat rectam $BH=CE$ sive supra sive infra constituendo, binae existent sectionum circularium series, quae ad axem *CD* oblique erunt inclinatae; ex quo huiusmodi cylindri scaleni appellantur.

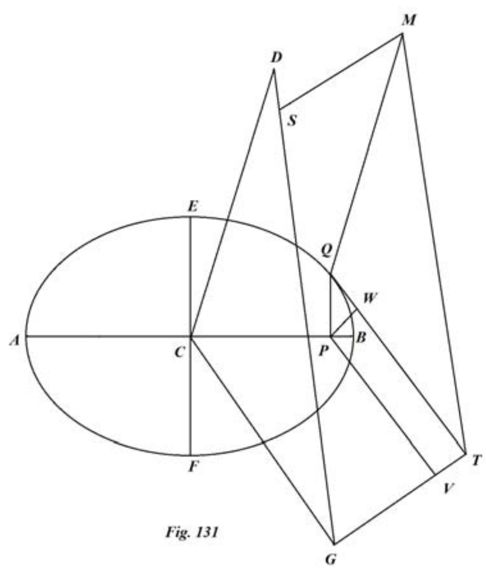

Fig. 131

58. Sit nunc (Fig. 131) recta *GT*, utcunque oblique posita, intersectio plani secantis cum basi, ad quam ex centro basis *C* demittatur perpendiculum $GC=f$, et ponatur angulus $BCG=\theta$; sitque angulus inclinationis $CGD=\varphi$, cui aequalis erit angulus *QTM*, ducta *QT* ad *GT* normali. Erit

ergo

$$DG = \frac{f}{\cos.\varphi} \quad \text{et} \quad CD = \frac{f \cdot \sin.\varphi}{\cos.\varphi},$$

Sit M punctum in sectione quaesita, unde ad basin perpendiculum MQ hincque porro ad axem QP demittatur ita ut, vocatis $CP = x$, $PQ = y$ et $QM = z$, sit $aacc = aayy + ccxx$. Ducantur porro ad intersectionem GT normales PV, QT; erit

$$GV = x \cdot \sin.\theta, \quad PV = f - x \cdot \cos.\theta;$$

et ob angulum $QPW = \theta$ fiet $QW = y \cdot \sin.\theta$, $PW = VT = y \cdot \cos.\theta$

et $QT = f - x \cdot \cos.\theta + y \cdot \sin.\theta$. Denique, ducta MT, ob angulum $MTQ = \varphi$ erit

$$TM = \frac{z}{\sin.\varphi} \quad \text{et} \quad QT = \frac{z \cdot \cos.\varphi}{\sin.\varphi}.$$

59. Compleatur parallelogrammum rectangulum $GSMT$ et vocetur $DS = t$, $SM = GT = u$ eritque

$$u = GV + VT = x \cdot \sin.\theta + y \cdot \cos.\theta.$$

At ob

$$QT = f - x \cdot \cos.\theta + y \cdot \sin.\theta$$

erit

$$QT - CG = y \cdot \sin.\theta - x \cdot \cos.\theta,$$

ex quo fit

$$DS = TM - DG = \frac{y.\sin.\theta - x.\cos.\theta}{\cos.\varphi} = t.$$

Cum igitur sit

$$x \cdot \sin.\theta + y \cdot \cos.\theta = u \quad \text{et} \quad y \cdot \sin.\theta - x \cdot \cos.\theta = t \cdot \cos.\varphi,$$

habebitur

$$y = u \cdot \cos.\theta + t \cdot \sin.\theta \cdot cos\varphi \quad \text{et} \quad x = u \cdot \sin.\theta - t \cdot \cos.\theta \cdot \cos.\varphi,$$

Qui valores in aequatione $aacc = aayy + ccxx$ loco x et y substituti dabunt

$$\begin{aligned} aacc = {} & aauu \cos.\theta^2 + 2aaut \cdot \sin.\theta \cdot \cos.\theta \cdot \cos.\varphi + aatt \sin.\theta^2 \cdot \cos.\varphi^2 \\ & + ccuu \sin.\theta^2 - 2ccut \cdot \sin.\theta \cdot \cos.\theta \cdot \cos.\varphi + cctt \cos.\theta^2 \cdot \cos.\varphi^2, \end{aligned}$$

quam aequationem patet esse ad ellipsin, cuius centrum sit in *D* ; at coordinatae *DS* et *SM* ad axes principales non sint normales, nisi sit $a = c$ seu cylindrus rectus.

60. Ad hanc sectionem propius (Fig.132) cognoscendam sit *aMebf* curva, cuius aequatio est inventa inter coordinatas $DS = t$ et $MS = u$, sitque brevitatis ergo ista aequatio

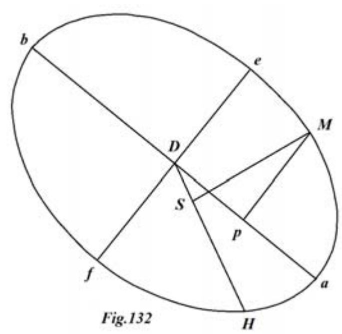

Fig.132

$$aacc = \alpha uu + 2\beta tu + \gamma tt,$$

ita ut pro casu praesente habeatur

$$\alpha = aa\cos.\theta^2 + cc\sin.\theta^2$$

et

$$\beta = (aa - cc)\cdot\sin.\theta\cdot\cos.\theta\cdot\cos.\varphi$$

atque

$$\gamma = aa\sin.\theta^2\cos.\varphi^2 + cc\cos.\theta^2\cos.\varphi^2.$$

Sint huius sectionis *ab* et *ef* axes principales coniugati ductaque ad eorum alterutrum applicata *Mp* vocetur $Dp = p$ et $Mp = q$ ac ponatur angulus, $aDH = \zeta$; erit

$$u = p\cdot\sin.\zeta + q\cdot\cos.\zeta \quad \text{et} \quad t = p\cdot\cos.\zeta - q\cdot\sin.\zeta,$$

quibus valoribus substitutis fiet

$$\begin{array}{llll} aacc = & +\alpha\cdot\sin.\zeta^2 \quad pp & +2\alpha\cdot\sin.\zeta\cdot\cos.\zeta \quad pq & +\alpha\cdot\cos.\zeta^2\cdot qq \\ & +2\beta\cdot\sin.\zeta\cdot\cos.\zeta & +2\beta\cdot(\cos.\zeta^2 - \sin.\zeta^2) & -2\beta\cdot\sin.\zeta\cdot\cos.\zeta \\ & +\gamma\cdot\cos.\zeta^2 & -2\gamma\cdot\sin.\zeta\cdot\cos.\zeta & +\gamma\cdot\sin.\zeta^2 \end{array}$$

61. Haec iam aequatio cum referatur ad diametrum orthogonalem, coefficiens ipsius *pq* debet esse $= 0$, unde ob

$$2\cdot\sin.\zeta\cdot\cos.\zeta = \sin.2\zeta \quad \text{et} \quad \cos.\zeta^2 - \sin.\zeta^2 = \cos.2\zeta$$

fiet

$$(\alpha - \gamma)\cdot\sin.2\zeta + 2\beta\cdot\cos.2\zeta = 0 \quad \text{ideoque} \quad \text{tang}.2\zeta = \frac{2\beta}{\gamma - \alpha},$$

unde angulus *aDH* ac proinde positio diametrorum principalium cognoscitur. Hinc porro ipsi semiaxes definiuntur hoc modo

$$aD = \frac{ac}{\sqrt{(\alpha \cdot \sin.\zeta^2 + 2\beta \cdot \sin.\zeta \cdot \cos.\zeta + \gamma \cdot \cos.\zeta^2)}}$$

et

$$eD = \frac{ac}{\sqrt{(\alpha \cdot \cos.\zeta^2 - 2\beta \cdot \sin.\zeta \cdot \cos.\zeta + \gamma \cdot \sin.\zeta^2)}}.$$

62. Quia est

$$2\beta = \frac{2(\gamma - \alpha) \cdot \sin.\zeta \cdot \cos.\zeta}{\cos.\zeta^2 - \sin.\zeta^2}$$

erit valore hoc in expressionibus inventis substitute

$$aD = \frac{ac\sqrt{(\cos.\zeta^2 - \sin.\zeta^2)}}{\sqrt{(\gamma \cdot \cos.\zeta^2 - \alpha \cdot \sin.\zeta^2)}} = \frac{ac\sqrt{2 \cdot \cos.2\zeta}}{\sqrt{((\alpha+\gamma) \cdot \cos.2\zeta - \alpha + \gamma)}}$$

et

$$eD = \frac{ac\sqrt{(\cos.\zeta^2 - \sin.\zeta^2)}}{\sqrt{(\alpha \cdot \cos.\zeta^2 - \gamma \cdot \sin.\zeta^2)}} = \frac{ac\sqrt{2 \cdot \cos.2\zeta}}{\sqrt{((\alpha+\gamma) \cdot \cos.2\zeta + \alpha - \gamma)}}$$

Horum ergo semiaxium productum erit

$$aD.eD = \frac{2aacc \cdot \cos.2\zeta}{\sqrt{\left(2\alpha\gamma(1+\cos.2\zeta^2) - (\alpha\alpha + \gamma\gamma) \cdot \sin.2\zeta^2\right)}}.$$

At cum sit

$$(\gamma - \alpha) \cdot \sin.2\zeta = 2\beta \cdot \cos.2\zeta ,$$

erit

$$(\alpha\alpha + \gamma\gamma)\sin.\ 2\zeta^2 = 4\beta\beta \cdot \cos.2\zeta^2 + \ 2\alpha\gamma\sin.\ 2\zeta^2$$

ideoque

$$aD.eD = \frac{2aacc \cdot \cos.2\zeta}{\sqrt{\left(4\alpha\gamma\cos.2\zeta^2 - 4\beta\beta\cos.2\zeta^2\right)}} = \frac{aacc}{\sqrt{(a\gamma - \beta\beta)}} = \frac{ac}{\cos.\varphi}.$$

63. Simili modo, cum sint quadrata

$$aD^2 = \frac{2aacc \cdot \cos.2\zeta}{(\alpha+\gamma)\cdot\cos.2\zeta - \alpha + \gamma}$$

et

$$eD^2 = \frac{2aacc \cdot \cos.2\zeta}{(\alpha+\gamma)\cdot\cos.2\zeta + \alpha - \gamma}.$$

erit

$$aD^2 + eD^2 = \frac{4aacc\cdot(\alpha+\gamma)\cos.2\zeta^2}{4\alpha\gamma\cos.2\zeta^2 - 4\beta\beta\cos.2\zeta^2} = \frac{(\alpha+\gamma)aacc}{\alpha\gamma-\beta\beta}.$$

Hincque elicitor

$$aD + eD = \frac{ac(\alpha+\gamma+2\sqrt{(\alpha\gamma-\beta\beta)})}{\sqrt{(\alpha\gamma-\beta\beta)}}$$

et

$$aD - eD = \frac{ac(\alpha+\gamma-2\sqrt{(\alpha\gamma-\beta\beta)})}{\sqrt{(\alpha\gamma-\beta\beta)}}.$$

Semiaxes ergo *aD* et *eD* erunt radices huius aequationis

$$\left(\alpha\gamma-\beta\beta\right)x^4 - \left(\alpha+\gamma\right)aaccxx + a^4c^4 = 0,$$

at est

$$\sqrt{(a\gamma-\beta\beta)} = ac\cdot\cos.\varphi.$$

64. Cum sit $aD.eD = \dfrac{ac}{\cos.\varphi}$ atque φ sit angulus, quem planum secans cum plano basis constituit, hinc sequens elegans theorema consequimur:

THEOREMA

Si cylindrus quicunque secetur plano quocunque, erit rectangulum axium sectionis ad rectangulum axium basis cylindri uti secans anguli, quem planum sectionis cum plano basis constituit, ad sinum totum.

Quare cum omnia parallelogramma circa diametros coniugatas descripta aequalia sint rectangulis ex axibus formatis, etiam parallelogramma ista circa basin et sectionem quamcunque cylindri formata eandem inter se tenebunt rationem.

65. Natura autem huiusmodi sectionum obliquarum cylindri commodius sequenti modo definiri poterit. Si fuerit basis cylindri ellipsis (Fig. 133) *AEBF*, cuius

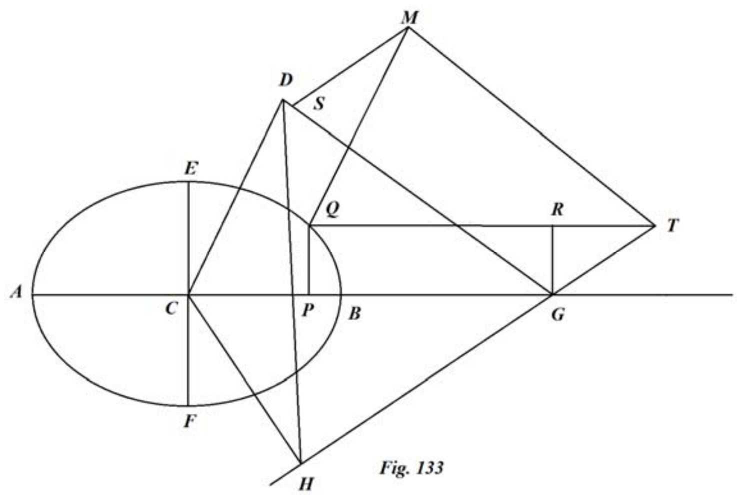

Fig. 133

semiaxes $AC = BC = a,\ EC = CF = c$, atque recta *CD* ad centrum basis *C* perpendicularis axis cylindri, secetur iste cylindrus plano, cuius cum plano basis intersectio sit recta *TH* ad axem *AB* productum utcunque oblique posita, ad quam ex *C* perpendiculum demittatur *CH*, sitque angulus $GCH = \theta$, transeat planum secans per axis cylindri punctum *D*, erit ducta *DH* angulus *CHD* inclinatio plani secantis ad planum basis, qui angulus vocetur $= \varphi$. Posita ergo $CG = f$ erit

$$GH = f \cdot \sin.\theta,\ CH = f \cdot \cos.\theta,\ DH = \frac{f \cdot \cos.\theta}{\cos.\varphi},\ \text{et } CD = \frac{f \cdot \cos.\theta \cdot \sin.\varphi}{\cos.\varphi}.$$

Hinc ob triangulum *DCG* ad *C* rectangulum erit

$$DG = \frac{f \cdot \sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}}{\cos.\varphi} \text{ et anguli } DGH \text{ sinus } = \frac{\cos.\theta}{\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}},$$

$$\text{cosinus} = \frac{\sin.\theta \cos.\varphi}{\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}} \text{ et tangens} = \frac{\cos.\theta}{\sin.\theta \cdot \cos.\varphi}.$$

66. Iam ex sectionis quaesitae puncto quovis *M* in basin demittatur perpendicularis *MQ* ductaque applicata *QP* sit $GP = x,\ PQ = y$, erit $aacc = aayy + ccxx$. Ducatur *QT* ipsi *CG* parallela in eamque ex *G* normalis *GR;* erit $GR = y$ et $QR = f - x$. Quoniam igitur angulus $TGR = GCH = \theta$, erit

$$GT = \frac{y}{\cos.\theta} \text{ et } TR = \frac{y \cdot \sin.\theta}{cos.\theta},$$

unde fit

$$QT = f - x + \frac{y \cdot \sin.\theta}{\cos.\theta}.$$

Ideoque ob triangula *CDG* et *QMT* similia erit

$$CG : DG = QT : MT \quad \text{et} \quad CG : (CG - QT) = DG : DS,$$

ducta *MS* parallela *GT*. Hinc erit

$$DS = \frac{(x \cdot \cos.\theta - y \cdot \sin.\theta)\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}}{\cos.\theta \cdot \cos.\varphi}$$

Positis ergo $DS = t$, $MS = u$, erit

$$x \cdot \cos.\theta - y \cdot \sin.\theta = \frac{t \cdot \cos.\theta \cdot \cos.\varphi}{\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}}, \quad y = u \cdot \cos.\theta,$$

unde aequatio inter *t* et *u* reperietur, quae adhuc erit satis complicata.

67. Quodsi autem loco axium principalium basis ducatur diameter *EF* (Fig. 133') intersectioni *TH* parallela ad eamque diameter coniugata *AB*, quae producta ipsi *TH* occurrat in *G*, tum vero maneant eadem, quae ante posuimus

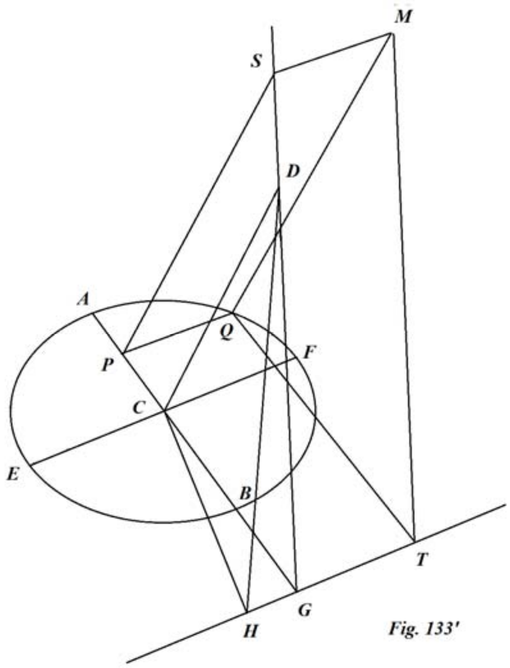

Fig. 133'

$$CG = f,\ GCH = \theta,\ CHD = \varphi,$$
$$CA = CB = m,\ CE = CF = n,$$

fueritque ducta *QP* diametro *EF* parallela, et positis

$$CP = x,\ PQ = y,$$

ut sit $mmnn = mmyy + nnxx$, erit

$$GT = MS = y \ \text{ et } \ DS = \frac{DG \cdot x}{CG} = \frac{x\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}}{\cos.\varphi}.$$

Quare, positis $DS = t$ et $MS = u$, fiet

$$x = \frac{t \cdot \cos.\varphi}{\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}} \ \text{ et } \ y = u,$$

erit vero $\frac{CG}{DG}$ cosinus anguli *CGD;* unde, si ponatur angulus $CGD = \eta$, erit $x = t \cdot \cos.\eta$; ideoque pro sectione quaesita erit

$$mmnn = mmuu + nntt \cdot \cos.\eta^2,$$

ad diametros coniugatas, centra existente in *D*; eritque semidiameter in directione $DS = \frac{m}{\cos.\eta}$ et alter $= n$. Anguli vero, quo hae diametri invicem inclinantur, *GSM*

$$\text{tangens erit} = \frac{\cos.\theta}{\sin.\theta \cos.\varphi} \ \text{ et cosinus} = \frac{\sin.\theta \cdot \cos.\varphi}{\sqrt{(1 - \sin.\theta^2 \cdot \sin.\varphi^2)}} = \sin.\theta \cos.\eta.$$

Hocque pacto natura sectionis facillime cognoscitur.

68. Expositis ergo sectionibus cylindri ad conum progrediamur sive rectum sive scalenum; eo vero tantum conum scalenum a recto differre considero, quod in scaleno sectiones ad axem coni normales sint ellipses sua centra in axe coni habentes, dum in recto hae sectiones sunt circuli. Sit igitur (Fig. 134) *OaebfO*

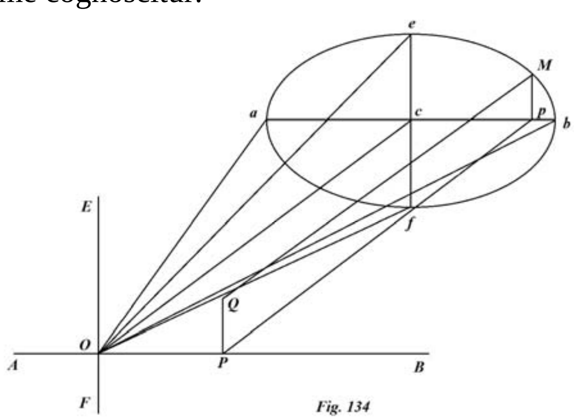

Fig. 134

conus quicunque verticem in O et axem Oc habens, quem ad planum tabulae pono normalem, ita ut tabula repraesentet planum per coni verticem O ductum et ad axem coni Oc normale. Ducantur per O in plano tabulae rectae AB, EF axibus ab et ef cuiusque sectionis axi normalis parallelae. Quodsi ergo ex sectionis $aebf$ puncto quocunque M ad planum tabulae demittatur normalis MQ et ex Q ad AB perpendiculum PQ, si ponantur $OP = x,\ PQ = y,\ QM = z$, erit quoque sectionis abscissa $cp = x$, applicata $pM = y$, unde, cum axes ab, ef ad $Oc = QM = z$ constantem teneant rationem, si ponatur $ac = bc = mz$ et $ec = fc = n \cdot z$, erit

$$mmnnzz = mmyy + nnxx,$$

quae est aequatio naturam superficiei conicae exprimens inter tres variabiles x, y et z.

69. Cum igitur omnes sectiones axi Oc normales sint ellipses, uti ex aequatione $mmnnzz = mmyy + nnxx$ (tribuendo ipsi z valorem constantem) apparet, simili modo facile cognoscentur sectiones, quae vel ad rectam AB vel EF erunt normales. Si enim iste conus secetur plano ad AB normali et per punctum P transeunte, posito $OP = a$ ista pro sectione habebitur aequatio $mmnnzz = mmyy + nnaa$ inter coordinatas $Pp = z$ et $pM = y$, quam propterea patet esse hyperbolam centrum in P habentem, cuius semiaxis transversus erit $= \frac{a}{m}$ et semi axis coniugatus $= \frac{na}{m}$. Pari modo, si y ponatur constans, sectio rectae EF normalis intelligetur esse hyperbola centrum habens in ipsa recta EF.

70. Si planum (Fig. 135), quo conus secatur, sit quidem perpendiculare ad planum $AEBF$, at vero ad neutram linearum AB, EF normale, facile quoque sectio coni definitur. Secet enim hoc planum basin $AEBF$ recta BE ac vocetur $OB = a, OE = b$. Iam ex puncto sectionis quovis M demittatur normalis MQ et ex Q applicata QP, ut sit $OP = x,\ PQ = y$ et $QM = z$ atque ex natura coni $mmnnzz = mmyy + nnxx$. Erit ergo

$$a : b = (a - x) : y \quad \text{seu} \quad y = b - \frac{bx}{a}.$$

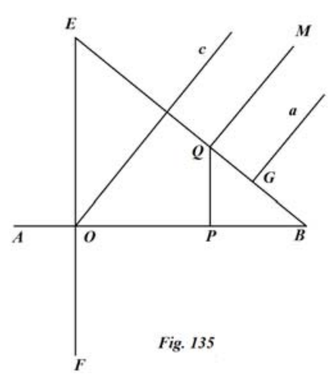

Fig. 135

Ponantur sectionis coordinatae $BQ = t$ et $QM = z$, erit

$$b : \sqrt{(aa + bb)} = y : t,$$

ideoque

$$y = \frac{bt}{\sqrt{(aa+bb)}} \text{ et } a - x = \frac{at}{\sqrt{(aa+bb)}}.$$

Sit $\sqrt{(aa+bb)} = c$; erit

$$y = \frac{bt}{c}, \quad x = a - \frac{at}{c}$$

atque prodibit inter t et z sequens aequatio

$$mmnncczz = mmbbtt + nnaacc - 2nnaact + nnaatt.$$

Fiat $t - \dfrac{nnaac}{mmbb+nnaa} = GQ = u$, existente $BG = \dfrac{nnaac}{mmbb+nnaa}$, et erit

$$mmnncczz = (mmbb+nnaa)uu + \frac{mmnnaabbcc}{mmbb+nnaa}.$$

71. Erit ergo haec coni sectio hyperbola centrum habens in puncto G, cuius semiaxis transversus erit

$$Ga = \frac{ab}{\sqrt{(mmbb+nnaa)}}$$

et semiaxis coniugatus $= \dfrac{mnabc}{mmbb+nnaa}$. Asymptotae vero huius hyperbolae, quae axem Ga in centro G decussabunt, cum axe Ga facient angulum, cuius tangens est $= \dfrac{mnc}{\sqrt{(mmbb+nnaa)}}$. Quo ergo sectio fiat hyperbola aequilatera, oportet esse

$$mmnnaa + mmnnbb = mmbb + nnaa,$$

seu

$$\frac{b}{a} = \text{tang.}OBE = \frac{n\sqrt{(mm-1)}}{m\sqrt{(1-nn)}}.$$

Nisi ergo sit $\frac{mm-1}{1-nn}$ maior nihilo, hyperbola aequilatera hoc modo oriri nequit. In cono recto quidem, ubi est $m=n$, anguli, quem asymptotae cum axe sectionis constituunt, tangens erit $=m$ et angulus $=$ angulo aOc.

72. Sit nunc (Fig. 136) sectio obliqua, ita tamen, ut eius intersectio *BT* cum plano *AEBF* sit normalis ad rectam *AB*. Ponatur $OB=f$ et *M* angulus inclinationis plani ad planum basis seu angulus $OBC=\varphi$, ita ut hoc planum secans axem coni *OC* in puncto *C* traiiciat; erit

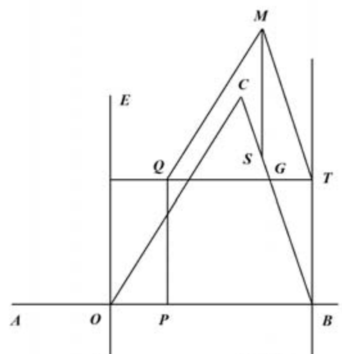

Fig. 136

$$BC=\frac{f}{\cos.\varphi} \text{ et } OC=\frac{f\cdot\sin.\varphi}{\cos.\varphi}.$$

Ex sectionis quaesitae puncto quovis *M* ad *BT* ducatur perpendicularis *MT;* tum vero ad planum basis perpendiculum *MQ* et ex *Q* ad *OB* normalis *QP*, ita ut positis $OP=x,\ PQ=y$ et $QM=z$ habeatur $mmnnzz=mmyy+nnxx.$ Ponantur pro sectione coordinatae $BT=t, TM=u$; erit ob angulum

$$QTM=\varphi,\ QM=z=u\cdot\sin.\varphi$$
$$TQ=u\cdot\cos.\varphi=f-x\ ;$$

unde fit

$$y=t,\ z=u\cdot\sin.\varphi \text{ et } x=f-u\cdot\cos.\varphi$$

ideo que

$$mmnnuu\cdot\sin.\varphi^2=mmtt+nn(f-u\cdot\cos.\varphi)^2.$$

73. Ponatur $BC=\frac{f}{\cos.\varphi}=g$, ut fiat $f=g\cdot\cos.\varphi$, erit

$$x=(g-u)\cdot\cos.\varphi$$

atque pro sectione erit

$$mmnnuu\cdot\sin.\varphi^2=mmtt+nngg\cdot\cos.\varphi^2-2nngu\cdot\cos.\varphi^2+nnuu\cdot\cos.\varphi^2.$$

Statuatur

$$u-\frac{g.\cos.\varphi^2}{\cos.\varphi^2-mm\cdot\sin.\varphi^2}=SG=s,$$

ducta *MS* parallela ipsi *BT* sumtaque

$$BG = \frac{g \cdot \cos.\varphi^2}{\cos.\varphi^2 - mm \cdot \sin.\varphi^2} = \frac{f \cdot \cos.\varphi}{\cos.\varphi^2 - mm \cdot \sin.\varphi^2} = \frac{f \cdot \cos.\varphi}{1-(1+mm)\cdot \sin.\varphi^2},$$

ita ut coordinatae sint $GS = s$ et $SM = t$, atque nascetur haec aequatio

$$mmtt + nn(\cos.\varphi^2 - mm \cdot \sin.\varphi^2)ss - \frac{mmnnff \cdot \sin.\varphi^2}{(\cos.\varphi^2 - mm \cdot \sin.\varphi^2)} = 0.$$

Erit ergo curva sectio conica centrum habens in *G*. Eritque ergo parabola, si centrum *G* in infinitum abit, quod fit, si $\tang.\varphi = \frac{1}{m}$ seu si recta *BC* fuerit lateri coni *Oa* (Fig.134) parallela. Hoc vero casu erit

$$mmtt + nnff - 2nnfu \cdot \cos.\varphi = 0;$$

vertex parabolae erit in *G* sumta $BG = \frac{f}{2\cos.\varphi}$, et latus rectum erit (Fig. 136)

$$= \frac{2nnf \cdot \cos.\varphi}{mm}.$$

74. Quoniam sectio est parabola, si fuerit $\cos.\varphi^2 - mm \cdot \sin.\varphi^2 = 0$, manifestum est eam fore ellipsin, si sit $\cos.\varphi^2$ maior quam $mm \cdot \sin.\varphi^2$ seu $\tang.\varphi$ maior quam $\frac{1}{m}$, quo quidem casu recta *BC* sursum converget cum latere coni opposito *Oa*. Cum igitur sit

$$BG = \frac{g}{1 - mm \cdot \tang.\varphi^2},$$

erit *BG* maior quam *BC*, existente *G* sectionis quaesitae centro. Erit ergo sectionis quaesitae semiaxis in directione *BC* positus

$$= \frac{mf \cdot \sin.\varphi}{\cos.\varphi^2 - mm \cdot \sin.\varphi^2},$$

alter vero semiaxis coniugatus

$$= \frac{nf \cdot \sin.\varphi}{\sqrt{(\cos.\varphi^2 - mm \cdot \sin.\varphi^2)}},$$

et semilatus rectum

$$= \frac{nn}{m} f \cdot \sin.\varphi.$$

Unde sectio erit circulus, si fuerit

$$m = n\sqrt{(\cos.\varphi^2 - mm\cdot\sin.\varphi^2)} \text{ seu } mm = nn - nn(1+mm)\cdot\sin.\varphi^2;$$

hincque fit

$$\sin.\varphi = \frac{\sqrt{(nn-mm)}}{\sqrt{(1+mm)}} = \sin.OBC \text{ et } \cos.\varphi = \frac{\sqrt{(1+nn)}}{\sqrt{(1+mm)}}.$$

Nisi ergo sit *n* maior quam *m*, nulla huiusmodi sectio esse poterit circulus.

75. Si fuerit $mm\cdot\sin.\varphi^2$ maior quam $\cos.\varphi^2$ seu tang. φ maior quam $\frac{1}{m}$, ita ut recta *BC* cum latere coni opposito *Oa* sursum divergat, sectio erit hyperbola, cuius semilatus transversum erit

$$= \frac{mf\cdot\sin.\varphi}{-\cos.\varphi^2 + mm\cdot\sin.\varphi^2}$$

et semilatus coniugatum

$$= \frac{nf\cdot\sin.\varphi}{\sqrt{mm\cdot\sin.\varphi^2 - \cos.\varphi^2}}$$

ac semilatus rectum

$$= \frac{nn}{m} f\cdot\sin.\varphi$$

et anguli, sub quo asymptotae axem in centro *G* decussant, tangens erit

$$= \frac{n}{m}\sqrt{mm\cdot\sin.\varphi^2 - \cos.\varphi^2}.$$

Quare hyperbola erit aequilatera, si fuerit

$$mmnn\cdot\sin.\varphi^2 - nn\cdot\cos.\varphi^2 = mm = (mm+1)nn\cdot\sin.\varphi^2 - nn = mm,$$

seu

$$\sin.\varphi = \frac{\sqrt{(mm+nn)}}{n\sqrt{(1+mm)}} \text{ et } \cos.\varphi = \frac{m\sqrt{(nn-1)}}{n\sqrt{(1+mm)}}.$$

Ad hoc ergo necesse est, ut sit *n* maior unitate, alioquin hyperbola aequilatera per sectionem huiusmodi produci nequit.

76. Si conus est rectus seu $m = n$, tum omnes sectiones ad has, quas evolvimus, referri possunt, quia positio rectae *AB* ab arbitrio nostro pendet. At pro cono scaleno superest, ut investigemus sectiones, quae a plana utcunque oblique ad rectam *AB* posito formantur. Sit igitur (Fig.137) *BR* intersectio plani secantis cum plano basis *AEBF*. Ponatur $OB = f$, angulus $OBR = \theta$ et angulus inclinationis secantis ad basin $= \varphi$ erit demisso ex *O* in *BR* perpendiculo *OR*, $OR = f \cdot \sin.\theta$ et $BR = f \cdot \cos.\theta$. Tum ducta in plano secante recta *RC* erit ob angulum $ORC = \varphi$

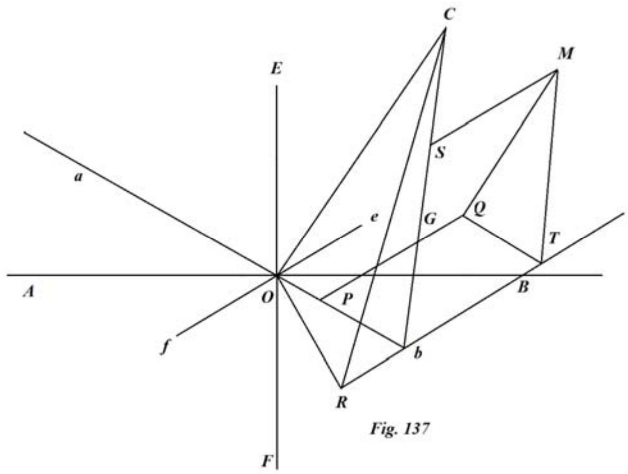

Fig. 137

$$RC = \frac{f \cdot \sin.\theta}{\cos.\varphi} \text{ et } OC = \frac{f \cdot \sin.\theta \cdot \sin.\varphi}{\cos.\varphi}.$$

Si iam sectio ad axem coni *OC* normalis in planum basis proiiciatur, erunt eius axes principales secundum rectas *AB* et *EF* dispositi alterque erit ut *m*, alter ut *n*.

77. In hac sectione proiecta ducatur diameter *ef* parallela ipsi *BR*: erit angulus $BOe = \theta$; sitque *aOb* positio diametri eius coniugatae. Ponatur semidiameter $Oa = \mu$, $Oe = \nu$, erit

$$\mu = \frac{\sqrt{m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2}}{\sqrt{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}}$$

et

$$\nu = \frac{mn}{\sqrt{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}},$$

atque

$$\text{tang.}BOb = \frac{nn \cdot \cos.\theta}{mm \cdot \sin.\theta},$$

cuius anguli propterea erit

$$\text{sinus} = \frac{nn \cdot \cos.\theta}{\sqrt{m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2}}$$

et

$$\text{cosinus} = \frac{mm \cdot \sin.\theta}{\sqrt{m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2}}.$$

Iam est angulus $ObR = \theta + BOb$, ergo

$$\sin.ObR = \frac{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}{\sqrt{m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2}}$$

et

$$\cos.ObR = \frac{(mm - nn) \cdot \sin.\theta \cdot \cos.\theta}{\sqrt{m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2}}$$

At est

$$\mu\nu = \frac{mn \cdot \sqrt{(m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2)}}{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}.$$

78. Cum igitur sit $OR = f \cdot \sin.\theta$, erit

$$Ob = \frac{OR}{\sin.ObR} = \frac{f \cdot \sin.\theta \sqrt{(m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2)}}{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}$$

et

$$Rb = \frac{(mm - nn) f \cdot \sin.\theta \cdot \cos.\theta}{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}.$$

Hinc ex triangulo *RbC* ad *R* rectangulo erit anguli *CbR*

$$\text{tangens} = \frac{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}{(mm - nn) \cdot \cos.\theta \cdot \cos.\varphi},$$

unde angulus *CbR* erit cognitus. Iam, ex puncto sectionis quovis *M* ad rectam *RT* ducatur *MT* parallela ipsi *Cb*, atque ex *M* ad *Cb* parallela *MS* ipsi *RT* vocenturque $bT = MS = t$, $bS = TM = u$, quae tanquam coordinatae obliquangulae sectionis quaesitae spectentur, existente

$$\text{anguli } bSM \ [= CbR] \text{ tang.} = \frac{mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2}{(mm - nn) \cdot \cos.\theta \cdot \cos.\varphi}.$$

Patet ergo has coordinatas fieri orthogonales in cono recto, propterea quia fit $m = n$.

79. Ex puncto sectionis *M* ad planum *AEBF* demittatur perpendiculum *MQ* iunctaque *TQ* erit parallela diametro *ab;* tum ex *Q* ducatur ordinata *QP* alteri diametro *ef* parallela. Atque vocatis $OP = x$, $PQ = y$ et $QM = z$ erit ex natura coni

$$\mu\mu\nu\nu zz = \mu\mu yy + \nu\nu xx.$$

Namque si per punctum *M* concipiatur coni sectio basi parallela, erunt eius semidiametri rectis *ab* et *ef* parallelae μz et νz. At cum inventa sint trianguli rectanguli *COb* latera *OC* et *Ob,* erit hypothenusa

$$Cb = \frac{f \cdot \sin.\theta \sqrt{(m^4 \cdot \sin.\theta^2 + n^4 \cdot \cos.\theta^2 - (mm - nn)^2 \cdot \sin.\theta^2 \cdot \cos.\theta^2 \cdot \sin.\varphi^2)}}{(mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2) \cdot \cos.\varphi}$$

et ob triangula *TMQ*, *bCO* similia erit

$$TM\,(= u) : TQ (= Ob - x) : QM\,(= z) = bC : Ob : OC, ,$$

ergo $x = Ob - \dfrac{Ob \cdot u}{Cb}$, $z = \dfrac{OC \cdot u}{Cb}$ et $y = t$ ideoque

$$\mu\mu\nu\nu \cdot OC^2 \cdot uu = \mu\mu \cdot Cb^2 \cdot tt + \nu\nu \cdot Ob^2 (Cb - u)^2.$$

80. Aequatio haec evoluta dabit hanc

$$0 = \mu\mu \cdot Cb^2 \cdot tt + \nu\nu (Ob^2 - \mu\mu \cdot OC^2) uu - 2\nu\nu \cdot Ob^2 \cdot Cb \cdot u + \nu\nu \cdot Ob^2 \cdot Cb^2,$$

in qua si ponatur $u - \dfrac{Ob^2 \cdot Cb}{Ob^2 - \mu\mu \cdot OC^2} = s$ seu sumta

$$bG = \frac{Ob^2 \cdot Cb}{Ob^2 - \mu\mu \cdot OC^2} = \frac{Cb}{1 - (mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2) \cdot \text{tang.}\varphi^2}$$

et vocata $GS = s$, erit G centrum sectionis conicae, cuius aequatio inter coordinatas t et s erit

$$\mu\mu Cb^2 \cdot tt + vv(Ob^2 - \mu\mu \cdot OC^2)ss = \frac{\mu\mu \cdot vv \cdot Ob^2 \cdot OC^2 \cdot Cb^2}{Ob^2 - \mu\mu \cdot OC^2},$$

cuius semidiameter transversus erit $= \dfrac{\mu \cdot Ob \cdot OC \cdot Cb}{Ob^2 - \mu\mu \cdot OC^2}$

et semidiameter coniugatus $= \dfrac{v \cdot Ob \cdot OC}{\sqrt{(Ob^2 - \mu\mu \cdot OC^2)}}$

et semilatus rectum $= \dfrac{vv \cdot Ob \cdot OC}{\sqrt{(Ob^2 - \mu\mu \cdot OC^2)}}$.

Ceterum apparet, si sit tang.φ minor quam $\dfrac{1}{\sqrt{(mm \cdot \sin.\theta^2 + nn \cdot \cos.\theta^2)}}$

tang.φ minor quam $\dfrac{v}{mn}$, curvam fore ellipsin; si sit tang.$\varphi = \dfrac{v}{mn}$, parabolam et,

si tang.φ maior quam $\dfrac{v}{mn}$, hyperbolam.

81. Tertium corpus, cuius sectiones plano factas hic investigare constituimus, est globus, cuius quidem omnes sectiones planas circulos esse ex geometria elementari constat. Interim tamen quo methodus clarius perspiciatur, quemadmodum ex data aequatione pro solido quocunque eius sectiones quaevis erui debeant, idem negotium hic analytice absolvam, quod vulgo synthetice tradi solet. Sit igitur (Fig.138) C centrum globi, per quod planum tabulae transire concipiatur, ita ut sectio hoc plano facta sit circulus maximus, cuius radius $CA = CB$ ponatur $= a$, qui simul erit radius globi. Sit porro recta DP intersectio plani secantis cum isto plano tabulae, ad quam ex C ducatur normalis CD, quae sit $= f$, angulus autem inclinationis sit $= \varphi$.

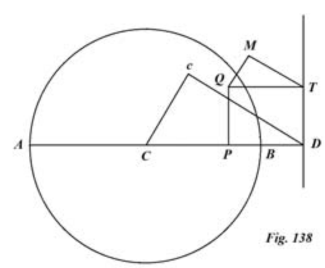

Fig. 138

82. Sit M punctum sectionis quaesitae quodcunque; unde ad planum tabulae demittatur perpendiculum MQ hincque ad rectam CD pro axe assumtam perpendicularis QP. Quodsi iam vocentur coordinatae $CP = x, PQ = y$ and $QM = z$, erit ex natura globi $xx + yy + zz = aa$. Ducatur

ex *M* pariter ad rectam *DT* normalis *MP*, et iuncta *QT*, ob ambas *QT* et *MP* ad *DP* normales, metietur angulus *MPQ* inclinationem plani secantis ad planum basis, quae est $=\varphi$. Quare, si *DT* et *MT* tanquam coordinatae sectionis quaesitae spectentur vocenturque $DT = t,\ TM = u$, fiet $MQ = u \cdot \sin.\varphi$ et $PQ = u \cdot \cos.\varphi$. Erit ergo

$$CP = x = f - u \cdot \cos.\varphi,\quad PQ = y = t \text{ et } QM = z = u \cdot \sin.\varphi .$$

Quibus valoribus substitutis emerget aequatio pro sectione globi quaesita haec

$$ff - 2fu \cdot \cos.\varphi + uu + tt = aa .$$

83. Perspicuum iam est hanc aequationem esse pro circulo. Namque si ponatur $u - f \cdot \cos.\varphi = s$, fiet

$$ff \cdot \sin.\varphi^2 + ss + tt = aa ,$$

unde radius sectionis erit $=\sqrt{(aa - ff \cdot \sin.\varphi^2)}$. Quare, si ex *D* applicatae *TM* parallela ducatur *Dc* in eamque ex centro *C* perpendiculum demittatur *Cc*, ob $CD = f$ et angulum $CDc = \varphi$ erit $Dc = f \cdot \cos.\varphi$ et $Cc = f \cdot \sin.\varphi$. Hinc, cum coordinatae *s* et *t* ad centrum referantur, erit punctum *c* centrum sectionis et $\sqrt{(CB^2 - Cc^2)}$ radius istius circuli, uti ex elementis est manifestum. Simili autem modo omnium aliorum solidorum, dummodo eorum natura sit aequatione inter tres variabiles expressa, sectiones quaecunque planis factae investigari poterunt.

84. Quo tamen tota operatio melius perspiciatur, proponatur (Fig. 139) solidum quodcunque, cuius natura sit expressa aequatione inter ternas coordinatas $AP = x,\ PQ = y$ et $QM = z$, quarum illae positae sint in plano tabulae, haec vero *z* sit ad planum normalis. Secetur iam hoc solidum plano quocunque, cuius cum plano tabulae intersectio sit recta *DT* et inclinationis angulus $=\varphi$. Ponatur recta $AD = f$, angulus $ADE = \theta$ eritque demisso ex *A* in *DE* perpendiculo *AE*

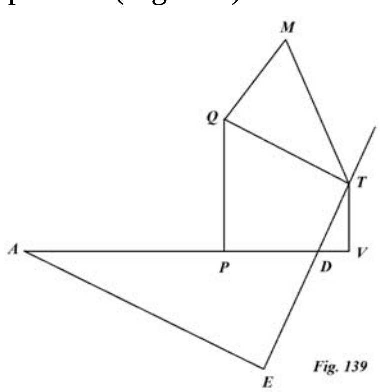

Fig. 139

$$AE = f \cdot \sin.\theta \text{ et } DE = f \cdot \cos.\theta .$$

Tum ex sectionis quaesitae puncto *M* ad *DT* ducatur perpendicularis *MP* iunctaque *QT* aequabitur angulus *MTQ* inclinationi datae φ. Quare si *DP* et *PM* pro coordinatis sectionis quaesitae accipiantur et vocentur $DT = t$, $TM = u$, erit

$$QM = u \cdot \sin.\varphi \text{ et } PQ = u \cdot \cos.\varphi .$$

85. Ex *T* ad axem *AD* demittatur perpendiculum *TV* atque ob angulum $TDV = \theta$ erit $PV = t \cdot \sin.\theta$ et $DV = t \cdot \cos.\theta$. Quia porro angulus *TQP* est $= s$, erit

$$TV = u \cdot \sin.\theta \cdot \cos.\varphi \text{ et } PQ - TV = u \cdot \cos.\theta \cdot \cos.\varphi .$$

Ex his itaque coordinatae *x, y* et *z* sequenti modo per *t* et *u* definientur, ut sit

$$AP = x = f + t \cdot \cos.\theta - u \cdot \sin.\theta \cdot \cos.\varphi ;$$

et

$$PQ = y = t \cdot \sin.\theta - u \cdot \cos.\theta \cdot \cos.\varphi ;$$

atque

$$QM = z = u \cdot \sin.\varphi.$$

Quare, si isti valores in aequatione inter *x*, yet *z* pro solido data substituantur, obtinebitur aequatio inter *t* et *u* seu coordinatas sectionis quaesitae, cuius adeo natura innotescet. Convenit autem hic modus fere cum eo, quo supra § 50 usi sumus.