APPENDIX

ON SURFACES.

CHAPTER I

ON THE SURFACES OF BODIES IN GENERAL

1. The matters which have been treated in the account regarding curved lines and representing these by equations in the above section, are indeed generally apparent and may be extended to all curved lines, all the points of which may be put in the same plane. Truly if a whole curved line were not in the same plane, then the precepts given above are not sufficient for eliciting the properties of curves of this kind. Curves of this kind have a two-fold curvature, and under the same name that most excellent of texts has been written by Clairaut, worthy of his geometrical acumen. But since here we shall be concerned especially with the nature of surfaces, which I have put in place to be explained in this section, I will not treat that aspect by itself, as its explanation will be given jointly with the following theory of surfaces.

2. Just as lines are either right or curves, thus surfaces are either planes or they are non-planar. Moreover I call surfaces non planar, which are either convex or concave, or of a nature sharing in each. Thus the outer surface of a globe, cylinder, and a cone, with the bases excepted, is convex ; but the inner surface of a large bowl is concave. Just as again a line is a right line, any three points of which are placed on the same direction, thus a surface is a plane, any four points of which have been placed on the same plane; from which it is evident a surface is not to be a plane, that is either convex or concave, no four points of which are situated on the same plane.

3. Therefore we shall have understood how such a surface may be non-planar most easily, if we may inquire how much it may disagree with a plane everywhere. Evidently in a similar manner we gather, how the nature of curved lines may be understood from the distances, by which each point of that are distant from a right line assumed as an axis ; thus it will be convenient to estimate the nature of surfaces from the separation of its individual points from a plane surface, taken as it pleases. Therefore for some surface proposed, whose nature it may be necessary to define, for a plane surface may be chosen arbitrarily, to which, from the individual points of the proposed surface, perpendiculars may be drawn; with which done, if the length of any of these perpendiculars can be determined from an equation, then we will agree that the nature of the surface to be expressed by this equation. For from such an equation in turn, all the points of the surface will be able to be assigned, and thus the surface itself will be determined.

4. The plane of the table may represent that plane surface, to which we may refer each individual point of the proposed surface (Fig. 119). *M* shall be some point of the proposed surface, which may be taken placed above the plane of the table, from which *MQ* may be sent to this perpendicular plane crossing at the point *Q*. Now at the position of this point *Q* expressed by a calculation, a certain right line *AB* is taken in the plane of the table for the axis, to which from the point *Q* a right line *QP* may be drawn normal. Finally on the axis itself *AB* some point *A* may be taken for the beginning of the axes ; with which done the position of the point *M* will become known, if we may know the lengths of the three lines *AP*, *PQ* and *QM* ; and thus from the three coordinates normal between themselves, the position of each point *M* on the surface will be determined in a like manner, from which the individual points of curved lines situated in the plane are accustomed to be shown by two coordinates normal between themselves.

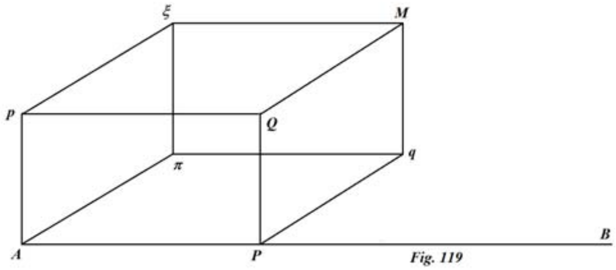

Fig. 119

5. Therefore since we may have the three coordinates *AP*, *PQ* and *QM*, we may put $AP = x,\ PQ = y$ and $QM = z$, and from these we may understand the nature of the proposed surface, if with the two *x* and *y* taken as it pleases we may know how great the third *z* shall become; for in this way we will be able to determine all the points *M* on the surface. Therefore the nature of any surface is expressed by an equation, from which the coordinate *z* is defined by the two remaining *x* and *y* together with constants. Hence for some proposed surface the variable *z* will be equal to a certain function of the two variable *x* and *y*. And in turn, if *z* were equal to some function of *x* and *y*, then that equation will show some surface, the nature of which will be known from that equation. For with everything being substituted for *x* and *y*, which they can receive, with both positive as well as negative values, all the points *Q* of the assumed plane will be obtained ; then truly from the equation of z by *x* and *y* the length of the perpendicular $QM = z$ will be constructed everywhere, then it may pertain to the surface ; which if the value of *z* were positive, will be a point of the surface *M* situated above the plane *APQ*, but if it were negative, this falls below the plane, if it may vanish, the point of the surface *M* will be found in this plane ; but if it were imaginary, then clearly no point of the surface *M* will correspond to the point *Q*. But if it may arise, so that *z* may have more real values, then the a right line drawn normal to the plane through the point *Q* will cross the surface in several points *M* .

6. Therefore so that it may relate to the various natures of surfaces, this itself at once offers a distinction between continuous or regular surfaces and discontinuous or irregular ones. Clearly a surface will be continuous, whose points may be expressed by the same equation between z , x and y, or where z is the same function of x and y for all the points of the surface. But a surface is irregular, whose various parts will be shown by different functions, as if a surface were proposed, which in one place shall be spherical, in another place conical, cylindrical, or planar. But here we have excluded irregular surfaces completely and we consider only regular surfaces, the nature of which may be expressed by a certain single constant equation. For with these treated, because irregular surfaces have been constructed from parts of various regular surfaces, these also will be able to be decided easily.

7. But the first division of regular surfaces into algebraic and transcending is put in place. Moreover a surface may be called algebraic, whose nature is expressed by an algebraic equation between the coordinates x, y and z, or when z is equal to an algebraic function of x and y. Contrarily therefore, if z were not an algebraic function of x and y or if in the equation between x, y and z transcending quantities should be present such as from logarithms and depending on the arcs of circles, then the surface, whose nature will be expressed by an equation of this kind, will be transcending. There will be such a surface, if there were $z = x \cdot l.y$ or $z = y^x$, or $z = y \cdot \sin.x$. But it will be understood easily that it is necessary to treat algebraic surface before we move on to transcending ones.

8. Then in the first place it is required to attend to recognizing the nature of the surface, it shall be such a function z of x and y on account of the number of values which it contains. Therefore here at first these surfaces occur, for which z is equal to a uniform function of x and y. P shall be a uniform or rational function of this kind of x and y; and, if there were $z = P$, just as many points of the surface will correspond to the individual points of the plane Q, or any right line normal to the plane APQ shall pierce the surface in a single point. Nor truly in this case will the value of the right line QM be able to be made imaginary anywhere, but all right lines of this kind will provide real points of the surface. Yet meanwhile this diversity of functions does not lead to an essential variety between the surfaces ; for it depends on the position of the plane APQ, which thus is arbitrary as well as the axis, thus so that, if the same surface may be referred to another plane, the function z, which was uniform, may emerge as some multiform.

9. Let P and Q be some uniform functions of x and y ; and, if there were $zz - Pz + Q = 0$, then the right lines drawn normally through the individual points of the plane Q will cut the surface either in two points or not any ; for z will have two values, which will be both real, or both imaginary. In a similar manner if with P, Q and R designating uniform functions of x and y there were

$z^3 - Pz^2 + Qz - R = 0$, then *z* will be a triform function and any right line *QM* will cut the surface either in three points, if all the roots were real or only in one, if two of the roots truly were imaginary. And in a similar manner it will be required to be judged, if *z* may be defined by an equation, in which more dimensions my prevail. Therefore as the function *z* may become multiform, it will be recognised easily, if the equation between *x* , *y* and *z* may be reduced to rationality.

10. For the rest, in the same manner as we have seen in equations for curved lines the two coordinates can be interchanged between themselves, thus in any equation for a surface the three coordinates *x, y* and *z* can be interchanged among themselves. For in the first place, if in the plane *APQ* another right line *Ap* normal to *AP* may be assumed for the axis, now there will be $Ap = y$ and $pQ = x$ and thus the two *x* and *y* have been interchanged themselves. All the remaining permutations will be understood by completing the rectangular parallelepiped $ApQM\xi\pi qPA$; in which in the first place the three planes to be looked at come fixed between the normals to themselves $APQp$, $APq\pi$ et $AP\xi\pi$; to which just as many individual proposed surfaces may be referred, whose point [of intersection] is *M*, may declare the same equation between *x, y* and *z*. But in any single plane a two-fold axis is given, each having beginning a beginning at the point *A*, from which six different relations between the three coordinates result.

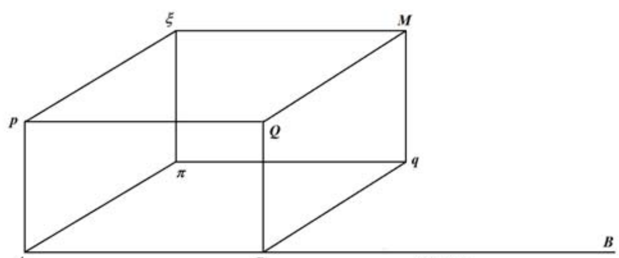

Fig. 119

The coordinates will be :

for the plane $APQp$

either $AP = x$, $PQ = y$, $QM = z$,

or $Ap = y$, $pQ = x$, $QM = z$,

for the plane $APq\pi$

either $AP = x$, $Pq = z$, $qM = y$,

or $A\pi = z$, $\pi q = x$, $qM = y$,

for the plane $AP\xi\pi$

either $Ap = y$, $p\xi = z$, $\xi M = x$,

or $A\pi = z$, $\pi\xi = y$, $\xi M = x$.

So that if moreover the right line *AM* may be drawn from the fixed point *A* to the point of the surface *M*, this will be $\sqrt{(xx + yy + zz)}$.

11. Therefore the same known equation is shown between the coordinates *x, y* and *z* of the surface for the three planes, which are normal between themselves and in turn cross over through the point *A*. Clearly in whatever manner the variable *z* shows the distance of each point M of the surface from the plane *APQ*, thus the variable *y* provides the distance of the same point *M* from the plane *APq* and the variable *x* from the plane $Ap\xi$. So that if moreover we may know, by how great intervals the point M may be distant from each of these three planes, then likewise its true position becomes known. Therefore these three planes, to which some surface is referred by the equation of the three variables *x, y* and *z*, must be noted especially ; if one of which, namely *APQ,* were horizontal, the two remaining will be vertical, clearly the one may be in place following the right line *AP* to the horizontal, and the other along the right line *Ap*.

12. Therefore with these three planes put in place normal between themselves, to which the proposed surface may be referred, the right normals *MQ*, *Mq* and $M\xi$ may be drawn from its individual points M to these planes *APQ, APq* and $A\pi\xi$, which will be $MQ = z,\ Mq = y$ and $M\xi = x$. Then by completing the parallelepiped three right lines will be had equal to these, which depart from the fixed point *A*, namely $AP = x,\ Ap = y$ and $An = z$, with which known the position of the point *M* may be determined. But it is evident, if these variables *x, y* and *z* run together, while in the directions which the figure indicates, they may be agreed to be positive, while the values of these are agreed to be negative, if they may be directed in different directions.

13. If in the equation between the three variables *x, y* and *z* that, which is normal to the plane *APQ*, surely *z,* may have even dimensions everywhere, then it will have equal twin values, the one positive and the other negative. Therefore the surface will be prepared thus, so that for each point of the plane *APQ* there shall be a part similar and equal to itself, and thus the body, which may be bounded by this surface, will be divided into two equal and similar parts by a cut made along the plane *APQ*. Therefore just as in plane figures that right line, by which a figure may be divided into two similar and equal parts, is called a diameter, thus with solids that plane, by which the body is cut into two similar parts, we will call *diametrical*. Whereby, if the variable *z* may have even dimensions everywhere in the equation, then the plane *APQ* will be diametrical.

14. It is understood in a similar manner, if in the equation for the surface the variable *y,* which is normal to the plane *APq*, shall have even dimensions everywhere, then the plane *APq* shall be diametrical. Moreover if the variable *x* shall have even dimensions everywhere, then the plane $Ap\xi$ will be diametrical. Therefore with the equation for some surface given between the three variables *x, y* and *z* it will be apparent at once, each shall be diametrical or cut by the three planes *APQ, APq,* $Ap\xi$. But it may come about, so that indeed all these three

planes shall be diametrical. Clearly for a globe, whose centre shall be at *A*, on account of the radius $AM = \sqrt{(xx+yy+zz)} = a$ there will be $xx + yy + zz = aa$, from which and by which three planes the globe will be apportioned into similar and equal parts.

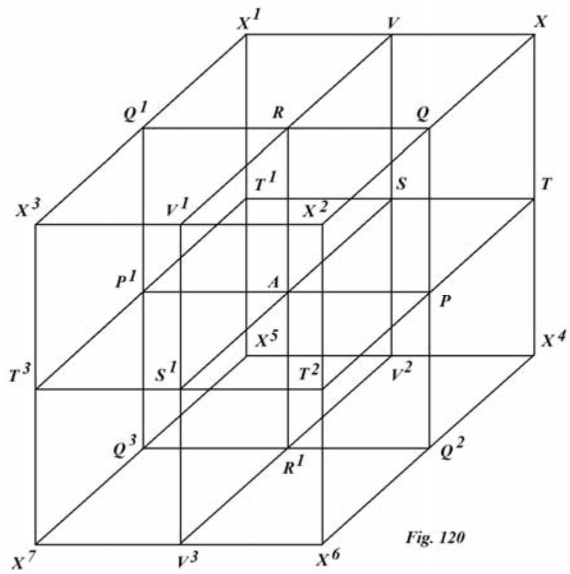

Fig. 120

15. For the figure of the surface (Fig. 120), which may be held in the proposed equation, it is necessary to attend in the first place to be familiar with the three planes normal between themselves, which may be represented in the figure by $QQ^1Q^2Q^3$, $TT^1T^2T^3$, and $VV^1V^2V^3$ and which intersect each other mutually at the point *A*. These three planes, if they may be considered to be extended to infinity in every which way, will divide the whole of space into eight regions, which may be shown in the figure by the letters AX, AX^1, AX^2, AX^3, AX^4, AX^5, AX^6 and AX^7. And if now in the first region *AX* the variables *x*, *y* and *z* may be put to have positive values, in the remaining regions either one or two, or all three become negative. But the account of these values may be seen most clearly from the following table :

Region AX	Region AX^1	Region AX^2	Region AX^3
$AP = +x$	$AP^1 = -x$	$AP = +x$	$AP^1 = -x$
$AR = +y$	$AR = +y$	$AR = +y$	$AR = +y$
$AS = +z$	$AS = +z$	$AS = -z$	$AS^1 = -z$
Region AX^4	Region AX^5	Region AX^6	Region AX^7
$AP = +x$	$AP^1 = -x$	$AP = +x$	$AP^1 = -x$
$AR^1 = -y$	$AR^1 = -y$	$AR^1 = -y$	$AR^1 = -y$
$AS = +z$	$AS = +z$	$AS^I = -z$	$AS^1 = -z$

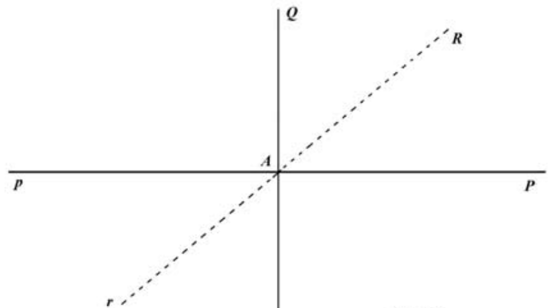
Fig. 121

16. But it will be more convenient to signify these eight diverse regions by numbers, by which we may be able to indicate these more easily in any discussion. Therefore since these eight regions (Fig. 121) shall be adjoining at the point *A* and may be distinguished among themselves by the intersection of the three normal planes, moreover these planes may be determined by the three right lines *Pp*, *Qq*, *Rr* themselves crossing normally at the point *A*, these regions can be defined by the letters *P*, *Q*, *R* either greater or smaller [*i.e.* capitals or ordinary text]. Evidently the principal or first region will be the space *PQR*, which the parallelepiped enfolds by the three right lines *AP*, *AQ*, *AR* extended to infinity ; and the region *Pqr* will be the space, which the parallelepiped formed out of the three right lines *AP*, *Aq*, *Ar* produced to infinity may enclose. Therefore with the three variable put in place
$AP = x,\ AQ = y,\ AR = z$, and as there will be $Ap = -x,\ Aq = -y$ and $Ar = -z$.
Therefore we may distinguish between these regions by numbers in the following manner, so that there shall be between the coordinates

	First I	second II	third Ill	fourth IV
	PQR	*PQr*	*PqR*	*pQR*
between coordinates	$AP = +x$	$AP = +x$	$AP = +x$	$Ap = -x$
	$AQ = +y$	$AQ = +y$	$Aq = -y$	$AQ = +y$
	$AR = +z$	$AR = -z$	$Ar = +z$	$AR = +z$
	fifth V	sixth VI	seventh VII	eighth VIII
	Pqr	*pQr*	*pqR*	*pqr*
between coordinates	$AP = +x$	$Ap = -x$	$Ap = -x$	$Ap = -x$
	$Aq = -y$	$AQ = +y$	$Aq = -y$	$Aq = -y$
	$Ar = -z$	$Ar = -z$	$AR = +z$	$Ar = -z$

17. These regions more or less disagree with each other. Without doubt the first gives two regions, which have two coordinates in common, with one disagreement ; and thus they touch each other in turn with a plane, which we may call *adjoined.* Then, if two coordinates were different and they may have a single one in common, the regions touch each other in a line only, which we may call *disjoined.* In the third place, if all the coordinates may disagree by sign, the regions touch each other at the point *A* only, and we will call these regions *opposite*. Now which regions for each shall be adjoined, disjoined, or opposite, the following table will show:

Region	*Adjoined*			*Disjoined*			*Opposite*
PQR I	*PQr* II	*PqR* Ill	*pQR* IV	*Pqr* V	*pQr* VI	*pqR* VII	*pqr* VIII
PQr II	*PQR* I	*Pqr* V	*pQr* VI	*PqR* Ill	*pQR* IV	*pqr* VIII	*pqR* VII
PqR III	*Pqr* V	*PQR* I	*pqR* VII	*PQr* II	*pqR* VIII	*pQR* IV	*pQr* VI
pQR IV	*pQr* VI	*pqR* VII	*PQR* I	*pqr* VIII	*PQr* II	*PqR* Ill	*Pqr* V
Pqr V	*PqR* Ill	*PQr* II	*pqr* VIII	*PQR* I	*pqR* VII	*pQr* VI	*pQR* IV
pQr VI	*pQR* IV	*pqr* VIII	*PQr* II	*pqR* VII	*PQR* I	*Pqr* V	*PqR* Ill
pqR VII	*pqr* VIII	*pQR* IV	*PqR* III	*pQr* VI	*Pqr* V	*PQR* I	*PQr* II
pqr VIII	*pqR* VII	*pQr* VI	*Pqr* V	*pQR* IV	*PqR* III	*PQr* II	*PQR* I

18. Therefore it is apparent any region has three regions adjoining to itself, just as many disjoining regions, and a single opposite region, and it may be seen at once from the preceding table, in what manner any region may be coupled together to some other region. But the order, which the numbers denoting the regions maintain in this table, is worthy of attention; which so that may be seen better, the same numbers in the same order are included in the following square :

1	2	3	4	5	6	7	8
2	1	5	6	3	4	8	7
3	5	1	7	2	8	4	6
4	6	7	1	8	2	3	5
5	3	2	8	1	7	6	4
6	4	8	2	7	1	5	3
7	8	4	3	6	5	1	2
8	7	6	5	4	3	2	1

The nature and properties of which may be seen with a little attention, truly the most general use may be may be made on this account.

19. Now we have noted before, if in the equation of the variable z it shall have even dimensions everywhere, then the surface has two similar and equal parts, clearly the part in the first region will be equal to the part in the second and in a similar manner the third and fifth, likewise the fourth and sixth, and finally the seventh and the eighth will agree between themselves, as the two square series

starting from 1 and 2 show. But if in the equation the variable *y* may have even dimensions everywhere, then the first region will agree with the third, the second with the fifth, the fourth with the seventh, and the sixth with the eighth. But if *x* may have equal dimensions everywhere in the equation, then the first region will agree with the fourth, the second with the sixth, the third with the seventh, and the fifth with the eight. Namely :

if the dimensions may have even parts everywhere in the equation of z, y, and x in turn, then the regions of the variable

z	*y*	*x*
1, 2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4, 5, 6, 7, 8
2, 1, 5, 6, 3, 4, 8, 7;	3, 5, 1, 7, 2, 8, 4, 6;	4, 6, 7, 1, 8, 2, 3, 5

become equal in turn.
[Refer to the table in §16 to see which variables are positive or negative in a given region.]

20. In order that the parts of the surface placed in disjointed regions shall have equal parts between themselves the first and the fifth shall be equal to each other, then it is required to prepare the equation thus, so that it may remain the same, even if the two variables *y* and *z* shall be taken negative. Therefore this will come about, if both *y* and *z* may be put in place taken jointly with either even or odd dimensions everywhere. Just as also the first region may agree with the fifth, as well as the second with the third, the fifth with the eighth and the sixth with the seventh. In a similar manner if in the equation for the surface the two variables *x* and *z* may be filled either by a number of even or odd dimensions everywhere, then the region from the first will agree with the sixth, the second with the fourth, the third with the eighth, and the fifth with the seventh. Namely :

If in the equation for a surface, either even or odd numbers are used everywhere to form the dimensions of the variables

y & *z*, *then the regions*	*x* & *z*, *then the regions*	*x* & *y*, *then the regions*
1, 2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4, 5, 6, 7, 8
5, 3, 2, 8, 1, 7, 6, 4	6, 4, 8, 2, 7, 1, 5, 3	7, 8, 4, 3, 6, 5, 1, 2

will be equal.

But if all three variables x, y and z considered everywhere may hold either even or odd dimensions everywhere, then the opposite regions will be equal :

1, 2, 3, 4, 5, 6, 7, 8
8, 7, 6, 5, 4, 3, 2, 1.

21. If from these two or three conditions are not to be taken at the same time in the equation, then either four or all eight parts of the surface will contain similar and equal parts. Clearly :

If both x and y considered separately may maintain even dimensions everywhere, then the following four regions will be congruent

1, 2, 3, 4, 5, 6, 7, 8
3, 5, 1, 7, 2, 8, 4, 6
4, 6, 7, 1, 8, 2, 3, 5
7, 8, 4, 3, 6, 5, 1, 2.

If both x and z considered separately may maintain even dimensions everywhere, then the following four regions will be congruent

1, 2, 3, 4, 5, 6, 7, 8
2, 1, 5, 6, 3, 4, 8, 7
4, 6, 7, 1, 8, 2, 3, 5
6, 4, 8, 2, 7, 1, 5, 3.

If both y and z considered separately may maintain even dimensions everywhere, then the following four regions will be congruent

1, 2, 3, 4, 5, 6, 7, 8
2, 1, 5, 6, 3, 4, 8, 7
3, 5, 1, 7, 2, 8, 4, 6
5, 3, 2, 8, 1, 7, 6, 4.

22. If a single one of the variables may have even dimensions everywhere, truly the remaining two considered at the same time either everywhere shall be constituted from even or odd dimensions everywhere, then the four regions also will be congruent in the following manner :

If z shall have even dimensions everywhere, and both x and y may be formed from either even or odd dimensions everywhere, then the following four regions will be in agreement :

1, 2, 3, 4, 5, 6, 7, 8
2, 1, 5, 6, 3, 4, 8, 7
7, 8, 4, 3, 6, 5, 1, 2
8, 7, 6, 5, 4, 3, 2, 1.

If y shall have even dimensions everywhere, and both x and z may be formed from either even or odd dimensions everywhere, then the following four regions will be in agreement :

1, 2, 3, 4, 5, 6, 7, 8
3, 5, 1, 7, 2, 8, 4, 6
6, 4, 8, 2, 7, 1, 5, 3
8, 7, 6, 5, 4, 3, 2, 1.

If x shall have even dimensions everywhere, and both y and z may be formed from either even or odd dimensions everywhere, then the following four regions will be in agreement :

1, 2, 3, 4, 5, 6, 7, 8
4, 6, 7, 1, 8, 2, 3, 5
5, 3, 2, 8, 1, 7, 6, 4
8, 7, 6, 5, 4, 3, 2, 1.

Therefore with these three cases considered together at the same time, all three variables x, y and z will be formed either from even or odd dimensions everywhere.

23. The following cases of four equal regions shall be present :

If both x and y, as well as y and z shall be formed everywhere either from even or odd dimensions, then the following four regions will be equal

1, 2, 3, 4, 5, 6, 7, 8
5, 3, 2, 8, 1, 7, 6, 4
7, 8, 4, 3, 6, 5, 1, 2
6, 4, 8, 2, 7, 1, 5, 3.

The same similitude will be produced, if the two remaining variables x and z everywhere above may be formed either from even or odd dimensions, thus so that this condition now may be contained in the proposition. Therefore the parts of the surface in the four disjoint regions will be equal to each other, if in the equation any two variables considered jointly may be formed either from even or odd dimensions everywhere. But since they may give rise to three combinations, it is to be observed, if two were provided with the proposed property, then likewise the third combination would be entertaining the same property.

24. But if the conditions, which have produced the four similar and equal regions, in addition a new condition not present in these may fall, which by itself introduces equality into two regions, then evidently all the regions are made equal to each other and the surface will be formed from eight parts equal and similar to each other. Therefore the equation for surfaces of this kind at this stage all will possess the mentioned properties jointly, namely the individual variables x, y, z considered separately everywhere will be formed from even dimensions ; from which now it follows any two considered jointly and also all three likewise taken everywhere will be constituted from even dimensions.

25. But whether or not an equation shall be proved between the three proposed variables with one or two or indeed with three of the properties shown, certainly that is shown easily, which may be held by the even dimensions of some variable. Nor is it more difficult to inquire, whether all the variables likewise considered everywhere may be formed from either even or odd dimensions. But whether only two shall be prepared according to that property, will be more difficult to examine. There may be put into the equation either $x = nz,$ $y = nz,$ or $x = ny$ and it may be considered, whether in one case or another an equation will result, in which the variable z in the first two cases or y in the final case may be led to even dimensions ; which if it may eventuate, the two variables taken jointly may establish either even or odd dimensions everywhere and hence it is necessary that the surface will have at least two parts equal and similar to each other.

APPENDIX
SUPERFICIEBUS.

CAPUT I

DE SUPERFICIEBUS CORPORUM IN GENERE

1. Quae in superiori sectione de lineis curvis sunt tradita earumque ad aequationes revocandarum ratione, latissime quidem patent atque ad omnes lineas curvas, quarum cuncta puncta in eodem plano sint posita, extenduntur. Verum si tota linea curva non fuerit in eodem plano sita, tum praecepta supra data non sufficiunt ad proprietates eiusmodi curvarum eruendas. Huius generis curvae duplicem habent curvaturam hocque nomine de iis eximium scripsit tractatum Acutissimus Geometra Clairaut. Cum autem haec materia maxime sit connexa cum natura superficierum, de qua hac sectione exponere constitui, seorsim eam non pertractabo, sed eius explicationem cum sequenti de superficiebus doctrina coniungam.

2. Quemadmodum lineae sunt vel rectae vel curvae, ita superficies sunt vel planae vel non planae. Non planas autem voco, quae vel convexae sunt vel concavae, vel utriusque naturae participes. Sic superficies externa globi, cylindri et coni, exceptis basibus, est convexa; interna autem catini superficies concava. Quemadmodum porro linea recta est, cuius terna quaeque puncta in directum sunt posita, ita superficies plana est, cuius quaterna quaeque puncta in eodem plano sunt posita; ex quo perspicuum est superficiem non planam, hoc est sive convexam sive concavam, esse, cuius non omnia quaterna puncta in eodem plano sunt sita.

3. Superficies igitur non plana qualis sit, facillime intelligetur, si, quantum a superficie plana ubique discrepet, cognoverimus. Simili scilicet modo, quo indolem linearum curvarum ex distantiis, quibus eius quaeque puncta a linea recta pro axe assumta distant, colligimus, ita naturam superficierum aestimari conveniet ex singulorum eius punctorum distantiis a superficie plana pro lubitu assumta. Proposita ergo quacunque superficie, cuius indolem definiri oporteat, pro arbitrio eligatur superficies plana, ad quam ex singulis superficiei propositae punctis perpendicula ducta concipiantur; quo facto, si cuiusvis horum perpendiculorum longitudo per aequationem determinari queat, naturam

superficiei hac ipsa aequatione exprimi censebimus. Ex tali enim aequatione vicissim omnia superficiei puncta assignari poterunt atque ideo ipsa superficies determinabitur.

4. Repraesentet (Fig. 119) planum tabulae eam superficiem planam, ad quam singula cuiusque superficiei propositae puncta referamus. Sit *M* punctum quodcunque superficiei propositae, quod extra planum tabulae situm concipiatur, unde ad hoc planum perpendicularis demittatur *MQ* plano in puncto *Q* occurrens. Iam ad situm huius puncti *Q* calculo exprimendum assumatur in plano tabulae recta quaepiam *AB* pro axe, ad quem ex puncto *Q* recta normalis ducatur *QP*. Denique in ipso axe *AB* sumatur punctum quodvis *A* pro initio abscissarum; quo facto situs puncti *M* innotescet, si noverimus longitudines trium istarum linearum *AP*, *PQ* et *QM* ; sicque tribus coordinatis inter se normalibus situs cuiusque superficiei puncti *M* simili modo determinabitur, quo linearum curvarum in plano sitarum singula puncta per duas coordinatas inter se normales exhiberi solent.

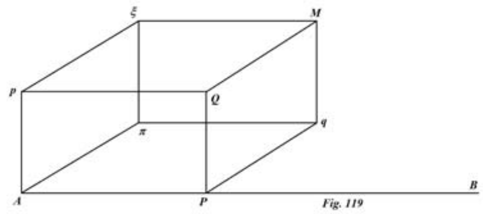

Fig. 119

5. Cum igitur habeamus tres coordinatas *AP*, *PQ* et *QM*, ponamus $AP = x,\ PQ = y$ et $QM = z$ ex hisque indolem superficiei propositae intelligemus, si sumtis pro lubitu binis *x* et *y* noverimus, quanta futura sit tertia *z;* hoc enim modo omnia superficiei puncta *M* determinare poterimus. Natura ergo cuiusvis superficiei exprimitur aequatione, qua coordinata *z* definitur per binas reliquas *x* et *y* una cum constantibus. Hinc pro quavis superficie proposita variabilis *z* aequabitur functioni cuidam binarum variabilium *x* et *y*. Atque vicissim, si *z* aequalis fuerit functioni cuicunque ipsarum *x* et *y*, tum ista aequatio exhibebit superficiem quampiam, cuius natura ex ipsa illa aequatione innotescet. Substituendis enim pro *x* et *y* omnibus, quos recipere possunt, valoribus, tam affirmativis quam negativis, omnia plani assumti puncta *Q* obtinebuntur; tum vero ex aequatione ipsius *z* per *x* et *y* constabit ubique longitudo perpendiculi $QM = z$, donec ad superficiem pertingat; qui ipsius *z* valor si fuerit affirmativus, punctum superficiei *M* supra planum *APQ* erit situm, sin autem sit negativus, infra hoc planum cadet, si evanescat, punctum superficiei *M* in hoc ipso plano reperietur; at si fuerit imaginarius, tum puncto *Q* nullum prorsus superficiei punctum *M* respondebit. Quodsi autem eveniat, ut *z* habeat plures valores reales, tum recta ad planum normalis per punctum *Q* ducta superficiem in pluribus punctis *M* traiiciet.

6. Quod igitur ad varias superficierum naturas attinet, hic statim se offert distinctio in continuas seu regulares et discontinuas seu irregulares. Superficies scilicet continua erit, cuius omnia puncta per eandem aequationem inter *z* et *x* et *y* exprimuntur, seu ubi *z* est eadem functio ipsarum *x* et *y* pro omnibus superficiei

punctis. Superficies autem irregularis est, cuius variae partes per diversas functiones exhibentur, uti si proposita fuerit superficies, quae in uno loco sit sphaerica, in alio conica seu cylindrica seu plana. Hic autem superficies irregulares penitus excludimus atque ad solas regulares, quarum natura una quadam constanti aequatione contineatur, respiciemus. His enim pertractatis, quoniam superficies irregulares ex partibus variarum regularium sunt conflatae, etiam istas facile diiudicare licebit.

7. Superficierum autem regularium primaria divisio instituitur in algebraicas et transcendentes. Superficies autem algebraica vocatur, cuius natura exprimitur per aequationem algebraicam inter coordinatas *x*, *y* et *z*, seu quando *z* aequalis est functioni algebraicae ipsarum *x* et *y*. Contra igitur, si *z* non fuerit functio algebraica ipsarum *x* et *y* seu si in aequatione inter *x*, y et *z* insint quantitates transcendentes veluti a logarithmis et arcubus circularibus pendentes, tum superficies, cuius natura huiusmodi aequatione exprimitur, erit transcendens. Talis erit superficies, si fuerit $z = x \cdot l.y$ seu $z = y^x$ seu $z = y \cdot \sin.x$. Facile autem intelligitur superficies algebraicas ante tractari oportere, quam ad transcendentes progrediamur.

8. Deinde ad naturam superficiei cognoscendam imprimis attendendum est, qualis sit functio *z* ipsarum *x* et *y* ratione numeri valorum, quos continet. Hic igitur primum occurrunt eae superficies, pro quibus *z* aequatur functioni uniformi ipsarum *x* et *y*. Sit *P* huiusmodi functio uniformis seu rationalis ipsarum *x* et *y;* atque, si fuerit $z = P$, singulis punctis plani *Q* totidem respondebunt superficiei puncta seu quaelibet recta ad planum *APQ* normalis superficiem in unico puncto traiiciet. Neque vero hoc casu usquam valor rectae *QM* fieri poterit imaginarius, sed omnes istiusmodi rectae puncta superficiei realia praebebunt. Interim tamen ista functionum diversitas non essentialem varietatem inter superficies producit; pendet enim a situ plani *APQ*, qui perinde ac axis est arbitrarius, ita ut, si superficies eadem ad aliud planum referatur, functio *z*, quae erat uniformis, evadere possit utcunque multiformis.

9. Sint *P* et *Q* functiones quaecunque uniformes ipsarum *x* et *y* ; atque, si fuerit $zz - Pz + Q = 0$, tum rectae per singula plani puncta *Q* normaliter ductae superficiem vel in duobus punctis secabunt vel nusquam; habebit enim *z* duos valores, qui vel ambo erunt reales vel ambo imaginarii. Simili modo si denotantibus *P*, *Q* et *R* functiones uniformes ipsarum *x* et *y* fuerit $z^3 - Pz^2 + Qz - R = 0$, tum erit *z* functio triformis et quaelibet recta *QM* superficiem secabit vel in tribus punctis, si omnes radices aequationis fuerint reales vel tantum in unico, si scilicet binae radices fuerint imaginariae. Similique modo erit iudicandum, si *z* definiatur per aequationem, in qua plures obtineat dimensiones. Quam multiformis igitur futura sit functio *z*, facillime cognoscetur, si aequatio inter *x* et *y* et *z* ad rationalitatem perducatur.

10. De cetero, sicuti in aequationibus pro lineis curvis binas coordinatas inter se permutari posse vidimus, ita in aequatione quavis pro superficie tres coordinatae *x, y* et *z* inter se sunt permutabiles. Primo enim, si in plano *APQ* altera recta *Ap* ad *AP* normalis pro axe assumatur, erit nunc $Ap = y$ et $pQ = x$ sicque binae *x* et *y* inter se sunt permutatae. Reliquae permutationes omnes intelligentur complendo parallelepipedon rectangulum $ApQM\xi\pi qPA$; in quo primum spectanda veniunt tria plana fixa inter se normalia $APQp$, $APq\pi$ et $AP\xi\pi$; ad quae singula quemadmodum referatur superficies proposita, cuius punctum est *M*, eadem aequatio inter *x, y* et *z* declarat. In unoquoque autem plano duplex datur axis, uterque initium habens in puncto *A*, unde sex diversae relationes inter tres coordinatas resultant.

Coordinatae erunt

pro plano $APQp$

vel $AP = x$, $PQ = y$, $QM = z$,

vel $Ap = y$, $pQ = x$, $QM = z$,

pro plano $APq\pi$

vel $AP = x$, $Pq = z$, $qM = y$,

vel $A\pi = z$, $\pi q = x$, $qM = y$,

pro plano $AP\xi\pi$

vel $Ap = y$, $p\xi = z$, $\xi M = x$,

vel $A\pi = z$, $\pi\xi = y$, $\xi M = x$.

Quodsi autem a puncto fixo *A* ad punctum superficiei *M* ducatur recta *AM*, erit ea $\sqrt{(xx + yy + zz)}$.

11. Eadem ergo aequatio inter coordinatas *x, y* et *z* cognitionem superficiei ad tria plana exhibet, quae inter se sunt normalia atque se invicem in puncto *A* decussant. Quemadmodum scilicet variabilis *z* distantiam cuiusque superficiei puncti *M* a plano *APQ* exhibet, ita variabilis *y* eiusdem puncti *M* distantiam a plano *APq* et variabilis *x* a plano $Ap\xi$ praebet. Quodsi autem noverimus, quantis intervallis punctum *M* distet ab unoquoque horum trium planorum, tum simul eius verus situs innotescit. Haec igitur tria plana, ad quae superficies quaevis per aequationem trium variabilium *x, y* et *z* refertur, imprimis notari debent; quorum si unum, uti *APQ*, fuerit horizontale, duo reliqua erunt verticalia, alterum scilicet horizontali secundum rectam *AP* alterum secundum rectam *Ap* insistet.

12. Constitutis ergo his tribus planis inter se normalibus, ad quae superficies proposita referatur, ex singulis eius punctis *M* ad ista plana *APQ, APq* et $A\pi\xi$ ducantur rectae normales *MQ, Mq* et $M\xi$, quae erunt $MQ = z$, $Mq = y$ et $M\xi = x$. Deinde completo parallelepipedo habebuntur tres rectae istis aequales, quae ex puncto fixo *A* egrediantur, scilicet

$AP = x$, $Ap = y$ et $An = z$, ex quibus cognitis situs puncti *M* determinatur. Manifestum autem est, si istae variabiles *x*, *y* et *z*, dum in plagas, quas figura indicat, vergunt, affirmativae censeantur, tum earum valores, si in plagas contrarias dirigantur, negativos censeri oportere.

13. Si in aequatione inter tres variabiles *x*, *y* et *z* ea, quae ad planum *APQ* est normalis, nempe *z*, ubique pares habeat dimensiones, tum geminos habebit valores aequales, alterum affirmativum alterum negativum. Superficies igitur ita erit comparata, ut ad utramque plani *APQ* partem sit sui similis et aequalis, atque adeo corpus, quod ista superficie terminatur, sectione secundum planum *APQ* facta in duas partes similes et aequales dividetur. Quemadmodum ergo in figuris planis ea linea recta, qua figura in duas partes similes et aequales dirimebatur, diameter est appellata, ita in solidis id planum, quo corpus in duas partes similes dividitur, *diametrale* vocemus. Quare, si variabilis *z* in aequatione ubique pares habeat dimensiones, tum planum *APQ* erit diametrale.

14. Simili modo intelligitur, si in aequatione pro superficie variabilis *y*, quae ad planum *APq* est normalis, ubique pares habeat dimensiones, tum planum *APq* fore diametrale. Sin autem variabilis *x* pares ubique habeat dimensiones, tum planum $Ap\xi$ erit diametrale. Ex aequatione ergo pro quavis superficie inter tres variabiles *x*, *y* et *z* data statim apparet, utrum ex tribus planis *APQ*, *APq*, $Ap\xi$ sit diametrale an secus. Fieri autem potest, ut duo imo omnia tria haec plana sint diametralia. Scilicet pro globo, cuius centrum sit in *A*, ob radium $AM = \sqrt{(xx + yy + zz)} = a$ erit $xx + yy + zz = aa$, unde singulis hisce tribus planis globus in duas partes similes et aequales dispertietur.

15. Ad figuram superficiei (Fig. 120), quae in proposita aequatione continetur, cognoscendam ad tria illa plana inter se normalia imprimis attendi oportet, quae in figura repraesentantur per $QQ^1Q^2Q^3$ et $TT^1T^2T^3$ atque $VV^1V^2V^3$ atque se mutuo in puncto *A* intersecant. Haec tria plana, si in infinitum quaquaversus producta concipiantur, universum

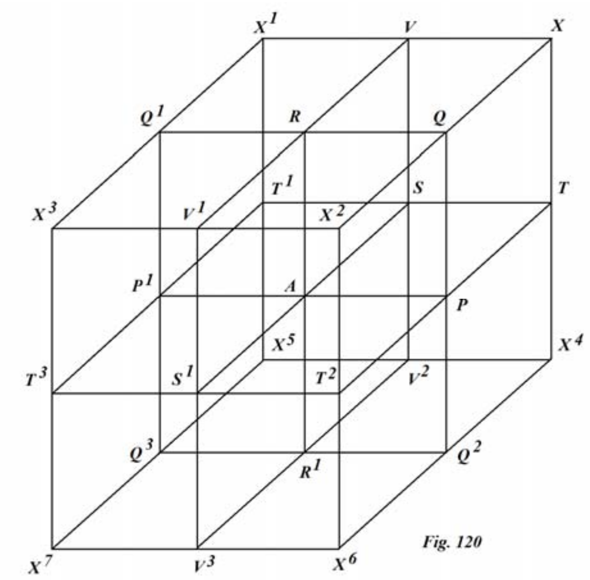

Fig. 120

spatium divident in octo regiones, quae in figura exhibentur litteris AX, AX^1, AX^2, AX^3, AX^4, AX^5, AX^6 et AX^7. Quodsi, iam in prima regione AX variabiles x, y et z affirmativos
valores habere ponantur, in reliquis regionibus una vel duae vel omnes tres fient negativae. Ratio autem horum valorum clarissime ex sequenti schemate perspicietur

Regio AX	Regio AX^1	Regia AX^2	Regia AX^3
$AP=+x$	$AP^1=-x$	$AP=+x$	$AP^1=-x$
$AR=+y$	$AR=+y$	$AR=+y$	$AR=+y$
$AS=+z$	$AS=+z$	$AS=-z$	$AS^1=-z$
Regio AX^4	Regio AX^5	Regia AX^6	Regio AX^7
$AP=+x$	$AP^1=-x$	$AP=+x$	$AP^1=-x$
$AR^1=-y$	$AR^1=-y$	$AR^1=-y$	$AR^1=-y$
$AS=+z$	$AS=+z$	$AS^1=-z$	$AS^1=-z$

16. Commodius autem erit octo has diversas regiones numeris insignire, quo facilius, de quanam sermo sit, indicare queamus. Cum igitur (Fig. 121) octo istae regiones in puncta A sint confines atque intersectione trium planorum inter se normalium distinguantur, plana autem haec tribus rectis Pp, Qq, Rr sese in puncta A normaliter decussantibus determinentur, regiones illae tribus litteris P, Q, R vel maiusculis vel minusculis definiri poterunt. Regio scilicet principalis seu prima PQR erit spatium, quod parallelepipedum ex tribus rectis AP, AQ, AR in infinitum productis formatum complectitur; et regio Pqr erit spatium, quod parallelepipedum ex tribus rectis AP, Aq, Ar in infinitum productis formatum includet. Positis ergo tribus variabilibus $AP=x$, $AQ=y$, $AR=z$ erit utique $Ap=-x$, $Aq=-y$ et $Ar=-z$. Sequenti ergo modo octo has regiones numeris distinguemus, ut sit inter coordinatas

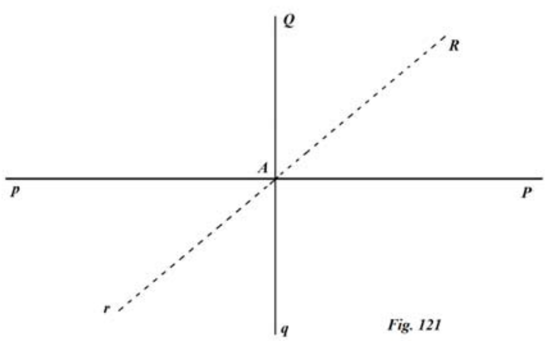

Fig. 121

prima I *PQR* inter coordinatas $\left\{\begin{matrix} AP=+x \\ AQ=+y \\ AR=+z \end{matrix}\right.$	secunda II *PQr* $\left.\begin{matrix} AP=+x \\ AQ=+y \\ AR=-z \end{matrix}\right\}$
tertia Ill *PqR* inter coordinatas $\left\{\begin{matrix} AP=+x \\ Aq=-y \\ AR=+z \end{matrix}\right.$	quarta IV *pQR* $\left.\begin{matrix} Ap=-x \\ AQ=+y \\ AR=+z \end{matrix}\right\}$
quinta V *Pqr* inter coordinatas $\left\{\begin{matrix} AP=+x \\ Aq=-y \\ Ar=-z \end{matrix}\right.$	sexta VI *pQr* $\left.\begin{matrix} Ap=-x \\ AQ=+y \\ Ar=-z \end{matrix}\right\}$
septima VII *pqR* inter coordinatas $\left\{\begin{matrix} Ap=-x \\ Aq=-y \\ AR=+z \end{matrix}\right.$	octava VIII *pqr* $\left.\begin{matrix} Ap=-x \\ Aq=-y \\ Ar=-z \end{matrix}\right\}$

17. Regiones istae vel magis vel minus a se invicem discrepant. Primo nimirum dantur binae regiones, quae duas coordinatas habent communes, unica discrepante; ideoque plano se invicem tangunt, quas vocemus *coniunctas*. Deinde, si duae coordinatae fuerint diversae unicamque habeant communem, regiones linea recta tantum se tangent, quas vocemus *disiunctas.* Tertio, si omnes coordinatae signis dissentiant, regiones tantummodo in puncto A se tangent hasque *oppositas* vocabimus. Quae iam regiones cuique sint coniunctae vel disiunctae vel oppositae, sequens tabella exhibebit:

Regio	*Coniunctae*			*Disiunctae*			*Oppositae*
PQR I	*PQr* II	*PqR* Ill	*pQR* IV	*Pqr* V	*pQr* VI	*pqR* VII	*pqr* VIII
PQr II	*PQR* I	*Pqr* V	*pQr* VI	*PqR* Ill	*pQR* IV	*pqr* VIII	*pqR* VII
PqR III	*Pqr* V	*PQR* I	*pqR* VII	*PQr* II	*pqR* VIII	*pQR* IV	*pQr* VI
pQR IV	*pQr* VI	*pqR* VII	*PQR* I	*pqr* VIII	*PQr* II	*PqR* Ill	*Pqr* V

Pqr V	*PqR* Ill	*PQr* II	*pqr* VIII	*PQR* I	*pqR* VII	*pQr* VI	*pQR* IV
pQr VI	*pQR* IV	*pqr* VIII	*PQr* II	*pqR* VII	*PQR* I	*Pqr* V	*PqR* Ill
pqR VII	*pqr* VIII	*pQR* IV	*PqR* III	*pQr* VI	*Pqr* V	*PQR* I	*PQr* II
pqr VIII	*pqR* VII	*PQr* VI	*Pqr* V	*pQR* IV	*PqR* III	*PQr* II	*PQR* I

18. Patet ergo quamlibet regionem habere tres sibi coniunctas, totidem disiunctas unicamque oppositam, atque ex tabula praecedente statim perspicitur, quemadmodum quaelibet regio ad aliam quamcunque sit comparata. Ordo autem, quem numeri regiones denotantes in ista tabula tenent, attentione est dignus; qui ut melius in oculos incurrat, eosdem numeros eodem ordine quadrato sequenti inclusi:

1	2	3	4	5	6	7	8
2	1	5	6	3	4	8	7
3	5	1	7	2	8	4	6
4	6	7	1	8	2	3	5
5	3	2	8	1	7	6	4
6	4	8	2	7	1	5	3
7	8	4	3	6	5	1	2
8	7	6	5	4	3	2	1

Cuius indoles et proprietates levi attentione percipientur, usus vero in sequentibus uberius ob oculos ponetur.

19. Ante iam annotavimus, si in aequatione variabilis z ubique habeat pares dimensiones, tum superficiem duas esse habituram partes similes et aequales, pars scilicet in regione prima aequalis erit parti in secunda similique modo regiones tertia et quinta item quarta et sexta ac denique septima et octava inter se convenient, uti quadrati binae series ab 1 et 2 incipientes exhibent. Sin autem in aequatione variabilis y ubique pares habeat dimensiones, tum regio prima cum tertia, secunda cum quinta, quarta cum septima et sexta cum octava congruet. Sed six in aequatione ubique pares habeat dimensiones, tum regio prima cum quarta, secunda cum sexta, tertia cum septima et quinta cum octava congruet. Scilicet

si in aequatione pares ubique habeat dimensiones variabilis

z	y	x
convenient regions	*convenient regiones*	*convenient regions*
1, 2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4, 5, 6, 7, 8
2, 1, 5, 6, 3, 4, 8, 7	3, 5, 1, 7, 2, 8, 4, 6	4, 6, 7, 1, 8, 2, 3, 5

20. Ut partes superficiei in regionibus disiunctis prima et quinta sitae inter se sint aequales, tum aequationem ita comparatam esse oportet, ut maneat eadem, etiamsi binae variabiles *y* et *z* negativae accipiantur. Hoc igitur eveniet, si ambae *y* et *z* in singulis aequationis terminis vel pares ubique vel impares dimensiones iunctim sumtae constituant. Quodsi autem regio prima congruat cum quinta, tum secunda cum tertia, quarta cum octava et sexta cum septima conveniet. Simili modo si in aequatione pro superficie binae variabiles *x* et *z* vel parem ubique dimensionum numerum vel imparem ubique adimpleant, tum regio prima cum sexta, secunda cum quarta, tertia cum octava et quinta cum septima congruet. Scilicet

Si in aequatione pro superficie ubique vel pares vel ubique impares ad impleant dimensiones variabiles

y et *z*	*x* et *z*	*x* et *y*
congruent regiones	*congruent regiones*	*congruent regiones*
1,2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4, 5, 6, 7, 8
5, 3, 2, 8, 1, 7, 6, 4	6, 4, 8, 2, 7, 1, 5, 3	7, 8, 4, 3, 6, 5, 1, 2

Quodsi autem omnes tres variabiles x, y et z iunctim consideratae ubique vel pares vel ubique impares teneant dimensiones, tum convenient regiones oppositae

1, 2, 3, 4, 5, 6, 7, 8
8, 7, 6, 5, 4, 3, 2, 1.

21. Si ex his conditionibus duae vel tres simul in aequatione inesse deprehendantur, tum vel quaternae vel omnes octo regiones partes superficiei similes et aequales continebunt. Scilicet

Si et x et y seorsim consideratae ubique pares obtineant dimensiones, tum sequentes quaternae regiones congruent

1, 2, 3, 4, 5, 6, 7, 8
3, 5, 1, 7, 2, 8, 4, 6
4, 6, 7, 1, 8, 2, 3, 5
7, 8, 4, 3, 6, 5, 1, 2.

Si et x et z seorsim consideratae ubique pares habeant dimensiones, tum sequentes quaternae regiones congruent

1, 2, 3, 4, 5, 6, 7, 8
2, 1, 5, 6, 3, 4, 8, 7
4, 6, 7, 1, 8, 2, 3, 5
6, 4, 8, 2, 7, 1, 5, 3.

Si variabiles y et z seorsim consideratae ubique pares habeant dimensiones, tum sequentes quaternae regiones congruent

1, 2, 3, 4, 5, 6, 7, 8
2, 1, 5, 6, 3, 4, 8, 7
3, 5, 1, 7, 2, 8, 4, 6
5, 3, 2, 8, 1, 7, 6, 4.

22. Si una variabilium ubique pares habeat dimensiones, reliquae vero binae simul consideratae vel ubique pares vel ubique impares constituant dimensiones, tum quoque quaternae regiones congruent sequenti modo:

Si z ubique pares habeat dimensiones et x et y ubique vel pares vel impares dimensiones constituant, tum sequentes quaternae regiones congruent

1, 2, 3, 4, 5, 6, 7, 8
2, 1, 5, 6, 3, 4, 8, 7
7, 8, 4, 3, 6, 5, 1, 2
8, 7, 6, 5, 4, 3, 2, 1.

Si y ubique pares habeat dimensiones atque x et z ubique vel pares vel impares dimensiones iunctim constituant, tum sequentes quaternae regiones congruent

1, 2, 3, 4, 5, 6, 7, 8
3, 5, 1, 7, 2, 8, 4, 6
6, 4, 8, 2, 7, 1, 5, 3
8, 7, 6, 5, 4, 3, 2, 1.

Si x ubique pares habeat dimensiones atque y et z iunctim consideratae ubique vel pares vel impares constituant dimensiones, tum sequentes quaternae regiones congruent

1, 2, 3, 4, 5, 6, 7, 8
4, 6, 7, 1, 8, 2, 3, 5
5, 3, 2, 8, 1, 7, 6, 4
8, 7, 6, 5, 4, 3, 2, 1.

His ergo tribus casibus simul omnes tres variabiles x, y et z iunctim consideratae ubique vel pares vel impares dimensiones adimplebunt.

23. Supersunt sequentes casus quaternarum regionum aequalium

Si x et y, et y et z ubique vel pares vel ubique impares dimensiones constituant, tum sequentes quaternae regiones congruent

1, 2, 3, 4, 5, 6, 7, 8
5, 3, 2, 8, 1, 7, 6, 4
7, 8, 4, 3, 6, 5, 1, 2
6, 4, 8, 2, 7, 1, 5, 3.

Eaedem ergo similitudines prodeunt, si insuper binae reliquae variabiles x et z ubique vel pares vel impares dimensiones constituant, ita ut haec conditio iam in proposita contineatur. Portiones ergo superficiei in quaternis disiunctis regionibus erunt inter se aequales, si in aequatione binae quaeque variabiles iunctim consideratae ubique vel pares vel impares dimensiones constituant. Cum autem tres dentur combinationes, notandum est, si duae exposita proprietate fuerint praeditae, tum simul tertiam combinationem eadem proprietate esse gavisuram.

24. Quodsi ad conditiones, quae quaternas regiones similes et aequales produxerant, nova insuper accedat in iis non contenta, quae per se aequalitatem in binas regiones inferret, tum omnes prorsus regiones inter se fient aequales atque superficies constabit ex octo partibus inter se aequalibus et similibus. Aequatio ergo pro huiusmodi superficiebus omnes hactenus memoratas proprietates coniunctim possidebit, scilicet singulae variabiles x, y, z seorsim consideratae ubique pares constituent dimensiones; ex quo iam sequitur binas quasque iunctim consideratas atque etiam omnes tres simul sumtas ubique pares esse constituturas dimensiones.

25. Utrum autem aequatio inter tres variabiles proposita una duabusve vel adeo tribus exhibitarum proprietatum sit praedita an non, id quidem, quod ad cuiusvis variabilis pares dimensiones attinet, facile perspicitur. Neque difficilius est inquirere, utrum omnes variabiles simul consideratae ubique vel pares vel impares constituant dimensiones. At utrum binae tantum ad hanc proprietatem sint comparatae, difficilius erit examinare. Ponatur in aequatione vel $x = nz$ vel $y = nz$ vel $x = ny$ ac dispiciatur, utrum uno alterove casu aequatio resultet, in qua variabilis z duobus prioribus casibus vel y postremo casu ubique induat pares dimensiones; quod si eveniat, duae variables iunctim sumtae ubique vel pares vel impares dimensiones constituant necesse est hincque superficies duas saltem habebit partes inter se similes et aequales.