INTRODUCTION
TO THE
ANALYSIS OF THE INFINITES.

FIRST BOOK.

PREFACE

I have noticed repeatedly that the greatest part of the difficulties, which students of mathematics are accustomed to strike in learning the Analysis of the infinites, arises thence, because they address the mind to that higher art with ordinary algebra scarcely understood ; so it happens, that not only do novices stop as if on the threshold, but also they may form for themselves perverse ideas of that infinity, the notion of which is called upon as an aid to their understanding. But though the analysis of the infinite has not been perfected, at this stage it requires a knowledge of all the common tricks of algebra to be found, yet very many questions remain unanswered, the understanding of which questions acts to drive the mind towards a higher knowledge, which yet, are either omitted or not handled with enough care in the elements of common algebra. On that account I have no reason to doubt, why these things which I have gathered together in these books, may not be able to make good the omission in abundance. Indeed I have provided the help, not only in order that I might explain more fully and more carefully these things which the analysis of the infinite requires completely, than is commonly accustomed to arise, but also as I have unravelled many questions sufficiently well, by which readers besides may adapt themselves slowly to the expected and as if familiar idea of infinity. Here also I have resolved many questions by the precepts of common algebra, which are treated commonly in the analsyis of the infinite, by which henceforth the greatest agreement of each method may be shown.

I have divided this work into two books, in the first of which I have embraced work that pertains to pure analysis ; truly in the second, I have explained works which are by necessity known from geometry, because the analysis of the infinite is accustomed to be treated also in this way, so that the application of this to geometry likewise may be shown. Moreover in each book I have passed over first principles and I have been led to expound only matters which are found elsewhere, which are either not entirely or less conveniently discussed, or which are desired from different principles.

Therefore in the first book, since it is involved with the general analysis of the infinite concerned with variable quantities and functions of these, I have set out more fully the argument about functions especially, and I have shown both the transformation as well as the resolution, and the expansion of functions by infinite series. I have specified many kinds of functions, an account of which is especially required in higher analysis. Initially, I have divided these into algebraic and transcendental functions ; the first of which are formed by the common operations of common algebraic from variable quantities, these truly either are composed by other accounts or are effected by the same operations repeated indefinitely. The first subdivision of algebraic functions shall be into rational and irrational functions ; I have shown how to resolve the former into simpler parts and well as into factors, which operation brings the greatest aid to integral calculus ; just as truly I have shown how the latter by suitable substitutions may be able to be led to a rational form. But the setting out by an infinite series pertains equally to each kind and also is accustomed to be applied with the greatest use to transcending functions ; but there is nobody ignorant about the extend the teaching of infinites series may have increased higher analysis.

Therefore some chapters have added been added, in which I have examined carefully the sums and properties of several infinite series, thus certain of which have been prepared that without the help of the analysis of the infinite, might scarcely be considered able to be investigated. The series are of such a kind that their sums are expressed by logarithms or by circular arcs ; which since they shall be transcending quantities, are shown in turn by the quadrature of the hyperbola and of the circle, the greater part have been treated finally in the usual way in the analysis of the infinite. But afterwards I have progressed from powers to exponential quantities, which are nothing other than powers, the exponents of which are variables, from the conversion of these especially I have arrived at the productive and natural idea of logarithms; from which not only their fullest use has followed at once, but also from all these infinite series follow, by which it is possible to elicit these common quantities that are accustomed to be represented ; and hence from that a way has itself been found of constructing tables of logarithms. In a similar manner I have been active in the consideration of the arcs of circles, because the kind of quantities, although especially different from logarithms, yet have so close a connecting link, that, while the one is seen to become imaginary, the other becomes transcending. Moreover with matters returned from geometry, which are related to the finding of multiples and submultiples of the sines and cosines of arcs, I have expressed both the sine and cosine of the smallest arc from the sine or cosine of each arc, even to the extent that they are vanishing, and from which I have deduced the infinite series themselves. In turn from which, since the arc of the vanishing sine shall become equal to the sine itself, and with the cosine truly becoming equal to the radius; thus, I have compared any arc with its sine and cosine, with the aid of infinite series. Then truly so that I have obtained various expressions both finite as well as infinite for quantities of this kind, as there shall be no further need to look further into the nature of these with the infinitesimal calculus. And just as logarithms require a particular algorithm, the greatest use of which plays an outstanding role in general analysis, so thus I have induced circular quantities to become standard algorithms also, in order that the quantities in a calculation may be treated with equal convenience, both by logarithms as well as by the algebraic quantities themselves. But how much of this benefit will remain hence for the resolution of the more difficult questions, as indicated splendidly by several chapters of this book, as well as by several examples proffered from the analysis of the infinite – unless the methods may become known well enough now, and used more and more in the days to come.

Moreover this investigaton has brought the maximum assistance to resolving fractional functions into real factors ; because the account shall be of direct necessity in the integral calculus, I have explained it most diligently and in the greatest detail. Afterwards I have subjected infinite series to an examination, which arise from the establishment of functions of this kind, and which have become known by the name of recurring series ; where I have shown both the sum of these as well as the general terms and other significant properties, and because it leads in turn to this resolution in factors, thus in turn, just as the products may be set forth from several, I have indeed also considered carefully infinite factors placed together by multiplication in series. Because not only has the work uncovered the way towards knowing innumerable series, but because in this way series will be allowed to be resolved in products of infinite constant factors, I have found convenient enough numerical expressions,with the help of which the logarithms of sines, cosines and tangents are most easily added up. Besides also from the same source, I have derived the solutions of several questions, which can be proposed about the partition of numbers, the forces of analysis may consider overcoming questions of this kind without this aid.

This diversity of so much material could easily be increased into several volumes, but I have proposed everything succinctly, as far as possible, so that everywhere indeed the foundations are laid plainly, and indeed further embelishment may be left to the industry of the readers, and by exerting their efforts they may manage to move forwards to the futher limis of analysis. Nor indeed do I fear offering not only much that clearly is new in this book, but also the sources are to be laid bare from which the more significant material found can be accessed at this stage.

INTRODUCTIO
I N
ANALYSIN INFINITORUM.

LLBER PRIMUS

PRAEFATIO

Saepenumero animadverti maximam difficultatum partem, quas Matheseos cultores in addiscenda Analysi infinitorum offendere solent, inde oriri, quod Algebra communi vix apprehensa animum ad illam sublimiorem artem appellant; quo fit, ut non solum quasi in limine subsistant, sed etiam perversas ideas illius infiniti, cuius notio in subsidium vocatur, sibi forment. Quanquam autem Analysis infinitorum non perfectam Algebrae communis omniumque artificiorum adhuc inventorum cognitionem requirit, tamen plurimae extant quaestiones, quarum evolutio discentium animos ad sublimiorem scientiam praeparare valet, quae tamen in communibus Algebrae elementis vel omittuntur vel non satis accurate pertractantur. Hanc ob rem non dubito, quin ea, quae in his libris congessi, hunc defectum abunde supplere queant. Non solum enim operam dedi, ut eas res, quas Analysis infinitorum absolute requirit, uberius atque distinctius exponerem, quam vulgo fieri solet, sed etiam satis multas quaestiones enodavi, quibus lectores sensim et quasi praeter expectationem ideam infiniti sibi familiarem reddent. Plures quoque quaestiones per praecepta communis Algebrae hic resolvi, quae vulgo in Analysi infinitorum tractantur, quo facilius deinceps utriusque methodi summus consensus eluceat.

Divisi hoc opus in duos libros, in quorum priori, quae ad meram Analysin pertinent, sum complexus; in posteriori vero, quae ex Geometria sunt scitu necessaria, explicavi, quoniam Analysis infinitorum ita quoque tradi solet, ut simul eius applicatio ad Geometriam ostendatur. In utroque autem prima elementa praetermisi eaque tantum exponenda duxi, quae alibi vel omnino non vel minus commode tractata vel ex diversis principiis petita reperiuntur.

In prima igitur libro, cum universa Analysis infinitorum circa quantitates variabiles earumque functiones versetur, hoc argumentum de functionibus inprimis fusius exposui atque functionum tam transformationem quam resolutionem et evolutionem per series infinitas demonstravi. Complures enumeravi functionum species, quarum in Analysi sublimiori praecipue ratio est habenda. Primum eas distinxi in algebraicas et transcendentes; quarum illae per operationes in Algebra communi usitatas ex quantitatibus variabilibus formantur, hae vero vel per alias rationes componuntur vel ex iisdem operationibus infinities repetitis efficiuntur. Algebraicarum functionum primaria subdivisio fit in rationales et irrationales; priores docui cum in partes simpliciores tum in factores resolvere, quae operatio in Calculo integrali maximum adiumentum affert; posteriores vero quemadmodum idoneis substitutionibus ad formam rationalem perduci queant, ostendi. Evolutio autem per series infinitas ad utrumque genus aeque pertinet atque etiam ad functiones transcendentes summa cum utilitate applicari solet; at quantopere doctrina de seriebus infinitis Analysin sublimiorem amplificaverit, nemo est, qui ignoret.

Nonnulla igitur adiunxi capita, quibus plurium serierum infinitarum proprietates atque summas sum scrutatus, quarum quaedam ita sunt comparatae, ut sine subsidio Analyseos infinitorum vix investigari posse videantur. Huiusmodi series sunt, quarum summae exprimuntur vel per logarithmos vel arcus circulares; quae quantitates cum sint transcendentes, dum per quadraturam hyperbolae et circuli exhibentur, maximam partem demum in Analysi infinitorum tractari sunt solitae. Postquam autem a potestatibus ad quantitates exponentiales essem progressus, quae nil aliud sunt nisi potestates, quarum exponentes sunt variabiles, ex earum conversione maxime naturalem ac secundam logarithmorum ideam sum adeptus; unde non solum amplissimus eorum usus sponte est consecutus, sed etiam ex ea cunctas series infinitas, quibus vulgo istae quantitates repraesentari solent, elicere licuit; hincque adeo facillimus se prodidit modus tabulas logarithmorum construendi. Simili modo in contemplatione arcuum circularium sum versatus, quod quantitatum genus, etsi a logarithmis maxime est diversum, tamen tam arcto vinculo est connexum, ut, dum alterum imaginarium fieri videtur, in alterum transeat. Repetitis autem ex Geometria, quae de inventione sinuum et cosinuum arcuum multiplorum ac submultiplorum traduntur, ex sinu vel cosinu cuiusque arcus expressi sinum cosinumque arcus minimi et quasi evanescentis, quo ipso ad series infinitas sum deductus; unde, cum arcus evanescens sinui suo sit aequalis, cosinus vero radio, quemvis arcum cum suo sinu et cosinu ope serierum infinitarum comparavi. Tum vero tam varias expressiones cum finitas tum infinitas pro huius generis quantitatibus obtinui, ut ad earum naturam perspiciendam Calculo infinitesimali prorsus non amplius esset opus. Atque quemadmodum logarithmi peculiarem algorithmum requirunt, cuius in universa Analysi summus extat usus, ita quantitates circulares ad certam quoque algorithmi normam perduxi, ut in calculo aeque commode ac logarithmi et ipsae quantitates algebraicae tractari possent. Quantum autem hinc utilitatis ad resolutionem difficillimarum quaestionum redundet, cum nonnulla capita huius libri luculenter declarant, tum ex Analysi infinitorum plurima specimina proferri possent, nisi iam satis essent cognita et in dies magis .multiplicarentur.

Maximum autem haec investigatio attulit adiumentum ad functiones fractas in factores reales resolvendas; quod argumentum, cum in Calculo integrali sit prorsus necessarium, diligentius enucleavi. Series postmodum infinitas, quae ex huiusmodi functionum evolutione nascuntur et quae recurrentium nomine innotuerunt, examini subieci; ubi earum tam summas quam terminos generales aliasque insignes proprietates exhibui, et quoniam ad haec resolutio in factores manuduxit, ita vicissim, quemadmodum producta ex pluribus, immo etiam infinitis factoribus conflata per multiplicationem in series explicentur, perpendi. Quod negotium non solum ad cognitionem innumerabilium serierum viam aperuit, sed quia hoc modo series in producta ex infinitis factoribus constantia resolvere licebat, satis commodas inveni expressiones numericas, quarum ope logarithmi sinuum, cosinuum et tangentium facillime supputari possunt. Praeterea quoque ex eodem fonte solutiones plurium quaestionum, quae circa partitionem numerorum proponi possunt, derivavi, cuiusmodi quaestiones sine hoc subsidio vires Analyseos superare videantur.

Haec tanta materiarum diversitas in plura volumina facile excrescere potuisset, sed omnia, quantum fieri potuit, tam succincte proposui, ut ubique fundamentum clarissime quidem explicaretur, uberior vero amplificatio industriae lectorum relinqueretur, quo habeant, quibus vires suas exerceant finesque Analyseos ulterius promoveant. Neque enim vereor profiteri in hoc libro non solum multa plane nova contineri, sed etiam fontes esse detectos, unde plurima insignia inventa adhuc hauriri queant.