CHAPTER IX

CONCERNING THE INVESTIGATION OF TRINOMIAL FACTORS.

143. To the extent that it may be required to find the simple factors of each whole function, we have shown above [§ 29] how this can happen through the resolution of equations. For if some whole function shall be proposed

$$\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$$

of which the simple factors of the form $p-qz$ are sought, it is clear, if $p-qz$ were a factor of the function $\alpha+\beta z+\gamma z^2+\text{etc.}$, then on putting $z=\frac{p}{q}$, in which case the factor $p-qz$ is made $=0$, also the proposed function itself must vanish. Hence if $p-qz$ shall be a factor or divisor of the function

$$\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.};$$

it follows that this expression arises

$$\alpha+\frac{\beta p}{q}+\frac{\gamma p^2}{q^2}+\frac{\delta p^3}{q^3}+\frac{\varepsilon p^4}{q^4}+\text{etc.}=0.$$

From which reciprocally, if all the roots $\frac{p}{q}$ of this equation may be elicited, just as many individual simple factors of the proposed whole function will be given

$$\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$$

evidently of the form $p-qz$. Moreover likewise it is apparent that the number of simple factors of this kind be defined from the maximum power of z.

144. But generally the imaginary factors are elicited with difficulty in this manner, on which account I examine a special method in this chapter, with the help of which on many occasions the simple imaginary factors may be able to be found. Truly because the simple imaginary factors are to be prepared thus, so that the product of the pairs become real, we will find those imaginary factors themselves, if we investigate certain real double factors of this form

$$p-qz+rzz,$$

but of which the simple factors shall be imaginary. Because if indeed all the real double factors of the function $\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$ may be agreed to be of this trinomial form $p-qz+rzz$, likewise all the imaginary factors will be obtained.

145. Moreover the trinomial $p-qz+rzz$ will have simple imaginary factors if there should be $4pr>qq$ or

$$\frac{q}{2\sqrt{pr}}<1.$$

Therefore since the sine and cosine of angles shall be less than one, the formula $p-qz+rzz$ will have simple imaginary factors if there were $\frac{q}{2\sqrt{pr}}=$ to the sine or cosine of a certain angle. Therefore let

$$\frac{q}{2\sqrt{pr}}=\cos.\varphi \text{ or } q=2\sqrt{pr}\cdot\cos.\varphi$$

and the trinomial $p-qz+rzz$ will contain simple imaginary factors. But in order that the irrationality may do no harm, I shall assume this form

$$pp-2pqz\cos.\varphi+qqzz,$$

the simple imaginary factors of which shall be these

$$qz-p\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right) \text{ and } qz-p\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right).$$

Where indeed it is apparent, if there should be $\cos.\varphi=\pm1$, then both factors on account of $\sin.\varphi=0$ become real and equal.

146. Therefore the simple imaginary factors may be elicited from the whole function of that $\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$, if the letters p and q with the angle φ may be determined, so that the trinomial $pp-2pqz\cos.\varphi+qqzz$ is made a factor of the function. Then indeed likewise these simple imaginary factors will be present

$$qz-p\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right) \text{ and } qz-p\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right).$$

On account of which the proposed function will vanish, if there may be put

$$z = \tfrac{p}{q}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)$$

as well as

$$z = \tfrac{p}{q}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right)$$

Hence with the substitution made, each two–fold factor arises from the equation, from which both the fraction $\tfrac{p}{q}$ as well as the arc φ will be able to be defined.

147. Moreover these substitutions being made in place of *z*, even if on first consideration may seem difficult, yet may be resolved readily enough through these, which have been treated in the previous chapter. For since we have shown to be

$$\left(\cos.\varphi \pm \sqrt{-1}\cdot\sin.\varphi\right)^n = \cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi,$$

the following formulas will be obtained substituted in place of the individual powers of *z*:

for the first factor	for the other factor
$z = \tfrac{p}{q}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)$	$z = \tfrac{p}{q}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right)$
$z^2 = \tfrac{p^2}{q^2}\left(\cos.2\varphi + \sqrt{-1}\cdot\sin.2\varphi\right)$	$z^2 = \tfrac{p^2}{q^2}\left(\cos.2\varphi - \sqrt{-1}\cdot\sin.2\varphi\right)$
$z^3 = \tfrac{p^3}{q^3}\left(\cos.3\varphi + \sqrt{-1}\cdot\sin.3\varphi\right)$	$z^3 = \tfrac{p^3}{q^3}\left(\cos.3\varphi - \sqrt{-1}\cdot\sin.3\varphi\right)$
$z^4 = \tfrac{p^4}{q^4}\left(\cos.4\varphi + \sqrt{-1}\cdot\sin.4\varphi\right)$	$z^4 = \tfrac{p^4}{q^4}\left(\cos.4\varphi - \sqrt{-1}\cdot\sin.4\varphi\right)$
etc.	etc.

For brevity's sake, put $\tfrac{p}{q} = r$, and with the substitution made the two following equations will be produced :

$$0 = \begin{Bmatrix} \alpha + \beta r\cos.\varphi + \gamma r^2\cos.2\varphi + \delta r^3\cos.2\varphi + \text{etc.} \\ + \beta r\sqrt{-1}\cdot\sin.\varphi + \gamma r^2\sqrt{-1}\cdot\sin.2\varphi + \delta r^3\sqrt{-1}\cdot\sin.3\varphi + \text{etc.} \end{Bmatrix}$$

$$0 = \begin{Bmatrix} \alpha + \beta r\cos.\varphi + \gamma r^2\cos.2\varphi + \delta r^3\cos.2\varphi + \text{etc.} \\ - \beta r\sqrt{-1}\cdot\sin.\varphi - \gamma r^2\sqrt{-1}\cdot\sin.2\varphi - \delta r^3\sqrt{-1}\cdot\sin.3\varphi - \text{etc.} \end{Bmatrix}$$

148. But if these two equations in turn may be added and subtracted and in the latter case divided by $2\sqrt{-1}$, these two real equations will be produced :

$$0=\alpha+\beta r\cos.\varphi \quad +\gamma r^2\cos.2\varphi \quad +\delta r^3\cos.2\varphi \quad +\text{etc.}$$
$$0= \quad \beta r\sqrt{-1}\cdot\sin.\varphi+\gamma r^2\sqrt{-1}\cdot\sin.2\varphi+\delta r^3\sqrt{-1}\cdot\sin.3\varphi+\text{etc.}$$

which at once are able to be formed from the form of the proposed function

$$\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$$

by putting first for each power of z

$$z^n=r^n\cos.n\varphi\,,$$

and then

$$z^n=r^n\sin.n\varphi\,.$$

Thus indeed on account of $\sin.0\varphi=0$ and $\cos.0\varphi=1$ for z^0 or 1 in the first case of the constant term 1 is put , but in the latter place 0.

Therefore if from these two equations the unknowns r and φ may be defined, on account of $r=\frac{p}{q}$ the factor of the trinomial function will be found

$$pp-2pqz\cos.\varphi+qqzz$$

involving two simple imaginary numbers.

149. If the first equation above may be multiplied by $\sin.m\varphi$, the second by $\cos.m\varphi$ and the products either added or subtracted, these two equations will be produced :

$$0=\alpha\sin.m\varphi+\beta r\sin.(m+1)\varphi+\gamma r^2\sin.(m+2)\varphi+\delta r^3\sin.(m+3)\varphi+\text{etc.},$$
$$0=\alpha\sin.m\varphi+\beta r\sin.(m-1)\varphi+\gamma r^2\sin.(m-2)\varphi+\delta r^3\sin.(m-3)\varphi+\text{etc.}$$

But if the first equation may be multiplied by $\cos.m\varphi$ and the second by $\sin.m\varphi$, on adding and subtracting the following equations will emerge :

$$0=\alpha\cos.m\varphi+\beta r\cos.(m-1)\varphi+\gamma r^2\cos.(m-2)\varphi+\delta r^3\cos.(m-3)\varphi+\text{etc.},$$
$$0=\alpha\cos.m\varphi+\beta r\cos.(m+1)\varphi+\gamma r^2\cos.(m+2)\varphi+\delta r^3\cos.(m+3)\varphi+\text{etc.}$$

Therefore any two equations of this kind taken together will determine the unknowns r and φ which can happen generally from several kinds, and likewise several trinomial factors are obtained, and thus all these which the proposed function contains within itself.

150. So that the use of these rules may be made clearer, we will investigate here the trinomial factors of certain functions that occur more often, so that hence it will be allowed to produce these, as often as the occasion may arise. Thus let this be the proposed function :

$$a^n+z^n,$$

the trinomial factors of which of the form

$$pp-2pqz\cos.\varphi+qqzz$$

may be able to be determined. Therefore on putting $r=\frac{p}{q}$ these two equations will be had

$$0=a^n+r^n\cos.n\varphi \quad \text{and} \quad 0=r^n\sin.n\varphi,$$

the latter of which gives

$$\sin.n\varphi=0;$$

from which $n\varphi$ will be the arc either of the form $(2k+1)\pi$ or $2k\pi$ with k denoting a whole number. I distinguish thus these cases, because the cosines of these are different ; for in the first case there will be $\cos.(2k+1)\pi=-1$, but in the second case $\cos.2k\pi=+1$. But it is apparent in the first case the form

$$n\varphi=(2k+1)\pi$$

must be taken, clearly which gives $\cos.n\varphi=-1$, from which there becomes

$$0=a^n-r^n$$

and hence again,

$$r=a=\frac{p}{q}.$$

Therefore there will be

$$p = a,\ q = 1$$

and

$$\varphi = \tfrac{2k+1}{n}\pi$$

from which a factor of the function $a^n + z^n$ will be

$$aa - 2az\cos.\tfrac{2k+1}{n}\pi + zz.$$

Therefore since each integer can be put in place for the number *k*, several factors not yet infinite emerge in this manner, because if $2k+1$ may be increased beyond *n*, previous factors are returned, which will be made clearer from examples, since there shall be $\cos.(2\pi \pm \varphi) = \cos.\varphi$. Then if *n* is an odd number, on putting $2k+1 = n$ the quadratic $aa + 2az + zz$ will be a factor; nor truly hence does it follow that the quadratic $(a+z)^2$ is a factor of the function $a^n + z^n$, because (in § 148) a single equation comes about, from which it is apparent only that $a + z$ is a divisor of the formula $a^n + z^n$; which rule is required to be maintained always, whenever $\cos.\varphi$ becomes either + 1 or –1.

EXAMPLE

We may set out some cases, by which these factors are made clearer on being viewed, and we may separate these into two classes, as *n* may be made an even or odd number.

If $n = 1$, a factor of the formula $a + z$, is $a + z$.	If $n = 2$, a factor of the formula $a^2 + z^2$, is $a^2 + z^2$
If $n = 3$, the factors of the formula $a^3 + z^3$, are $aa - 2az\cos.\frac{1}{3}\pi + zz$, $a + z$.	If $n = 4$, the factors of the formula $a^4 + z^4$, are $aa - 2az\cos.\frac{1}{4}\pi + zz$, $aa - 2az\cos.\frac{3}{4}\pi + zz$.
If $n = 5$, the factors of the formula $a^5 + z^5$ are, $aa - 2az\cos.\frac{1}{5}\pi + zz$, $aa - 2az\cos.\frac{3}{5}\pi + zz$, $a + z$.	If $n = 6$, the factors of the formula $a^6 + z^6$ are, $aa - 2az\cos.\frac{1}{6}\pi + zz$, $aa - 2az\cos.\frac{3}{6}\pi + zz$, $aa - 2az\cos.\frac{5}{6}\pi + zz$.

From which examples it is apparent that all the factors to be found, if in place of $2k+1$ all the odd numbers not greater than the exponent may be substituted, truly in these cases, in which a square factor will emerge, only the root of this must be enumerated from the factors.

151. If this function may be proposed

$$a^n - z^n,$$

a trinomial factor of this will be

$$pp - 2pqz\cos.\varphi + qqzz,$$

if on putting $r = \frac{p}{q}$ it became

$$0 = a^n - r^n\cos.n\varphi \text{ and } 0 = r^n\sin.n\varphi.$$

Therefore again there becomes

$$\sin.n\varphi = 0$$

and thus $n\varphi = (2k+1)\pi$ or $n\varphi = 2k\pi$. But in this case the latter value must be taken, so that there shall be $\cos.n\pi = +1$, which gives $0 = a^n - r^n$ and $r = \frac{p}{q} = a$.
And thus there will be found

$$p = a,\ q = 1$$

and

$$\varphi = \frac{2k}{n}\pi,$$

from which the trinomial factor of the proposed formula will be

$$aa - 2az\cos.\frac{2k}{n}\pi + zz\ ;$$

which form, if all the even numbers not greater than *n* may be put in place of $2k$, likewise it will give all the factors ; where with regard to the factors of squares the same condition is required to be held, which we mentioned before. And in the first place on putting $k = 0$ the factor $aa - 2az + zz$ will arise, for which truly the root $a - z$ must be taken. Similarly, if *n* were an even number and there may be put $2k = n$, $aa + 2az + zz$ will emerge, from which it is apparent that $a + z$ is a divisor of the form $a^n - z^n$.

EXAMPLE

The cases with the exponents n as treated before thus themselves will be considered, exactly as n were an even or an odd number.

If $n = 1$, a factor of the formula $a - z$, is itself $a - z$.	If $n = 2$, the factors of the formula $a^2 - z^2$, will be $a + z,\ a - z$
If $n = 3$, the factors of the formula $a^3 - z^3$, will be $a - z$, $aa - 2az\cos.\frac{2}{3}\pi + zz$	If $n = 4$, the factors of the formula $a^4 - z^4$, will be $a - z, a + z$, $aa - 2az\cos.\frac{2}{4}\pi + zz$.
If $n = 5$, the factors of the formula $a^5 - z^5$, will be $a - z$, $aa - 2az\cos.\frac{2}{5}\pi + zz$, $aa - 2az\cos.\frac{4}{5}\pi + zz$.	If $n = 6$, the factors of the formula $a^6 - z^6$, will be $a - z, a + z$, $aa - 2az\cos.\frac{2}{6}\pi + zz$, $aa - 2az\cos.\frac{4}{6}\pi + zz$.

152. Therefore from these it is confirmed, what we have intimated now above [§ 32], every integral function, if it cannot be resolved into simple real factors, yet is able to be resolved into two–fold real factors. For we have seen this function of indefinite dimensions $a^n \pm z^n$ is able to be resolved into two–fold real factors always besides the simple real factors. [These factors we would now call, of course, linear and quadratic.]

Therefore we may progress to more composite functions, to use $a + \beta z^n + rz^{2n}$, of which indeed the resolution to be abundantly clear from the preceding, if the two factors may have the form $\eta + \theta z^n$,. Therefore this will be required to be done, so that we may demonstrate the resolution into real factors, either simple or two–fold, in the case of the form $a + \beta z^n + rz^{2n}$, which does not have two real factors of the form $\eta + \theta z^n$.

153. Therefore we will consider this function $a^{2n} - 2a^n z^n \cos.g + z^{2n}$, which cannot be resolved into two real factors of the form $\eta + \theta z^n$. But if therefore we may put the real two–fold factor of this function to be

$$pp - 2pqz\cos.\varphi + qqzz,$$

on putting $r = \frac{p}{q}$ the two following equations will be required to be resolved :

$$0 = a^{2n} - 2a^n r^n \cos. g \cos. n\varphi + r^{2n} \cos. 2n\varphi$$

and

$$0 = -2a^n r^n \cos. g \sin. n\varphi + r^{2n} \sin. 2n\varphi .$$

Or in place of the first equation this may be taken from § 149 (on putting $m = 2n$)

$$0 = a^{2n} \sin. 2n\varphi - 2a^n r^n \cos. g \sin. n\varphi ,$$

which taken with the latter gives $r = a$; then truly there will be

$$\sin. 2n\varphi = 2\cos. g \sin. n\varphi .$$

But there is

$$\sin. 2n\varphi = 2\sin. n\varphi \cos. n\varphi ,$$

from which there becomes

$$\cos. n\varphi = \cos. g .$$

But there is always $\cos.(2k\pi \pm g) = \cos. g$, from which there is had

$$n\varphi = 2k\pi \pm g$$

and

$$\varphi = \frac{2k\pi \pm g}{n}$$

Hence therefore the two–fold factor of the general form will be

$$= aa - 2az\cos. \frac{2k\pi \pm g}{n} + zz$$

and all the factors will be produced, if for 2*k* all the even numbers not greater than *n* may be substituted successively, as will be able to be seen from the application to cases.

EXAMPLE

Therefore we may consider the case, in which *n* is 1, 2, 3, 4 etc., so that the account of the factors may be apparent. Therefore of the formula

$$aa - 2az\cos. g + zz$$

there will be the single factor

$$aa - 2az\cos. g + zz;$$

of the formula

$$a^4 - 2a^2z^2\cos. g + z^4$$

the two factors

$$aa - 2az\cos.\tfrac{g}{2} + zz,$$

$$aa - 2az\cos.\tfrac{2\pi \pm g}{2} + zz \text{ or } aa + 2az\cos.\tfrac{g}{2} + zz;$$

of the formula

$$a^6 - 2a^3z^3\cos. g + z^6$$

the three factors

$$aa - 2az\cos.\tfrac{g}{3} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi - g}{3} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi + g}{3} + zz;$$

of the formula

$$a^8 - 2a^4z^4\cos. g + z^8$$

the four factors

$$aa - 2az\cos.\tfrac{g}{4} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi - g}{4} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi + g}{4} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi \pm g}{4} + zz \text{ or } aa + 2az\cos.\tfrac{g}{4} + zz;$$

of the formula

$$a^{10} - 2a^5z^5\cos. g + z^{10}$$

the five factors

$$aa - 2a.z\cos.\tfrac{g}{5} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi-g}{5} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi+g}{5} + zz,$$
$$aa - 2az\cos.\tfrac{4\pi-g}{5} + zz,$$
$$aa - 2az\cos.\tfrac{4\pi+g}{5} + zz.$$

Therefore also from these examples it is confirmed every integral function can be resolved into real factors, either simple or two–fold.

154. Hence it is permissible to progress further to this function

$$\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n},$$

which surely will have one real factor of the form $\eta + \theta z^n$, the real factors of which therefore are able to be shown, either simple or two–fold; truly the other multiplier of the form $\iota + \kappa z^n + \lambda z^{2n}$, in whatever manner it should be prepared, will be resolved into factors in a like manner by the proceeding paragraph.
Then this function

$$\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n}$$

since it may have always two real factors of this form $\eta + \theta z^n + \iota z^{2n}$, similarly is resolved into either simple factors or two–fold real factors in factors.
Why not also be allowed to progress to the form

$$\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n} + \xi z^{5n}$$

which with certainty may have a single factor of the form $\eta + \theta z^n$, the other factor will be of the preceding form, from which also this function will be granted a resolution into real factors, either simple or two–fold.

Whereby if any doubt had remained about the resolution of whole functions, this now may be removed almost entirely.

155. Also it is possible to treat these in factors resolved into infinite series ; because of course we have seen above [§ 123] that there is

$$1 + x + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.} = e^x,$$

but truly to be

$$e^x = \left(1+\tfrac{x}{i}\right)^i$$

with *i* denoting some infinite number, it is seen that the series

$$1+\tfrac{x}{1}+\tfrac{x^2}{1\cdot2}+\tfrac{x^3}{1\cdot2\cdot3}+\text{etc.}$$

has infinitely many simple factors equal to each other, evidently $1+\tfrac{x}{i}$. But if from the same series the first term may be taken away, it will become

$$\tfrac{x}{1}+\tfrac{x^2}{1\cdot2}+\tfrac{x^3}{1\cdot2\cdot3}+\tfrac{x^4}{1\cdot2\cdot3\cdot4}+\text{etc.}=e^x-1=\left(1+\tfrac{x}{i}\right)^i-1$$

of which the form may be compared with §151 [*i.e.* a^n-z^n], so that there becomes

$$a=1+\tfrac{x}{i}, n=i \text{ and } z=1,$$

whichever factor will be

$$=\left(1+\tfrac{x}{i}\right)^2-2\left(1+\tfrac{x}{i}\right)\cos.\tfrac{2k}{i}\pi+1,$$

from which with all the even numbers being substituted for *2k* likewise all the factors will be produced.

Moreover on putting $2k=0$ the square factor $\tfrac{xx}{ii}$ will be produce but for which on account of the reasons mentioned only the root must be taken $\tfrac{x}{i}$; therefore there will be the factor *x* of the expression e^x-1, which indeed is apparent at once. Towards finding the remaining factors it is necessary to noted on account of the infinitely small arc $\tfrac{2k}{i}\pi$

$$\cos.\tfrac{2k}{i}\pi=1-\tfrac{2kk}{ii}\pi\pi$$

(§ 134) with the following terms on account of the infinite number *i* becoming nothing. Hence any factor will be

$$\tfrac{xx}{ii}+\tfrac{4kk}{ii}\pi\pi+\tfrac{4kk\pi\pi}{i^3}x$$

and thus the form e^x-1 will be divisible by $1+\tfrac{x}{i}+\tfrac{xx}{4kk\pi\pi}$.
Whereby the expression

$$e^x-1=x\left(1+\tfrac{x}{1\cdot2}+\tfrac{x^2}{1\cdot2\cdot3}+\tfrac{x^3}{1\cdot2\cdot3\cdot4}+\text{etc.}\right)=\left(1+\tfrac{x}{i}\right)^i-1$$

besides the factor *x*, will have these infinite factors

$$\left(1+\frac{x}{i}+\frac{xx}{4\pi\pi}\right)\left(1+\frac{x}{i}+\frac{xx}{36\pi\pi}\right)\left(1+\frac{x}{i}+\frac{xx}{64\pi\pi}\right) \text{ etc.}$$

156. But since these factors contain the infinitely small part $\frac{x}{i}$ which, since it shall be present in the individual terms and by multiplication of all, of which the number is $\frac{1}{2}i$, the term will be produced $\frac{x}{2}$ that cannot be omitted, towards avoiding this inconvenience we will consider this expression

$$e^x - e^{-x} = \left(1+\frac{x}{i}\right)^i - \left(1-\frac{x}{i}\right)^i = 2\left(\frac{x}{1}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.}\right);$$

for there is

$$e^{-x} = 1-\frac{x}{1}+\frac{x^2}{1\cdot 2}-\frac{x^3}{1\cdot 2\cdot 3}+\text{etc.}$$

Which with compared with §151 gives

$$n=i,\ a=1+\frac{x}{i} \text{ et } z=1-\frac{x}{i},$$

from which a factor of this expression will be

$$= aa-2az\cos.\frac{2k}{n}\pi+zz$$
$$=2+\frac{2xx}{ii}-2\left(1-\frac{xx}{ii}\right)\cos.\frac{2k}{i}\pi=\frac{4xx}{ii}+\frac{4kk}{ii}\pi\pi-\frac{4kk\pi\pi xx}{i^4}$$

on account of

$$\cos.\frac{2k}{i}\pi = 1-\frac{2kk}{ii}\pi\pi.$$

Therefore the function $e^x - e^{-x}$ will be divisible by

$$1+\frac{xx}{kk\pi\pi}-\frac{xx}{ii}$$

but when the term $\frac{xx}{ii}$ is omitted without risk, because, even if it may be multiplied by *i*, yet it may remain infinitely small. Besides truly as before, if $k=0$, the first factor will be $=x$. On account of which with these factors reduced in order there will be :

$$\frac{e^x - e^{-x}}{2} = x\left(1+\frac{xx}{\pi\pi}\right)\left(1+\frac{xx}{4\pi\pi}\right)\left(1+\frac{xx}{9\pi\pi}\right)\left(1+\frac{xx}{16\pi\pi}\right)\left(1+\frac{xx}{25\pi\pi}\right) \text{ etc.}$$
$$= x\left(1+\frac{xx}{1\cdot2\cdot3}+\frac{x^4}{1\cdot2\cdot3\cdot4\cdot5}+\frac{x^6}{1\cdot2\cdots7}+\text{etc.}\right).$$

It is evident I have given a form of this agreed kind through multiplication with the individual factors, so that by the actual multiplication the first term x may result.

157. In the same manner since there shall be

$$\frac{e^x + e^{-x}}{2} = 1+\frac{xx}{1\cdot2}+\frac{x^4}{1\cdot2\cdot3\cdot4}+\text{etc.} = \frac{\left(1+\frac{x}{i}\right)^i+\left(1-\frac{x}{i}\right)^i}{2},$$

the comparison of this expression with the above [§150] $a^n + z^n$ will give

$$a = 1+\frac{x}{i},\ z = 1-\frac{x}{i} \text{ and } n = i;$$

therefore any factor will be

$$= aa - 2az\cos.\frac{2k+1}{n}\pi + zz = 2+\frac{2xx}{ii} - 2\left(1-\frac{xx}{ii}\right)\cos.\frac{2k+1}{i}\pi$$

But there is

$$\cos.\frac{2k+1}{i}\pi = 1-\frac{(2k+1)^2}{2ii}\pi\pi,$$

from which the form of the factors will be

$$\frac{4xx}{ii}+\frac{(2k+1)^2}{ii}\pi\pi$$

with the term vanishing, of which the denominator is i^4. Therefore because the whole factor of the expression

$$1+\frac{xx}{1\cdot2}+\frac{x^4}{1\cdot2\cdot3\cdot4}+\text{etc.}$$

of this kind must have the form $1+axx$, so that the factor found may be reduced to this form, it must be divided by $\frac{(2k+1)^2\pi^2}{ii}$; hence a factor of the proposed form will be

$$1+\frac{4xx}{(2k+1)^2\pi\pi}$$

and from that all the factors may be found to infinity, if in place of $2k+1$ successively all the odd numbers may be substituted. On this account the original expansion will be

$$\frac{e^{x}+e^{-x}}{2}=1+\frac{xx}{1\cdot 2}+\frac{x^{4}}{1\cdot 2\cdot 3\cdot 4}+\frac{x^{6}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}$$
$$=\left(1+\frac{4xx}{\pi\pi}\right)\left(1+\frac{4xx}{9\pi\pi}\right)\left(1+\frac{4xx}{25\pi\pi}\right)\left(1+\frac{4xx}{49\pi\pi}\right)\text{ etc.}$$

158. If x becomes an imaginary quantity, these exponential formulas are changed into the sines and cosines of a certain real arc. Indeed let $x=z\sqrt{-1}$; there becomes

$$\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{2}=\sin .z=z-\frac{z^{3}}{1\cdot 2\cdot 3}+\frac{z^{5}}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{z^{7}}{1\cdot 2\cdot 3\cdots 7}+\text{etc.},$$

thus which expression has the factors infinite in number

$$z\left(1-\frac{zz}{\pi\pi}\right)\left(1-\frac{zz}{4\pi\pi}\right)\left(1-\frac{zz}{9\pi\pi}\right)\left(1-\frac{zz}{16\pi\pi}\right)\left(1-\frac{zz}{25\pi\pi}\right)\text{ etc.},$$

or there becomes

$$\sin .z=z\left(1-\frac{z}{\pi}\right)\left(1+\frac{z}{\pi}\right)\left(1-\frac{z}{2\pi}\right)\left(1+\frac{z}{2\pi}\right)\left(1-\frac{z}{3\pi}\right)\left(1+\frac{z}{3\pi}\right)\text{etc.}$$

Therefore as many times as the arc z thus has been prepared, so that a certain factor vanishes, which comes about, if $z=0, z=\pm\pi$, $z=\pm 2\pi$ and generally if $z=\pm k\pi$ with k denoting some whole number, likewise the sine of this arc must be $=0$, which indeed thus is apparent, so that hence those following factors may have been allowed to be elicited.

In a similar manner since there shall be

$$\frac{e^{z\sqrt{-1}}+e^{-z\sqrt{-1}}}{2}=\cos .z,$$

also there will be

$$\cos .z=\left(1-\frac{4zz}{\pi\pi}\right)\left(1-\frac{4zz}{9\pi\pi}\right)\left(1-\frac{4zz}{25\pi\pi}\right)\left(1-\frac{4zz}{49\pi\pi}\right)\text{ etc.}$$

or from these factors being resolved in pairs, the relation becomes also

$$\cos .z=\left(1-\frac{2z}{\pi}\right)\left(1+\frac{2z}{\pi}\right)\left(1-\frac{2z}{3\pi}\right)\left(1+\frac{2z}{3\pi}\right)\left(1-\frac{2z}{5\pi}\right)\left(1+\frac{2z}{5\pi}\right)\text{ etc.},$$

from which in an equal manner it becomes apparent, if there were $z=\pm\frac{2k+1}{2}\pi$, to become $\cos .z=0$, that which is also clear from the nature of the circle.

159. Also from § 153 the factors of this expression can be found :

$$e^x - 2\cos.g + e^{-x} = 2\left(1 - \cos.g + \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right).$$

For that expression can be changed into this

$$\left(1 + \frac{x}{i}\right)^i - 2\cos.g + \left(1 - \frac{x}{i}\right)^i$$

since which compared with that form gives

$$2n = i,\;\; a = 1 + \frac{x}{i} \text{ and } z = 1 - \frac{x}{i},$$

from which any factor of this formula will be

$$= aa - 2az\cos.\frac{2k\pi \pm g}{n} + zz = 2 + \frac{2xx}{ii} - 2\left(1 - \frac{xx}{ii}\right)\cos.\frac{2k\pi \pm g}{i}$$

but there is

$$\cos.\frac{2(2k\pi \pm g)}{i} = 1 - \frac{2(2k\pi \pm g)^2}{ii},$$

[since all the higher orders vanish,] from which the factor will be $= \frac{4xx}{ii} + \frac{4(2k\pi \pm g)^2}{ii}$ or of this form :

$$1 + \frac{xx}{(2k\pi \pm g)^2},$$

[since all the factors in the expression $1 + \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$ must be of the form $1 + \alpha xx$.]

Therefore if the expression may be divided by $2(1 - \cos.g)$, so that the constant term in the finite series shall be $= 1$, with all the factors taken there will be :

$$\frac{e^x - 2\cos.g + e^{-x}}{2(1 - \cos.g)}$$
$$= \left(1 + \frac{xx}{gg}\right)\left(1 + \frac{xx}{(2\pi - g)^2}\right)\left(1 + \frac{xx}{(2\pi + g)^2}\right)\left(1 + \frac{xx}{(4\pi - g)^2}\right)\left(1 + \frac{xx}{(4\pi + g)^2}\right)$$
$$\left(1 + \frac{xx}{(6\pi - g)^2}\right)\left(1 + \frac{xx}{(6\pi + g)^2}\right) \text{ etc.}$$

And if in place of x there is put $z\sqrt{-1}$, there becomes

$$\frac{\cos.z - \cos.g}{1-\cos.g}$$
$$= \left(1-\tfrac{z}{g}\right)\left(1+\tfrac{z}{g}\right)\left(1-\tfrac{z}{2\pi-g}\right)\left(1+\tfrac{z}{2\pi-g}\right)\left(1-\tfrac{z}{2\pi+g}\right)\left(1+\tfrac{z}{2\pi+g}\right)$$
$$\left(1-\tfrac{z}{4\pi-g}\right)\left(1+\tfrac{z}{4\pi+g}\right)\text{etc.}$$
$$= 1-\frac{zz}{1\cdot 2(1-\cos.g)}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4(1-\cos.g)}-\frac{z^6}{1\cdot 2\cdot 3\cdots 6(1-\cos.g)}+\text{etc.}$$

Therefore all the factors of this series continued to infinity are known.

160. Also all the factors of a function of this kind are able to be assigned conveniently

$$e^{b+x} \pm e^{c-x}.$$

For it may be transformed into this form

$$\left(1+\tfrac{b+x}{i}\right)^i \pm \left(1+\tfrac{c-x}{i}\right)^i,$$

which compared with the form $a^i \pm z^i$ will have a factor

$$aa - 2az\cos.\tfrac{m\pi}{i} + zz$$

with m denoting an odd number, if the upper sign may prevail, truly the contrary for an even number [§ 150 and 151]. But since on account of the number i being a number of infinite magnitude, there shall be

$$\cos.\tfrac{m\pi}{i} = 1-\tfrac{mm\pi\pi}{2ii},$$

that general factor will be

$$(a-z)^2 + \tfrac{mm\pi\pi}{ii}az.$$

But in that case there becomes

$$a = 1+\tfrac{b+x}{i} \quad \text{and} \quad z = 1+\tfrac{c-x}{i},$$

from which there becomes

$$(a-z)^2 = \frac{(b-c+2x)^2}{ii} \text{ and } az = 1 + \frac{b+c}{i} + \frac{bc+(c-b)x-xx}{ii};$$

and thus the factor will be multiplied by ii

$$= (b-c)^2 + 4(b-c)x + 4xx + mm\pi\pi$$

with the terms discarded divided by i or ii, because now with all the general terms present, before which these vanish. Therefore with the constant term reduced to unity the factor will be

$$= 1 + \frac{4(b-c)x+4xx}{mm\pi\pi+(b-c)^2}.$$

161. Now because in all the factors the constant term is $=1$, that function itself $e^{b+x} \pm e^{c-x}$ must be divided by a constant of this kind, so that the constant term is made $=1$ or so that its value on putting $x=0$ is made $=1$; such a divisor will be $e^b \pm e^c$ and on that account this expression

$$\frac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$$

will be able to be expressed by an infinite number of factors. Therefore the expression becomes, if the superior sign may prevail and m may denote an odd number,

$$\frac{e^{b+x}+e^{c-x}}{e^b+e^c}$$

$$= \left(1 + \frac{4(b-c)x+4xx}{\pi\pi+(b-c)^2}\right)\left(1 + \frac{4(b-c)x+4xx}{9\pi\pi+(b-c)^2}\right)\left(1 + \frac{4(b-c)x+4xx}{25\pi\pi+(b-c)^2}\right) \text{ etc.};$$

but if the lower sign may prevail and thus m denotes an even number, and in the case $m=0$, the root of the quadratic factor may be put in place, there will be

$$\frac{e^{b+x}-e^{c-x}}{e^b-e^c}$$

$$= \left(1 + \frac{2x}{b-c}\right)\left(1 + \frac{4(b-c)x+4xx}{4\pi\pi+(b-c)^2}\right)\left(1 + \frac{4(b-c)x+4xx}{16\pi\pi+(b-c)^2}\right)\left(1 + \frac{4(b-c)x+4xx}{36\pi\pi+(b-c)^2}\right) \text{ etc.};$$

162. Putting $b = 0$, because it can be done without general detriment, and there will be

$$\frac{e^x + e^c e^{-x}}{1+e^c} = \left(1 - \frac{4cx - 4xx}{\pi\pi + c^2}\right)\left(1 - \frac{4cx - 4xx}{9\pi\pi + c^2}\right)\left(1 - \frac{4cx - 4xx}{25\pi\pi + c^2}\right) \text{ etc.,}$$

$$\frac{e^x - e^c e^{-x}}{1-e^c} = \left(1 - \frac{2x}{c}\right)\left(1 - \frac{4cx - 4xx}{4\pi\pi + cc}\right)\left(1 - \frac{4cx - 4xx}{16\pi\pi + c^2}\right)\left(1 - \frac{4cx - 4xx}{36\pi\pi + c^2}\right) \text{ etc.,}$$

Now *c* may be made negative and these two equations will be obtained:

$$\frac{e^x + e^c e^{-x}}{1+e^c} = \left(1 + \frac{4cx + 4xx}{\pi\pi + c^2}\right)\left(1 + \frac{4cx + 4xx}{9\pi\pi + c^2}\right)\left(1 + \frac{4cx + 4xx}{25\pi\pi + c^2}\right) \text{ etc.,}$$

$$\frac{e^x - e^c e^{-x}}{1-e^c} = \left(1 + \frac{2x}{c}\right)\left(1 + \frac{4cx + 4xx}{4\pi\pi + cc}\right)\left(1 + \frac{4cx + 4xx}{16\pi\pi + c^2}\right)\left(1 + \frac{4cx + 4xx}{36\pi\pi + c^2}\right) \text{ etc.}$$

The first form may be multiplied by the third and there will be produced :

$$\frac{e^{2x} + e^{-2x} + e^c + e^{-c}}{2 + e^c + e^{-c}};$$

truly *y* may be put in place of *2x*

$$\frac{e^y + e^{-y} + e^c + e^{-c}}{2 + e^c + e^{-c}} = \left(1 - \frac{2cy - yy}{\pi\pi + cc}\right)\left(1 + \frac{2cy + yy}{\pi\pi + cc}\right)\left(1 - \frac{2cy - yy}{9\pi\pi + cc}\right)\left(1 + \frac{2cy + yy}{9\pi\pi + cc}\right)$$
$$\left(1 - \frac{2cy - yy}{25\pi\pi + cc}\right)\left(1 + \frac{2cy + yy}{25\pi\pi + cc}\right) \text{ etc.}$$

The first form may be multiplied by the fourth; it will give

$$\frac{e^{2x} - e^{-2x} + e^c - e^{-c}}{e^c - e^{-c}};$$

put *y* for *2x* and it becomes

$$\frac{e^{y}-e^{-y}+e^{c}-e^{-c}}{e^{c}-e^{-c}}$$
$$=\left(1+\frac{y}{c}\right)\left(1-\frac{2cy-yy}{\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{9\pi\pi+cc}\right)$$
$$\left(1+\frac{2cy+yy}{16\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{25\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{36\pi\pi+cc}\right)\text{ etc.,}$$

If the second form may be multiplied by the third, the same equation will be produced, unless *c* shall be taken negative ; evidently there will be

$$\frac{e^{c}-e^{-c}-e^{y}+e^{-y}}{e^{c}-e^{-c}}$$
$$=\left(1-\frac{y}{c}\right)\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{4\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)$$
$$\left(1-\frac{2cy-yy}{16\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{36\pi\pi+cc}\right)\text{ etc.}$$

Finally the second form may be multiplied by the fourth and there will be

$$\frac{e^{y}+e^{-y}-e^{c}-e^{-c}}{2-e^{c}-e^{-c}}$$
$$=\left(1-\frac{yy}{cc}\right)\left(1-\frac{2cy-yy}{4\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{16\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{16\pi\pi+cc}\right)$$
$$\left(1-\frac{2cy-yy}{36\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{36\pi\pi+cc}\right)\text{ etc.}$$

163. These four combinations now can be transferred to the circle conveniently on putting

$$c=g\sqrt{-1}\ \text{ and }\ y=v\sqrt{-1};\ ;$$

for indeed there becomes

$$e^{v\sqrt{-1}}+e^{-v\sqrt{-1}}=2\cos.v,\ \ e^{v\sqrt{-1}}-e^{-v\sqrt{-1}}=2\sqrt{-1}\cdot\sin.v.$$

and

$$e^{g\sqrt{-1}}+e^{-g\sqrt{-1}}=2\cos.g,\ \ e^{g\sqrt{-1}}-e^{-g\sqrt{-1}}=2\sqrt{-1}\cdot\sin.g.$$

Hence the first combination will give

$$\frac{\cos.v+\cos.g}{1+\cos.g}$$

$$=1-\frac{vv}{1\cdot 2(1+\cos.g)}+\frac{v^4}{1\cdot 2\cdot 3\cdot 4(1+\cos.g)}-\frac{v^6}{1\cdot 2\cdot 3\cdots 6(1+\cos.g)}+\text{etc.}$$

$$=\left(1+\frac{2gv-vv}{\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{\pi\pi-gg}\right)\left(1+\frac{2gv-vv}{9\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{9\pi\pi-gg}\right)$$
$$\left(1+\frac{2gv-vv}{25\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{25\pi\pi-gg}\right)\text{ etc.}$$

$$=\left(1+\frac{v}{\pi-g}\right)\left(1-\frac{v}{\pi+g}\right)\left(1-\frac{v}{\pi-g}\right)\left(1+\frac{v}{\pi+g}\right)$$
$$\left(1+\frac{v}{3\pi-g}\right)\left(1-\frac{v}{3\pi+g}\right)\left(1-\frac{v}{3\pi-g}\right)\left(1+\frac{v}{3\pi+g}\right)\text{ etc.}$$

$$=\left(1-\frac{vv}{(\pi-g)^2}\right)\left(1-\frac{vv}{(\pi+g)^2}\right)\left(1-\frac{vv}{(3\pi-g)^2}\right)\left(1-\frac{vv}{(3\pi+g)^2}\right)\left(1-\frac{vv}{(5\pi-g)^2}\right)\text{ etc.}$$

Truly the fourth combination gives

$$\frac{\cos.v-\cos.g}{1-\cos.g}$$

$$=1-\frac{vv}{1\cdot 2(1-\cos.g)}+\frac{v^4}{1\cdot 2\cdot 3\cdot 4(1-\cos.g)}-\frac{v^6}{1\cdot 2\cdot 3\cdots 6(1-\cos.g)}+\text{etc.}$$

$$=\left(1-\frac{vv}{gg}\right)\left(1+\frac{2gv-vv}{4\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{4\pi\pi-gg}\right)\left(1+\frac{2gv-vv}{16\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{16\pi\pi-gg}\right)\text{ etc.}$$

$$=\left(1-\frac{v}{g}\right)\left(1+\frac{v}{g}\right)\left(1+\frac{v}{2\pi-g}\right)\left(1-\frac{v}{2\pi+g}\right)\left(1-\frac{v}{2\pi-g}\right)\left(1+\frac{v}{2\pi+g}\right)$$
$$\left(1+\frac{v}{4\pi-g}\right)\left(1-\frac{v}{4\pi+g}\right)\text{ etc.}$$

$$=\left(1-\frac{vv}{gg}\right)\left(1-\frac{vv}{(2\pi+g)^2}\right)\left(1-\frac{vv}{(2\pi+g)^2}\right)\left(1-\frac{vv}{(4\pi-g)^2}\right)\left(1-\frac{vv}{(4\pi+g)^2}\right)\text{ etc.}$$

The second combination gives

$$\frac{\sin.g+\sin.v}{\sin.g}$$

$$=1+\frac{v}{\sin.g}-\frac{v^3}{1\cdot 2\cdot 3\sin.g}+\frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\sin.g}-\text{etc.}$$

$$=\left(1+\frac{v}{g}\right)\left(1+\frac{2gv-vv}{\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{4\pi\pi-gg}\right)\left(1+\frac{2gv-vv}{9\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{16\pi\pi-gg}\right)\text{ etc.}$$

$$=\left(1+\frac{v}{g}\right)\left(1+\frac{v}{\pi-g}\right)\left(1-\frac{v}{\pi+g}\right)\left(1-\frac{v}{2\pi-g}\right)\left(1+\frac{v}{2\pi+g}\right)$$
$$\left(1+\frac{v}{3\pi-g}\right)\left(1-\frac{v}{3\pi+g}\right)\left(1-\frac{v}{4\pi-g}\right)\text{ etc.}$$

And with *v* taken negative the third combination emerges.

164. Indeed these expressions also first found in § 162 can be reduced to circular arcs in this manner. Since there shall be

$$\frac{e^{x}+e^{c}e^{-x}}{1+e^{c}}=\frac{\left(1+e^{-c}\right)\left(e^{x}+e^{c}e^{-x}\right)}{2+e^{c}+e^{-c}}=\frac{e^{x}+e^{-x}+e^{c-x}+e^{-c+x}}{2+e^{c}+e^{-c}},$$

if we may put

$$c=g\sqrt{-1} \text{ and } x = z\sqrt{-1},$$

this expression will change into this :

$$\frac{\cos.z+\cos.(g-z)}{1+\cos.g}=\cos.z+\frac{\sin.g\sin.z}{1+\cos.g}.$$

Therefore there will be on account of $\frac{\sin.g}{1+\cos.g}=\tan.\frac{1}{2}g$

$$\cos.z+\text{tang.}\tfrac{1}{2}g\cdot\sin.z$$
$$=1+\tfrac{z}{1}\text{tang.}\tfrac{1}{2}g-\tfrac{zz}{1\cdot2}-\tfrac{z^{3}}{1\cdot2\cdot3}\text{tang.}\tfrac{1}{2}g+\tfrac{z^{4}}{1\cdot2\cdot3\cdot4}+\tfrac{z^{5}}{1\cdot2\cdots5}\text{tang.}\tfrac{1}{2}g-\text{etc.}$$
$$=\left(1+\frac{4gz-4zz}{\pi\pi-gg}\right)\left(1+\frac{2gz-zz}{9\pi\pi-gg}\right)\left(1+\frac{4gz-4zz}{25\pi\pi-gg}\right)\text{ etc.}$$
$$=\left(1+\frac{2z}{\pi-g}\right)\left(1-\frac{2z}{\pi+g}\right)\left(1+\frac{2z}{3\pi-g}\right)\left(1-\frac{2z}{3\pi+g}\right)\left(1+\frac{2z}{5\pi-g}\right)\left(1-\frac{2z}{5\pi+g}\right)\text{ etc.}$$

In a similar manner with the other expression, if the numerator and denominator may be multiplied by $1-e^{-c}$, it will be changed into

$$\frac{e^{x}+e^{-x}-e^{c-x}-e^{-c+x}}{2-e^{c}-e^{-c}},$$

with which done $c=g\sqrt{-1}$ and $x=z\sqrt{-1}$ gives

$$\frac{\cos.z-\cos.(g-z)}{1-\cos.g}=\cos.z-\frac{\sin.g\ \sin.z}{1-\cos.g}=\cos.z-\frac{\sin.z}{\text{tang.}\frac{1}{2}g}.$$

Therefore there will be

$$\cos.z - \cot.\tfrac{1}{2}g\ \sin.z$$
$$= 1 - \tfrac{z}{1}\cot.\tfrac{1}{2}g - \tfrac{zz}{1\cdot 2} + \tfrac{z^3}{1\cdot 2\cdot 3}\cot.\tfrac{1}{2}g + \tfrac{z^4}{1\cdot 2\cdot 3\cdot 4} - \tfrac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cot.\tfrac{1}{2}g + \text{etc.}$$
$$= \left(1 - \tfrac{2z}{g}\right)\left(1 + \tfrac{4gz - 4zz}{4\pi\pi - gg}\right)\left(1 + \tfrac{4gz - zz}{16\pi\pi - gg}\right)\left(1 + \tfrac{4gz - 4zz}{36\pi\pi - gg}\right)\ \text{etc.}$$
$$= \left(1 - \tfrac{2z}{g}\right)\left(1 + \tfrac{2z}{2\pi - g}\right)\left(1 - \tfrac{2z}{2\pi + g}\right)\left(1 + \tfrac{2z}{4\pi - g}\right)\left(1 - \tfrac{2z}{4\pi + g}\right)\ \text{etc.}$$

But if therefore there may be put $v = 2z$ or $z = \tfrac{1}{2}v$, there will be had

$$\frac{\cos.\frac{1}{2}(g - v)}{\cos.\frac{1}{2}g} = \cos.\tfrac{1}{2}v + \text{tang}.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1 + \tfrac{v}{\pi - g}\right)\left(1 - \tfrac{v}{\pi + g}\right)\left(1 + \tfrac{v}{3\pi - g}\right)\left(1 - \tfrac{v}{3\pi + g}\right)\ \text{etc}$$
$$\frac{\cos.\frac{1}{2}(g + v)}{\cos.\frac{1}{2}g} = \cos.\tfrac{1}{2}v - \text{tang}.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1 - \tfrac{v}{\pi - g}\right)\left(1 + \tfrac{v}{\pi + g}\right)\left(1 - \tfrac{v}{3\pi - g}\right)\left(1 + \tfrac{v}{3\pi + g}\right)\ \text{etc.}$$
$$\frac{\sin.\frac{1}{2}(g - v)}{\sin.\frac{1}{2}g} = \cos.\tfrac{1}{2}v - \cot.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1 - \tfrac{v}{g}\right)\left(1 + \tfrac{v}{2\pi - g}\right)\left(1 - \tfrac{v}{2\pi + g}\right)\left(1 + \tfrac{v}{4\pi - g}\right)\left(1 - \tfrac{v}{4\pi + g}\right)\ \text{etc.}$$
$$\frac{\sin.\frac{1}{2}(g + v)}{\sin.\frac{1}{2}g} = \cos.\tfrac{1}{2}v + \cot.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1 + \tfrac{v}{g}\right)\left(1 - \tfrac{v}{2\pi - g}\right)\left(1 + \tfrac{v}{2\pi + g}\right)\left(1 - \tfrac{v}{4\pi - g}\right)\left(1 + \tfrac{v}{4\pi + g}\right)\ \text{etc.}$$

The law of the progression of which factors is simple and uniform enough ; and from these expressions these expressions themselves arise by multiplication, which have been found in the preceding paragraph.

CAPUT IX

DE INVESTIGATIONE FACTORUM TRINOMIALIUM

143. Quemadmodum factores simplices cuiusque functionis integrae inveniri oporteat, supra [§ 29] quidem ostendimus hoc fieri per resolutionem aequationum. Si enim proposita sit functio quaecunque integra

$$\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$$

huiusque quaerantur factores simplices formae $p-qz$, manifestum est, si $p-qz$ fuerit factor functionis $\alpha+\beta z+\gamma z^2+\text{etc.}$, tum posito $z=\frac{p}{q}$, quo casu factor $p-qz$ fit $=0$, etiam ipsam functionem propositam evanescere debere. Hinc $p-qz$ erit factor vel divisor functionis

$$\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$$

sequitur fore hanc expressionem

$$\alpha+\frac{\beta p}{q}+\frac{\gamma p^2}{q^2}+\frac{\delta p^3}{q^3}+\frac{\varepsilon p^4}{q^4}+\text{etc.}=0.$$

Unde vicissim, si omnes radices $\frac{p}{q}$ huius aequationis eruantur, singulae dabunt totidem factores simplices functionis integrae propositae

$$\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$$

nempe $p-qz$. Patet autem simul numerum factorum huiusmodi simplicium ex maxima potestate ipsius z definiri.

144. Hoc autem modo plerumque difficulter factores imaginarii eruuntur, quamobrem hoc capite methodum peculiarem tradam, cuius ope saepenumero factores simplices imaginarii inveniri queant. Quoniam vero factores simplices imaginarii ita sunt comparati, ut binorum productum fiat reale, hos ipsos factores imaginarios reperiemus, si factores investigemus duplices seu huius formae

$$p-qz+rzz,$$

reales quidem, sed quorum factores simplices sint imaginarii. Quodsi enim functionis $\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$ constent omnes factores reales duplices huius formae trinomiales $p-qz+rzz$, simul omnes factores imaginarii habebuntur.

145. Trinomium autem $p-qz+rzz$ factores simplices habebit imaginarios, si fuerit $4pr>qq$ seu

$$\frac{q}{2\sqrt{pr}}<1.$$

Cum igitur sinus et cosinus angulorum sint unitate minores, formula $p-qz+rzz$ factores simplices habebit imaginarios, si fuerit $\frac{q}{2\sqrt{pr}}=$ sinui vel cosinui cuiuspiam anguli. Sit ergo

$$\frac{q}{2\sqrt{pr}}=\cos.\varphi \text{ seu } q=2\sqrt{pr}.\cos.\varphi$$

atque trinomium $p-qz+rzz$ continebit factores simplices imaginarios. Ne autem irrationalitas molestiam facessat, assumo hanc formam

$$pp-2pqz\cos.\varphi+qqzz,$$

cuius factores simplices imaginarii erunt hi

$$qz-p\left(\cos.\varphi+\sqrt{-1}.\sin.\varphi\right) \text{ et } qz-p\left(\cos.\varphi-\sqrt{-1}.\sin.\varphi\right).$$

Ubi quidem patet, si fuerit $\cos.\varphi=\pm1$, tum ambos factores ob $\sin.\varphi=0$ fieri aequales et reales.

146. Proposita ergo functione integra $\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.}$ eius factores simplices imaginarii eruentur, si determinentur litterae *p* et *q* cum angulo φ, ut hoc trinomium $pp-2pqz\cos.\varphi+qqzz$ fiat factor functionis.
Tum enim simul inerunt isti factores simplices imaginarii

$$qz-p\left(\cos.\varphi+\sqrt{-1}.\sin.\varphi\right) \text{ et } qz-p\left(\cos.\varphi-\sqrt{-1}.\sin.\varphi\right).$$

Quamobrem functio proposita evanescet, si ponatur tam

$$z=\frac{p}{q}\left(\cos.\varphi+\sqrt{-1}.\sin.\varphi\right)$$

quam

$$z=\frac{p}{q}\left(\cos.\varphi-\sqrt{-1}.\sin.\varphi\right)$$

Hinc facta substitutione utraque duplex nascetur aequatio, ex quibus tam fractio $\frac{p}{q}$ quam arcus φ definiri poterunt.

147. Hae autem substitutiones loco z faciendae, etiamsi primo intuitu difficiles videantur, tamen per ea, quae in capite praecedente sunt tradita, satis expedite absolventur. Cum enim fuerit ostensum esse

$$\left(\cos.\varphi \pm \sqrt{-1}.\sin.\varphi\right)^n = \cos.n\varphi \pm \sqrt{-1}.\sin.n\varphi\ ,$$

sequentes formulae loco singularum ipsius z potestatum habebuntur substituendae:

pro priori factore	pro altero factore
$z = \frac{p}{q}\left(\cos.\varphi + \sqrt{-1}.\sin.\varphi\right)$	$z = \frac{p}{q}\left(\cos.\varphi - \sqrt{-1}.\sin.\varphi\right)$
$z^2 = \frac{p^2}{q^2}\left(\cos.2\varphi + \sqrt{-1}.\sin.2\varphi\right)$	$z^2 = \frac{p^2}{q^2}\left(\cos.2\varphi - \sqrt{-1}.\sin.2\varphi\right)$
$z^3 = \frac{p^3}{q^3}\left(\cos.3\varphi + \sqrt{-1}.\sin.3\varphi\right)$	$z^3 = \frac{p^3}{q^3}\left(\cos.3\varphi - \sqrt{-1}.\sin.3\varphi\right)$
$z^4 = \frac{p^4}{q^4}\left(\cos.4\varphi + \sqrt{-1}.\sin.4\varphi\right)$	$z^4 = \frac{p^4}{q^4}\left(\cos.4\varphi - \sqrt{-1}.\sin.4\varphi\right)$
etc.	etc.

Ponatur brevitatis gratia $\frac{p}{q} = r$ factaque substitutione sequentes duae nascentur aequationes:

$$0 = \left\{\begin{array}{l} \alpha + \beta r\cos.\varphi \qquad + \gamma r^2\cos.2\varphi \qquad + \delta r^3\cos.2\varphi \qquad + \text{etc.} \\ \quad + \beta r\sqrt{-1}.\sin.\varphi + \gamma r^2\sqrt{-1}.\sin.2\varphi + \delta r^3\sqrt{-1}.\sin.3\varphi + \text{etc.} \end{array}\right\}$$

$$0 = \left\{\begin{array}{l} \alpha + \beta r\cos.\varphi \qquad + \gamma r^2\cos.2\varphi \qquad + \delta r^3\cos.2\varphi \qquad + \text{etc.} \\ \quad - \beta r\sqrt{-1}.\sin.\varphi - \gamma r^2\sqrt{-1}.\sin.2\varphi - \delta r^3\sqrt{-1}.\sin.3\varphi - \text{etc.} \end{array}\right\}$$

148. Quodsi hae duae aequationes invicem addantur et subtrahantur et posteriori casu per $2\sqrt{-1}$ dividantur, prodibunt hae duae aequationes reales:

$$0 = \alpha + \beta r\cos.\varphi \quad + \gamma r^2\cos.2\varphi \quad + \delta r^3\cos.2\varphi \quad + \text{etc.}$$
$$0 = \quad \beta r\sqrt{-1}.\sin.\varphi + \gamma r^2\sqrt{-1}.\sin.2\varphi + \delta r^3\sqrt{-1}.\sin.3\varphi + \text{etc.}$$

quae statim ex forma functionis propositae

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \text{etc.}$$

formari possunt ponendo primum pro unaquaque ipsius z potestate

$$z^n = r^n\cos.n\varphi,$$

deinceps

$$z^n = r^n\sin.n\varphi.$$

Sic enim ob $\sin.0\varphi = 0$ et $\cos.0\varphi = 1$ pro z^0 seu 1 in termino constanti priori casu ponitur 1, posteriori autem 0.

Si ergo ex his duabus aequationibus definiantur incognitae r et φ, ob $r = \frac{p}{q}$ habebitur factor functionis trinomialis

$$pp - 2pqz\cos.\varphi + qqzz$$

duos factores simplices imaginarios involvens.

149. Si aequatio prior multiplicetur per $\sin.m\varphi$, posterior per $\cos.m\varphi$ atque producta vel addantur vel subtrahantur, prodibunt istae duae aequationes:

$$0 = \alpha\sin.m\varphi + \beta r\sin.(m+1)\varphi + \gamma r^2\sin.(m+2)\varphi + \delta r^3\sin.(m+3)\varphi + \text{etc.},$$
$$0 = \alpha\sin.m\varphi + \beta r\sin.(m-1)\varphi + \gamma r^2\sin.(m-2)\varphi + \delta r^3\sin.(m-3)\varphi + \text{etc.}$$

Sin autem aequatio prior multiplicetur per $\cos.m\varphi$ et posterior per $\sin.m\varphi$, per additionem ac subtractionem sequentes emergent aequationes:

$$0 = \alpha\cos.m\varphi + \beta r\cos.(m-1)\varphi + \gamma r^2\cos.(m-2)\varphi + \delta r^3\cos.(m-3)\varphi + \text{etc.},$$
$$0 = \alpha\cos.m\varphi + \beta r\cos.(m+1)\varphi + \gamma r^2\cos.(m+2)\varphi + \delta r^3\cos.(m+3)\varphi + \text{etc.}$$

Huiusmodi ergo duae aequationes quaecunque coniunctae determinabunt incognitas r et φ quod cum plerumque pluribus modis fieri possit, simul plures factores trinomiales obtinentur iique adeo omnes, quos functio proposita in se complectitur.

150. Quo usus harum regularum clarius appareat, quarumdam functionum saepius occurrentium factores trinomiales hic indagabimus, ut eos, quoties occasio postulaverit, hinc depromere liceat. Sit itaque proposita haec functio

$$a^n + z^n,$$

cuius factores trinomiales formae

$$pp - 2pqz\cos.\varphi + qqzz$$

determinari oporteat. Posito ergo $r = \frac{p}{q}$ habebuntur hae duae aequationes

$$0 = a^n + r^n \cos.n\varphi \ \text{ et } \ 0 = r^n \sin.n\varphi.,$$

quarum posterior dat

$$\sin.n\varphi = 0;$$

unde erit $n\varphi$ arcus vel huius formae $(2k+1)\pi$ vel $2k\pi$ denotante k numerum integrum. Casus hos ideo distinguo, quod eorum cosinus sint differentes; priori enim casu erit $\cos.(2k+1)\pi = -1$, posteriori casu autem $\cos.\,2k\pi = +1$.
Patet autem priorem formam

$$n\varphi = (2k+1)\pi$$

sumi debere, quippe quae dat $\cos.n\varphi = -1$, unde fit

$$0 = a^n - r^n$$

hincque porro

$$r = a = \frac{p}{q}.$$

Erit ergo

$$p = a,\ q = 1$$

et

$$\varphi = \tfrac{2k+1}{n}\pi$$

unde functionis $a^n + z^n$ factor erit

$$aa - 2az\cos.\tfrac{2k+1}{n}\pi + zz.$$

Cum igitur pro *k* numerum quemque integrum ponere liceat, prodeunt hoc modo plures factores neque tamen infiniti, quoniam, si $2k+1$ ultra *n* augetur, factores priores recurrunt, quod ex exemplis clarius patebit, cum sit $\cos.(2\pi \pm \varphi) = \cos.\varphi$. Deinde si *n* est numerus impar, posito $2k+1 = n$ erit factor quadratus $aa + 2az + zz$; neque vero hinc sequitur quadratum $(a+z)^2$ esse factorem functionis $a^n + z^n$, quoniam (in § 148) unica aequatio resultat, qua tantum patet $a+z$ esse divisorem formulae $a^n + z^n$; quae regula semper est tenenda, quoties $\cos.\varphi$ fit vel + 1 vel –1.

EXEMPLUM

Evolvamus aliquot casus, quo isti factores clarius ob oculos ponantur, atque hos casus in duas classes distribuamus, prout *n* fuerit numerus vel par vel impar.

Si $n = 1$, formulae $a + z$, factor est $a + z$.	Si $n = 2$, formulae $a^2 + z^2$, factor est $a^2 + z^2$
Si $n = 3$, formulae $a^3 + z^3$, factores sunt $aa - 2az\cos.\tfrac{1}{3}\pi + zz$, $a + z$.	Si $n = 4$, formulae $a^4 + z^4$, factores sunt $aa - 2az\cos.\tfrac{1}{4}\pi + zz$, $aa - 2az\cos.\tfrac{3}{4}\pi + zz$.
Si $n = 5$, formulae $a^5 + z^5$, factores sunt $aa - 2az\cos.\tfrac{1}{5}\pi + zz$, $aa - 2az\cos.\tfrac{3}{5}\pi + zz$, $a + z$.	Si $n = 6$, formulae $a^6 + z^6$, factores sunt $aa - 2az\cos.\tfrac{1}{6}\pi + zz$, $aa - 2az\cos.\tfrac{3}{6}\pi + zz$, $aa - 2az\cos.\tfrac{5}{6}\pi + zz$.

Ex quibus exemplis patet omnes factores obtineri, si loco $2k+1$ omnes numeri impares non maiores quam exponens *n* substituantur, iis vero casibus, quibus factor quadratus prodit, tantum eius radicem factoribus annumerari debere.

151. Si proposita sit haec functio

$$a^n - z^n,$$

eius factor trinomialis erit

$$pp - 2pqz\cos.\varphi + qqzz,$$

si posito $r = \frac{p}{q}$ fuerit

$$0 = a^n - r^n\cos.n\varphi \text{ et } 0 = r^n\sin.n\varphi.$$

Erit ergo iterum

$$\sin.n\varphi = 0$$

ideoque $n\varphi = (2k+1)\pi$ vel $n\varphi = 2k\pi$. Hoc autem casu valor posterior sumi debet, ut sit $\cos.n\pi = +1$, qui dat $0 = a^n - r^n$ et $r = \frac{p}{q} = a$.
Habebitur itaque

$$p = a,\ q = 1$$

et

$$\varphi = \frac{2k}{n}\pi,$$

unde factor trinomialis formulae propositae erit

$$aa - 2az\cos.\tfrac{2k}{n}\pi + zz\ ;$$

quae forma, si loco *2k* omnes numeri pares non maiores quam *n* ponantur, simul dabit omnes factores; ubi de factoribus quadratis idem est tenendum, quod ante monuimus. Ac primo quidem posito $k = 0$ prodit factor $aa - 2az + zz$, pro quo vero radix $a - z$ capi debet. Similiter, si *n* fuerit numerus par et ponatur $2k = n$, prodit $aa + 2az + zz$, unde patet $a + z$ esse divisorem formae $a^n - z^n$.

EXEMPLUM

Casus exponentis *n* ut ante tractati ita se habebunt, prout *n* fuerit numerus vel impar vel par.

Si $n = 1$, formulae $a - z$, ipsa factor est $a - z$.	Si $n = 2$, formulae $a^2 - z^2$, factores erunt $a + z$, $a - z$
Si $n = 3$, formulae $a^3 - z^3$, factores sunt	Si $n = 4$, formulae $a^4 - z^4$, factores sunt

$a-z$,	$a-z, a+z$,
$aa-2az\cos.\frac{2}{3}\pi+zz$	$aa-2az\cos.\frac{2}{4}\pi+zz$.
Si $n=5$, formulae	Si $n=6$, formulae
a^5-z^5,	a^6-z^6,
factores erunt	factores erunt
$a-z$,	$a-z, a+z$,
$aa-2az\cos.\frac{2}{5}\pi+zz$,	$aa-2az\cos.\frac{2}{6}\pi+zz$,
$aa-2az\cos.\frac{4}{5}\pi+zz$.	$aa-2az\cos.\frac{4}{6}\pi+zz$.

152. His igitur confirmatur id, quod supra [§ 32] iam innuimus, omnem functionem integram, si non in factores simplices reales, tamen in factores duplices reales resolvi posse. Vidimus enim hanc functionem indefinitae dimensionis a^n+z^n semper in factores duplices reales praeter simplices reales resolvi posse.

Progrediamur ergo ad functiones magis compositas, uti $a+\beta z^n+rz^{2n}$, cuius quidem, si duos habeat factores formae $\eta+\theta z^n$ resolutio ex praecedentibus abunde patet. Hoc ergo tantum erit efficiendum, ut formae $a+\beta z^n+rz^{2n}$ eo casu, quo non habet duos factores reales formae $\eta+\theta z^n$, resolutionem in factores reales, vel simplices vel duplices, doceamus.

153. Consideremus ergo hanc functionem $a^{2n}-2a^nz^n\cos.g+z^{2n}$, quae in duos factores formae $\eta+\theta z^n$ reales resolvi nequit. Quodsi ergo ponamus huius functionis factorem duplicem realem esse

$$pp-2pqz\cos.\varphi+qqzz,$$

posito $r=\frac{p}{q}$ duae sequentes aequationes erunt resolvendae

$$0=a^{2n}-2a^nr^n\cos.g\cos.n\varphi+r^{2n}\cos.2n\varphi$$

et

$$0=-2a^nr^n\cos.g\sin.n\varphi+r^{2n}\sin.2n\varphi.$$

Vel loco prioris aequationis sumatur ex § 149 (ponendo $m=2n$) haec

$$0=a^{2n}\sin.2n\varphi-2a^nr^n\cos.g\sin.n\varphi,$$

quae cum posteriori collata dat $r = a$; tum vero erit

$$\sin.2n\varphi = 2\cos.g \sin.n\varphi .$$

At est

$$\sin.2n\varphi = 2\sin.n\varphi \cos.n\varphi ,$$

unde fit

$$\cos. n\varphi = \cos. g .$$

At est semper $\cos.(2k\pi \pm g) = \cos.g$, ex quo habetur

$$n\varphi = 2k\pi \pm g$$

et

$$\varphi = \frac{2k\pi \pm g}{n}$$

Hinc ergo factor generalis duplex formae propositae erit

$$= aa - 2az\cos.\frac{2k\pi \pm g}{n} + zz$$

atque omnes factores prodibunt, si pro $2k$ omnes numeri pares non maiores quam n successive substituantur, uti ex applicatione ad casus videre licebit.

EXEMPLUM

Consideremus ergo casus, quibus n est 1, 2, 3, 4 etc., ut ratio factorum appareat. Erit ergo formulae

$$aa - 2az\cos.g + zz$$

unicus factor

$$aa - 2az \cos. g + zz ;$$

formulae

$$a^4 - 2a^2z^2\cos.g + z^4$$

factores duo

$$aa - 2az \cos.\frac{g}{2} + zz ,$$

$$aa - 2az\cos.\frac{2\pi \pm g}{2} + zz \quad \text{seu} \quad aa + 2az \cos.\frac{g}{2} + zz ;$$

formulae

$$a^6 - 2a^3z^3\cos.g + z^6$$

factores tres

$$aa - 2az\cos.\tfrac{g}{3} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi - g}{3} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi + g}{3} + zz;$$

formulae

$$a^8 - 2a^4 z^4 \cos.g + z^8$$

factores quatuor

$$aa - 2az\cos.\tfrac{g}{4} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi - g}{4} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi + g}{4} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi \pm g}{4} + zz \text{ seu } aa + 2az\cos.\tfrac{g}{4} + zz;$$

formulae

$$a^{10} - 2a^5 z^5 \cos.g + z^{10}$$

factores quinque

$$aa - 2a.z\cos.\tfrac{g}{5} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi - g}{5} + zz,$$
$$aa - 2az\cos.\tfrac{2\pi + g}{5} + zz,$$
$$aa - 2az\cos.\tfrac{4\pi - g}{5} + zz,$$
$$aa - 2az\cos.\tfrac{4\pi + g}{5} + zz.$$

Confirmatur ergo etiam his exemplis omnem functionem integram in factores reales sive simplices sive duplices resolvi posse.

154. Hinc ulterius progredi licebit ad functionem hanc

$$\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n},$$

quae certo habebit unum factorem realem formae $\eta + \theta z^n$, cuius igitur factores reales vel simplices vel duplices exhiberi possunt; alter vero multiplicator formae $\iota + \kappa z^n + \lambda z^{2n}$, utcunque fuerit comparatus, per paragraphum praecedentem pari modo in factores resolvi poterit.
Deinde haec functio

$$\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}+\varepsilon z^{4n}$$

cum perpetuo habeat duos factores reales formae huius $\eta+\theta z^{n}+\iota z^{2n}$, similiter in factores vel simplices vel duplices reales resolvitur.
Quin etiam progredi licet ad formam

$$\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}+\varepsilon z^{4n}+\xi z^{5n}$$

quae cum certo habeat unum factorem formae $\eta+\theta z^{n}$, alter factor erit formae praecedentis, unde etiam haec functio resolutionem in factores reales vel simplices vel duplices admittet.

Quare si ullum dubium mansisset circa huiusmodi resolutionem omnium functionum integrarum, hoc nunc fere penitus tolletur.

155. Traduci vero etiam potest haec in factores resolutio ad series infinitas; scilicet quia vidimus supra [§ 123] esse

$$1+x+\tfrac{x^{2}}{1\cdot 2}+\tfrac{x^{3}}{1\cdot 2\cdot 3}+\tfrac{x^{4}}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}=e^{x},$$

at vero esse

$$e^{x}=\left(1+\tfrac{x}{i}\right)^{i}$$

denotante *i* numerum infinitum, perspicuum est seriem

$$1+\tfrac{x}{1}+\tfrac{x^{2}}{1\cdot 2}+\tfrac{x^{3}}{1\cdot 2\cdot 3}+$$

habere factores infinitos simplices inter se aequales nempe $1+\frac{x}{i}$. At si ab eadem serie primus terminus dematur, erit

$$\tfrac{x}{1}+\tfrac{x^{2}}{1\cdot 2}+\tfrac{x^{3}}{1\cdot 2\cdot 3}+\tfrac{x^{4}}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}=e^{x}-1=\left(1+\tfrac{x}{i}\right)^{i}-1$$

cuius formae cum § 151 comparatae, quo fit

$$a=1+\tfrac{x}{i},n=i\text{ et }z=1,$$

factor quicunque erit

$$=\left(1+\tfrac{x}{i}\right)^{2}-2\left(1+\tfrac{x}{i}\right)\cos.\tfrac{2k}{i}\pi+1,$$

unde substituendo pro *2k* omnes numeros pares simul omnes factores prodibunt.

Posito autem $2k = 0$ prodit factor quadratus $\frac{xx}{ii}$ pro quo autem tantum ob rationes allegatas radix $\frac{x}{i}$ sumi debet; erit ergo x factor expressionis $e^x - 1$, quod quidem sponte patet. Ad reliquos factores inveniendos notari oportet esse ob arcum $\frac{2k}{i}\pi$ infinite parvum

$$\cos.\tfrac{2k}{i}\pi = 1 - \tfrac{2kk}{ii}\pi\pi$$

(§ 134) terminis sequentibus ob i numerum infinitum in nihilum abeuntibus.
Hinc erit factor quilibet

$$\tfrac{xx}{ii} + \tfrac{4kk}{ii}\pi\pi + \tfrac{4kk\pi\pi}{i^3}x$$

atque adeo forma $e^x - 1$ erit divisibilis per $1 + \frac{x}{i} + \frac{xx}{4kk\pi\pi}$.
Quare expressio

$$e^x - 1 = x\left(1 + \tfrac{x}{1\cdot 2} + \tfrac{x^2}{1\cdot 2\cdot 3} + \tfrac{x^3}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right) = \left(1 + \tfrac{x}{i}\right)^i - 1$$

praeter factorem x habebit hos infinitos

$$\left(1 + \tfrac{x}{i} + \tfrac{xx}{4\pi\pi}\right)\left(1 + \tfrac{x}{i} + \tfrac{xx}{36\pi\pi}\right)\left(1 + \tfrac{x}{i} + \tfrac{xx}{64\pi\pi}\right) \text{ etc.}$$

156. Cum autem hi factores contineant partem infinite parvam $\frac{x}{i}$ quae, cum in singulis insit atque per multiplicationem omnium, quorum numerus est $\frac{1}{2}i$,producat terminum $\frac{x}{2}$ omitti non potest, ad hoc ergo incommodum vitandum consideremus hanc expressionem

$$e^x - e^{-x} = \left(1 + \tfrac{x}{i}\right)^i - \left(1 - \tfrac{x}{i}\right)^i = 2\left(\tfrac{x}{1} + \tfrac{x^3}{1\cdot 2\cdot 3} + \tfrac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.}\right);$$

est enim

$$e^{-x} = 1 - \tfrac{x}{1} + \tfrac{x^2}{1\cdot 2} - \tfrac{x^3}{1\cdot 2\cdot 3} + \text{etc.}$$

Quae cum § 151 comparata dat

$$n = i,\ a = 1 + \tfrac{x}{i} \text{ et } z = 1 - \tfrac{x}{i},$$

unde huius expressionis factor erit

$$= aa - 2az\cos.\tfrac{2k}{n}\pi + zz$$
$$= 2 + \tfrac{2xx}{ii} - 2\left(1 - \tfrac{xx}{ii}\right)\cos.\tfrac{2k}{i}\pi = \tfrac{4xx}{ii} + \tfrac{4kk}{ii}\pi\pi - \tfrac{4kk\pi\pi xx}{i^4}$$

ob

$$\cos.\tfrac{2k}{i}\pi = 1 - \tfrac{2kk}{ii}\pi\pi.$$

Functio ergo $e^x - e^{-x}$ divisibilis erit per

$$1 + \tfrac{xx}{kk\pi\pi} - \tfrac{xx}{ii}$$

ubi autem terminus $\tfrac{xx}{ii}$ tuto omittitur, quia, etsi per i multiplicetur, tamen manet infinite parvus. Praeterea vero ut ante, si $k = 0$, erit primus factor $= x$. Quocirca his factoribus in ordinem redactis erit

$$\tfrac{e^x - e^{-x}}{2} = x\left(1 + \tfrac{xx}{\pi\pi}\right)\left(1 + \tfrac{xx}{4\pi\pi}\right)\left(1 + \tfrac{xx}{9\pi\pi}\right)\left(1 + \tfrac{xx}{16\pi\pi}\right)\left(1 + \tfrac{xx}{25\pi\pi}\right) \text{ etc.}$$
$$= x\left(1 + \tfrac{xx}{1\cdot 2\cdot 3} + \tfrac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \tfrac{x^6}{1\cdot 2\cdots 7} + \text{etc.}\right).$$

Singulis scilicet factoribus per multiplicationem constantis eiusmodi formam dedi, ut per actualem multiplicationem primus terminus x resultet.

157. Eodem modo cum sit

$$\tfrac{e^x + e^{-x}}{2} = 1 + \tfrac{xx}{1\cdot 2} + \tfrac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.} = \tfrac{\left(1+\frac{x}{i}\right)^i + \left(1-\frac{x}{i}\right)^i}{2},$$

huius expressionis cum superiori [§150] $a^n + z^n$ comparatio dabit

$$a = 1 + \tfrac{x}{i},\ z = 1 - \tfrac{x}{i} \text{ et } n = i;$$

erit ergo factor quicunque

$$= aa - 2az\cos.\tfrac{2k+1}{n}\pi + zz = 2 + \tfrac{2xx}{ii} - 2\left(1 - \tfrac{xx}{ii}\right)\cos.\tfrac{2k+1}{i}\pi$$

Est autem

$$\cos.\tfrac{2k+1}{i}\pi = 1 - \tfrac{(2k+1)^2}{2ii}\pi\pi,$$

unde forma factoris erit

$$\frac{4xx}{ii} + \frac{(2k+1)^2}{ii}\pi\pi$$

evanescente termino, cuius denominator est i^4. Quoniam ergo omnis factor expressionis

$$1 + \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

huiusmodi formam habere debet $1 + axx$, quo factor inventus ad hanc formam reducatur, dividi debet per $\frac{(2k+1)^2\pi^2}{ii}$; hinc factor formae propositae erit

$$1 + \frac{4xx}{(2k+1)^2\pi\pi}$$

ex eoque omnes factores infiniti invenientur, si loco $2k+1$ successive omnes numeri impares substituantur. Hanc ob rem erit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$
$$= \left(1 + \frac{4xx}{\pi\pi}\right)\left(1 + \frac{4xx}{9\pi\pi}\right)\left(1 + \frac{4xx}{25\pi\pi}\right)\left(1 + \frac{4xx}{49\pi\pi}\right) \text{ etc.}$$

158. Si x fiat quantitas imaginaria, formulae hae exponentiales in sinum et cosinum cuiuspiam arcus realis abeunt. Sit enim $x = z\sqrt{-1}$; erit

$$\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2} = \sin.z = z - \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{z^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.},$$

quae adeo expressio hos habet factores numero infinitos

$$z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)\left(1 - \frac{zz}{25\pi\pi}\right) \text{ etc.},$$

seu erit

$$\sin.z = z\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)\text{etc.}$$

Quoties ergo arcus z ita est comparatus, ut quispiam factor evanescat, quod fit, si $z = 0, z = \pm\pi$, $z = \pm 2\pi$ et generaliter si $z = \pm k\pi$ denotante k numerum quemcunque integrum, simul sinus eius arcus debet esse $= 0$, quod quidem ita patet, ut hinc istos factores a posteriori eruere licuisset.

Simili modo cum sit

$$\frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2} = \cos.z,$$

erit quoque

$$\cos.z = \left(1-\tfrac{4zz}{\pi\pi}\right)\left(1-\tfrac{4zz}{9\pi\pi}\right)\left(1-\tfrac{4zz}{25\pi\pi}\right)\left(1-\tfrac{4zz}{49\pi\pi}\right) \text{ etc.}$$

seu his factoribus in binos resolvendis erit quoque

$$\cos.z = \left(1-\tfrac{2z}{\pi}\right)\left(1+\tfrac{2z}{\pi}\right)\left(1-\tfrac{2z}{3\pi}\right)\left(1+\tfrac{2z}{3\pi}\right)\left(1-\tfrac{2z}{5\pi}\right)\left(1+\tfrac{2z}{5\pi}\right) \text{ etc.},$$

ex qua pari modo patet, si fuerit $z = \pm\frac{2k+1}{2}\pi$,fore $\cos.z = 0$, id quod etiam ex natura circuli liquet.

159. Ex § 153 etiam inveniri possunt factores huius expressionis

$$e^x - 2\cos.g + e^{-x} = 2\left(1-\cos.g + \tfrac{xx}{1\cdot 2} + \tfrac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right).$$

Transit enim haec expressio in hanc

$$\left(1+\tfrac{x}{i}\right)^i - 2\cos.g + \left(1-\tfrac{x}{i}\right)^i$$

quae cum illa forma comparata dat

$$2n = i,\ \ a = 1+\tfrac{x}{i} \text{ et } z = 1-\tfrac{x}{i},$$

unde factor quicunque huius formulae erit

$$= aa - 2az\cos.\tfrac{2k\pi\pm g}{n} + zz = 2 + \tfrac{2xx}{ii} - 2\left(1-\tfrac{xx}{ii}\right)\cos.\tfrac{2k\pi\pm g}{i}$$

at est

$$\cos.\tfrac{2(2k\pi\pm g)}{i} = 1 - \tfrac{2(2k\pi\pm g)^2}{ii},$$

unde factor erit $= \frac{4xx}{ii} + \frac{4(2k\pi\pm g)^2}{ii}$ seu huius formae

$$1 + \frac{xx}{(2k\pi\pm g)^2}$$

Si ergo expressio per $2(1-\cos.g)$ dividatur, ut in serie infinita terminus constans sit $=1$, erit sumendis omnibus factoribus

$$\frac{e^x-2\cos.g+e^{-x}}{2(1-\cos.g)}$$
$$=\left(1+\frac{xx}{gg}\right)\left(1+\frac{xx}{(2\pi-g)^2}\right)\left(1+\frac{xx}{(2\pi+g)^2}\right)\left(1+\frac{xx}{(4\pi-g)^2}\right)\left(1+\frac{xx}{(4\pi+g)^2}\right)$$
$$\left(1+\frac{xx}{(6\pi-g)^2}\right)\left(1+\frac{xx}{(6\pi+g)^2}\right)\text{ etc.}$$

Atque si loco x ponatur $z\sqrt{-1}$, erit

$$\frac{\cos.z-\cos.g}{1-\cos.g}$$
$$=\left(1-\frac{z}{g}\right)\left(1+\frac{z}{g}\right)\left(1-\frac{z}{2\pi-g}\right)\left(1+\frac{z}{2\pi-g}\right)\left(1-\frac{z}{2\pi+g}\right)\left(1+\frac{z}{2\pi+g}\right)$$
$$\left(1-\frac{z}{4\pi-g}\right)\left(1+\frac{z}{4\pi+g}\right)\text{etc.}$$
$$=1-\frac{zz}{1\cdot 2(1-\cos.g)}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4(1-\cos.g)}-\frac{z^6}{1\cdot 2\cdot 3\cdots 6(1-\cos.g)}+\text{etc.}$$

Huius adeo seriei in infinitum continuatae factores omnes cognoscuntur.

160. Commode etiam huiusmodi functionis

$$e^{b+x}\pm e^{c-x}$$

factores inveniri omnesque assignari possunt. Transmutatur enim in hanc formam

$$\left(1+\frac{b+x}{i}\right)^i\pm\left(1+\frac{c-x}{i}\right)^i,$$

quae comparata cum forma $a^i\pm z^i$ factorem habebit

$$aa-2az\cos.\frac{m\pi}{i}+zz$$

denotante m numerum imparem, si valeat signum superius, contra vero numerum parem [§ 150 et 151]. Cum autem ob i numerum infinite magnum
sit

$$\cos.\tfrac{m\pi}{i} = 1 - \tfrac{mm\pi\pi}{2ii},$$

erit factor ille generalis

$$(a-z)^2 + \tfrac{mm\pi\pi}{ii}az.$$

At hoc casu erit

$$a = 1 + \tfrac{b+x}{i} \quad \text{et} \quad z = 1 + \tfrac{c-x}{i},$$

unde fit

$$(a-z)^2 = \tfrac{(b-c+2x)^2}{ii} \text{ et } az = 1 + \tfrac{b+c}{i} + \tfrac{bc+(c-b)x-xx}{ii};$$

ideoque factor erit per *ii* multiplicatus

$$= (b-c)^2 + 4(b-c)x + 4xx + mm\pi\pi$$

neglectis terminis per *i* vel *ii* divisis, quoniam iam omnis generis termini adsunt, prae quibus hi evanescerent. Termino ergo constante ad unitatem per divisionem reducto erit factor

$$= 1 + \tfrac{4(b-c)x+4xx}{mm\pi\pi+(b-c)^2}.$$

161. Nunc quoniam in omnibus factoribus terminus constans est $=1$, ipsa functio $e^{b+x} \pm e^{c-x}$; per eiusmodi constantem dividi debet, ut terminus constans fiat $=1$ seu ut eius valor posito $x=0$ fiat $=1$; talis divisor erit $e^b \pm e^c$ et hanc ob rem expressio haec

$$\tfrac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$$

per factores numero infinitos exponi poterit. Erit ergo, si valeat signum superius atque *m* denotet numerum imparem,

$$\tfrac{e^{b+x} + e^{c-x}}{e^b + e^c}$$

$$= \left(1 + \tfrac{4(b-c)x+4xx}{\pi\pi+(b-c)^2}\right)\left(1 + \tfrac{4(b-c)x+4xx}{9\pi\pi+(b-c)^2}\right)\left(1 + \tfrac{4(b-c)x+4xx}{25\pi\pi+(b-c)^2}\right) \text{etc.};$$

sin autem signum inferius valeat atque ideo *m* denotet numerum parem casuque $m=0$ radix factoris quadrati ponatur, erit

$$\frac{e^{b+x}-e^{c-x}}{e^{b}-e^{c}}$$
$$=\left(1+\frac{2x}{b-c}\right)\left(1+\frac{4(b-c)x+4xx}{4\pi\pi+(b-c)^{2}}\right)\left(1+\frac{4(b-c)x+4xx}{16\pi\pi+(b-c)^{2}}\right)\left(1+\frac{4(b-c)x+4xx}{36\pi\pi+(b-c)^{2}}\right)\text{ etc.};$$

162. Ponatur $b=0$, quod sine detrimento universalitatis fieri potest, eritque

$$\frac{e^{x}+e^{c}e^{-x}}{1+e^{c}}$$
$$=\left(1-\frac{4cx-4xx}{\pi\pi+c^{2}}\right)\left(1-\frac{4cx-4xx}{9\pi\pi+c^{2}}\right)\left(1-\frac{4cx-4xx}{25\pi\pi+c^{2}}\right)\text{ etc.},$$

$$\frac{e^{x}-e^{c}e^{-x}}{1-e^{c}}$$
$$=\left(1-\frac{2x}{c}\right)\left(1-\frac{4cx-4xx}{4\pi\pi+cc}\right)\left(1-\frac{4cx-4xx}{16\pi\pi+c^{2}}\right)\left(1-\frac{4cx-4xx}{36\pi\pi+c^{2}}\right)\text{ etc.},$$

Iam ponatur *c* negativum atque habebuntur hae duae aequationes

$$\frac{e^{x}+e^{c}e^{-x}}{1+e^{c}}$$
$$=\left(1+\frac{4cx+4xx}{\pi\pi+c^{2}}\right)\left(1+\frac{4cx+4xx}{9\pi\pi+c^{2}}\right)\left(1+\frac{4cx+4xx}{25\pi\pi+c^{2}}\right)\text{ etc.},$$

$$\frac{e^{x}-e^{c}e^{-x}}{1-e^{c}}$$
$$=\left(1+\frac{2x}{c}\right)\left(1+\frac{4cx+4xx}{4\pi\pi+cc}\right)\left(1+\frac{4cx+4xx}{16\pi\pi+c^{2}}\right)\left(1+\frac{4cx+4xx}{36\pi\pi+c^{2}}\right)\text{ etc.}$$

Multiplicetur forma prima per tertiam ac prodibit

$$\frac{e^{2x}+e^{-2x}+e^{c}+e^{-c}}{2+e^{c}+e^{-c}};$$

ponatur vero *y* loco *2x* eritque

$$\frac{e^{y}+e^{-y}+e^{c}+e^{-c}}{2+e^{c}+e^{-c}}$$
$$=\left(1-\frac{2cy-yy}{\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{9\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)$$
$$\left(1-\frac{2cy-yy}{25\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)\text{ etc.}$$

Multiplicetur prima forma per quartam; erit productum

$$\frac{e^{2x}-e^{-2x}+e^{c}-e^{-c}}{e^{c}-e^{-c}};$$

ponatur *y* pro *2x* eritque

$$\frac{e^{y}-e^{-y}+e^{c}-e^{-c}}{e^{c}-e^{-c}}$$
$$=\left(1+\frac{y}{c}\right)\left(1-\frac{2cy-yy}{\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{9\pi\pi+cc}\right)$$
$$\left(1+\frac{2cy+yy}{16\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{25\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{36\pi\pi+cc}\right) \text{ etc.},$$

Si secunda forma per tertiam multiplicetur, prodibit eadem aequatio, nisi quod *c* capiendum sit negativum; erit nempe

$$\frac{e^{c}-e^{-c}-e^{y}+e^{-y}}{e^{c}-e^{-c}}$$
$$=\left(1-\frac{y}{c}\right)\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{4\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)$$
$$\left(1-\frac{2cy-yy}{16\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{36\pi\pi+cc}\right) \text{ etc.}$$

Multiplicetur denique forma secunda per quartam eritque

$$\frac{e^{y}+e^{-y}-e^{c}-e^{-c}}{2-e^{c}-e^{-c}}$$
$$=\left(1-\frac{yy}{cc}\right)\left(1-\frac{2cy-yy}{4\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{16\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{16\pi\pi+cc}\right)$$
$$\left(1-\frac{2cy-yy}{36\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{36\pi\pi+cc}\right) \text{ etc.}$$

163. Hae quatuor combinationes nunc commode ad circulum transferri possunt ponendo

$$c=g\sqrt{-1} \text{ et } y=v\sqrt{-1};\ ;$$

erit enim

$$e^{v\sqrt{-1}}+e^{-v\sqrt{-1}}=2\cos.v,\ e^{v\sqrt{-1}}-e^{-v\sqrt{-1}}=2\sqrt{-1}\cdot\sin.v.$$

et

$$e^{g\sqrt{-1}}+e^{-g\sqrt{-1}}=2\cos.g,\ e^{g\sqrt{-1}}-e^{-g\sqrt{-1}}=2\sqrt{-1}\cdot\sin.g.$$

Hinc prima combinatio dabit

$$\frac{\cos.v+\cos.g}{1+\cos.g}$$

$$=1-\frac{vv}{1\cdot 2(1+\cos.g)}+\frac{v^4}{1\cdot 2\cdot 3\cdot 4(1+\cos.g)}-\frac{v^6}{1\cdot 2\cdot 3\cdots 6(1+\cos.g)}+\text{etc.}$$

$$=\left(1+\frac{2gv-vv}{\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{\pi\pi-gg}\right)\left(1+\frac{2gv-vv}{9\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{9\pi\pi-gg}\right)$$

$$\left(1+\frac{2gv-vv}{25\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{25\pi\pi-gg}\right)\text{ etc.}$$

$$=\left(1+\frac{v}{\pi-g}\right)\left(1-\frac{v}{\pi+g}\right)\left(1-\frac{v}{\pi-g}\right)\left(1+\frac{v}{\pi+g}\right)$$

$$\left(1+\frac{v}{3\pi-g}\right)\left(1-\frac{v}{3\pi+g}\right)\left(1-\frac{v}{3\pi-g}\right)\left(1+\frac{v}{3\pi+g}\right)\text{ etc.}$$

$$=\left(1-\frac{vv}{(\pi-g)^2}\right)\left(1-\frac{vv}{(\pi+g)^2}\right)\left(1-\frac{vv}{(3\pi-g)^2}\right)\left(1-\frac{vv}{(3\pi+g)^2}\right)\left(1-\frac{vv}{(5\pi-g)^2}\right)\text{ etc.}$$

Quarta vero combinatio dat

$$\frac{\cos.v-\cos.g}{1-\cos.g}$$

$$=1-\frac{vv}{1\cdot 2(1-\cos.g)}+\frac{v^4}{1\cdot 2\cdot 3\cdot 4(1-\cos.g)}-\frac{v^6}{1\cdot 2\cdot 3\cdots 6(1-\cos.g)}+\text{etc.}$$

$$=\left(1-\frac{vv}{gg}\right)\left(1+\frac{2gv-vv}{4\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{4\pi\pi-gg}\right)\left(1+\frac{2gv-vv}{16\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{16\pi\pi-gg}\right)\text{ etc.}$$

$$=\left(1-\frac{v}{g}\right)\left(1+\frac{v}{g}\right)\left(1+\frac{v}{2\pi-g}\right)\left(1-\frac{v}{2\pi+g}\right)\left(1-\frac{v}{2\pi-g}\right)\left(1+\frac{v}{2\pi+g}\right)$$

$$\left(1+\frac{v}{4\pi-g}\right)\left(1-\frac{v}{4\pi+g}\right)\text{ etc.}$$

$$=\left(1-\frac{vv}{gg}\right)\left(1-\frac{vv}{(2\pi+g)^2}\right)\left(1-\frac{vv}{(2\pi+g)^2}\right)\left(1-\frac{vv}{(4\pi-g)^2}\right)\left(1-\frac{vv}{(4\pi+g)^2}\right)\text{ etc.}$$

Secunda combinatio dat

$$\frac{\sin.g+\sin.v}{\sin.g}$$

$$=1+\frac{v}{\sin.g}-\frac{v^3}{1\cdot 2\cdot 3\sin.g}+\frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\sin.g}-\text{etc.}$$

$$=\left(1+\frac{v}{g}\right)\left(1+\frac{2gv-vv}{\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{4\pi\pi-gg}\right)\left(1+\frac{2gv-vv}{9\pi\pi-gg}\right)\left(1-\frac{2gv+vv}{16\pi\pi-gg}\right)\text{ etc.}$$

$$=\left(1+\frac{v}{g}\right)\left(1+\frac{v}{\pi-g}\right)\left(1-\frac{v}{\pi+g}\right)\left(1-\frac{v}{2\pi-g}\right)\left(1+\frac{v}{2\pi+g}\right)$$

$$\left(1+\frac{v}{3\pi-g}\right)\left(1-\frac{v}{3\pi+g}\right)\left(1-\frac{v}{4\pi-g}\right)\text{ etc.}$$

Ac sumto v negativo prodit tertia combinatio.

164. Ipsae vero etiam expressiones in § 162 primum inventae ad arcus circulares traduci possunt hoc modo. Cum sit

$$\frac{e^x+e^c e^{-x}}{1+e^c}=\frac{(1+e^{-c})(e^x+e^c e^{-x})}{2+e^c+e^{-c}}=\frac{e^x+e^{-x}+e^{c-x}+e^{-c+x}}{2+e^c+e^{-c}},$$

si ponamus

$$c=g\sqrt{-1} \text{ et } x = z\sqrt{-1},$$

haec expressio abit in hanc

$$\frac{\cos.z+\cos.(g-z)}{1+\cos.g}=\cos.z+\frac{\sin.g\sin.z}{1+\cos.g}.$$

Erit ergo ob $\frac{\sin.g}{1+\cos.g}=\tan.\frac{1}{2}g$

$$\cos.z+\text{tang.}\tfrac{1}{2}g\cdot\sin.z$$

$$=1+\frac{z}{1}\text{tang.}\tfrac{1}{2}g-\frac{zz}{1\cdot2}-\frac{z^3}{1\cdot2\cdot3}\text{tang.}\tfrac{1}{2}g+\frac{z^4}{1\cdot2\cdot3\cdot4}+\frac{z^5}{1\cdot2\cdots5}\text{tang.}\tfrac{1}{2}g-\text{etc.}$$

$$=\left(1+\frac{4gz-4zz}{\pi\pi-gg}\right)\left(1+\frac{2gz-zz}{9\pi\pi-gg}\right)\left(1+\frac{4gz-4zz}{25\pi\pi-gg}\right)\text{ etc.}$$

$$=\left(1+\frac{2z}{\pi-g}\right)\left(1-\frac{2z}{\pi+g}\right)\left(1+\frac{2z}{3\pi-g}\right)\left(1-\frac{2z}{3\pi+g}\right)\left(1+\frac{2z}{5\pi-g}\right)\left(1-\frac{2z}{5\pi+g}\right)\text{ etc.}$$

Simili modo altera expressio, si numerator et denominator per $1-e^{-c}$ multiplicetur, abit in

$$\frac{e^x+e^{-x}-e^{c-x}-e^{-c+x}}{2-e^c-e^{-c}},$$

quae facto $c=g\sqrt{-1}$ et $x=z\sqrt{-1}$ dat

$$\frac{\cos.z-\cos.(g-z)}{1-\cos.g}=\cos.z-\frac{\sin.g\ \sin.z}{1-\cos.g}=\cos.z-\frac{\sin.z}{\text{tang.}\frac{1}{2}g}.$$

Erit ergo

$$\cos.z - \cot.\tfrac{1}{2}g\ \sin.z$$
$$= 1 - \tfrac{z}{1}\cot.\tfrac{1}{2}g - \tfrac{zz}{1\cdot 2} + \tfrac{z^3}{1\cdot 2\cdot 3}\cot.\tfrac{1}{2}g + \tfrac{z^4}{1\cdot 2\cdot 3\cdot 4} - \tfrac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cot.\tfrac{1}{2}g + \text{etc.}$$
$$= \left(1-\tfrac{2z}{g}\right)\left(1+\tfrac{4gz-4zz}{4\pi\pi-gg}\right)\left(1+\tfrac{4gz-zz}{16\pi\pi-gg}\right)\left(1+\tfrac{4gz-4zz}{36\pi\pi-gg}\right)\ \text{etc.}$$
$$= \left(1-\tfrac{2z}{g}\right)\left(1+\tfrac{2z}{2\pi-g}\right)\left(1-\tfrac{2z}{2\pi+g}\right)\left(1+\tfrac{2z}{4\pi-g}\right)\left(1-\tfrac{2z}{4\pi+g}\right)\ \text{etc.}$$

Quodsi ergo ponatur $v = 2z$ seu $z = \tfrac{1}{2}v$, habebitur

$$\frac{\cos.\frac{1}{2}(g-v)}{\cos.\frac{1}{2}g} = \cos.\tfrac{1}{2}v + \text{tang}.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1+\tfrac{v}{\pi-g}\right)\left(1-\tfrac{v}{\pi+g}\right)\left(1+\tfrac{v}{3\pi-g}\right)\left(1-\tfrac{v}{3\pi+g}\right)\ \text{etc}$$
$$\frac{\cos.\frac{1}{2}(g+v)}{\cos.\frac{1}{2}g} = \cos.\tfrac{1}{2}v - \text{tang}.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1-\tfrac{v}{\pi-g}\right)\left(1+\tfrac{v}{\pi+g}\right)\left(1-\tfrac{v}{3\pi-g}\right)\left(1+\tfrac{v}{3\pi+g}\right)\ \text{etc.}$$
$$\frac{\sin.\frac{1}{2}(g-v)}{\sin.\frac{1}{2}g} = \cos.\tfrac{1}{2}v - \cot.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1-\tfrac{v}{g}\right)\left(1+\tfrac{v}{2\pi-g}\right)\left(1-\tfrac{v}{2\pi+g}\right)\left(1+\tfrac{v}{4\pi-g}\right)\left(1-\tfrac{v}{4\pi+g}\right)\ \text{etc.}$$
$$\frac{\sin.\frac{1}{2}(g+v)}{\sin.\frac{1}{2}g} = \cos.\tfrac{1}{2}v + \cot.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1+\tfrac{v}{g}\right)\left(1-\tfrac{v}{2\pi-g}\right)\left(1+\tfrac{v}{2\pi+g}\right)\left(1-\tfrac{v}{4\pi-g}\right)\left(1+\tfrac{v}{4\pi+g}\right)\ \text{etc.}$$

Quorum factorum lex progressionis satis est simplex et uniformis; atque ex his expressionibus per multiplicationem oriuntur eae ipsae, quae paragrapho praecedente sunt inventae.