CHAPTER VII

ESTABLISHING EXPONENTIAL AND LOGARITHMIC QUANTITIES IN SERIES

114. Because $a^0 = 1$ and with the exponent of *a* increasing, the value of the power likewise is increased, if indeed *a* is a number greater than one, it follows, if the exponent may exceed zero by an infinitely small amount, the power itself also is an infinitely small amount greater than one. Let ω be an infinitely small number or so small a fraction, so that it is only not equal to zero; there will be

$$a^{\omega} = 1 + \psi$$

with ψ also being an infinitely small number. For from the preceding chapter it is agreed, unless ψ should be an infinitely small number, neither is it possible for ω to be such. Therefore either $\psi = \omega$, $\psi > \omega$, or $\psi < \omega$, which ratio [between ψ and ω] certainly depends on the letter *a*; which since at this stage it shall be unknown, the relation is put as $\psi = k\omega$, thus so that there shall be

$$a^{\omega} = 1 + k\omega,$$

and with *a* taken for the logarithmic base there will be

$$\omega = l(1 + k\omega).$$

EXAMPLE

So that it may be made clearer, just as the number k may depend on the base *a*, we may put $a = 10$ and from the common tables we may look for the logarithm of a number that minimally is greater than one, for example $1 + \frac{1}{1\,000\,000}$, thus so that it shall be $k\omega = \frac{1}{1\,000\,000}$; there will be

$$l\left(1 + \frac{1}{1\,000\,000}\right) = l\,\frac{1000001}{1000000} = 0{,}00000043429 = \omega.$$

Hence on account of $k\omega = 0{,}00000100000$, there will be

$$\frac{1}{k} = \frac{43429}{100000}$$

and

$$k = \frac{100000}{43429} = 2{,}30258;$$

from which it is apparent that *k* is a finite number depending on the value of the base *a*. If indeed another number may be put in place for the base *a*, then the logarithm of the same number $1+k\omega$ will hold the same ratio to the first given, from which likewise another value of the *k* may appear.

115. Since $a^{\omega}=1+k\omega$, there will be $a^{i\omega}=(1+k\omega)^{i}$, whatever number may be substituted in place of *i*. Therefore there becomes

$$a^{i\omega}=1+\tfrac{i}{1}k\omega+\tfrac{i(i-1)}{1\cdot 2}k^2\omega^2+\tfrac{i(i-1)(i-2)}{1\cdot 2\cdot 3}k^3\omega^3+\text{etc.}$$

But if therefore $i=\frac{z}{\omega}$ may be put in place and *z* may denote some finite number, on account of the infinitely small number ω, *i* becomes infinitely great and hence $\omega=\frac{z}{i}$ thus so that ω shall be a fraction having an infinite denominator and thus will be infinitely small, such as has been assumed. Therefore $\frac{z}{\omega}$ may be substituted in place of *i* and the equation becomes

$$a^{z}=\left(1+\tfrac{kz}{i}\right)^{i}=1+\tfrac{1}{1}kz+\tfrac{1(i-1)}{1\cdot 2i}k^2z^2+\tfrac{1(i-1)(i-2)}{1\cdot 2i\cdot 3i}k^3z^3+\tfrac{1(i-1)(i-2)(i-3)}{1\cdot 2i\cdot 3i\cdot 4i}k^4z^4+\text{etc.},$$

which equation will be true, if an infinitely great number be substituted for *i*. Then truly *k* is a finite number depending on *a*, just as we have seen.

116. But since *i* shall be an infinitely great number, there will be

$$\tfrac{i-1}{i}=1;$$

for it is clear, so that the greater the number may be substituted in place of *i*, the closer to that value of the fraction $\frac{i-1}{i}$ will be approaching to one ; hence, if *i* shall be a number greater than all assignable numbers, also the fraction $\frac{i-1}{i}$ will itself be equal to one. On account of similar reasoning there will be

$$\tfrac{i-2}{i}=1,\ \tfrac{i-3}{i}=1$$

and thus henceforth; hence it follows that

$$\tfrac{i-1}{2i}=\tfrac{1}{2},\ \tfrac{i-2}{3i}=\tfrac{1}{3},\ \tfrac{i-3}{4i}=\tfrac{1}{4}$$

and thus henceforth. Therefore with these values substituted there will be

$$a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1\cdot 2} + \frac{k^3z^3}{1\cdot 2\cdot 3} + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc., indefinitely.}$$

But this equation will show a similar relation between the numbers a and k;
for by putting $z = 1$ there will be

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \frac{k^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

and, so that *a* shall be $= 10$, it is necessary that $k = 2{,}30258$ approximately, as we have found before.

117. We may put

$$b = a^n;$$

with the number *a* taken for the logarithmic base it will become $lb = n$. Hence, since there shall be $b^z = a^{nz}$, there will be by the infinite series

$$b^z = 1 + \frac{knz}{1} + \frac{k^2n^2z^2}{1\cdot 2} + \frac{k^3n^3z^3}{1\cdot 2\cdot 3} + \frac{k^4n^4z^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.};$$

truly on putting lb for *n*, the equation becomes

$$b^z = 1 + \frac{kz}{1} lb + \frac{k^2z^2}{1\cdot 2}(lb)^2 + \frac{k^3z^3}{1\cdot 2\cdot 3}(lb)^3 + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4}(lb)^4 + \text{etc.}$$

Therefore with the value of the letter *k* known from the given value of the base *a* it will be possible to express any quantity of the exponential b^z by an infinite series, whose terms proceed following the powers of *z*. From these set out we have shown also, how logarithms may be set out in an infinite series.

118. Since there shall be $a^\omega = 1 + k\omega$ with ω being an infinitely small fraction and the ratio between *a* and *k* may be defined by this equation

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \frac{k^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

if *a* may be taken for the logarithmic base, the above equation becomes

$$\omega = l(1 + k\omega) \text{ and } i\omega = l(1 + k\omega)^i.$$

Moreover it is clear, by how much greater the number for *i* may be taken, for the power $(1+k\omega)^i$ to become much greater than one and on putting in place $i =$ an infinitely large number, the value of the power $(1+k\omega)^i$ will ascend to some number greater than one. But if therefore there may be put

$$(1+k\omega)^i = 1+x,$$

there becomes

$$l(1+x) = i\omega,$$

from which, since $i\omega$ shall be a finite number, evidently the logarithm of the number $1+x$, seen to be i, must become an infinitely large number ; for otherwise $i\omega$ cannot have a finite value.

119. Moreover since the equation may be put in place :

$$(1+k\omega)^i = 1+x,$$

there will be

$$1+k\omega = (1+x)^{\frac{1}{i}} \text{ and } k\omega = (1+x)^{\frac{1}{i}} - 1,$$

from which there becomes

$$i\omega = \tfrac{i}{k}\left((1+x)^{\frac{1}{i}} - 1\right).$$

Because truly there is $i\omega = l(1+x)$, the equation becomes

$$l(1+x) = \tfrac{i}{k}(1+x)^{\frac{1}{i}} - \tfrac{i}{k}$$

on putting the number *i* infinitely large. But

$$(1+x)^{\frac{1}{i}} = 1+\tfrac{1}{i}x - \tfrac{1(i-1)}{i\cdot 2i}x^2 + \tfrac{1(i-1)(2i-1)}{i\cdot 2i\cdot 3i}x^3 - \tfrac{1(i-1)(2i-1)(3i-1)}{i\cdot 2i\cdot 3i\cdot 4i}x^4 + \text{etc.}$$

But on account of *i* being an infinite number there will be

$$\tfrac{i-1}{2i} = \tfrac{1}{2}, \quad \tfrac{2i-1}{3i} = \tfrac{2}{3}, \quad \tfrac{3i-1}{4i} = \tfrac{3}{4} \text{ etc.};$$

hence the above equation becomes

$$i(1+x)^{\frac{1}{i}} = i + \tfrac{x}{1} - \tfrac{xx}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \text{etc.}$$

and consequently

$$l(1+x) = \tfrac{1}{k}\left(\tfrac{x}{1} - \tfrac{xx}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \text{etc.}\right)$$

with the logarithmic base $=a$ and with k denoting a number agreeing with this base, so that clearly there shall be

$$a = 1 + \tfrac{k}{1} + \tfrac{k^2}{1\cdot 2} + \tfrac{k^3}{1\cdot 2\cdot 3} + \text{etc.},$$

120. Therefore since we may have a series equal to the logarithm of the number $1+x$, with the aid of this we will be able to define the value of the number k from the given base a. For if we may put
$1+x=a$, on account of $la=1$, there will be

$$1 = \tfrac{1}{k}\left(\tfrac{a-1}{1} - \tfrac{(a-1)^2}{2} + \tfrac{(a-1)^3}{3} - \tfrac{(a-1)^4}{4} + \text{etc.}\right),$$

and hence there will be found

$$k = \tfrac{a-1}{1} - \tfrac{(a-1)^2}{2} + \tfrac{(a-1)^3}{3} - \tfrac{(a-1)^4}{4} + \text{etc.}$$

thus the value of the infinite series of which , if there is put $a=10$, must be approximately $=2{,}30258$, though it will be understood with difficulty

$$2{,}30258 = \tfrac{9}{1} - \tfrac{9^2}{2} + \tfrac{9^3}{3} - \tfrac{9^4}{4} + \text{etc.},$$

because the terms of this series continually become larger nor can the sum be had truly approximately by summing a number of terms; to which inconvenience a remedy will be produced soon.

121. Therefore because there is

$$l(1+x) = \tfrac{1}{k}\left(\tfrac{x}{1} - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \text{etc.}\right)$$

it will be on putting x negative

$$l(1-x) = -\tfrac{1}{k}\left(\tfrac{x}{1} + \tfrac{x^2}{2} + \tfrac{x^3}{3} + \tfrac{x^4}{4} + \text{etc.}\right)$$

The latter series is taken from the former ; there becomes

$$l\frac{1+x}{1-x}=\frac{2}{k}\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\text{etc.}\right).$$

Now there is put

$$\frac{1+x}{1-x}=a,$$

so that there shall be

$$x=\frac{a+1}{a-1};$$

on account of $la=1$, there will be

$$k=2\left(\frac{a-1}{a+1}+\frac{(a-1)^3}{3(a+1)^3}+\frac{(a-1)^5}{5(a+1)^5}+\text{etc.}\right),$$

from which equation the value of the number *k* will be able to be found from the base *a*. If therefore the base *a* may be put $=10$, it becomes

$$k=2\left(\frac{9}{11}+\frac{9^3}{3\cdot 11^3}+\frac{9^5}{5\cdot 11^5}+\frac{9^7}{7\cdot 11^7}+\text{etc.}\right),$$

the terms of which series decrease sensibly and thus soon will show the value for *k* near enough.

122. Because it is permitted to take a system of logarithms constructed from any base it pleases, so that the constant can become $k=1$. Therefore we may put $k=1$ and *a* will be found by the series found above (§ 116) :

$$a=1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\text{etc.},$$

which terms, if they may be changed into decimal fractions and with the series added, will provide this value for *a*

2,71828182845904523536028,

the final figure of which is agreed to be true.

But if now logarithms may be constructed from this base, these are accustomed to be called *natural* or *hyperbolic*, because the quadrature of the hyperbola can be expressed by logarithms of this kind. For the sake of brevity moreover, we may put steadily the letter *e* for this number 2,71828 1828459 etc., which therefore will denote the base of natural or hyperbolic logarithms, to which the value of the letter $k=1$ corresponds ; or this letter *e* also will express the sum of this series

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\text{etc. to infinity.}$$

123. Therefore hyperbolic logarithms will have this property, that the logarithm of this number $1+\omega$ shall be $=\omega$, with ω an infinitely small quantity, and since from this property the value $k=1$ may become known, the logarithms of all the hyperbolic numbers are able to be shown. Therefore on putting e for the number found above there will be always

$$e^z = 1+\frac{z}{1}+\frac{z^2}{1\cdot 2}+\frac{z^3}{1\cdot 2\cdot 3}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}$$

Truly the hyperbolic logarithms themselves may be found from these series, from which there becomes

$$l(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\text{etc.}$$

and

$$l\,\frac{1+x}{1-x}=\frac{2x}{1}+\frac{2x^3}{3}+\frac{2x^5}{5}+\frac{2x^7}{7}+\frac{x^5}{5}+\frac{2x^9}{9}+\text{etc.},$$

which series converge strongly, if a very small fraction may put in place for *x*. Thus from this latter series the logarithms of numbers not much greater than unity can be found easily by calculation. For on putting $x=\frac{1}{5}$ there will be

$$l\,\frac{6}{4}=l\,\frac{3}{2}=\frac{2}{1\cdot 5}+\frac{2}{3\cdot 5^3}+\frac{2}{5\cdot 5^5}+\frac{2}{7\cdot 5^7}+\text{etc.}$$

and on making $x=\frac{1}{7}$ there will be

$$l\,\frac{4}{3}=\frac{2}{1\cdot 7}+\frac{2}{3\cdot 7^3}+\frac{2}{5\cdot 7^5}+\frac{2}{7\cdot 5^7}+\text{etc.}$$

and by putting $x=\frac{1}{9}$ there will be

$$l\,\frac{5}{4}=\frac{2}{1\cdot 9}+\frac{2}{3\cdot 9^3}+\frac{2}{5\cdot 9^5}+\frac{2}{7\cdot 9^7}+\text{etc.}$$

Truly from the logarithms of these factors the logarithms of whole numbers can be found ; for from the nature of logarithms

$$l\,\frac{3}{2}+l\,\frac{4}{3}=l2,$$

then

$$l\,\frac{3}{2}+l2=l3 \text{ and } 2l2=l4,$$

again

$$l\tfrac{5}{4}+l4=l5,\ l2+l3=l6,\ 3l2=l8,\ 2l3=l9 \text{ et } l2+l5=l10.$$

EXAMPLE

Hence the hyperbolic logarithms of the numbers from 1 as far as 10 thus may be found, so that there shall be

*l*1 = 0,00000 00000 00000 00000 00000
*l*2 = 0,69314 71805 59945 30941 72321
*l*3 = 1,09861 22886 68109 69139 52452
*l*4 = 1,38629 43611 19890 61883 44642
*l*5 = 1,60943 79124 34100 37460 07593
*l*6 = 1,79175 94692 28055 00812 47741
*l*7 = 1,94591 01490 55313 30510 53527
*l*8 = 2,07944 15416 79835 92825 16964
*l*9 = 2,19722 45773 36219 38279 04905
*l*10 = 2,30258 50929 94045 68401 79915

Evidently all these logarithms have been deduced from the above three series except *l*7, which I have attended according to this manner. I have put $x=\frac{1}{99}$ into the latter series and found without doubt

$$l\tfrac{100}{98}=l\tfrac{50}{49}=0{,}02020\,27073\,17519\,44840\,80453,$$

which subtracted from

$$l50=2l5+l2=3{,}91202\,30054\,28146\,05861\,87508$$

leaves *l*49, the half of which gives *l*7.

124. The hyperbolic logarithm of $1+x$ itself or $l(1+x)$ may be put $=y$; there becomes

$$y=\tfrac{x}{1}-\tfrac{x^2}{2}+\tfrac{x^3}{3}-\tfrac{x^4}{4}+\text{etc.}$$

But with the number *a* taken for the logarithmic base the logarithm of the same number $1+x$ will be $=v$; there becomes, as we have seen,

$$v=\tfrac{1}{k}\left(x-\tfrac{xx}{2}+\tfrac{x^3}{3}-\tfrac{x^4}{4}+\text{etc.}\right)=\tfrac{y}{k}$$, and hence

$$k=\tfrac{y}{v};$$

from which the value of *k* to the corresponding base *a* thus is found most conveniently, so that it shall equal to the hyperbolic logarithm of some number divided by the logarithm of the same number formed from the base *a*. Therefore on putting this number $=a$ there becomes $v=1$ and hence *k* is equal to the hyperbolic logarithm of the base *a*. Therefore in the system of common logarithms, where a is equal to 10, *k* is equal to the hyperbolic logarithm of 10, from which

$$k=2{,}30258\,50929\,94045\,6840179915,$$

which value we have now deduced closely enough above. Therefore if the individual hyperbolic logarithms may be divided by this number *k* or, what returns the same, they may be multiplied by this decimal fraction

$$0{,}43429\,44819\,03251\,82765\,11289\,,$$

the agreeing common logarithms to the base $a=10$ will be produced.

125. Since there shall be

$$e^{z}=1+\frac{z}{1}+\frac{z^{2}}{1\cdot 2}+\frac{z^{3}}{1\cdot 2\cdot 3}+\text{etc.},$$

if there may be put $a^{y}=e^{z}$, with the hyperbolic logarithms taken, there will be $yla=z$, because $le=1$; with which value substituted in place of *z*

$$a^{y}=1+\frac{yla}{1}+\frac{y^{2}(la)^{2}}{1\cdot 2}+\frac{y^{3}(la)^{3}}{1\cdot 2\cdot 3}+\text{etc.},$$

from which any exponential quantity can be set out in an infinite series with the aid of hyperbolic logarithms.

Then truly with *i* denoting an infinitely large magnitude both the exponential quantities as well as the logarithms can be expressed by the powers of the exponent. For there will be

$$e^{z}=\left(1+\tfrac{z}{i}\right)^{i}$$

and hence

$$a^{y}=\left(1+\frac{yla}{i}\right)^{i},$$

then, for the hyperbolic logarithms, there will be found

$$l(1+x)=i\left((1+x)^{\frac{1}{i}}-1\right).$$

The remaining uses of hyperbolic logarithms will be shown in more detail in the integral calculus.

CAPUT VII

DE QUANTITATUM EXPONENTIALIUM AC LOGARITHMORUM PER SERIES EXPLICATIONE

114. Quia est $a^0 = 1$ atque crescente exponente ipsius *a* simul valor potestatis augetur, si quidem *a* est numerus unitate maior, sequitur, si exponens infinite parum cyphram excedat, potestatem ipsam quoque infinite parum unitatem esse superaturam. Sit ω numerus infinite parvus seu fractio tam exigua, ut tantum non nihilo sit aequalis; erit

$$a^{\omega} = 1 + \psi$$

existente ψ quoque numero infinite parvo. Ex praecedente enim capite constat, nisi ψ esset numerus infinite parvus, neque ω talem esse posse. Erit ergo vel $\psi = \omega$ vel $\psi > \omega$ vel $\psi < \omega$, quae ratio utique a quantitate litterae *a* pendebit; quae cum adhuc sit incognita, ponatur $\psi = k\omega$, ita ut sit

$$a^{\omega} = 1 + k\omega,$$

et sumpta *a* pro basi logarithmica erit

$$\omega = l(1 + k\omega).$$

EXEMPLUM

Quo clarius appareat, quemadmodum numerus *k* pendeat a basi *a*, ponamus esse $a = 10$ atque ex tabulis vulgaribus quaeramus logarithmum numeri quam minime unitatem superantis, puta $1 + \frac{1}{1\,000\,000}$, ita ut sit $k\omega = \frac{1}{1\,000\,000}$;
erit

$$l\left(1 + \frac{1}{1\,000\,000}\right) = l\,\frac{1000001}{1000000} = 0{,}00000043429 = \omega.$$

Hinc ob $k\omega = 0{,}00000100000$ erit

$$\frac{1}{k} = \frac{43429}{100000}$$

et

$$k = \frac{100000}{43429} = 2{,}30258;$$

unde patet *k* esse numerum finitum pendentem a valore basis *a*. Si enim alius numerus basi *a* statuatur, tum logarithmus eiusdem numeri $1 + k\omega$ ad priorern datam tenebit rationem, unde· simul alius valor litterae *k* prodiret.

115. Cum sit $a^{\omega}=1+k\omega$, erit $a^{i\omega}=(1+k\omega)^{i}$, quicunque numerus loco *i* substituatur. Erit ergo

$$a^{i\omega}=1+\tfrac{i}{1}k\omega+\tfrac{i(i-1)}{1\cdot2}k^{2}\omega^{2}+\tfrac{i(i-1)(i-2)}{1\cdot2\cdot3}k^{3}\omega^{3}+\text{etc.}$$

Quodsi ergo statuatur $i=\frac{z}{\omega}$ et *z* denotet numerum quemcunque finitum, ob ω numerum infinite parvum fiet *i* numerus infinite magnus hincque $\omega=\frac{z}{i}$ ita ut sit ω fractio denominatorem habens infinitum adeoque infinite parva, qualis est assumpta. Substituatur ergo $\frac{z}{\omega}$ loco ω eritque

$$a^{z}=\left(1+\tfrac{kz}{i}\right)^{i}=1+\tfrac{1}{1}kz+\tfrac{1(i-1)}{1\cdot2i}k^{2}z^{2}+\tfrac{1(i-1)(i-2)}{1\cdot2i\cdot3i}k^{3}z^{3}+\tfrac{1(i-1)(i-2)(i-3)}{1\cdot2i\cdot3i\cdot4i}k^{4}z^{4}+\text{etc.},$$

quae aequatio erit vera, si pro *i* numerus infinite magnus substituatur. Tum vero est *k* numerus finitus ab *a* pendens, uti modo vidimus.

116. Cum autem *i* sit numerus infinite magnus, erit

$$\tfrac{i-1}{i}=1;$$

patet enim, quomaior numerus loco *i* substituatur, eo propius valorem fractionis $\frac{i-1}{i}$ ad unitatem esse accessurum; hinc, si *i* sit numerus omni assignabili maior, fractio quoque $\frac{i-1}{i}$ ipsam unitatem adaequabit. Ob similem autem rationem erit

$$\tfrac{i-2}{i}=1,\ \tfrac{i-3}{i}=1$$

et ita porro; hinc sequitur fore

$$\tfrac{i-1}{2i}=\tfrac{1}{2},\ \tfrac{i-2}{3i}=\tfrac{1}{3},\ \tfrac{i-3}{4i}=\tfrac{1}{4}$$

et ita porro. His igitur valoribus substitutis erit

$$a^{z}=1+\tfrac{kz}{1}+\tfrac{k^{2}z^{2}}{1\cdot2}+\tfrac{k^{3}z^{3}}{1\cdot2\cdot3}+\tfrac{k^{4}z^{4}}{1\cdot2\cdot3\cdot4}+\text{etc., in infinitum.}$$

Haec autem aequatio simul relationem inter numeros *a* et *k* ostendit; posito enim $z=1$ erit

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \frac{k^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

atque, ut a sit $= 10$, necesse est, ut sit circiter $k = 2{,}30258$, uti ante invenimus.

117. Ponamus esse

$$b = a^n ;$$

erit sumpto numero a pro basi logarithmica $lb = n$. Hinc, cum sit $b^z = a^{nz}$, ,
erit per seriem infinitam

$$b^z = 1 + \frac{knz}{1} + \frac{k^2n^2z^2}{1\cdot 2} + \frac{k^3n^3z^3}{1\cdot 2\cdot 3} + \frac{k^4n^4z^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.};$$

posito vero lb pro n erit

$$b^z = 1 + \frac{kz}{1} lb + \frac{k^2z^2}{1\cdot 2}(lb)^2 + \frac{k^3z^3}{1\cdot 2\cdot 3}(lb)^3 + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4}(lb)^4 + \text{etc.}$$

Cognito ergo valore litterae k ex dato valore basis a quantitas exponentialis quaecunque b^z per seriem infinitam exprimi poterit, cuius termini secundum potestates ipsius z procedant. His expositis ostendamus quoque, quomodo logarithmi per series infinitas explicari possint.

118. Cum sit $a^\omega = 1 + k\omega$ existente ω fractione infinite parva atque ratio inter a et k definiatur per hanc aequationem

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \frac{k^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

si a sumatur pro basi logarithmica, erit

$$\omega = l(1 + k\omega) \quad \text{et} \quad i\omega = l(1 + k\omega)^i .$$

Manifestum autem est, quo maior numerus pro i sumatur, eo magis potestatem $(1 + k\omega)^i$ unitatem esse superaturam atque statuendo $i =$ numero infinito valorem potestatis $(1 + k\omega)^i$ ad quemvis numerum unitate maiorem ascendere.
Quodsi ergo ponatur

$$(1 + k\omega)^i = 1 + x,$$

erit

$$l(1+x)=i\omega,$$

unde, cum sit $i\omega$ numerus finitus, logarithmus scilicet numeri $1+x$, perspicuum est i esse debere numerum infinite magnum; alioquin enim $i\omega$ valorem finitum habere non posset.

119. Cum autem positum sit

$$(1+k\omega)^i=1+x,$$

erit

$$1+k\omega=(1+x)^{\frac{1}{i}} \text{ et } k\omega=(1+x)^{\frac{1}{i}}-1,$$

unde fit

$$i\omega=\tfrac{i}{k}\left((1+x)^{\frac{1}{i}}-1\right).$$

Quia vero est $i\omega=l(1+x)$, erit

$$l(1+x)=\tfrac{i}{k}(1+x)^{\frac{1}{i}}-\tfrac{i}{k}$$

posito i numero infinite magno. Est autem

$$(1+x)^{\frac{1}{i}}=1+\tfrac{1}{i}x-\tfrac{1(i-1)}{i\cdot 2i}x^2+\tfrac{1(i-1)(2i-1)}{i\cdot 2i\cdot 3i}x^3-\tfrac{1(i-1)(2i-1)(3i-1)}{i\cdot 2i\cdot 3i\cdot 4i}x^4+\text{etc.}$$

Ob i autem numerum infinitum erit

$$\tfrac{i-1}{2i}=\tfrac{1}{2},\quad \tfrac{2i-1}{3i}=\tfrac{2}{3},\quad \tfrac{3i-1}{4i}=\tfrac{3}{4}\ \text{etc.};$$

hinc erit

$$i(1+x)^{\frac{1}{i}}=i+\tfrac{x}{1}-\tfrac{xx}{2}+\tfrac{x^3}{3}-\tfrac{x^4}{4}+\text{etc.}$$

et consequenter

$$l(1+x)=\tfrac{1}{k}\left(\tfrac{x}{1}-\tfrac{xx}{2}+\tfrac{x^3}{3}-\tfrac{x^4}{4}+\text{etc.}\right)$$

posita basi logarithmica $=a$ ac denotante k numerum huic basi convenientem, ut scilicet sit

$$a=1+\tfrac{k}{1}+\tfrac{k^2}{1\cdot 2}+\tfrac{k^3}{1\cdot 2\cdot 3}+\text{etc.},$$

120. Cum igitur habeamus seriem logarithmo numeri $1+x$ aequalem, eius ope ex data basi *a* definire poterimus valorem numeri *k*. Si enim ponamus
$1+x=a$, ob $la=1$ erit

$$1=\frac{1}{k}\left(\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\text{etc.}\right)$$

hincque habebitur

$$k=\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\text{etc.}$$

cuius ideo seriei infinitae valor, si ponatur $a=10$, circiter esse debebit $=2{,}30258$, quanquam difficulter intelligi potest esse

$$2{,}30258=\frac{9}{1}-\frac{9^2}{2}+\frac{9^3}{3}-\frac{9^4}{4}+\text{etc.},$$

quoniam huius seriei termini continuo fiunt maiores neque adeo aliquot terminis sumendis summa vero propinqua haberi potest; cui incommodo mox remedium afferetur.

121. Quoniam igitur est

$$l(1+x)=\frac{1}{k}\left(\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\text{etc.}\right)$$

erit posito *x* negativo

$$l(1-x)=-\frac{1}{k}\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\text{etc.}\right)$$

Subtrahatur series posterior a priori; erit

$$l\frac{1+x}{1-x}=\frac{2}{k}\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\text{etc.}\right).$$

Nunc ponatur

$$\frac{1+x}{1-x}=a,$$

ut sit

$$x=\frac{a+1}{a-1};$$

ob $la=1$ erit

$$k=2\left(\frac{a-1}{a+1}+\frac{(a-1)^3}{3(a+1)^3}+\frac{(a-1)^5}{5(a+1)^5}+\text{etc.}\right),$$

ex qua aequatione valor numeri k ex basi a inveniri poterit. Si ergo basis a ponatur $=10$, erit

$$k=2\left(\tfrac{9}{11}+\tfrac{9^3}{3\cdot 11^3}+\tfrac{9^5}{5\cdot 11^5}+\tfrac{9^7}{7\cdot 11^7}+\text{etc.}\right),$$

cuius seriei termini sensibiliter decrescunt ideoque mox valorem pro k satis propinquum exhibent.

122. Quoniam ad systema logarithmorum condendum basin a pro lubitu accipere licet, ea ita assumi poterit, ut fiat $k=1$. Ponamus ergo esse $k=1$ eritque per seriem supra (§ 116) inventam

$$a=1+\tfrac{1}{1}+\tfrac{1}{1\cdot 2}+\tfrac{1}{1\cdot 2\cdot 3}+\tfrac{1}{1\cdot 2\cdot 3\cdot 4}+\text{etc.},$$

qui termini, si in fractiones decimales convertantur atque actu addantur, praebebunt hunc valorem pro a

2,71828182845904523536028,

cuius ultima adhuc nota veritati est consentanea.

Quodsi iam ex hac basi logarithmi construantur, ii vocari solent logarithmi *naturales* seu *hyperbolici*, quoniam quadratura hyperbolae per istiusmodi logarithmos exprimi potest. Ponamus autem brevitatis gratia pro numero hoc 2,71828 1828459 etc. constanter litteram

$$e,$$

quae ergo denotabit basin logarithmorum naturalium seu hyperbolicorum, cui respondet valor litterae $k=1$; sive haec littera e quoque exprimet summam huius seriei

$$1+\tfrac{1}{1}+\tfrac{1}{1\cdot 2}+\tfrac{1}{1\cdot 2\cdot 3}+\tfrac{1}{1\cdot 2\cdot 3\cdot 4}+\text{etc. in infinitum.}$$

123. Logarithmi ergo hyperbolici hanc habebunt proprietatem, ut numeri $1+\omega$ logarithmus sit $=\omega$ denotante ω quantitatem infinite parvam, atque cum ex hac proprietate valor $k=1$ innotescat, omnium numerorum logarithmi hyperbolici exhiberi poterunt. Erit ergo posita e pro numero supra invento perpetuo

$$e^z=1+\tfrac{z}{1}+\tfrac{z^2}{1\cdot 2}+\tfrac{z^3}{1\cdot 2\cdot 3}+\tfrac{z^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}$$

Ipsi vero logarithmi hyperbolici ex his seriebus invenientur, quibus est

$$l(1+x)=x-\tfrac{x^2}{2}+\tfrac{x^3}{3}-\tfrac{x^4}{4}+\tfrac{x^5}{5}-\tfrac{x^6}{6}+\text{etc.}$$

et

$$l\frac{1+x}{1-x}=\frac{2x}{1}+\frac{2x^3}{3}+\frac{2x^5}{5}+\frac{2x^7}{7}+\frac{x^5}{5}+\frac{2x^9}{9}+\text{etc.},$$

quae series vehementer convergunt, si pro x statuatur fractio valde parva. Ita ex serie posteriori facile negotio inveniuntur logarithmi numerorum unitate non multo maiorum. Posito namque $x=\frac{1}{5}$ erit

$$l\frac{6}{4}=l\frac{3}{2}=\frac{2}{1\cdot5}+\frac{2}{3\cdot5^3}+\frac{2}{5\cdot5^5}+\frac{2}{7\cdot5^7}+\text{etc.}$$

et facto $x=\frac{1}{7}$ erit

$$l\frac{4}{3}=\frac{2}{1\cdot7}+\frac{2}{3\cdot7^3}+\frac{2}{5\cdot7^5}+\frac{2}{7\cdot5^7}+\text{etc.}$$

et facto $x=\frac{1}{9}$ erit

$$l\frac{5}{4}=\frac{2}{1\cdot9}+\frac{2}{3\cdot9^3}+\frac{2}{5\cdot9^5}+\frac{2}{7\cdot9^7}+\text{etc.}$$

Ex logarithmis vero harum fractionum reperientur logarithmi numerorum integrorum; erit enim ex natura logarithmorum

$$l\frac{3}{2}+l\frac{4}{3}=l2,$$

tum

$$l\frac{3}{2}+l2=l3 \text{ et } 2l2=l4,$$

porro

$$l\frac{5}{4}+l4=l5,\ l2+l3=l6,\ 3l2=l8,\ 2l3=l9 \text{ et } l2+l5=l10.$$

EXEMPLUM

Hinc logarithmi hyperbolici numerorum ab 1 usque ad 10 ita se habebunt, ut sit

*l*1 = 0,00000 00000 00000 00000 00000
*l*2 = 0,69314 71805 59945 30941 72321
*l*3 = 1,09861 22886 68109 69139 52452
*l*4 = 1,38629 43611 19890 61883 44642
*l*5 = 1,60943 79124 34100 37460 07593
l6 = 1,79175 94692 28055 00812 47741
*l*7 = 1,94591 01490 55313 30510 53527
l8 = 2,07944 15416 79835 92825 16964
*l*9 = 2,19722 45773 36219 38279 04905
*l*10 = 2,30258 50929 94045 68401 79915

Hi scilicet logarithmi omnes ex superioribus tribus seriebus sunt deducti praeter $l7$, quem hoc compendio sum assecutus. Posui nimirum in serie posteriori $x=\frac{1}{99}$ sicque obtinui

$$l\frac{100}{98}=l\frac{50}{49}=0{,}02020\,27073\,17519\,44840\,80453,$$

qui subtractus a

$$l50=2l5+l2=3{,}91202\,30054\,28146\,05861\,87508$$

relinquit $l49$, cuius semissis dat $l7$.

124. Ponatur logarithmus hyperbolicus ipsius $1+x$ seu $l(1+x)=y$; erit

$$y=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\text{etc.}$$

Sumpto autem numero a pro basi logarithmica sit numeri eiusdem $1+x$ logarithmus $=v$; erit, ut vidimus,

$$v=\frac{1}{k}\left(x-\frac{xx}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\text{etc.}\right)=\frac{y}{k}$$

hincque

$$k=\frac{y}{v};$$

ex quo commodissime valor ipsius k basi a respondens ita definitur, ut sit aequalis cuiusvis numeri logarithmo hyperbolico diviso per logarithmum eiusdem numeri ex basi a formati. Posito ergo numero hoc $=a$ erit $v=1$ hincque fit $k=$ logarithmo hyperbolico basis a. In systemate ergo logarithmorum communium, ubi est $a=10$, erit $k=$ logarithmo hyperbolico ipsius 10, unde fit

$$k=2{,}30258\,50929\,94045\,68401\,79915,$$

quem valorem iam supra satis prope collegimus. Si ergo singuli logarithmi hyperbolici per hunc numerum k dividantur vel, quod eodem redit, multiplicentur per hanc fractionem decimalem

$$0{,}43429\,44819\,03251\,82765\,11289,$$

prodibuut logarithmi vulgares basi $a=10$ convenientes.

125. Cum sit

$$e^z = 1 + \tfrac{z}{1} + \tfrac{z^2}{1\cdot 2} + \tfrac{z^3}{1\cdot 2\cdot 3} + \text{etc.},$$

si ponatur $a^y = e^z$, erit sumptis logarithmis hyperbolicis $yla = z$, quia est $le = 1$; quo valore loco z substituto erit

$$a^y = 1 + \frac{yla}{1} + \frac{y^2(la)^2}{1\cdot 2} + \frac{y^3(la)^3}{1\cdot 2\cdot 3} + \text{etc.},$$

unde quaelibet quantitas exponentialis ope logarithmorum hyperbolicorum per seriem infinitam explicari potest.

Tum vero denotante i numerum infinite magnum tam quantitates exponentiales quam logarithmi per potestates exponi possuut. Erit enim

$$e^z = \left(1 + \tfrac{z}{i}\right)^i$$

hincque

$$a^y = \left(1 + \frac{yla}{i}\right)^i,$$

deinde pro logarithmis hyperbolicis habetur

$$l(1+x) = i\left((1+x)^{\frac{1}{i}} - 1\right).$$

De cetero logarithmorum hyperbolicorum usus in calculo integrali fusius demonstrabitur.