CHAPTER V

FUNCTIONS OF TWO OR MORE VARIABLES

77. Although thus far we have considered several variable quantities, yet these were prepared thus, so that all were functions of one [variable] and by a single determination the others likewise could be determined. But now we will consider variable quantities of this kind, which may not depend on each other in turn, thus so that, whatever the determined value attributed to one variable, the rest nevertheless still remain undetermined and variable. Therefore variable quantities of this kind – such as x, y, z – will agree by reason of the assigned values, since any assigned values you please may themselves be combined together; moreover, if they may be prepared individually, they will be especially diverse, since, whatever determined value you please may be substituted for one value of z, yet the remaining x and y may extend out just as widely as before. Therefore the distinction is shifted from variable quantities depending on each other to this case with variable quantities not depending on each other, as in the first case, if one value may be determined, likewise the rest will be determined; truly in the later case the determination of one variable will restrain minimally the assigned values of the others.

78. *Therefore a function of two or of several variable quantities x, y, z is some kind of expression composed from these quantities.*

Thus

$$x^3 + xyz + az^2$$

will be a function of the three variable quantities x, y, z. Therefore this function, if one variable may be determined, e.g. z, that is, in place of this a constant number may be substituted, at this point will remain a variable quantity, evidently a function of x and y. And if besides z, y also may be determined, then at this stage it will be a function of x. Therefore a function of several variables of this kind will not be given any determined value before the individual variable quantities should be determined. Therefore since one variable quantity shall be able to be determined in an infinite number of ways, a function of two variables, because it is possible to take infinitely many determinations for the determination of the one, generally allows an infinitude of infinite determinations. And in a function of three variables the number of determinations at this stage will be infinitely greater ; and thus it will increase again for more variables.

79. *Functions of this kind of several variables and likewise functions of one variable are divided most conveniently into algebraic and transcending forms.*

Of which the former, in which the account of the composition has been put in place in terms of algebraic operations only; the latter truly, in the formation of which transcending operations also are present. In these kinds may be observed anew, provided the transcending operations either implicate all the variable quantities or some or only one. Thus this expression

$$zz + y\log.z\,,$$

because the logarithm of z itself is present, will be indeed a transcending function of y and of z themselves, truly thus it is required to be thought less transcending, because, if the variable z will be determined, it will become an algebraic function of y. Yet meanwhile it will not be expedient to clarify the treatment by subdivisions of this kind.

80. *Algebraic functions then are subdivided into rationals and irrationals, moreover the rational again into whole and fractional.*

The account of these denominations from the first chapter is now understood amply. Clearly a rational function generally is free from all irrationality, of which a function is said to be affected; and this function will be whole, if it is not beset by fractions, otherwise it will be a fractional function. Thus this will be the general form of an integral function of the two variables y and z :

$$\alpha + \beta y + \gamma z + \delta y^2 + \varepsilon yz + \zeta z^2 + \eta y^3 + \theta y^2 z + \iota yz^2 + \chi z^3 + \text{etc.}$$

Therefore if P and Q may denote whole functions of this kind, either of two or of more variables, $\frac{P}{Q}$ will be the general form of fractional functions.

Finally an irrational function is either explicit or implicit ; the former by a root sign now have been completely resolved, but the latter are shown by an irresolvable equation. Thus V will be an implicit irrational function of y and z, if it were

$$V^5 = \left(ayz + z^3\right)V^2 + \left(y^4 + z^4\right)V + y^5 + 2ayz^3 + z^5.$$

81. *Multiform functions of several variables thence must be viewed equally with these [multiform functions], which depend on a single variable.*

Thus rational functions will be uniform, because with single variable quantities determined they exhibit a single determined value. P, Q, R, S etc. may denote rational functions or uniform functions of the variables x, y, z and V will be a biform function of the same variables, if there were

$$V^2 - PV + Q = 0$$

for whatever values are attributed to the determined quantities *x, y* and *z*, the function *V* always will have not one but a double value determined. In a like manner *V* will be a tri–form function, if there were

$$V^3 - PV^2 + QV - R = 0,$$

and a function quadri–form, if it had the form

$$V^4 - PV^3 + QV^2 - RV + S = 0;$$

and in this manner an account of multiform functions of higher degrees will be provided.

82. Just as, if a function of one variable *z* is put equal to zero, the value of the variable quantity *z* determined follows to be either simple or multiple, thus, if a function of the two variables *y* et *z* is put equal to zero, then either variable is defined by the other and thus a function of this variable emerges, since before they were not mutually dependent. In a similar manner, if a function of three variables *x, y, z* is put in place equal to zero, then one variable is defined by the two remaining variables and a function of these exists. Likewise it comes about, if a function may not be put equal to zero, but to some constant quantity or also equal to other functions; for from any equation, however many variables it involves, always one variable is defined by the remaining and it shall be a function of these ; moreover two diverse equations arising between the same variables define two variables by the others, and thus henceforth.

83. *Moreover the division of functions of two or more variables into homogeneous and heterogeneous forms is especially noteworthy.*

A homogeneous function is one that has the same number of dimensions everywhere ; but a *heterogeneous function* is one in which diverse numbers of dimensions occur. Truly each variable is agreed to constitute a single dimension ; and of each square term produced from two variables, two dimensions ; three dimensions are produced from three variable, either from the same [repeated] or diverse variables, and so on thus ; but constant quantities are not admitted into the numeration of dimensions. Thus in these formulas

$$\alpha y, \ \beta z$$

a single dimension is said to be present; in these

$$\alpha y^2, \ \beta yz, \ \gamma z^2$$

indeed two dimensions are present; in these

$$\alpha y^3,\ \beta y^2 z,\ \gamma yz^2,\ \delta z^3$$

three; in these truly

$$\alpha y^4,\ \beta y^3 z,\ \gamma y^2 z^2,\ \delta yz^3,\ \varepsilon z^4$$

four and thus henceforth.

84. We will apply this distinction first to whole functions and we will consider only two variables to be present, because the account of several variables is the same.

Therefore a whole function will be homogeneous, in the individual terms of which the same number of dimensions arises.

Therefore functions of this kind can be subdivided most conveniently following the number of dimensions, which the variables constitute in these everywhere. Thus

$$\alpha y + \beta z$$

will be the general form of a whole function of one dimension ; truly this expression

$$\alpha y^2 + \beta yz + \gamma z^2$$

will be the general form of a function of two dimensions ; then the general form of a [homogeneous] function of three variables will be

$$\alpha y^3 + \beta y^2 z + \gamma yz^2 + \delta z^3$$

of four dimensions truly this

$$\alpha y^4 + \beta y^3 z + \gamma y^2 z^2 + \delta yz^3 + \varepsilon z^4$$

and thus henceforth. Therefore in analogy a constant quantity α alone will be a quantity of zero dimensions.

85. *Again a fractional function will be homogeneous, if its numerator and denominator should be homogeneous functions.*

Thus this fraction

$$\frac{ayz+bzz}{\alpha y+\beta z}$$

will be a homogeneous function of *y* and *z;* moreover the number of dimensions will be found if the number of the dimensions of the denominator is taken from the number of the dimensions of the numerator, and for that reason the proposed fractional function will be of one dimension. Truly this fractional function

$$\frac{y^5+z^5}{yy+zz}$$

will be a [homogeneous] function of three dimensions. Therefore when the same number of dimensions arises in the numerator and in the denominator, then the fraction will be a function of zero dimensions, as happens in this fraction

$$\frac{y^3+z^3}{yyz}$$

or also in these

$$\frac{y}{z},\ \frac{\alpha zz}{yy},\ \frac{\beta y^3}{z^3}$$

But if therefore the dimensions in the denominator shall be more than in the numerator, the number of dimensions of the fraction will be negative ; thus

$$\frac{y}{zz}$$

will be a function of -1 dimensions,

$$\frac{y+z}{y^4+z^4}$$

will be a function of -3 dimensions,

$$\frac{1}{y^5+ayz^4}$$

will be a function of -5 dimensions, because in the numerator no dimension is present. Moreover, it is understood spontaneously how several homogeneous functions, in which individual function the same number of dimensions rules, either added or subtracted producing a homogeneous function also of the same number of dimensions. Thus this expression

$$\alpha y+\frac{\beta zz}{y}+\frac{\gamma y^4-\delta z^4}{yyz+yzz}$$

will be a function of one dimension ; but this

$$\alpha+\frac{\beta y}{z}+\frac{\gamma zz}{yy}+\frac{y+zz}{yy-zz}$$

will be a function of zero dimensions.

86. The nature of homogeneous functions also can be extended to irrational functions. For if P were some homogeneous function, for example of dimensions n, then $\sqrt{P}$ will be a function of dimensions $\frac{1}{2}n$, $\sqrt[3]{P}$ will be a function of dimensions $\frac{1}{3}n$ and generally $P^{\frac{\mu}{\nu}}$ will be a function of dimensions $\frac{\mu}{\nu}n$. Thus $\sqrt{(yy+zz)}$ will be a function of one dimension, $\sqrt[3]{\left(y^9+z^9\right)}$ will be a function of three dimensions, $(yz+zz)^{\frac{3}{4}}$ will be of $\frac{3}{2}$ dimensions and $\frac{yy+zz}{\sqrt{\left(y^4+z^4\right)}}$ will be a function of zero dimensions. Therefore from these with the preceding jointly, this expression is understood

$$\frac{1}{y}+\frac{y\sqrt{(yy+zz)}}{z^3}-\frac{y}{\sqrt[3]{\left(y^6-z^6\right)}}+\frac{y\sqrt{z}}{zz\sqrt{y}+\sqrt{\left(y^5+z^5\right)}}$$

to be a homogeneous function of dimensions -1.

87. Whether an implicit irrational function shall be homogeneous or not, can be deduced from these easily. Let V be an implicit function of this kind and

$$V^3+PV^2+QV+R=0$$

with P, Q and R present functions of y and z themselves. In the first place therefore it is apparent that V cannot be a homogeneous function, unless P, Q and R shall be homogeneous functions. In addition truly, if we may put V to be a function of n dimensions, V^2 will be a function of $2n$ and V^3 a function of $3n$ dimensions ; therefore since it must have the same number of dimensions everywhere, it requires that P shall be a function of n dimensions, Q a function of $2n$ dimensions and R a function of $3n$ dimensions. Therefore if in turn the letters P, Q, R shall be homogeneous functions respectively of $n, 2n, 3n$ dimensions, hence it may be concluded V to be a function of n dimensions. Thus if there were

$$V^5+\left(y^4+z^4\right)V^3+\alpha y^8V-z^{10}=0,$$

V would be a homogeneous function of two dimensions of y and z themselves.

88. *If V were a homogeneous function of y and z of n dimensions, and in that there is put everywhere $y = uz$, the function V will change into a product of the power z^n into a certain function of the variable u.*

For by this substitution $y = uz$ powers of *z* of such a size will be introduced into the individual term, as they were present before of *y* itself. Therefore since in the individual terms the dimensions of *y* and *z* jointly shall equal the number *n*, now the variable *z* alone everywhere will have *n* dimensions and thus the power z^n of this will be present everywhere. Therefore the function *V* becomes divisible by this power, and the quotient will be a variable function involving only *u*.

This will be apparent initially with whole functions. For if there shall be

$$V = \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3,$$

on putting $y = uz$ it becomes

$$V = z^3\left(\alpha u^3 + \beta u^2 + \gamma u + \delta\right).$$

Then truly likewise it is clear with fractional functions. For let

$$V = \tfrac{\alpha y + \beta z}{yy + zz},$$

evidently a function of -1 dimensions; with the substitution $y = uz$ made it becomes

$$V = z^{-1} \cdot \tfrac{\alpha u + \beta}{uu + 1}$$

Nor also are irrational functions thus being excepted. Indeed if there shall be

$$v = \frac{y + \sqrt{(yy + zz)}}{z\sqrt{\left(y^3 + z^3\right)}},$$

which is a function of $-\frac{3}{2}$ dimensions, on putting $y = uz$ it will produce

$$v = z^{-\frac{3}{2}} \cdot \frac{u + \sqrt{(uu + 1)}}{\sqrt{\left(u^3 + 1\right)}}.$$

And thus in this manner homogeneous functions of only two variables will be reduced to functions of one variable ; nor indeed the power of *z*, which is a factor, corrupts that function of *u*.

89. *Therefore a homogeneous function V of the two variables y and z of zero dimensions on putting $y = uz$ will be transformed into a pure function of the variable u.*

For since the number of dimensions shall be zero, the power of *z*, which will multiply the function of *u*, will be $z^0 = 1$ and in this case the variable z of course will leave the calculation. Thus if there were

$$V = \frac{y+z}{y-z}$$

on making $y = uz$ there will arise

$$V = \frac{u+1}{u-1}$$

and with irrational functions if there shall be

$$V = \frac{y-\sqrt{(yy-zz)}}{z},$$

on putting $y = uz$ the function becomes

$$V = u - \sqrt{(uu-1)}.$$

90. *A whole homogeneous function of the two variables y and z will be able to be resolved into just as many simple factors of the form $\alpha y + \beta z$, as there will be dimensions.*

For since the function shall be homogeneous, on putting $y = uz$ it will be changed into a product from z^n into a certain whole function of *u*, which function therefore will be able to be resolved into simple factors of the form $\alpha u + \beta$. These individual factors will be multiplied by *z* and each one will have the form $\alpha uz + \beta z = \alpha y + \beta z$ on account of $uz = y$. Moreover because of the multiplier z^n, just as many factors of this kind will arise, as the exponent *n* may contain units; but these simple factors will be either real or imaginary, that is, the coefficients α and β will be either real or imaginary.

And so from this it follows that a function of two dimensions

$$ayy + byz + czz$$

has two simple factors of the form $\alpha y + \beta z$; moreover the function

$$ay^3 + by^2 z + cyz^2 + dz^3$$

will have three simple factors of the form $\alpha y + \beta z$; and thus again the natural constitution of whole homogeneous functions will be acquired, which have more variables.

91. Therefore just as this expression $\alpha y + \beta z$ contains the general form of whole functions of one dimension, thus

$$(\alpha y + \beta z)(\gamma y + \delta z)$$

will be the general form of whole functions of two dimensions ; and all the whole functions of three dimensions will be contained in this form

$$(\alpha y + \beta z)(\gamma y + \delta z)(\varepsilon y + \zeta z);$$

and thus all the whole homogeneous functions will be able to be shown by the products from just as many factors of this kind $\alpha y + \beta z$, as these functions may contain dimensions. But these factors in the same manner are found from the resolution of the equations, by which above in § 29 we have shown how to find the simple factors of whole functions of one variable. Moreover these properties of homogeneous functions of two variables may not be extended to homogeneous functions of three or of more variables ; indeed the general form of functions of this kind of two dimensions only, which is

$$ayy + byz + cyx + dxx + exx + fzz ,$$

generally cannot be reduced to a product of this form

$$(\alpha y + \beta z + \gamma x)(\delta y + \varepsilon z + \zeta x)$$

and functions of more dimensions much less are able to be recalled as products of this kind.

92. From these things, which have been said about homogeneous functions, likewise it is understood, what a heterogeneous function shall be ; evidently in the terms of which the same number of dimensions is not taken everywhere. But heterogeneous functions can be subdivided by the multiplicity of the dimensions, which occur in these. Thus a function will be *bifid*, in which a twofold number of dimensions occurs, and thus it will be a sum of two homogeneous functions, the number of dimensions of which differ ; thus

$$y^5 + 2y^3z^2 + yy + zz$$

will be a bifid, because one part contains five, and the other part contains two dimensions.
Moreover a function is *trifid*, in which three diverse numbers of dimensions are present, or which can be distributed into three homogeneous functions, as

$$y^6 + y^2z^2 + z^4 + y - z .$$

But besides heterogeneous functions either fractional or irrational are given mixed together to such an extend, which cannot be resolved into homogeneous functions, functions of this kind are

$$\frac{y^3+ayz}{by+zz}, \quad \frac{a+\sqrt{(yy+zz)}}{yy-bz}.$$

93. Meanwhile a heterogeneous function can be reduced to a homogeneous function with the aid of a suitable substitution, made either in place of one or of the other variable ; which is not allowed to be shown so easily, in which cases it may be done. Therefore it may suffice to present a certain example, in which a reduction of this kind has a place. Certainly if this function shall be proposed

$$y^5 + zzy + y^3z + \frac{z^3}{y}$$

after a rise in attention it will be apparent that can be led to a homogeneity on putting $z = xx$; for a homogeneous function of 5 dimensions of x and y. Then this function

$$y + y^2x + y^3xx + y^5x^4 + \frac{a}{x}$$

is reduced to homogeneity by putting $x = \frac{1}{z}$; for it will produce a function of one dimension

$$y + \frac{yy}{z} + \frac{y^3}{zz} + \frac{y^5}{z^4} + az.$$

But the cases are much more difficult, in which it is not permitted to arrive at homogeneity by as simple a substitution.

94. Finally the division of whole functions of the second order commonly used merits especially to be noted, following which the order is defined from the number of the maximum dimension, which is present in the function. Thus

$$xx + yy + zz + ay - aa$$

is a function of the second order, because two dimensions occur. And

$$y^4 + yz^3 - ay^2z + abyz - aayy + b^4$$

belongs to functions of the fourth order. It is customary to consider this division chiefly in the theory of curved lines, from which at this stage the division by a single whole function comes to be remembered.

95. Certainly the division of whole functions into *multiplicative* [Latin : *complexas*] and *non–multiplicative* [Latin : *incomplexas*] remains. Moreover a function is multiplicative, which can be resolved into rational factors or which is the product from two or more rational functions; of this kind is

$$y^4 - z^4 + 2az^3 - 2byzz - aazz + 2abzy - bbyy,$$

which is the product from these two factors

$$(yy + zz - az + by)(yy - zz + az - by).$$

Thus we have seen every whole homogeneous function, which includes only two variables, to be a multiplicative function, because it has just as many simple factors of the form $\alpha y + \beta z$, as it contains dimensions. Therefore a whole function is non–multiplicative, if it cannot be resolved generally into rational factors, as

$$yy + zz - aa,$$

of which no rational factors are easily understood to be given. From the search of the divisors it will be apparent, whether a proposed function will be multiplicative or non–multiplicative.

CAPUT V

DE FUNCTIONIBUS DUARUM PLURIUMVE VARIABILIUM

77. Quanquam plures hactenus quantitates variabiles sumus contemplati, tamen eae ita erant comparatae, ut omnes unius essent functiones unaque determinata reliquae simul determinarentur. Nunc autem eiusmodi considerabimus quantitates variabiles, quae a se invicem non pendeant, ita ut, quamvis uni determinatus valor tribuatur, reliquae tamen nihilominus maneant indeterminatae ac variabiles. Eiusmodi ergo quantitates variabiles, cuiusmodi sint x, y, z, ratione significationis convenient, cum quaelibet omnes valores determinatos in se complectatur; at, si inter se comparentur, maxime erunt diversae, cum, licet pro una z valor quicunque determinatus substituatur, reliquae tamen x et y aeque late pateant atque ante. Discrimen ergo inter quantitates variabiles a se pendentes et non pendentes in hoc versatur, ut priori casu, si una determinetur, simul reliquae determinentur, posteriori vero determinatio unius significationes reliquarum minime restringat.

78. *Functio· ergo duarum pluriumve quantitatum variabilium x, y, z est expressio quomodocunque ex his quantitatibus composita.*

Ita erit

$$x^3 + xyz + az^2$$

functio quantitatum variabilium trium x, y, z. Haec ergo functio, si una determinetur variabilis, puta z, hoc est, eius loco constans numerus substituatur, manebit adhuc quantitas variabilis, scilicet functio ipsarum x et y. Atque si praeter z quoque y determinetur, tum erit adhuc functio ipsius x. Huiusmodi ergo plurium variabilium functio non ante valorem determinatum obtinebit, quam singulae quantitates variabiles fuerint determinatae. Cum igitur una quantitas variabilis infinitis modis determinari possit, functio duarum variabilium, quia pro quavis determinatione unius infinitas determinationes suscipere potest, omnino infinities infinitas determinationes admittet. Atque in functione trium variabilium numerus determinationum erit adhuc infinities maior; sicque porro crescet pro pluribus variabilibus.

79. *Huiusmodi functiones plurium variabilium perinde atque functiones unius variabilis commodissime dividuntur in algebraicas ac transcendentes.*

Quarum illae sunt, in quibus ratio compositionis in solis Algebrae operationibus est posita; hae vera, in quarum formationem quoque operationes transcendentes ingrediuntur. In his denuo species notari possent, prout operationes transcendentes vel omnes quantitates variabiles implicant vel aliquot vel tantum unicam. Sic ista expressio

$$zz + y\log.z\,,$$

quia logarithmus ipsius z inest, erit quidem functio transcendens ipsarum y et z, verum ideo minus transcendens est putanda, quod, si variabilis z determinetur, supersit functio algebraica ipsius y. Interim tamen non expedit huiusmodi subdivisionibus tractationem amplificari.

80. *Functiones deinde algebraicae subdividiuntur in rationales et irrationales, rationales autem porro in integras ac fractas.*

Ratio harum denominationum ex capite prima iam abunde intelligitur. Functio scilicet rationalis omnino est libera ab omni irrationalitate quantitates variabiles, quarum functio dicitur, afficiente; haecque erit integra, si nullis fractionibus inquinetur, contra vero fracta. Sic functionis integrae duarum variabilium y et z haec erit forma generalis

$$\alpha + \beta y + \gamma z + \delta y^2 + \varepsilon yz + \zeta z^2 + \eta y^3 + \theta y^2 z + \iota yz^2 + \chi z^3 + \text{etc.}$$

Quodsi ergo P et Q denotent huiusmodi functiones integras, sive duarum sive plurium variabilium, erit $\frac{P}{Q}$ forma generalis functionum fractarum.

Functio denique irrationalis est vel explicita vel implicita; illa per signa radicalia iam penitus est evoluta, haec autem per aequationem irresolubilem exhibetur. Sic V erit functio implicita irrationalis ipsarum y et z, si fuerit

$$V^5 = \left(ayz + z^3\right)V^2 + \left(y^4 + z^4\right)V + y^5 + 2ayz^3 + z^5.$$

81. *Multiformitas deinde in his functionibus aeque notari debet atque in iis, quae ex unica variabili constant.*

Sic functiones rationales erunt uniformes, quia singulis quantitatibus variabilibus determinatis unicum valorem determinatum exhibent. Denotent P, Q, R, S etc. functiones rationales seu uniformes variabilium x, y, z eritque V functio biformis earundem variabilium, si fuerit

$$V^2 - PV + Q = 0$$

quicunque enim valores determinati quantitatibus *x*, *y* et *z* tribuuntur, functio *V* non unum sed duplicem perpetuo habebit valorem determinatum. Simili modo erit *V* functio triformis, si fuerit

$$V^3 - PV^2 + QV - R = 0,$$

atque functio quadriformis, si fuerit

$$V^4 - PV^3 + QV^2 - RV + S = 0;$$

hocque modo ratio functionum multiformium ulteriorum erit comparata.

82. Quemadmodum, si functio unius variabilis *z* nihilo aequalis ponitur, quantitas variabilis *z* valorem consequitum determinatum vel simplicem vel multiplicem, ita, si functio duarum variabilium *y* et *z* nihilo aequalis ponitur, tum altera variabilis per alteram definitur eiusque ideo functio evadit, cum ante a se mutuo non penderent. Simili modo si functio trium variabilium *x, y, z* nihilo aequalis statuatur, tum una variabilis per duas reliquas definitur earumque functio existit. Idem evenit, si functio non nihilo, sed quantitati constanti vel etiam alii functioni aequalis ponatur; ex omni enim aequatione, quotcunque variabiles involvat, semper una variabilis per reliquas definitur earumque fit functio; duae autem aequationes diversae inter easdem variabiles ortae binas per reliquas definient atque ita porro.

83. *Functionum autem duarum pluriumve variabilium divisio maxime notatu digna est in homogeneas et heterogeneas.*

Functio homogenea est, per quam ubique idem regnat variabilium numerus dimensionum; *functio* autem *heterogenea* est, in qua diversi occurrunt dimensionum numeri. Censetur vero unaquaeque variabilis unam dimensionem constituere; quadratum uniuscuiusque atque productum ex duabus duas; productum ex tribus variabilibus, sive iisdem sive diversis, tres, et ita porro; quantitates autem constantes ad dimensionum numerationem non admittuntur. Ita in his formulis

$$\alpha y,\ \beta z$$

unica dimensio inesse dicitur; in his vero

$$\alpha y^2,\ \beta yz,\ \gamma z^2$$

duae insunt dimensiones; in his

$$\alpha y^3,\ \beta y^2 z,\ \gamma yz^2,\ \delta z^3$$

tres; in his vero

$$\alpha y^4,\ \beta y^3 z,\ \gamma y^2 z^2,\ \delta yz^3,\ \varepsilon z^4$$

quatuor sicque porro.

84. Applicemus primum hanc distinctionem ad functiones integras atque duas tantum variabiles inesse ponamus, quoniam plurium par est ratio.

Functio igitur integra erit homogenea, in cuius singulis terminis idem existit dimensionum. numerus.

Subdividentur ergo huiusmodi functiones commodissime secundum numerum dimensionum, quem variabiles in ipsis ubique constituunt. Sic erit

$$\alpha y+\beta z$$

forma generalis functionum integrarum unius dimensionis; haec vero expressio

$$\alpha y^2+\beta yz+\gamma z^2$$

erit forma generalis functionum duarum dimensionum; tum forma generalis functionum trium dimensionum erit

$$\alpha y^3+\beta y^2z+\gamma yz^2+\delta z^3$$

quatuor dimensionum vero haec

$$\alpha y^4+\beta y^3z+\gamma y^2z^2+\delta yz^3+\varepsilon z^4$$

et ita porro. Ad analogiam igitur erit quantitas constans sola α functio nullius dimensionis.

85. *Functio porro fracta erit homogenea,* si *eius numerator ac denominator fuerint functiones homogeneae.*

Sic haec fractio

$$\frac{ayz+bzz}{\alpha y+\beta z}$$

erit functio homogenea ipsarum *y* et *z;* numerus dimensionum autem habebitur, si a numero dimensionum numeratoris subtrahatur numerus dimensionum denominatoris, atque ob hanc rationem fractio allata erit functio unius dimensionis. Haec vero fractio

$$\frac{y^5+z^5}{yy+zz}$$

erit functio trium dimensionum. Quando ergo in numeratore ac denominatore idem dimensionum numerus inest, tum fractio erit functio nullius dimensionis, uti evenit in hac fractione

$$\frac{y^3+z^3}{yyz}$$

vel etiam in his

$$\frac{y}{z},\ \frac{\alpha zz}{yy},\ \frac{\beta y^3}{z^3}$$

Quodsi igitur in denominatore plures sint dimensiones quam in numeratore, numerus dimensionum fractionis erit negativus; sic

$$\frac{y}{zz}$$

erit functio –1 dimensionis,

$$\frac{y+z}{y^4+z^4}$$

erit functio –3 dimensionum,

$$\frac{1}{y^5+ayz^4}$$

erit functio – 5 dimensionum, quia in numeratore nulla inest dimensio. Ceterum sponte intelligitur plures functiones homogeneas, in quibus singulis idem regnat dimensionum numerus, sive additas sive subtractas praebere functionem quoque homogeneam eiusdem dimensionum numeri. Sic haec expressio

$$\alpha y+\frac{\beta zz}{y}+\frac{\gamma y^4-\delta z^4}{yyz+yzz}$$

erit functio unius dimensionis; haec autem

$$\alpha+\frac{\beta y}{z}+\frac{\gamma zz}{yy}+\frac{y+zz}{yy-zz}$$

erit functio nullius dimensionis.

86. Natura functionum homogenearum quoque ad expressiones irrationales extenditur. Si enim fuerit P functio quaecunque homogenea, puta n dimensionum, tum $\sqrt{P}$ erit functio $\frac{1}{2}n$ dimensionum, $\sqrt[3]{P}$ erit functio $\frac{1}{3}n$ dimensionum et generatim $P^{\frac{\mu}{\nu}}$ erit functio $\frac{\mu}{\nu}n$ dimensionum. Sic $\sqrt{(yy+zz)}$ erit functio unius dimensionis, $\sqrt[3]{\left(y^9+z^9\right)}$ erit functio trium dimensionum, $(yz+zz)^{\frac{3}{4}}$ erit functio $\frac{3}{2}$ dimensionum atque $\frac{yy+zz}{\sqrt{\left(y^4+z^4\right)}}$ erit functio nullius dimensionis. His ergo cum praecedentibus coniunctis intelligetur haec expressio

$$\frac{1}{y}+\frac{y\sqrt{(yy+zz)}}{z^3}-\frac{y}{\sqrt[3]{(y^6-z^6)}}+\frac{y\sqrt{z}}{zz\sqrt{y}+\sqrt{(y^5+z^5)}}$$

esse functio homogenea –1 dimensionis.

87. Utrum functio irrationalis implicita sit homogenea necne, ex his facile colligi potest. Sit *V* huiusmodi functio implicita ac

$$V^3+PV^2+QV+R=0$$

existentibus *P*, *Q* et *R* functionibus ipsarum *y* et z. Primum igitur patet *V* functionem homogeneam esse non posse, nisi *P*, *Q* et *R* sint functiones homogeneae. Praeterea vero, si ponamus *V* esse functionem *n* dimensionum, erit V^2 functio *2n* et V^3 functio *3n* dimensionum; cum igitur ubique idem debeat esse numerus dimensionum, oportet, ut *P* sit functio *n* dimensionum, *Q* functio *2n* dimensionum et *R* functio *3n* dimensionum. Si ergo vicissim litterae *P*, *Q*, *R* [sint] functiones homogeneae respective *n*, *2n*, *3n* dimensionum, hinc concludetur fore *V* functionem *n* dimensionum. Ita si fuerit

$$V^5+\left(y^4+z^4\right)V^3+\alpha y^8V-z^{10}=0,$$

erit *V* functio homogenea duarum dimensionum ipsarum *y* et *z*.

88. *Si fuerit V functio homogenea n dimensionum ipsarum y et z in eaque ponatur ubique* $y=uz$ *, functio V abibit in productum ex potestate* z^n *in functionem quandam variabilis u.*

Per hanc enim substitutionem $y=uz$ in singulos terminos tantae inducentur potestates ipsius *z*, quantae ante inerant ipsius *y*. Cum igitur in singulis terminis dimensiones ipsarum *y* et *z* coniunctim aequassent numerum *n*, nunc sola variabilis *z* ubique habebit *n* dimensiones ideoque ubique inerit eius potestas z^n. Per hanc ergo potestatem functio *V* fiet divisibilis et quotus erit functio variabilem tantum *u* involvens. Hoc primum patebit in functionibus integris. Si enim sit

$$V=\alpha y^2+\beta y^2z+\gamma yz^2+\delta z^3,$$

posito $y=uz$ fiet

$$V=z^3\left(\alpha u^3+\beta u^2+\gamma u+\delta\right).$$

Deinde vero idem manifestum est in fractis. Sit enim

$$V = \frac{\alpha y + \beta z}{yy + zz},$$

nempe functio –1 dimensionis; facto $y = uz$ fiet

$$V = z^{-1} \cdot \frac{\alpha u + \beta}{uu + 1}$$

Neque etiam functiones irrationales hinc excipiuntur. Si enim sit

$$v = z^{-\frac{3}{2}} \cdot \frac{u + \sqrt{(uu+1)}}{\sqrt{(u^3+1)}}.$$

quae est functio $-\frac{3}{2}$ dimensionum, posito $y = uz$ prodibit

$$v = z^{-\frac{3}{2}} \cdot \frac{u + \sqrt{(uu+1)}}{\sqrt{(u^3+1^3)}}.$$

Hoc itaque modo functiones homogeneae duarum tantum variabilium reducentur ad functiones unius variabilis; neque enim potestas ipsius *z*, quia est factor, functionem illam ipsius *u* inquinat.

89. *Functio ergo homogenea V duarum variabilium y et z nullius dimensionis posito $y = uz$ transmutabitur in functionem, unicae variabilis u puram.*

Cum enim numerus dimensionum sit nullus, potestas ipsius *z*, quae functionem ipsius *u* multiplicabit, erit $z^0 = 1$ hocque casu variabilis *z* prorsus ex computo egredietur. Ita si fuerit

$$V = \frac{y+z}{y-z}$$

facto $y = uz$ orietur

$$V = \frac{u+1}{u-1}$$

atque in irrationalibus si sit

$$V = \frac{y - \sqrt{(yy - zz)}}{z},$$

posito $y = uz$ erit

$$V = u - \sqrt{(uu - 1)}.$$

90. *Functio integra homogenea duarum variabilium y et z resolvi poterit in*

tot factores simplices formae $\alpha y+\beta z$, quot habuerit dimensiones.

Cum enim functio sit homogenea, posito $y=uz$ transibit in productum ex z^n in functionem quandam ipsius *u* integram, quae functio propterea in factores simplices formae $\alpha u+\beta$ resolvi poterit. Multiplicentur singuli factores hi per *z* eritque uniuscuiusque forma $\alpha uz+\beta z=\alpha y+\beta z$ ob $uz=y$. Propter multiplicatorem autem z^n, tot huiusmodi factores nascentur, quot exponens *n* contineat unitates; factores autem hi simplices erunt vel reales vel imaginarii, hoc est, coefficientes α et β erunt vel reales vel imaginarii.

Ex hoc itaque sequitur functionem duarum dimensionum

$$ayy+byz+czz$$

duos habere factores simplices formae $\alpha y+\beta z$; functio autem

$$ay^3+by^2z+cyz^2+dz^3$$

habebit tres factores simplices formae $\alpha y+\beta z$; sicque porro functionum homogenearum integrarum, quae plures habent dimensiones, natura erit comparata.

91. Quemadmodum ergo haec expressio $\alpha y+\beta z$ continet formam generalem functionum integrarum unius dimensionis, ita

$$(\alpha y+\beta z)(\gamma y+\delta z)$$

erit forma generalis functionum integrarum duarum dimensionum; atque in hac forma

$$(\alpha y+\beta z)(\gamma y+\delta z)(\varepsilon y+\zeta z)$$

continebuntur omnes functiones integrae trium dimensionum. sicque omnes functiones integrae homogeneae per producta ex tot huiusmodi factoribus $\alpha y+\beta z$ exhiberi poterunt, quot functiones illae contineant dimensiones. Isti autem factores eodem modo per resolutionem aequationum reperiuntur, quo supra factores simplices functionum integrarum unius variabilis invenire docuimus. Ceterum haec proprietas functionum homogenearum duarum variabilium non extenditur ad functiones homogeneas trium pluriumve variabilium; forma enim generalis huiusmodi functionum duarum tantum dimensionum, quae est

$$ayy+byz+cyx+dxx+exx+fzz,$$

generaliter non reduci potest ad huiusmodi productum

$$(\alpha y+\beta z+\gamma x)(\delta y+\varepsilon z+\zeta x)$$

multoque minus functiones plurium dimensionum ad huiusmodi producta revocari possunt.

92. Ex his, quae de functionibus homogeneis sunt dicta, simul intelligitur, quid sit functio heterogenea; in cuius scilicet terminis non ubique idem dimensionum numerus deprehenditur. Possunt autem functiones heterogeneae subdividi pro multiplicitate dimensionum, quae in ipsis occurrunt. Sic functio *bifida* erit, in qua duplex dimensionum numerus occurrit, eritque adeo aggregatum duarum functionum homogenearum, quarum numeri dimensionum differunt; ita

$$y^5+2y^3z^2+yy+zz$$

erit functio bifida, quia partim quinque,partim duas continet dimensiones.
Functio autem *trifida* est, in qua tres diversi dimensionum numeri insunt seu quae in tres functiones homogeneas distribui possunt, uti

$$y^6+y^2z^2+z^4+y-z\,.$$

Praeterea autem dantur functiones heterogeneae fractae vel irrationales tantopere permixtae, quae in functiones homogeneas resolvi non possunt, cuiusmodi sunt

$$\frac{y^3+ayz}{by+zz},\quad \frac{a+\sqrt{(yy+zz)}}{yy-bz}\,.$$

93. Interdum functio heterogenea ope substitutionis idoneae, vel loco unius vel utriusque variabilis factae, ad homogeneam reduci potest; quod quibus casibus fieri queat, non tam facile indicare licet. Sufficiet ergo exempla quaedam attulisse, quibus eiusmodi reductio locum habet. Si scilicet haec proposita sit functio

$$y^5+zzy+y^3z+\frac{z^3}{y}$$

post levem attentionem apparebit eam ad homogeneitatem perduci posito $z=xx$; prodibit enim functio homogenea 5 dimensionum ipsarum *x* et *y*. Deinde haec functio

$$y+y^2x+y^3xx+y^5x^4+\frac{a}{x}$$

ad homogeneitatem reducitur ponendo $x = \frac{1}{z}$; prodit enim functio unius dimensionis

$$y + \frac{yy}{z} + \frac{y^3}{zz} + \frac{y^5}{z^4} + az.$$

Multo difficiliores autem sunt casus, quibus non per tam simplicem substitutionem ad homogeneitatem pervenire licet.

94. Tandem inprimis notari meretur functionum integrarum secundum ordines divisio satis usitata, secundum quam ordo definitur ex maximo dimensionum numero, qui in functione inest. Sic

$$xx + yy + zz + ay - aa$$

est functio secundi ordinis, quia duae dimensiones occurrunt. Et

$$y^4 + yz^3 - ay^2z + abyz - aayy + b^4$$

pertinet ad functiones quarti ordinis. Ad hanc divisionem potissimum in doctrina de lineis curvis respici solet, unde adhuc una functionum integrarum divisio commemoranda venit.

95. Superest scilicet divisio functionum integrarum in *complexas* atque *incomplexas.* Functio autem complexa est, quae in factores rationales resolvi potest seu quae est productum ex duabus functionibus pluribusve rationalibus; cuiusmodi est

$$y^4 - z^4 + 2az^3 - 2byzz - aazz + 2abzy - bbyy,$$

quae est productum ex his duabus functionibus

$$(yy + zz - az + by)(yy - zz + az - by).$$

Ita vidimus omnem functionem integram homogeneam, quae tantum duas variabiles complectatur, esse functionem complexam, quoniam tot factores simplices formae $\alpha y + \beta z$ habet, quot continet dimensiones. Functio igitur integra erit incomplexa, si in factores rationales resolvi omnino nequeat, uti

$$yy + zz - aa,$$

cuius nullos dari factores rationales facile intelligitur. Ex inquisitione divisorum patebit, utrum functio proposita sit complexa an incomplexa.