CHAPTER IV

THE DEVELOPMENT OF FUNCTIONS INTO INFINITE SERIES

[The Latin word used by Euler is *explication* in English, which means an unfolding or understanding; however, we have used the English equivalent of Labey's French translation of the title.]

59. Since fractional and irrational functions of *z* may not be contained in the whole form $A+Bz+Cz^2+Dz^3+\text{etc.}$, so that thus the number of terms shall be finite, expressions of this kind going off to infinity are accustomed to be sought, which may show the value of any function, either fractional or irrational; why indeed also may it not be agreed to understand the nature of transcending functions better, if they may be expressed by a form of this kind, even if infinite. For since the nature of whole functions may be made seen best, if the following different powers of *z* may be expanded out and thus may be reduced to the form $A+Bz+Cz^2+Dz^3+\text{etc.}$, thus likewise the form is considered most suitable towards the nature of all the remaining functions to be represented brought to mind, even if the number of terms shall actually be infinite. But it is evident that no non integral function of *z* can be put in place by a finite number of term of this kind $A+Bz+Cz^2+Dz^3+\text{etc.}$; indeed for that the function itself must become whole [or integral]. Truly whether it can exhibit an infinite series of terms of this kind, if for which it may be in doubt, this doubt may be removed by the expansion of each function itself. Moreover so that this explication may appear wider, besides the powers of *z* having integral positive exponents any powers must be allowed. Thus there will be no reason, why every function of *z* cannot be changed into an infinite expression of this kind:

$$Az^{\alpha}+Bz^{\beta}+Cz^{\gamma}+Dz^{\delta}+\text{etc.},$$

with the exponents $\alpha,\beta,\gamma,\delta$ etc. denoting any numbers whatsoever.

60. *Moreover, it is understood by continuous division that the fraction*

$$\frac{a}{\alpha+\beta z}$$

be resolved into this infinite series

$$\frac{a}{\alpha}-\frac{a\beta z}{\alpha^2}+\frac{a\beta^2z^2}{\alpha^3}-\frac{a\beta^3z^3}{\alpha^4}+\frac{a\beta^4z^4}{\alpha^5}-\text{etc.}$$

which, since any term may have the constant ratio $1:-\frac{\beta z}{\alpha}$ *to the following term, is called a geometric series.*

Truly also, this series can be found thus, as that itself may be had initially for unknowns ; for there may be put

$$\frac{a}{\alpha+\beta z} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

and towards producing an equality the coefficients A, B, C, D etc. are sought. Therefore the equation becomes

$$a = (\alpha + \beta z)\left(A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}\right)$$

and with the multiplication acted on completed it becomes

$$\begin{aligned} a = \alpha A + \alpha Bz + \alpha Cz^2 &+ \alpha Dz^3 + \alpha Ez^4 + \text{etc.} \\ + \beta Az + \beta Bz^2 &+ \beta Cz^3 + \beta Dz^4 + \text{etc.} \end{aligned}$$

On account of which there must become

$$a = \alpha A \quad \text{and thus} \quad A = \frac{a}{\alpha}$$

and with the sum of coefficients of each and every power of z put equal to nothing, from which these equations are produced

$$\begin{gathered} \alpha B + \beta A = 0, \\ \alpha C + \beta B = 0, \\ \alpha D + \beta C = 0, \\ \alpha E + \beta D = 0 \\ \text{etc.;} \end{gathered}$$

therefore for any known coefficient the following is found easily ; for if the coefficient of each term were $= P$ and the following $= Q$, there will be

$$\alpha Q + \beta P = 0 \text{ or } Q = -\frac{\beta P}{\alpha}.$$

Therefore since the first term A shall be determined $= \frac{a}{\alpha}$, from that the following letters B, C, D etc. are defined in the same manner, from which they have arisen from division. Moreover by inspection it is seen the coefficient of the power z^n found in the infinite series for $\frac{a}{\alpha+\beta z}$ to be $\pm\frac{a\beta^n}{\alpha^{n+1}}$ where it has the + sign, if n shall be even, but the – sign, if n shall be odd, or the coefficient will be $=\frac{a}{\alpha}\left(\frac{-\beta}{\alpha}\right)^n$.

61. *In a similar manner with the aid of continued division this fractional function*

$$\frac{a+bz}{\alpha+\beta z+\gamma zz}$$

is able to be converted into an infinite series.

But since the division shall be wearisome nor may the nature of the infinite series be shown so easily, it will be more convenient to devise a certain series, and to be determined in the manner handled before. Therefore let there be

$$\frac{a+bz}{\alpha+\beta z+\gamma zz}=A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.};$$

and both to be multiplied by $\alpha+\beta z+\gamma zz$ and there becomes

$$\begin{aligned} a+bz=\alpha A+\alpha Bz+\alpha Cz^2&+\alpha Dz^3+\alpha Ez^4+\text{ etc.}\\ +\beta Az+\beta Bz^2&+\beta Cz^3+\beta Dz^4+\text{ etc.}\\ +\gamma Az^2&+\gamma Bz^3+\gamma Cz^4+\text{ etc.}\end{aligned}$$

Hence there will be

$$aA=a,\ aB+\beta A=b,$$

from which there is found

$$A=\frac{a}{\alpha}\ \text{ et }\ B=\frac{b}{\alpha}-\frac{a\beta}{\alpha\alpha}\ ;$$

truly the remaining letters will be determined from the following equations

$$\begin{aligned}\alpha C+\beta B+\gamma A&=0,\\ \alpha D+\beta C+\gamma B&=0,\\ \alpha E+\beta D+\gamma C&=0,\\ \alpha F+\beta E+\gamma D&=0\\ &\text{etc.};\end{aligned}$$

hence therefore from the two adjoining coefficients the following is found. Thus, if two adjoining coefficients were P, Q and the following R, there will be

$$\alpha R+\beta Q+\gamma P=0 \quad \text{or} \quad R=\frac{-\beta Q-\gamma P}{\alpha}.$$

Therefore since the two first letters A and B shall now have been found, all the following C, D, E, F etc. will be found from these and thus the infinite series $A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}$ will be found of proposed equal fractional function $\frac{a+bz}{\alpha+\beta z+\gamma zz}$.

EXAMPLE

If this fraction were proposed

$$\frac{1+2z}{1-z-zz}$$

and to this the series may be put in place

$$A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}$$

on account of

$$a=1,\ b=2,\ \alpha=1,\ \beta=-1,\ \gamma=-1$$

there will be

$$A=1,\ B=3;$$

then truly there will be

$$C=B+A,$$
$$D=C+B,$$
$$E=D+C,$$
$$F=E+D$$
$$\text{etc.}$$

Therefore any coefficient is equal to the sum of the two preceding ; whereby if two contiguous coefficients P and Q were known, the following will be $R=P+Q$. Therefore since the two first coefficients A and B shall be known, the proposed fraction

$$\frac{1+2z}{1-z-zz}$$

will be changed into this infinite series

$$1+3z+4z^2+7z^3+11z^4+18z^5+\text{etc.},$$

which can be continued as long as it is desired with no trouble.

62. Now from these the natures of infinite series is understood well enough, into which fractional functions are transformed; for they support a law of this kind, so that any term will be able to be determined from some number preceding.

Evidently, it the denominator of the proposed fraction were

$$\alpha+\beta z$$

and the infinite series were put in place

$$A+Bz+Cz^2+\cdots+Pz^n+Qz^{n+1}+Rz^{n+2}+Sz^{n+3}+\text{ etc.}$$

any coefficient Q thus will be defined from the preceding P alone thus, so that there shall be

$$\alpha Q+\beta P=0\,.$$

But if the denominator were the trinomial

$$\alpha+\beta z+\gamma zz\,,$$

any coefficient R of the series will be defined thus from the two preceding Q and P, thus so that there shall be

$$\alpha R+\beta Q+\gamma P=0\,.$$

In a similar manner, if the denominator were a quadrinomial, so that

$$\alpha+\beta z+\gamma zz+\delta z^3$$

whatever coefficient S of the series will be determined thus from the three preceding R, Q and P, so that there shall be

$$aS+\beta R+\gamma Q+\delta P=0\,,$$

and thus with the others.

Therefore in these series any term is determined from some number preceding following a certain constant law, which law appears at once from the denominator of the fraction producing this series. Moreover these series are accustomed to be called *recurring*, by the most celebrated de Moivre, who has examined the nature of these most carefully [see the original Proc. Royal Soc., 32, *De fractionibus algebraicis* .... (1724); this is currently available online from JSTOR.], therefore what is recurring according to the preceding terms is required if we wish to investigate the following terms.

63. Again it is required towards the formation of the series, that the constant term of the denominator α shall not be $=0$; for since the first term of the series found shall be $A=\frac{a}{\alpha}$, as that and all the following shall become infinite, if there should be $\alpha=0$. Therefore with this case excluded, that I shall set out hence, a fractional function being transformed into an infinite recurring series of this kind will have the form

$$\frac{a+bz+cz^2+dz^3+\text{etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{etc.}},$$

where I put the first term of the denominator $=1$; for this fraction can be reduced always, unless this shall be $=0$; moreover I consider all the remaining terms of the denominator as negative, so that hence all the forms of the series become positive. In effect, the recurring series hence arising may be put hence

$$A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.},$$

the coefficients will be determined thus, so that there shall be

$$\begin{aligned}
&A=a,\\
&B=\alpha A+b,\\
&C=\alpha B+\beta A+c,\\
&D=\alpha C+\beta B+\gamma A+d,\\
&E=\alpha D+\beta C+\gamma B+\delta A+e\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

Therefore any coefficient you wish is equal to a sum from multiples of the preceding, together with a certain number, that the numerator provides. Moreover unless the numerator progresses to infinity, these to be added soon will cease and some term will be determined following a constant law from some preceding numbers. Therefore, so that the law of the progression will not be troublesome anywhere, it may be agreed that a natural fractional function be used ; for if an improper fraction may be taken, then the whole part contained in that may be added to that series and in these terms which it either increases or diminishes, it interrupts the law of the progression. For the sake of an example this spurious fraction [*i.e.* improper fraction, relating to the Fibonacci sequence]

$$\frac{1+2z-z^3}{1-z-zz}\ \left[=z-1+\frac{2}{1-z-zz}\right]$$

will provide this series

$$1+3z+4zz+6z^3+10z^4+16z^5+26z^6+42z^7+\text{etc.},$$

where from the law, by which any coefficient is the sum of the two preceding, the fourth term $6z^3$ is excepted.

64. Particular recurring series merit contemplation, if the denominator of the fraction, from which it originates, were a power. Thus, if this fraction

$$\frac{a+bz}{(1-\alpha z)^2}$$

is resolved into a series, it produces

$$\begin{array}{l} a+2\alpha az+3\alpha^2az^2+4\alpha^3az^3+5\alpha^4az^4+\text{etc.}, \\ \quad + \quad b \;+2\alpha b \qquad +3\alpha^2 b \quad +4\alpha^3 b \end{array}$$

In which the coefficient of the power z^n will be

$$(n+1)\alpha^n a+n\alpha^{n-1}b.$$

Yet this recurring series will be, for which any term may be determined from the two preceding terms, the law of which the determination is seen from the denominator expanded out $1-2\alpha z+\alpha\alpha zz$.

If there is put $\alpha=1$ and $z=1$, the series will change into the general arithmetical progression

$$a+(2a+b)+(3a+2b)+(4a+3b)+\text{etc.},$$

the differences of which are constants. Therefore every arithmetical progression is a recurring series; for if

$$A+B+C+D+E+F+\text{etc}.$$

shall be an arithmetical progression, it will be

$$C=2B-A,\ D=2C-B,\ E=2D-C \ \text{ etc}.$$

65. From which this fraction

$$\frac{a+bz+czz}{(1-\alpha z)^3}$$

on account of

$$\frac{1}{(1-\alpha z)^3}=(1-\alpha z)^{-3}=1+3\alpha z+6\alpha^2z^2+10\alpha^3z^3+15\alpha^4z^4+\text{etc.}$$

will be transformed into this infinite series

$$\begin{array}{lllll} a+3\alpha az+6\alpha^2az^2 & +10\alpha^3az^3 & +15\alpha^4az^4+ \text{ etc.,} \\ \quad + \; b \quad +3\alpha b & + \; 6\alpha^2 b & +10\alpha^3 b \\ \qquad\qquad + \; c & + \; 3\alpha c & + \; 6\alpha^2 c \end{array}$$

in which the power z^n will have the coefficient

$$\tfrac{(n+1)(n+2)}{1\cdot 2}\alpha^n a+\tfrac{n(n+1)}{1\cdot 2}\alpha^{n-1}b+\tfrac{n(n-1)}{1\cdot 2}\alpha^{n-2}c.$$

But if there may be put $\alpha=1$ and $z=1$, this series will be changed into the general progression of the second order, the second order differences of which are constants. Let

$$A+B+C+D+E+F+\text{etc.}$$

describe a progression of this kind ; this likewise will be a recurring series, any term of which is determined from the three preceding terms thus, thus it shall be

$$D=3C-3B+A,\ E=3D-3C+B,\ F=3E-3D+C \ \text{ etc}.$$

Therefore since the second differences of the preceding terms also shall be equal in an arithmetic progression, evidently$=0$, this property also is extended to arithmetical progressions.

66. In a similar manner this fraction

$$\tfrac{a+bz+czz+dz^3}{(1-\alpha z)^4}$$

will give an infinite series, in which some power of z, z^n, hence will have the coefficient

$$\tfrac{(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3}\alpha^n a+\tfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\alpha^{n-1}b+\tfrac{(n-1)n(n+1)}{1\cdot 2\cdot 3}\alpha^{n-2}c+\tfrac{(n-2)(n-1)n}{1\cdot 2\cdot 3}\alpha^{n-3}d.$$

Therefore by putting $\alpha=1$ and $z=1$ this series includes within itself all algebraic progressions of the third order, of which the third order differences are constant ; therefore all the progressions of this order, shall be of this kind

$$A+B+C+D+E+F+\text{etc.},$$

likewise they are recurrent originating from the denominator $1-4z+6zz-4z^3+z^4$, from which there will be

$$E=4D-6C+4B-A,\ F=4E-6D+4C-B \text{ etc.},$$

which property likewise agrees with all the progressions of lower order.

67. In this manner it will be shown that all the algebraic progressions of any order, which finally are described by constant differences, to be recurring series, of which the law may be defined from the denominator $(1-z)^n$ with the number n present greater than that, which the order of the progression indicates. Therefore since

$$a^m+(a+b)^m+(a+2b)^m+(a+3b)^m+\text{ etc}$$

.
shows a progression of the order m, there will be on account of the recurring series

$$0=a^m-\tfrac{n}{1}(a+b)^m+\tfrac{n(n-1)}{1\cdot 2}(a+2b)^m-\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(a+3b)^m$$
$$+\cdots\mp\tfrac{n}{1}\left(a+(n-1)b\right)^m\pm(a+nb)^m,$$

where the upper signs prevail, if n shall be an even number, but the lower, if n shall be an odd number. Therefore this equation is true always, if n therefore is understood to be an integer greater than m. Hence therefore it is understood, just how wide the theory of recurring series may extend.

68. If the denominator were not a binomial but a multinomial power, the nature of the series can also be set out by another method. Truly this fraction shall be proposed

$$\frac{1}{\left(1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{etc.}\right)^{m+1}};$$

it will give rise to this infinite series

$$\begin{array}{llll} 1+\frac{(m+1)}{1}\alpha z & +\frac{(m+1)(m+2)}{1\cdot 2}\alpha^2 z^2 & +\frac{(m+1)(m+2)(m+3)}{1\cdot 2\cdot 3}\alpha^3 z^3 & +\text{etc.}\\ & +\quad \frac{(m+1)}{1}\beta & +\quad \frac{(m+1)(m+2)}{1\cdot 2}2\alpha\beta & \\ & & +\quad \frac{(m+1)}{1}\gamma & \end{array}$$

[Note that the powers of z are assumed in the lower lines.]

Towards a closer examination of the nature of this series, this series may be set out by general letters in this fashion

$$1+Az+Bz^2+Cz^3+\cdots+Kz^{n-3}+Lz^{n-2}+Mz^{n-1}+Nz^n+\text{etc.}$$

and any coefficient N thus will be determined from just as many preceding, as there are letters $\alpha,\beta,\gamma,\ \delta$ etc., so that there shall be

$$N=\tfrac{m+n}{n}\alpha M+\tfrac{2m+n}{n}\beta L+\tfrac{3m+n}{n}\gamma K+\tfrac{4m+n}{n}\delta I+\text{etc.};$$

which law of continuation even if it is not constant, but depends on the exponent of the power of z, yet it agrees with another constant law of progression of the same series, as the denominator expanded out provides the agreement of the recurring nature of the series. Truly that non–constant law only has a place, if the numerator of the fraction were unity or some constant quantity; for if it contained some power of z also, then that law would become more complicated, which will be shown easier after the treatment of the principles of the differential calculus. [See the latter part of Ch. IV in these translations.]

69. Because up to this stage we have assumed the first term of the denominator not to be a constant $=0$ and in place of which we have called unity, now we may see, what kind of series may arise, if the constant term in the denominator may vanish. From these cases therefore a fractional function of this kind of form will be had :

$$\frac{a+bz+czz+\text{etc.}}{z\left(1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.}\right)};$$

.
therefore with the factor z of the denominator ignored the remaining fraction may be converted

$$\frac{a+bz+czz+\text{etc.}}{1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.}}$$

into the recurring series

$$A+Bz+Cz^2+Dz^3+\text{etc}.$$

and it is evident to become:

$$\frac{a+bz+czz+\text{etc.}}{z\left(1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.}\right)}=\frac{A}{z}+B+Cz+Dz^2+Ez^3+\text{etc.}$$

In a similar manner there will be:

$$\frac{a+bz+czz+\text{etc.}}{z^2\left(1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.}\right)}=\frac{A}{zz}+\frac{B}{z}+C+Dz+Ez^2+\text{etc.}$$

and generally it becomes:

$$\frac{a+bz+czz+\text{etc.}}{z^m\left(1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.}\right)}=\frac{A}{z^m}+\frac{B}{z^{m-1}}+\frac{C}{z^{m-2}}+\frac{D}{z^{m-3}}+\text{etc.},$$

whatever the exponent number m may be.

70. Because another variable x is be introduced into the fractional function by substitution in place of z, and with this agreed upon some fractional function can be transformed in innumerable diverse ways, in this way the same fractional function can be expanded in an indefinite number of ways by recurring series. To wit let this fraction be proposed

$$y=\frac{1+z}{1-z-zz}$$

and by the recurring series

$$y=1+2z+3z^2+5z^3+8z^4+\text{etc.};$$

putting $z=\frac{1}{x}$; it will become

$$y=\frac{xx+x}{xx-x-1}=\frac{-x(1+x)}{1+x-xx}$$

Now

$$\frac{1+x}{1+x-xx}=1+0x+xx-x^3+2x^4-3x^5+5x^6-\text{etc.},$$

from which there becomes

$$y=-x+0x^2-x^3+x^4-2x^5+3x^6-5x^7+\text{etc.}$$

Or there is put

$$z=\frac{1-x}{1+x};$$

then there will be

$$y=\frac{-2-2x}{1-4x-xx},$$

from which there becomes

$$y=-2-10x-42xx-178x^3-754x^4-\text{etc.},$$

recurring series of this kind are able to be found for innumerable forms of y.

71. Irrational functions are accustomed to be transformed into infinite series from this general theorem, which shall be :

$$(P+Q)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}P^{\frac{m-n}{n}}Q + \frac{m(m-n)}{n\cdot 2n}P^{\frac{m-2n}{n}}Q^2 + \frac{m(m-n)(m-2n)}{n\cdot 2n\cdot 3n}P^{\frac{m-3n}{n}}Q^3 + \text{etc.};$$

for these terms, unless $\frac{m}{n}$ should be a positive integer, run off to infinity.
Thus it will be for m and n by writing the defined numbers

$$(P+Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2}P^{-\frac{1}{2}}Q - \frac{1\cdot 1}{2\cdot 4}P^{-\frac{3}{2}}Q^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}P^{-\frac{5}{2}}Q^3 + \text{etc.},$$
$$(P+Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2}P^{-\frac{3}{2}}Q + \frac{1\cdot 3}{2\cdot 4}P^{-\frac{5}{2}}Q^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}P^{-\frac{7}{2}}Q^3 + \text{etc.},$$
$$(P+Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3}P^{-\frac{3}{3}}Q - \frac{1\cdot 2}{3\cdot 6}P^{-\frac{5}{3}}Q^2 + \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}P^{-\frac{8}{3}}Q^3 - \text{etc.},$$
$$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3}P^{-\frac{1}{3}}Q - \frac{2\cdot 1}{3\cdot 6}P^{-\frac{4}{3}}Q^2 + \frac{2\cdot 1\cdot 4}{3\cdot 6\cdot 9}P^{-\frac{7}{2}}Q^3 - \text{etc.}$$
etc.

72. Therefore the terms of series of this kind may be progressing thus, so that any can be formed from the antecedent terms. For let any term of the series, which arises from $(P+Q)^{\frac{m}{n}}$, be

$$= MP^{\frac{m-kn}{n}}Q^k;$$

the following will be

$$= \frac{m-kn}{(k+1)n}MP^{\frac{m-(k+1)n}{n}}Q^{k+1}.$$

But it is required to be noted in any following term the exponent of P decreases by one, truly contrary to the exponent of Q which increases by one. Moreover so that these will be fitted more easily to some particular case, the general form, $(P+Q)^{\frac{m}{n}}$ can be set out thus $P^{\frac{m}{n}}\left(1+\frac{Q}{P}\right)^{\frac{m}{n}}$; for with the formula $\left(1+\frac{Q}{P}\right)^{\frac{m}{n}}$ developed and with the resulting series multiplied by $P^{\frac{m}{n}}$ that series will be produced given before. Then truly, if m may denote not only a whole number, but also fractions, it will be possible to put one in place for n without risk. With which done if for $\frac{Q}{P}$, which is a function of z, there is put Z, there will be had

$$(1+Z)^m = 1+\tfrac{m}{1}Z+\tfrac{m(m-1)}{1\cdot 2}Z^2+\tfrac{m(m-1)(m-2)}{1\cdot 2\cdot 3}Z^3+\text{etc.}$$

Moreover it will be agreed for the following laws of the progressions, that this conversion of the general formula into a series be noted :

$$(1+Z)^{m-1} = 1+\tfrac{m-1}{1}Z+\tfrac{(m-1)(m-2)}{1\cdot 2}Z^2+\tfrac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}Z^3+\text{etc.}$$

73. If therefore at first there is put $Z=\alpha z$, and the series will be

$$(1+\alpha z)^{m-1} = 1+\tfrac{m-1}{1}\alpha z+\tfrac{(m-1)(m-2)}{1\cdot 2}\alpha^2 z^2+\tfrac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3 z^3+\text{etc.}$$

This general form will be written for this series

$$=1+Az+Bz^2+Cz^3+\cdots+Mz^{n-1}+Nz^n+\text{etc.}$$

and any coefficient N thus will be determined from the preceding M, so that there shall be

$$N=\tfrac{m-n}{n}\alpha M.$$

Thus on putting $n=1$, since there shall be $M=1$, and

$$N=A=\tfrac{m-1}{1}\alpha\,;$$

then for the factor $n=2$ on account of $M=A=\tfrac{m-1}{1}\alpha$ there will be

$$N=B=\tfrac{m-2}{2}\alpha M=\tfrac{(m-1)(m-2)}{1\cdot 2}\alpha^2$$

and again in a similar manner

$$C=\tfrac{m-3}{3}\alpha B=\tfrac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3.$$

as the series found before declared.

74. Let $Z = \alpha z + \beta zz$ and there becomes

$$\left(1+\alpha z+\beta zz\right)^{m-1} = 1+\tfrac{m-1}{1}\left(\alpha z+\beta zz\right)+\tfrac{(m-1)(m-2)}{1\cdot 2}\left(\alpha z+\beta zz\right)^{2}+\text{etc.}$$

But if therefore the terms following the powers of z may be set out, there will be

$$\begin{aligned}\left(1+\alpha z+\beta zz\right)^{m-1} = 1+\tfrac{m-1}{1}\alpha z &+\tfrac{(m-1)(m-2)}{1\cdot 2}\alpha^{2}z^{2}+\tfrac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^{3}z^{3}+\text{etc.}\\ &+\tfrac{m-1}{1}\beta z^{2} \quad +\tfrac{(m-1)(m-2)}{1\cdot 2}2\alpha\beta z^{3}\end{aligned}$$

The general form for this series is written

$$1+Az+Bz^{2}+Cz^{3}+\cdots+Lz^{n-2}+Mz^{n-1}+Nz^{n}+\text{etc.}$$

and any coefficient thus will be defined from the two preceding terms, so that there shall be

$$N=\tfrac{m-n}{n}\alpha M+\tfrac{2m-n}{n}\beta L,$$

from which all the terms will be able to be defined from the first, which is 1. Certainly there will be

$$\begin{aligned}A &= \tfrac{\mathrm{m}-1}{1}\alpha,\\ B &= \tfrac{m-2}{2}\alpha A+\tfrac{2m-2}{2}\beta,\\ C &= \tfrac{m-3}{3}\alpha B+\tfrac{2m-3}{3}\beta A,\\ D &= \tfrac{m-4}{4}\alpha C+\tfrac{2m-4}{4}\beta B\\ &\text{etc.}\end{aligned}$$

75. If there should be $Z=\alpha z+\beta z^{2}+\gamma z^{3}$

there becomes

$$\left(1+\alpha z+\beta z^{2}+\gamma z^{3}\right)^{m-1} = 1+\tfrac{m-1}{1}\left(\alpha z+\beta z^{2}+\gamma z^{3}\right)+\tfrac{(m-1)(m-2)}{1\cdot 2}\left(\alpha z+\beta z^{2}+\gamma z^{3}\right)^{2}+\text{etc.},$$

which expression, if all the terms following the powers of z will be in order, it will change into this series :

$$
\begin{aligned}
1+\tfrac{m-1}{1}\alpha z+\tfrac{(m-1)(m-2)}{1\cdot 2}\alpha^2 z^2 &+\tfrac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3 z^3+\text{etc.};\\
+\tfrac{m-1}{1}\beta z^2 &+\tfrac{(m-1)(m-2)}{1\cdot 2}2\alpha\beta z^3\\
&+\tfrac{m-1}{1}\gamma z^3
\end{aligned}
$$

the law of the progression of which, so that it may become clearer, is put in place of this

$$1+Az+Bz^2+Cz^3+\cdots+Lz^{n-2}+Mz^{n-1}+Nz^n+\text{etc.},$$

any coefficient of which series thus is determined from the three proceeding coefficients, so that it there shall be

$$N=\tfrac{m-n}{n}\alpha M+\tfrac{2m-n}{n}\beta L+\tfrac{3m-n}{n}\gamma K.$$

Therefore since the first term shall be $=1$ and the preceding equal to nothing, there will be :

$$
\begin{aligned}
A&=\tfrac{m-1}{1}\alpha,\\
B&=\tfrac{m-2}{2}\alpha A+\tfrac{2m-2}{2}\beta,\\
C&=\tfrac{m-3}{3}\alpha B+\tfrac{2m-3}{3}\beta A+\tfrac{3m-3}{3}\gamma,\\
D&=\tfrac{m-4}{4}\alpha C+\tfrac{2m-4}{4}\beta B+\tfrac{3m-4}{4}\gamma A,\\
E&=\tfrac{m-5}{5}\alpha D+\tfrac{2m-5}{5}\beta C+\tfrac{3m-5}{5}\gamma B\\
&\text{etc.}
\end{aligned}
$$

76. Therefore generally, if there may be put

$$\left(1+\alpha z+\beta z^2+\gamma z^3+\delta z^4\text{ etc.}\right)^{m-1}=1+Az+Bz^2+Cz^3+Dz^4+Ez^5+\text{etc.},$$

the individual terms of this series thus will be defined from the preceding terms, so that there shall be :

$$A = \tfrac{m-1}{1}\alpha,$$
$$B = \tfrac{m-2}{2}\alpha A + \tfrac{2m-2}{2}\beta,$$
$$C = \tfrac{m-3}{3}\alpha B + \tfrac{2m-3}{3}\beta A + \tfrac{3m-3}{3}\gamma,$$
$$D = \tfrac{m-4}{4}\alpha C + \tfrac{2m-4}{4}\beta B + \tfrac{3m-4}{4}\gamma A + \tfrac{4m-4}{4}\delta,$$
$$E = \tfrac{m-5}{5}\alpha D + \tfrac{2m-5}{5}\beta C + \tfrac{3m-5}{5}\gamma B + \tfrac{4m-5}{5}\delta A + \tfrac{5m-5}{5}\varepsilon$$
etc.

clearly any term is determined by just as many preceding terms, as the number of letters a,β,γ,δ etc. being present in the function of *z*, the power of which is converted into the series. The remaining account of this rule agrees with that, which we have found above in § 68, where we resolved a similar form $\left(1-\alpha z-\beta z^2-\gamma z^3-\text{etc.}\right)^{-m-1}$ into an infinite series ; for if indeed in place of m there may be written $-m$ and the letters a,β,γ,δ etc. may be taken negative, the series found are congruent at once. For the rest, an account of this law of the progression is not allowed to be demonstrated here from the former, because that finally will be able to be given conveniently by the principles of the calculus; therefore meanwhile the truth by application will suffice to have proved the examples of all kinds.

CAPUT IV

DE EXPLICATIONE FUNCTIONUM PER SERIES INFINITAS

59. Cum functiones fractae atque irrationales ipsius z non in forma integra $A+Bz+Cz^2+Dz^3+\text{etc.}$ contineantur, ita ut terminorum numerus sit finitus, quaeri solent huiusmodi expressiones in infinitum excurrentes, quae valorem cuiusvis functionis sive fractae sive irrationalis exhibeant; quin etiam natura functionum transcendentium melius intelligi censetur, si per eiusmodi formam etsi infinitam exprimantur. Cum enim natura functionis integrae optime perspiciatur, si secundum diversas potestates ipsius z explicetur atque adeo ad formam $A+Bz+Cz^2+Dz^3+\text{etc.}$ reducatur, ita eadem forma aptissima videtur ad reliquarum functionum omnium indolem menti repraesentandam, etiamsi terminorum numerus sit revera infinitus. Perspicuum autem est nullam functionem non integram ipsius z per numerum huiusmodi terminorum $A+Bz+Cz^2+Dz^3+\text{etc.}$ finitum exponi posse; eo ipso enim functio foret integra. Num vero per huiusmodi terminorum seriem infinitam exhiberi possit, si quis dubitet, hoc dubium per ipsam evolutionem cuiusque functionis tolletur. Quo autem haec explicatio latius pateat, praeter potestates ipsius z exponentes integros affirmativos habentes admitti debent potestates quaecunque. Sic dubium erit nullum, quin omnis functio ipsius z in huiusmodi expressionem infinitam transmutari possit $Az^\alpha+Bz^\beta+Cz^\gamma+Dz^\delta+\text{etc.}$ denotantibus exponentibus $\alpha, \beta, \gamma, \delta$ etc. numeros quoscunque.

60. *Per divisionem autem continuam intelligitur fractionem*

$$\frac{a}{\alpha+\beta z}$$

resolvi in hanc seriem infinitam

$$\frac{a}{\alpha}-\frac{a\beta z}{\alpha^2}+\frac{a\beta^2z^2}{\alpha^3}-\frac{a\beta^3z^3}{\alpha^4}+\frac{a\beta^4z^4}{\alpha^5}-\text{etc.}$$

quae, cum, quilibet terminus ad sequentem habeat rationem constantem $1:-\frac{\beta z}{\alpha}$, *vocatur series geometrica.*

Potest vero quoque haec series ita inveniri, ut ipsa initio pro incognita habeatur; ponatur enim

$$\frac{a}{\alpha+\beta z}=A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}$$

atque ad aequalitatem producendam quaerantur coefficientes A, B, C, D etc.

Erit ergo

$$a=(\alpha+\beta z)\left(A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}\right)$$

et multiplicatione actu peracta fiet

$$\begin{aligned}a=\alpha A+\alpha Bz+\alpha Cz^2 &+\alpha Dz^3+\alpha Ez^4+\text{etc.}\\ +\beta Az+\beta Bz^2 &+\beta Cz^3+\beta Dz^4+\text{etc.}\end{aligned}$$

Quamobrem esse debet

$$a=\alpha A \quad \text{ideoque} \quad A=\tfrac{a}{\alpha}$$

et coefficientium uniuscuiusque potestatis ipsius z summa nihilo aequalis est ponenda, unde prodibunt hae aequationes

$$\begin{aligned}&\alpha B+\beta A=0,\\ &\alpha C+\beta B=0,\\ &\alpha D+\beta C=0,\\ &\alpha E+\beta D=0\\ &\quad\text{etc.};\end{aligned}$$

cognito ergo quovis coefficiente facile reperitur sequens; si enim fuerit coefficiens termini cuiusque $=P$ et sequens $=Q$, erit

$$\alpha Q+\beta P=0 \text{ sive } Q=-\tfrac{\beta P}{\alpha}.$$

Cum igitur terminus primus A sit determinatus $=\frac{a}{\alpha}$, ex eo sequentes litterae B, C, D etc. definiuntur eodem modo, quo ex divisione sunt orti. Ceterum ex inspectione perspicitur in serie infinita pro $\frac{a}{\alpha+\beta z}$ inventa potestatis z^n coefficientem fore $\pm\frac{a\beta^n}{\alpha^{n+1}}$ ubi signum + locum habet, si n sit numerus par, signum – autem, si n sit numerus impar, seu coefficiens erit $=\frac{a}{\alpha}\left(\frac{-\beta}{\alpha}\right)^n$.

61. *Simili modo ope divisionis continuatae haec functio fracta*

$$\frac{a+bz}{\alpha+\beta z+\gamma zz}$$

in seriem infinitam converti potest.

Cum autem divisio sit taediosa neque tam facile naturam seriei infinitae ostendat, commodius erit seriem quaesitam fingere atque modo ante tradito determinare. Sit igitur

$$\frac{a+bz}{\alpha+\beta z+\gamma zz}=A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.};$$

multiplicetur utrinque per $\alpha+\beta z+\gamma zz$ atque fiet

$$\begin{aligned} a+bz=\alpha A+\alpha Bz+\alpha Cz^2+\alpha Dz^3+\alpha Ez^4+&\text{ etc.}\\ +\beta Az+\beta Bz^2+\beta Cz^3+\beta Dz^4+&\text{ etc.}\\ +\gamma Az^2+\gamma Bz^3+\gamma Cz^4+&\text{ etc.}\end{aligned}$$

Hinc erit

$$aA=a,\ aB+\beta A=b,$$

unde reperitur

$$A=\frac{a}{\alpha}\ \text{ et }\ B=\frac{b}{\alpha}-\frac{a\beta}{\alpha\alpha}\ ;$$

reliquae vero litterae ex sequentibus aequationibus determinabuntur

$$\begin{gathered}\alpha C+\beta B+\gamma A=0,\\ \alpha D+\beta C+\gamma B=0,\\ \alpha E+\beta D+\gamma C=0,\\ \alpha F+\beta E+\gamma D=0\\ \text{etc.};\end{gathered}$$

hinc ergo ex binis quibusque coefficientibus contiguis sequens reperitur. Sic, si duo coefficientes contigui fuerint *P, Q* et sequens *R,* erit

$$\alpha R+\beta Q+\gamma P=0\quad\text{seu}\quad R=\frac{-\beta Q-\gamma P}{\alpha}.$$

Cum igitur duae litterae primae *A* et *B* iam sint inventae, sequentes *C, D, E, F* etc. omnes successive ex iis invenientur sicque reperietur series infinita

$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$ functioni fractae propositae $\frac{a+bz}{\alpha+\beta z+\gamma zz}$ aequalis.

EXEMPLUM

Si fuerit proposita haec fractio

$$\frac{1+2z}{1-z-zz}$$

huicque aequalis statuatur series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

ob

$$a = 1,\ b = 2,\ \alpha = 1,\ \beta = -1,\ \gamma = -1$$

erit

$$A = 1,\ B = 3;$$

tum vero erit

$$C = B + A,$$
$$D = C + B,$$
$$E = D + C,$$
$$F = E + D$$
etc.

Quilibet ergo coefficiens aequalis est summae duorum praecedentium; quare si cogniti fuerint duo coefficientes contigui P et Q, erit sequens $R = P + Q$.

Cum igitur duo coefficientes primi A et B sint cogniti, fractio proposita

$$\frac{1+2z}{1-z-zz}$$

in hanc seriem infinitam transmutatur

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \text{etc.},$$

quae nullo negotio, quousque libuerit, continuari potest.

62. Ex his iam satis intelligitur indoles serierum infinitarum, in quas functiones fractae transmutantur; tenent enim eiusmodi legem, ut quilibet terminus ex aliquot praecedentibus determinari possit.

Scilicet, si denominator fractionis propositae fuerit

$$\alpha + \beta z$$

atque series infinita statuatur

$$A+Bz+Cz^2+\cdots+Pz^n+Qz^{n+1}+Rz^{n+2}+Sz^{n+3}+\text{ etc.}$$

quilibet coefficiens Q ex praecedente P solo ita definietur, ut sit

$$\alpha Q+\beta P=0\,.$$

Sin denominator fuerit trinomium

$$\alpha+\beta z+\gamma zz\,,$$

quilibet coefficiens seriei R ex duobus praecedentibus Q et P ita definietur, ut sit

$$\alpha R+\beta Q+\gamma P=0\,.$$

Simili modo, si denominator fuerit quadrinomium, ut

$$\alpha+\beta z+\gamma zz+\delta z^3$$

quilibet coefficiens seriei S ex tribus antecedentibus R, Q et P ita determinabitur, ut sit

$$aS+\beta R+\gamma Q+\delta P=0\,,$$

sicque de ceteris.

In his ergo seriebus quilibet terminus determinatur ex aliquot antecedentibus secundum legem quandam constantem, quae lex ex denominatore fractionis hanc seriem producentis sponte apparet. Vocari autem hae series a Celeberrimo MOIVREO, qui earum naturam maxime est scrutatus, solent *recurrentes*, propterea quod ad terminos antecedentes est recurrendum, si sequentes investigare velimus.

63. Ad harum porro serierum formationem requiritur, ut denominatoris terminus constans α non sit $=0$; cum enim inventus sit terminus seriei primus $A=\frac{a}{\alpha}$, tum is tum omnes sequentes fierent infiniti, si esset $\alpha=0$. Hoc ergo casu excluso, quem deinceps evolvam, functio fracta in seriem infinitam recurrentem transmutanda huiusmodi habebit formam

$$\frac{a+bz+cz^2+dz^3+\text{etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{etc.}}\,,$$

ubi primum denominatoris terminum pono $=1$; huc enim semper fractio reduci potest, nisi is sit $=0$; reliquos autem denominatoris terminos omnes tanquam negativos contemplor, ut seriei hinc formatae omnes termini fiant affirmativi. Quodsi enim series recurrens hinc orta ponatur

$$A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}\,,$$

coefficientes ita determinabuntur, ut sit

$$A = a,$$
$$B = \alpha A + b,$$
$$C = \alpha B + \beta A + c,$$
$$D = \alpha C + \beta B + \gamma A + d,$$
$$E = \alpha D + \beta C + \gamma B + \delta A + e$$
etc.

Quilibet ergo coefficiens aequalis est aggregato ex multiplis aliquot praecedentium una cum numero quodam, quem numerator praebet. Nisi autem numerator in infinitum progrediatur, haec addito mox cessabit atque quivis terminus secundum legem constantem ex aliquot praecedentibus determinabitur. Ne ergo lex progressionis usquam turbetur, conveniet functionem fractam genuinam adhibere; si enim fractio spuria accipiatur, tum pars integra in ea contenta ad seriem accedet atque in illis terminis, quos vel auget vel minuit, legem progressionis interrumpet. Exempli gratia haec fractio spuria

$$\frac{1+2z-z^3}{1-z-zz}$$

praebebit hanc seriem

$$1+3z+4zz+6z^3+10z^4+16z^5+26z^6+42z^7+\text{etc.},$$

ubi a lege, qua quivis coefficiens est summa duorum praecedentium, terminus quartus $6z^3$ excipitur.

64. Peculiarem contemplationem series recurrentes merentur, si denominator fractionis, unde oriuntur, fuerit potestas. Sic, si ista fractio

$$\frac{a+bz}{(1-\alpha z)^2}$$

in seriem resolvatur, prodit

$$\begin{array}{l} a+2\alpha az+3\alpha^2az^2+4\alpha^3az^3+5\alpha^4az^4+\text{etc.}, \\ \quad +\quad b\ \ +2\alpha b\quad\ +3\alpha^2b\quad +4\alpha^3b \end{array}$$

In qua coefficiens potestatis z^n erit

$$(n+1)\alpha^n a + n\alpha^{n-1}b.$$

Erit tamen haec series recurrens, quia quilibet terminus ex duobus praecedentibus determinatur, cuius determinationis lex perspicitur ex denominatore evoluto $1-2\alpha z+\alpha\alpha zz$.

Si ponatur $\alpha=1$ et $z=1$, abit series in progressionem arithmeticam generalem

$$a+(2a+b)+(3a+2b)+(4a+3b)+\text{etc.},$$

cuius differentiae sunt constantes. Omnis ergo progressio arithmetica est series recurrens; si enim sit

$$A+B+C+D+E+F+\text{etc}.$$

progressio arithmetica, erit

$$C=2B-A,\ D=2C-B,\ E=2D-C\ \text{ etc}.$$

65. Deinde haec fractio

$$\frac{a+bz+czz}{(1-\alpha z)^3}$$

ob

$$\frac{1}{(1-\alpha z)^3}=(1-\alpha z)^{-3}=1+3\alpha z+6\alpha^2z^2+10\alpha^3z^3+15\alpha^4z^4+\text{etc.}$$

transmutabitur in hanc seriem infinitam

$$\begin{array}{llllll} a & +3\alpha az & +6\alpha^2az^2 & +10\alpha^3az^3 & +15\alpha^4az^4 & +\text{ etc.},\\ & +\ b & +3\alpha b & +\ 6\alpha^2b & +10\alpha^3b & \\ & & +\ c & +\ 3\alpha c & +\ 6\alpha^2c & \end{array}$$

in qua potestas z^n coefficientem habebit

$$\frac{(n+1)(n+2)}{1\cdot2}\alpha^na+\frac{n(n+1)}{1\cdot2}\alpha^{n-1}b+\frac{n(n-1)}{1\cdot2}\alpha^{n-2}c.$$

Quodsi autem ponatur $\alpha=1$ et $z=1$, series haec abibit in progressionem generalem secundi ordinis, cuius differentiae secundae sunt constantes. Designet

$$A+B+C+D+E+F+\text{etc.}.$$

huiusmodi progressionem; erit ea simul series recurrens, cuius quilibet terminus ex tribus antecedentibus ita determinatur, ut sit

$$D = 3C - 3B + A,\ E = 3D - 3C + B,\ F = 3E - 3D + C \text{ etc}.$$

Cum igitur terminorum in progressione arithmetica procedentium secundae differentiae quoque sint aequales, nempe $= 0$, haec proprietas quoque ad progressiones arithmeticas extenditur.

66. Simili modo haec fractio

$$\frac{a+bz+czz+dz^3}{(1-\alpha z)^4}$$

dabit seriem infinitam, in qua potestas ipsius z quaecunque z^n hunc habebit coefficientem

$$\frac{(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3}\alpha^n a + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\alpha^{n-1} b + \frac{(n-1)n(n+1)}{1\cdot 2\cdot 3}\alpha^{n-2} c + \frac{(n-2)(n-1)n}{1\cdot 2\cdot 3}\alpha^{n-3} d.$$

Posito ergo $\alpha = 1$ et $z = 1$ haec series in se complectetur omnes progressiones algebraicas tertii ordinis, quarum differentiae tertiae sunt constantes; omnes ergo huius ordinis progressiones, cuiusmodi sit

$$A + B + C + D + E + F + \text{etc.},$$

erunt simul recurrentes ex denominatore $1 - 4z + 6zz - 4z^3 + z^4$ ortae, unde erit

$$E = 4D - 6C + 4B - A,\ F = 4E - 6D + 4C - B \text{ etc.},$$

quae proprietas simul in omnes progressiones inferiorum ordinum competit.

67. Hoc modo ostendetur omnes progressiones algebraicas cuiuscunque ordinis, quae tandem ad differentias constantes deducunt, esse series recurrentes, quarum lex definiatur ex denominatore $(1-z)^n$ existente n numero maiore quam is, qui ordinem progressionis indicat. Cum igitur

$$a^m + (a+b)^m + (a+2b)^m + (a+3b)^m + \text{ etc}$$

.
exhibeat progressionem ordinis m, erit ob naturam serierum recurrentium

$$0 = a^m - \frac{n}{1}(a+b)^m + \frac{n(n-1)}{1\cdot 2}(a+2b)^m - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(a+3b)^m$$
$$+ \cdots \mp \frac{n}{1}\left(a+(n-1)b\right)^m \pm (a+nb)^m,$$

ubi signa superiora valent, si n sit numerus par, inferiora autem, si n sit numerus impar. Haec ergo aequatio semper est vera, si fuerit n numerus integer maior quam m. Hinc ergo intelligitur, quam late pateat doctrina de seriebus recurrentibus.

68. Si denominator fuerit potestas non binomii sed multinomii, natura seriei quoque alio modo explicari potest. Sit nempe haec fractio

$$\frac{1}{(1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{etc.})^{m+1}}$$

proposita; erit series infinita hinc nata

$$\begin{aligned}1+\frac{(m+1)}{1}\alpha z+\frac{(m+1)(m+2)}{1\cdot 2}\alpha^2z^2+\frac{(m+1)(m+2)(m+3)}{1\cdot 2\cdot 3}\alpha^3z^3+\text{etc.}\\ +\quad\frac{(m+1)}{1}\beta\quad+\quad\frac{(m+1)(m+2)}{1\cdot 2}2\alpha\beta\\ +\quad\frac{(m+1)}{1}\gamma\end{aligned}$$

Ad naturam huius seriei penitius inspiciendam exponatur haec series per litteras generales hoc modo

$$1+Az+Bz^2+Cz^3+\cdots+Kz^{n-3}+Lz^{n-2}+Mz^{n-1}+Nz^n+\text{etc.}$$

ae quilibet coefficiens N ex tot praecedentibus, quot sunt litterae $\alpha,\beta,\gamma,\ \delta$ etc., ita determinabitur, ut sit

$$N=\ \frac{m+n}{n}\alpha M+\frac{2m+n}{n}\beta L+\frac{3m+n}{n}\gamma K+\frac{4m+n}{n}\delta I+\text{etc.};$$

quae lex continuationis etsi non est constans sed ab exponente potestatis z pendet, tamen eidem seriei alia convenit lex progressionis constans, quam denominator evolutus praebet naturae serierum recurrentium consentaneam. Illa vero lex non constans tantum locum habet, si numerator fractionis fuerit unitas seu quantitas constans; si enim quoque aliquot potestates ipsius z contineret, tum illa lex multo magis fieret complicata, id quod post tradita calculi differentialis principia facilius patebit.

69. Quoniam hactenus posuimus primum denominatoris terminum constantem non esse $=0$ eiusque loco unitatem collocavimus, nunc videamus, cuiusmodi series oriantur, si in denominatore terminus constans evanescat. His casibus ergo functio fracta huiusmodi formam habebit

$$\frac{a+bz+czz+\text{etc.}}{z(1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.})};$$

.

convertatur ergo neglecto denominatoris factore z reliqua fractio

$$\frac{a+bz+czz+\text{etc.}}{1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.}}$$

in seriem recurrentem

$$A+Bz+Cz^2+Dz^3+\text{etc}.$$

atque manifestum est fore

$$\frac{a+bz+czz+\text{etc.}}{z(1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.})}=\frac{A}{z}+B+Cz+Dz^2+Ez^3+\text{etc.}$$

Simili modo erit

$$\frac{a+bz+czz+\text{etc.}}{z^2(1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.})}=\frac{A}{zz}+\frac{B}{z}+C+Dz+Ez^2+\text{etc.}$$

atque generatim erit

$$\frac{a+bz+czz+\text{etc.}}{z^m(1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.})}=\frac{A}{z^m}+\frac{B}{z^{m-1}}+\frac{C}{z^{m-2}}+\frac{D}{z^{m-3}}+\text{etc.},$$

quicunque numerus fuerit exponens m.

70. Quoniam per substitutionem loco z alia variabilis x in functionem fractam introduci hocque pacto functio fracta quaevis in innumerabiles formas diversas transmutari potest, hoc modo eadem functio fracta infinitis modis per series recurrentes explicari poterit. Sit scilicet proposita haec fractio

$$y=\frac{1+z}{1-z-zz}$$

et per seriem recurrentem

$$y=1+2z+3z^2+5z^3+8z^4+\text{etc.};$$

ponatur $z=\frac{1}{x}$; erit

$$y=\frac{xx+x}{xx-x-1}=\frac{-x(1+x)}{1+x-xx}$$

Iam

$$\frac{1+x}{1+x-xx}=1+0x+xx-x^3+2x^4-3x^5+5x^6-\text{etc.},$$

unde erit

$$y=-x+0x^2-x^3+x^4-2x^5+3x^6-5x^7+\text{etc.}$$

Vel ponatur

$$z=\frac{1-x}{1+x};$$

erit

$$y = \frac{-2-2x}{1-4x-xx},$$

unde fit

$$y = -2 - 10x - 42xx - 178x^3 - 754x^4 - \text{etc.},$$

cuiusmodi series recurrentes pro y innumerabiles inveniri possunt.

71. Functiones irrationales ex hoc theoremate universali in series infinitas transformari solent, quod sit

$$(P+Q)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}P^{\frac{m-n}{n}}Q + \frac{m(m-n)}{n\cdot 2n}P^{\frac{m-2n}{n}}Q^2 + \frac{m(m-n)(m-2n)}{n\cdot 2n\cdot 3n}P^{\frac{m-3n}{n}}Q^3 + \text{etc.};$$

hi. enim termini, nisi fuerit $\frac{m}{n}$ numerus integer affirmativus, in infinitum excurrunt. Sic erit pro m et n numeros definitos scribendo

$$(P+Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2}P^{-\frac{1}{2}}Q + \frac{1\cdot 1}{2\cdot 4}P^{-\frac{3}{2}}Q^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}P^{-\frac{5}{2}}Q^3 + \text{etc.},$$
$$(P+Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2}P^{-\frac{3}{2}}Q + \frac{1\cdot 3}{2\cdot 4}P^{-\frac{5}{2}}Q^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}P^{-\frac{7}{2}}Q^3 + \text{etc.},$$
$$(P+Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3}P^{-\frac{3}{3}}Q - \frac{1\cdot 2}{3\cdot 6}P^{-\frac{5}{3}}Q^2 + \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}P^{-\frac{8}{3}}Q^3 - \text{etc.},$$
$$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3}P^{-\frac{1}{3}}Q - \frac{2\cdot 1}{3\cdot 6}P^{-\frac{4}{3}}Q^2 + \frac{2\cdot 1\cdot 4}{3\cdot 6\cdot 9}P^{-\frac{7}{2}}Q^3 - \text{etc.}$$
etc.

72. Huiusmodi ergo serierum termini ita progrediuntur, ut quilibet ex antecedente formari possit. Sit enim seriei, quae ex $(P+Q)^{\frac{m}{n}}$ nascitur, terminus quilibet

$$= MP^{\frac{m-kn}{n}}Q^k;$$

erit sequens

$$= \frac{m-kn}{(k+1)n}MP^{\frac{m-(k+1)n}{n}}Q^{k+1}.$$

Notandum autem est in quovis termino sequente exponentem ipsius P unitate decrescere, contra vero exponentem ipsius Q unitate crescere. Quo autem haec facilius ad quemvis casum accommodentur, forma generalis $(P+Q)^{\frac{m}{n}}$ ita exponi potest $P^{\frac{m}{n}}\left(1+\frac{Q}{P}\right)^{\frac{m}{n}}$; evoluta enim formula $\left(1+\frac{Q}{P}\right)^{\frac{m}{n}}$ serieque resultante per $P^{\frac{m}{n}}$ multiplicata prodibit ipsa series ante data. Tum vero, si *m* non solum numeros integros denotet, sed etiam fractos, pro *n* tuto unitas collocari poterit. Quibus factis si pro $\frac{Q}{P}$, quae est functio ipsius *z*, ponatur *Z*, habebitur

$$(1+Z)^{m} = 1+\tfrac{m}{1}Z+\frac{m(m-1)}{1\cdot 2}Z^{2}+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}Z^{3}+\text{etc.}$$

Ad sequentes progressionum leges autem observandas conveniet hanc formulae generalis in seriem conversionem notasse

$$(1+Z)^{m-1} = 1+\tfrac{m-1}{1}Z+\frac{(m-1)(m-2)}{1\cdot 2}Z^{2}+\frac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}Z^{3}+\text{etc.}$$

73. Sit igitur primum $Z=\alpha z$, eritque

$$(1+\alpha z)^{m-1} = 1+\tfrac{m-1}{1}\alpha z+\frac{(m-1)(m-2)}{1\cdot 2}\alpha^{2}z^{2}+\frac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^{3}z^{3}+\text{etc.}$$

Scribatur pro hac serie ista forma generalis

$$=1+Az+Bz^{2}+Cz^{3}+\cdots+Mz^{n-1}+Nz^{n}+\text{etc.}$$

atque quilibet coefficiens *N* ex praecedente *M* ita determinabitur, ut sit

$$N=\tfrac{m-n}{n}\alpha M.$$

Sic posito $n=1$, cum sit $M=1$, erit

$$N=A=\tfrac{m-1}{1}\alpha;$$

tum facto $n=2$ ob $M=A=\tfrac{m-1}{1}\alpha$ erit

$$N=B=\tfrac{m-2}{2}\alpha M=\frac{(m-1)(m-2)}{1\cdot 2}\alpha^{2}$$

similique modo porro

$$C=\tfrac{m-3}{3}\alpha B=\frac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^{3}.$$

ut series ante inventa declarat.

74. Sit $Z = \alpha z + \beta zz$ eritque

$$(1+\alpha z+\beta zz)^{m-1} = 1 + \tfrac{m-1}{1}(\alpha z+\beta zz) + \tfrac{(m-1)(m-2)}{1\cdot 2}(\alpha z+\beta zz)^2 + \text{etc.}$$

Quodsi ergo termini secundum potestates ipsius z disponantur, erit

$$\begin{aligned}(1+\alpha z+\beta zz)^{m-1} = 1 + \tfrac{m-1}{1}\alpha z &+ \tfrac{(m-1)(m-2)}{1\cdot 2}\alpha^2 z^2 + \tfrac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3 z^3 + \text{etc.}\\ &+ \tfrac{m-1}{1}\beta z^2 \quad + \tfrac{(m-1)(m-2)}{1\cdot 2} 2\alpha\beta z^3\end{aligned}$$

Scribatur pro hac serie ista forma generalis

$$1 + Az + Bz^2 + Cz^3 + \cdots + Lz^{n-2} + Mz^{n-1} + Nz^n + \text{etc.}$$

atque quilibet coefficiens ex duobus antecedentibus ita definietur, ut sit

$$N = \tfrac{m-n}{n}\alpha M + \tfrac{2m-n}{n}\beta L,$$

unde omnes termini ex primo, qui est 1, definiri poterunt. Erit nempe

$$\begin{aligned}A &= \tfrac{m-1}{1}\alpha,\\ B &= \tfrac{m-2}{2}\alpha A + \tfrac{2m-2}{2}\beta,\\ C &= \tfrac{m-3}{3}\alpha B + \tfrac{2m-3}{3}\beta A,\\ D &= \tfrac{m-4}{4}\alpha C + \tfrac{2m-4}{4}\beta B\\ &\text{etc.}\end{aligned}$$

75. Si fuerit $Z = \alpha z + \beta z^2 + \gamma z^3$ erit

$$\left(\alpha z+\beta z^2+\gamma z^3\right)^{m-1} = 1 + \tfrac{m-1}{1}\left(\alpha z+\beta z^2+\gamma z^3\right) + \tfrac{(m-1)(m-2)}{1\cdot 2}\left(\alpha z+\beta z^2+\gamma z^3\right)^2 + \text{etc.},$$

quae expressio, si omnes termini secundum potestates ipsius z ordinentur, abibit in hanc seriem

$$\begin{aligned}1+\tfrac{m-1}{1}\alpha z+\tfrac{(m-1)(m-2)}{1\cdot 2}\alpha^2z^2&+\tfrac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3z^3+\text{etc.}\\ +\tfrac{m-1}{1}\beta z^2\quad&+\tfrac{(m-1)(m-2)}{1\cdot 2}2\alpha\beta z^3\\ &+\tfrac{m-1}{1}\gamma z^3\end{aligned}$$

cuius lex progressionis ut melius patescat, ponatur eius loco

$$1+Az+Bz^2+Cz^3+\cdots+Lz^{n-2}+Mz^{n-1}+Nz^n+\text{etc.},$$

cuius seriei quilibet coefficiens ex tribus antecedentibus ita determinatur, ut sit

$$N=\tfrac{m-n}{n}\alpha M+\tfrac{2m-n}{n}\beta L+\tfrac{3m-n}{n}\gamma K.$$

Cum igitur primus terminus sit $=1$ et antecedentes nulli, erit

$$\begin{aligned}A&=\tfrac{m-1}{1}\alpha,\\ B&=\tfrac{m-2}{2}\alpha A+\tfrac{2m-2}{2}\beta,\\ C&=\tfrac{m-3}{3}\alpha B+\tfrac{2m-3}{3}\beta A+\tfrac{3m-3}{3}\gamma,\\ D&=\tfrac{m-4}{4}\alpha C+\tfrac{2m-4}{4}\beta B+\tfrac{3m-4}{4}\gamma A,\\ E&=\tfrac{m-5}{5}\alpha D+\tfrac{2m-5}{5}\beta C+\tfrac{3m-5}{5}\gamma B\\ &\text{etc.}\end{aligned}$$

76. Generaliter ergo si ponatur

$$\left(1+\alpha z+\beta z^2+\gamma z^3+\delta z^4\ \text{etc.}\right)^{m-1}=1+Az+Bz^2+Cz^3+Dz^4+Ez^5+\text{etc.},$$

huius seriei singuli termini ita ex praecedentibus definientur, ut sit

$$\begin{aligned}A&=\tfrac{m-1}{1}\alpha,\\ B&=\tfrac{m-2}{2}\alpha A+\tfrac{2m-2}{2}\beta,\\ C&=\tfrac{m-3}{3}\alpha B+\tfrac{2m-3}{3}\beta A+\tfrac{3m-3}{3}\gamma,\\ D&=\tfrac{m-4}{4}\alpha C+\tfrac{2m-4}{4}\beta B+\tfrac{3m-4}{4}\gamma A+\tfrac{4m-4}{4}\delta,\\ E&=\tfrac{m-5}{5}\alpha D+\tfrac{2m-5}{5}\beta C+\tfrac{3m-5}{5}\gamma B+\tfrac{4m-5}{5}\delta A+\tfrac{5m-5}{5}\varepsilon\\ &\text{etc.}\end{aligned}$$

quilibet scilicet terminus per tot praecedentes determinatur, quot habentur litterae a, β, γ, δ etc. in functione ipsius z, cuius potestas in seriem convertitur. Ceterum ratio huius legis convenit cum ea, quam supra § 68 [invenimus], ubi similem formam $\left(1-\alpha z-\beta z^{2}-\gamma z^{3}-\text{etc.}\right)^{-m-1}$ in seriem infinitam resolvimus; si enim loco m scribatur $-m$ atque litterae a, β, γ, δ etc. negative accipiantur, series inventae prorsus congruent. Interim hoc loco non licet rationem huius progressionis legis a priori demonstrare, id quod per principia Calculi differentialis demum commode fieri poterit; interea ergo sufficiet veritatem per applicationem ad omnis generis exempla comprobasse.