# CHAPTER III

# THE TRANSFORMATION OF FUNCTIONS BY SUBSTITUTION

46. *If y were some function of z and z may be defined by the new variable x, then y also will be able to be defined by x.*

Therefore since before $y$ was a function of $z$, now the new variable quantity $x$ is introduced, by which each of the previous $y$ and $z$ was defined. Thus, if there were

$$y = \frac{1-zz}{1+zz}$$

and there is put

$$z = \frac{1-x}{1+x},$$

with this value substituted in place of $z$ there will be

$$y = \frac{2x}{1+xx}.$$

Therefore with any determined value taken for $x$, from that the values to be determined for $z$ and $y$ will be found, and thus the value of $y$ corresponding to that value of $z$, which will be produced likewise. As, if there shall be $x = \frac{1}{2}$, then $z = \frac{1}{3}$ and $y = \frac{4}{5}$ is produced; but $y = \frac{4}{5}$ is found also, if $z = \frac{1}{3}$ is put into $\frac{1-zz}{1+zz}$, to which expression $y$ is equal.

But this introduction of a new variable is used for a double end : indeed either in this manner irrationality is removed, by which the expression given of $y$ in terms of $z$ may labour; or when on account of an equation of higher grade, by which the relation between $y$ and $z$ is expressed, it is not possible to show an explicit function of $z$ equal to $y$ itself, the new variable $x$ is introduced, from which each $y$ and $z$ can be defined conveniently ; from which the use of the designated substitution will be brought out well enough, and truly will be seen much more clearly from the following.

47. *If there were* $y = \sqrt{(a+bz)}$*, the new variable x, by which each z and y may be expressed rationally, may be found in the following manner.*

Because both $z$ as well as $y$ must be a rational function of $x$, it is evident that this is obtained, if there is put

$$\sqrt{(a+bz)} = bx.$$

Indeed in the first place, it becomes

$$y = bx \quad \text{and} \quad a + bz = bbxx$$

and hence

$$z = bxx - \tfrac{a}{b}.$$

On account of which each quantity $y$ and $z$ is expressed by a rational function of $x$ ; evidently since the equation shall be $y = \sqrt{(a + bz)}$, making $z = bxx - \tfrac{a}{b}$; it will become $y = bx$.

48. *If the function were $y = (a + bz)^{m:n}$, the new variable x, by which both y as well as z may be expressed rationally, thus may be found.*

The substitution $y = x^m$ may be put in place, and the equation becomes $(a + bz)^{1:n} = x$, therefore $a + bz = x^n$ and $z = \frac{x^n - a}{b}$.
Therefore thus each quantity $y$ and $z$ will be defined rationally by $x$, clearly with the aid of the substitution $z = \frac{x^n - a}{b}$, which $y = x^m$ brings about.
Therefore although neither $y$ by $z$ nor in turn $z$ by $y$ can be expressed rationally, yet each has been returned a rational function of the new variable quantity $x$ by the substitution introduced, with the scope of the substitution entirely suitable.

49. *If the function were $y = \left(\frac{a+bz}{f+gz}\right)^{m:n}$, a new variable quantity x is required, by which both y and z may be expressed rationally.*

In the first place it is evident, if $y = x^m$ is put in place, for the question to be satisfied ; indeed it becomes $\left(\frac{a+bz}{f+gz}\right)^{m:n} = x^m$ and thus $\frac{a+bz}{f+gz} = x^n$; from which equation $z = \frac{a - fx^n}{gx^n - b}$ is elicited, which the substitution $y = x^m$ produces.
Hence it is understood also, if there were

$$\left(\frac{\alpha+\beta y}{\gamma+\delta y}\right)^n = \left(\frac{a+bz}{f+gz}\right)^m$$

both $y$ as well as $z$ are going to be expressed rationally by $x$, if each formula is put $= x^{mn}$; for it will be found that $y = \frac{\alpha - \gamma x^m}{\delta x^m - \beta}$ and $z = \frac{a - fx^n}{gx^n - b}$; which cases have nothing of difficulty.

50. *If the function were*

$$y = \sqrt{(a+bz)(c+dz)}\,,$$

*a suitable substitution will be found, by which $y$ and $z$ are expressed rationally, in this manner.*

Put $\sqrt{(a+bz)(c+dz)} = (a+bz)x$;
for it is easily seen hence a rational value for $z$ is to be produced, because the value of $z$ itself is determined by a simple equation. Therefore there will be

$$c+dz = (a+bz)xx$$

and hence

$$z = \tfrac{c-axx}{bxx-d}.$$

Whereby again there becomes

$$a+bz = \tfrac{bc-ad}{bxx-d}$$

and on account of $y = \sqrt{(a+bz)(c+dz)} = (a+bz)x$ there will be had

$$y = \tfrac{(bc-ad)x}{bxx-d}.$$

Therefore the irrational function $y = \sqrt{(a+bz)(c+dz)}$ is lead to being rational with the help of the substitution

$$z = \tfrac{c-axx}{bxx-d}\,,$$

which will give certainly

$$y = \tfrac{(bc-ad)x}{bxx-d}.$$

Thus, it the function were

$$y = \sqrt{(aa-zz)} = \sqrt{(a+z)(a-z)}$$

on account of $b=+1, c=a,\ d=-1$ there is put

$$z = \tfrac{a-axx}{1+xx}$$

and the function becomes

$$y = \tfrac{2ax}{1+xx}.$$

Therefore as often as the quantity after the sign √ will have had two simple real factors, it may be resolved by reduction in this manner; but without two simple factors it would be imaginary, as is evident in the following manner.

51. *Let* $y=\sqrt{(p+qz+rzz)}$ *, so that by making a suitable substitution z, the value of* $y$ *may become rational.*

This can happen is several way, provided $p$ and $q$ are either positive or negative quantities. In the first place let the quantity $p$ be positive, and for $p$ put $aa$ ; for even if $p$ shall not be square, the irrational nature of the quantity present still does not disturb the calculation. Therefore there shall be :

I. $y=\sqrt{(p+qz+rzz)}$ and putting

$$\sqrt{(aa+bz+czz)}=a+xz\,;$$

there will be

$$b+cz=2ax+xxz\,,$$

from which the equation arises

$$z=\tfrac{b-2ax}{xx-c}\,;$$

then truly the original relation shall be

$$y=a+xz=\tfrac{bx-axx-ac}{xx-c}\,,$$

where $z$ and $y$ are rational functions of $x$. Now let there be :

II. $y=\sqrt{(aazz+bz+c)}$ and putting

$$\sqrt{(aazz+bz+c)}=az+x:$$

there will be

$$bz+c=2azx+xx$$

and

$$z=\tfrac{xx-c}{b-2ax}.$$

Moreover then there becomes

$$y=az+x=\tfrac{-ac+bx-axx}{b-2ax}\,.$$

III. If $p$ and $r$ were negative quantities, then, unless there shall be $qq>4pr$, the value of $y$ itself will always be imaginary. Moreover if there should b $qq>4pr$, the expression

$p+qz+rzz$ will be able to be resolved into two factors, which case is reduced to the preceding paragraph. But on many occasions it is reduced more suitably to this form :

$$y=\sqrt{aa+(b+cz)(d+ez)};$$

for which to be induced to a rational form, there is put

$$y=a+(b+cz)x$$

and it becomes

$$d+ez=2ax+bxx+cxxz,$$

from which there is produced

$$z=\frac{d-2ax-bxx}{cxx-e}$$

and

$$y=\frac{-ae+(cd-be)x-acxx}{cxx-e}.$$

Meanwhile it is more suitable to be reduced to this form

$$y=\sqrt{aazz+(b+cz)(d+ez)}.$$

Then putting

$$y=az+(b+cz)x;$$

it becomes

$$d+ez=2axz+bxx+cxxz$$

and

$$z=\frac{bxx-d}{e-2ax-cxx}$$

and finally

$$y=\frac{-ad+(be-cd)x-abxx}{e-2ax-cxx}.$$

EXAMPLE

If this irrational function of $z$ may be treated

$$y=\sqrt{(-1+3z-zz)},$$

which, since it can be reduced to this form

$$y=\sqrt{(1-2+3z-zz)}=\sqrt{1-(1-z)(2-z)},$$

putting

$$y = 1 - (1 - z)x\,,$$

it becomes

$$-\ 2\ + z = -2x + xx - xxz$$

and

$$z = \tfrac{2 - 2x + xx}{1 + xx}$$

Then there is

$$1 - z = \tfrac{-1 + 2x}{1 + xx}$$

and

$$y = 1 - (1 - z)x = \tfrac{1 + x - xx}{1 + xx}.$$

And these are almost the cases, which indeterminate algebra or the method of Diophantus supports, no other cases are understood in these tracts which permit reduction to rationality by rational substitution. On account of which I will progress to another use of substitution to be shown.

52. *If y were a function of z of this kind, so that it shall be*

$$ay^{\alpha} + bz^{\beta} + cy^{\gamma} z^{\delta} = 0\,,$$

*to find a new variable x, by which the values of y and z may be assigned explicitly.*

Because the resolution of the general equation is not had, from the proposed equation $ay^{\alpha} + bz^{\beta} + cy^{\gamma} z^{\delta} = 0$ neither $y$ nor $z$ in turn can be shown by $y$. To which inconvenience therefore a remedy may be produced, there is put

$$y = x^m z^n$$

and the equation becomes

$$ax^{\alpha m} z^{\alpha n} + bz^{\beta} + cx^{\gamma m} z^{\gamma n + \delta} = 0\,.$$

Now the exponent $n$ may be determined thus, so that from that equation the value of $z$ may be able to be defined, which can be performed in three ways.
I. Let
$an = \beta$ and thus $n = \frac{\beta}{a}$ ; the equation divided by $z^{\alpha n} = z^{\beta}$ will be:

$$ax^{\alpha m} + b + cx^{\gamma m} z^{\gamma n - \beta + \delta} = 0\,,$$

from which the equation arises:

$$z = \left(\tfrac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{1}{\gamma n - \beta + \delta}} \text{ or } z = \left(\tfrac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\alpha}{\beta\gamma - \alpha\beta + \alpha\delta}}$$

and

$$y = x^m \left(\frac{-ax^{\alpha m}-b}{cx^{\gamma m}}\right)^{\frac{\beta}{\beta n-\alpha\beta+\alpha\delta}}$$

II. Let

$$\beta = \gamma n+\delta \text{ or } n = \frac{\beta-\delta}{\gamma};$$

with the equation divided by $z^{\beta}$, it will be

$$ax^{\alpha m}z^{\alpha n-\beta}+b+cx^{\gamma m}=0,$$

from which it becomes

$$z = \left(\frac{-b-cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{1}{\alpha n-\beta}} = \left(\frac{-b-cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\gamma}{\alpha\beta-\alpha\delta-\beta\gamma}}$$

and

$$y = x^m \left(\frac{-b-cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\beta-\delta}{\alpha\beta-\alpha\delta-\beta\gamma}}$$

III. Let

$$an = \gamma n+\delta \text{ or } n = \frac{\delta}{\alpha-\gamma};$$

with the equation divided by $z^{\alpha n}$, there will be

$$ax^{\alpha m}+bz^{\beta-\alpha n}+cx^{\gamma m}=0,$$

from which it becomes

$$z = \left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{1}{\beta-\alpha n}} = \left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\alpha-\gamma}{\alpha\beta-\beta\gamma-\alpha\delta}}$$

and

$$y = x^m \left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\delta}{\alpha\beta-\beta\gamma-\alpha\delta}}$$

Therefore the functions of $x$, which are equal to $z$ and $y$ themselves, have been extracted by three different methods. Truly besides it is allowed to substitute any value for the number $m$ as it pleases with the number zero excepted, and thus the formulas will be able to be reduced to the most suitable expression.

EXAMPLE

The nature of the function *y* may be expressed by this equation

$$y^3 + z^3 - cyz = 0$$

and the functions of *x* with *y* and *z* may be found.
Therefore there will be

$$a = -1, \quad b = -1, \quad \alpha = 3, \quad \beta = 3, \quad \gamma = 1 \text{ et } \delta = 1.$$

Hence the first way will give with $m = 1$

$$z = \left(\tfrac{x^3+1}{cx}\right)^{-1} \text{ and } y = x\left(\tfrac{x^3+1}{cx}\right)^{-1}$$

or

$$z = \tfrac{cx}{x^3+1} \text{ and } y = \tfrac{cxx}{1+x^3},$$

thus each of the expressions is rational.

Truly the second way will give these values

$$z = \left(\tfrac{cx-1}{x^3}\right)^{1:3} \text{ and } y = x\left(\tfrac{cx-1}{x^3}\right)^{2:3}$$

or

$$z = \tfrac{1}{x}\sqrt[3]{(cx-1)} \text{ and } y = \tfrac{1}{x}\sqrt[3]{(cx-1)^2}\ .$$

The third way may be expedited thus, so that there shall be

$$z = \left(cx - x^3\right)^{2:3} \text{ and } y = x\left(cx - x^3\right)^{1:3}.$$

53. *Hence from the latter it is understood, equations of this kind, in which the value of the function y is determined by z, are able to be resolved by introducing a new variable x.*

Indeed we may put now these determinations in the resolution now put in place

$$z = \left(\tfrac{ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}}{A + Bx^\mu + Cx^\nu + \text{etc.}}\right)^{p:r}$$

and

$$y = x\left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{\nu}+\text{etc.}}\right)^{q:r}$$

and there will be

$$y^p = x^p z^q$$

and hence

$$x = yz^{-q:p}$$

Since therefore there shall be

$$z^{r:p} = \frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{\nu}+\text{etc.}},$$

if in place of $x$ we may substitute the value of this $yz^{-q:p}$, this equation will be produced

$$z^{r:p} = \frac{ay^{\alpha}z^{-\alpha q:p}+bx^{\beta}z^{-\beta q:p}+cx^{\gamma}z^{-\gamma q:p}+\text{etc.}}{A+By^{\mu}z^{-\mu q:p}+Cy^{\nu}z^{-\nu q:p}+\text{etc.}},$$

which is reduced to this

$$Az^{r:p} + By^{\mu}z^{(r-\mu q):p} + Cy^{\nu}z^{(r-\nu q):p} + \text{etc} = ay^{\alpha}z^{-\alpha q:p} + bx^{\beta}z^{-\beta q:p} + cx^{\gamma}z^{-\gamma q:p} + \text{etc.}$$

which multiplied by $z^{\alpha q:p}$ will be changed into this

$$Az^{(\alpha q+\nu):p} + By^{\mu}z^{(\alpha q-\mu q+\nu):p} + Cy^{\nu}z^{(\alpha q-\nu q+\nu):p} + \text{etc.}$$
$$= ay^{\alpha} + by^{\beta}z^{(\alpha q-\beta q):p} + cy^{\gamma}z^{(\alpha q-\gamma q):p} + \text{etc.}$$

Putting

$$\frac{aq+r}{p} = m \text{ and } \frac{aq-\beta q}{p} = n;$$

there becomes

$$p = \alpha-\beta,\ q = n \quad \text{and} \quad r = am-\beta m-\alpha n$$

and this equation arises

$$Az^{m} + By^{\mu}z^{m\text{-}\mu n:(\alpha-\beta)} + Cy^{\nu}z^{m\text{-}\nu n:(\alpha-\beta)} + \text{etc.}$$
$$= ay^{\alpha} + by^{\beta}z^{n} + cy^{\gamma}z^{(\alpha-\gamma)n:(\alpha-\beta)} + \text{etc.},$$

which therefore will be resolved thus, so that it shall be

$$z = \left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{\nu}+\text{etc.}}\right)^{\frac{\alpha-\beta}{\alpha m-\beta m-\alpha n}}$$

and

$$y = x\left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{\nu}+\text{etc.}}\right)^{\frac{n}{\alpha m-\beta m-\alpha n}}$$

Or putting

$$\frac{\alpha q+r}{p} = m \quad \text{and} \quad \frac{\alpha q-\mu q+r}{p} = n\,;$$

there will be

$$m-n = \frac{\mu q}{p}, \qquad \frac{q}{p} = \frac{m-n}{\mu} \ \text{ and } \ \frac{r}{p} = m-\frac{\alpha m-\alpha n}{\mu}.$$

Hence there becomes

$$p = \mu, \quad q = m-n \quad \text{and} \quad r = \mu m-\alpha m+\alpha n$$

and this equation will result

$$Az^{m} + By^{\mu}z^{n} + Cy^{\nu}z^{m\text{-}\nu(m-n):\mu} + \text{etc.}$$
$$= ay^{\alpha} + by^{\beta}z^{(\alpha-\beta)(m-n):\mu} + cy^{\gamma}z^{(\alpha-\gamma)(m-n):\mu} + \text{etc.},$$

which thus will be resolved, so that

$$z = \left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{\nu}+\text{etc.}}\right)^{\frac{\mu}{\mu m-\alpha m+\alpha n}}$$

and

$$y = x\left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{\nu}+\text{etc.}}\right)^{\frac{m-n}{\mu m-\alpha m+\alpha n}}$$

54. *If $y$ may depend thus on $z$, so that there shall be*

$$ayy + byz + czz + dy + ez = 0\,,$$

*both $y$ as well as $z$ will be expressed in the following manner by a new variable $x$.*

Putting $y = xz$*;* it will become, on dividing by $z$

$$axxz + bxz + cz + dx + e = 0\,,$$

from which there is found

$$z = \frac{-dx-e}{axx+bx+c}$$

and

$$y = \frac{-dxx-ex}{axx+bx+c}$$

But truly it is possible to reduce this equation to the proposed form between $y$ and $z$

$$ayy + byz + czz + dy + ez + f = 0$$

by decreasing or increasing each variable by a certain fixed constant quantity, from which also this equation can be expressed rationally by the new variable *x*.

55. *Thus if y shall depend on z, so that there shall be*

$$ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = 0,$$

*it will be possible to express both y as well as z rationally by the new variable x, in the following manner.*

Putting $y = xz$ and with the substitution made the whole equation can be divided by *zz*; moreover it will produce

$$ax^3z + bxxz + cxz + dz + exx + fx + g = 0.$$

From which the equation arises

$$z = \frac{-cxx - fx - g}{ax^3 + bxx + cx + d},$$

from which there becomes

$$y = \frac{-ex^3 - fxx - gx}{ax^3 + bxx + cx + d}$$

From these cases it is easily understood, how equations of higher order, by which *y* is defined by *z*, ought to be prepared, so that a resolution of this kind may find a place. Moreover these [following] cases are contained in the above formulas of §53, but, because the general formulas are not so easily adapted to cases of this kind occurring more often, it has been seen fit to set some of these out separately.

56. *If y may depend thus on z, so that there shall be*

$$ayy + byz + czz = d,$$

*each quantity y and z may be expressed by a new variable x.*

Put $y = xz$ and there becomes

$$(axx + bx + c)zz = d$$

and thus

$$z = \sqrt{\frac{d}{axx + bx + c}}$$

and

$$y = x\sqrt{\tfrac{d}{axx+bx+c}}.$$

In a similar manner, if there were

$$ay^3 + by^2z + cz^2 + dz^3 = ey + fz,$$

on putting $y = xz$, the whole equation divided by $z$ will give

$$\left(ax^3 + bxx + cz^2 + d\right)zz = ex + f,$$

from which there arises

$$z = \sqrt{\tfrac{ex+f}{ax^3+bxx+cx+d}}$$

and

$$y = x\sqrt{\tfrac{ex+f}{ax^3+bxx+cx+d}}.$$

Moreover these other similar cases admitting a resolution are to be handled in the following paragraph.

57. *If y may depend on z thus, so that it shall be*

$$ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \text{etc.} = \alpha y^n + \beta y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \text{etc.},$$

*both z as well as y will be expressed by the new variable x in the following manner.*

Let $y = xz$ and with the substitution made the whole equation will be able to be divided by $z^n$, if indeed the exponent $m$ shall be greater than $n$, and there will be

$$\left(ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \text{etc.}\right)z^{m-n} = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.},$$

from which there will be found

$$z = \left(\tfrac{\alpha x^n+\beta x^{n-1}+\gamma x^{n-2}+\delta x^{n-3}+\text{etc.}}{ay^m+by^{m-1}z+cy^{m-2}z^2+dy^{m-3}z^3+\text{etc.}}\right)^{1:m-n}$$

and

$$y = x\left(\tfrac{\alpha x^n+\beta x^{n-1}+\gamma x^{n-2}+\delta x^{n-3}+\text{etc.}}{ax^m+bx^{m-1}+cx^{m-2}+dx^{m-3}+\text{etc.}}\right)^{1:m-n}$$

This resolution clearly has a place, if in the equation expressing $y$ by $z$ the nature of the function occurs only twice everywhere, the number of the dimensions taken from $y$ and $z$, and just as in the case treated, the number of dimensions in the individual terms is of dimensions $m$ or $n$. [This rather mysterious paragraph means, as we see below, that the relation can be reduced to a soluble quadratic equation.]

58. *If in an equation between y and z, dimensions of the third kind occur, the dimensions of the greatest term of which surpasses the middle term by just as much as this middle term surpasses the lowest term, then by the aid of the resolution of a quadratic equation, the variables y and z will be able to be expressed by a new variable x.*

For if there is put $y = xz$, with a division made by the lowest power of $z$, the value of $z$ in terms of $x$ will be shown with the aid of square root extracted, which will become apparent from the following example.

EXAMPLE 1

Let

$$ay^3 + byyz + cyzz + dz^3 = 2eyy + 2fyz + 2gzz + hy + iz$$

be the equation, and there is put $y = xz$; from the division made by $z$:

$$\left(ax^3 + bxx + ex + d\right)zz = 2\left(exx + fx + g\right)z + hx + i ,$$

from which the following value of $z$ will be obtained:

$$z = \frac{exx+fx+g \pm \sqrt{\left(\left(exx+fx+g\right)^2 + \left(ax^3+bxx+cx+d\right)\left(hx+i\right)\right)}}{ax^3+bxx+cx+d},$$

with which found, there will be $y = xz$.

EXAMPLE 2

Let

$$y^5 = 2az^3 + by + cz$$

be the equation, and by putting $y = xz$ it becomes

$$x^5z^4 = 2azz + bx + c$$

from which there is found

$$zz = \frac{a \pm \sqrt{(aa + bx^6 + cx^5)}}{x^5}$$

and

$$z = \frac{\sqrt{\left(a \pm \sqrt{(aa + bx^6 + cx^5)}\right)}}{xx\sqrt{x}}$$

and also

$$y = \frac{\sqrt{\left(a \pm \sqrt{(aa + bx^6 + cx^5)}\right)}}{x\sqrt{x}}.$$

EXAMPLE 3

Let there be

$$y^{10} = 2az^6 + byz^3 + cz^4;$$

in which since the dimensions shall be 10, 7 et 4, there is put $y = xz$ and the equation divided by $z^4$ will change into this :

$$z^{10}z^6 = 2axz^3 + bx + c$$

or

$$z^6 = \frac{2axz^3 + bx + c}{z^{10}}$$

from which it is found

$$z^3 = \frac{ax \pm x\sqrt{(aa + bx^9 + cx^8)}}{x^{10}}$$

and thus there will be

$$z = \frac{\sqrt[3]{a \pm \sqrt{(aa + bx^9 + cx^8)}}}{x^3}$$

and

$$y = \frac{\sqrt[3]{a \pm \sqrt{(aa + bx^9 + cx^8)}}}{x^2}$$

From which examples the use of this kind of substitution is made abundantly clear.

## CAPUT III

# DE TRANSFORMATIONE FUNCTIONUM PER SUBSTITUTIONEM

46. *Si fuerit y functio quaecunque ipsius z atque z definiatur per novam variabilem x, tum quoque y per x definiri poterit.*

Cum ergo antea $y$ fuisset functio ipsius $z$, nunc nova quantitas variabilis $x$ inducitur, per quam utraque priorum $y$ et $z$ definiatur. Sic, si fuerit

$$y = \frac{1-zz}{1+zz}$$

atque ponatur

$$z = \frac{1-x}{1+x},$$

hoc valore loco $z$ substituto erit

$$y = \frac{2x}{1+xx}.$$

Sumpto ergo pro $x$ valore quocunque determinato ex eo reperientur valores determinati pro $z$ et $y$ sicque invenitur valor ipsius $y$ respondens illi valori ipsius $z$, qui simul prodit. Uti, si sit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$ et $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, si in $\frac{1-zz}{1+zz}$ cui expressioni $y$ aequatur, ponatur $z = \frac{1}{3}$.

Adhibetur autem haec novae variabilis introductio ad duplicem finem: vel enim hoc modo irrationalitas, qua expressio ipsius $y$ per $z$ data laborat, tollitur; vel quando ob aequationem altioris gradus, qua relatio inter $y$ et $z$ exprimitur, non licet functionem explicitam ipsius $z$ ipsi $y$ aequalem exhibere, nova variabilis $x$ introducitur, ex qua utraque $y$ et z commode definiri queat; unde insignis substitutionum usus iam satis elucet, ex sequentibus vero multo clarius perspicietur.

47. *Si fuerit* $y = \sqrt{(a+bz)}$*, nova variabilis x, per quam utraque z et y rationaliter exprimatur, sequenti modo invenietur.*

Quoniam tam $z$ quam $y$ debet esse functio rationalis ipsius $x$, perspicuum est hoc obtineri, si ponatur

$$\sqrt{(a+bz)} = bx.$$

Fiet enim primo $y = bx$ et $a+bz = bbxx$

hincque

$$z = bxx - \frac{a}{b}.$$

Quocirca utraque quantitas $y$ et $z$ per functionem rationalem ipsius $x$ exprimitur;

scilicet cum sit $y=\sqrt{(a+bz)}$, fiat $z=bxx-\frac{a}{b}$; erit $y=bx$.

48. *Si fuerit* $y=(a+bz)^{m:n}$ *, nova variabilis x, per quam tam y quam z rationaliter exprimatur, sic reperietur.*

Ponatur $y=x^m$ fietque $(a+bz)^{1:n}=x$, ergo $a+bz=x^n$ et $z=\frac{x^n-a}{b}$.
Sic ergo utraque quantitas *y* et *z* rationaliter per *x* definietur, ope scilicet substitutionis $z=\frac{x^n-a}{b}$, quae praebet $y=x^m$.
Quamvis igitur neque *y* per *z* neque vicissim *z* per *y* rationaliter exprimi possit, tamen utraque reddita est functio rationalis novae quantitatis variabilis *x* per substitutionem introductae, scopo substitutionis omnino convenienter.

49. *Si fuerit* $y=\left(\frac{a+bz}{f+gz}\right)^{m:n}$ *, requiritur nova quantitas variabilis x, per quam utraque y et z rationaliter exprimatur.*

Manifestum primo est, si ponatur $y=x^m$ quaesito satisfieri; erit enim $\left(\frac{a+bz}{f+gz}\right)^{m:n}=x^m$ ideoque $\frac{a+bz}{f+gz}=x^n$; ex qua aequatione elicitur $z=\frac{a-fx^n}{gx^n-b}$, quae substitutio praebet $y=x^m$.
Hinc quoque intelligitur, si fuerit

$$\left(\frac{\alpha+\beta y}{\gamma+\delta y}\right)^n=\left(\frac{a+bz}{f+gz}\right)^m$$

tam *y* quam *z* rationaliter per *x* expressum iri, si utraque formula ponatur $=x^{mn}$; reperietur enim $y=\frac{\alpha-\gamma x^m}{\delta x^m-\beta}$ et $z=\frac{a-fx^n}{gx^n-b}$; qui casus nil habent difficultatis.

50. *Si fuerit*

$$y=\sqrt{(a+bz)(c+dz)}\,,$$

*substitutio idonea invenietur, qua y et z rationaliter exprimuntur, hoc modo.*

Ponatur $\sqrt{(a+bz)(c+dz)}=(a+bz)x$;
facile enim perspicitur hinc valorem rationalem pro *z* esse proditurum, quia valor ipsius *z* per aequationem simplicem determinatur. Erit ergo

$$c+dz=(a+bz)xx$$

hincque

$$z=\frac{c-axx}{bxx-d}.$$

Quare porro fiet

$$a+bz=\frac{bc-ad}{bxx-d}$$

et ob $y=\sqrt{(a+bz)(c+dz)}=(a+bz)x$ habebitur

$$y=\frac{(bc-ad)x}{bxx-d}.$$

Functio ergo irrationalis $y=\sqrt{(a+bz)(c+dz)}$ ad rationalitatem perducitur ope substitutionis

$$z=\frac{c-axx}{bxx-d},$$

quippe quae dabit

$$y=\frac{(bc-ad)x}{bxx-d}.$$

Sic, si fuerit

$$y=\sqrt{(aa-zz)}=\sqrt{(a+z)(a-z)}$$

ob $b=+1, c=a,\ d=-1$ ponatur

$$z=\frac{a-axx}{1+xx}$$

eritque

$$y=\frac{2ax}{1+xx}.$$

Quoties ergo quantitas post signum √ habuerit duos factores simplices reales, hoc modo reductio ad rationalitatem absolvetur; sin autem factores bini simplices fuerint imaginarii, sequenti modo uti praestabit.

51. *Sit*

$$y=\sqrt{(p+qz+rzz)}$$

*atque requiritur substitutio idonea pro z facienda, ut valor ipsius y fiat rationalis.*

Pluribus modis hoc fieri potest, prout *p* et *q* fuerint quantitates vel affirmativae vel negativae. Sit primo *p* quantitas affirmativa ac ponatur *aa* pro *p;* etiamsi enim *p* non sit quadratum, tamen irrationalitas quantitatum constantium praesens negotium non turbat. Sit igitur

I. $y = \sqrt{(p + qz + rzz)}$ ac ponatur

$$\sqrt{(aa + bz + czz)} = a + xz\,;$$

erit

$$b + cz = 2ax + xxz\,,$$

unde fit

$$z = \tfrac{b-2ax}{xx-c}\,;$$

tum vero erit

$$y = a + xz = \tfrac{bx-axx-ac}{xx-c}\,,$$

ubi $z$ et $y$ sunt functiones rationales ipsius $x$. Sit iam

II. $y = \sqrt{(aazz + bz + c)}$ ac ponatur

$$\sqrt{(aazz + bz + c)} = az + x:$$

erit

$$bz + c = 2azx + xx$$

et

$$z = \tfrac{xx-c}{b-2ax}.$$

Tum autem fit

$$y = az + x = \tfrac{-ac+bx-axx}{b-2ax}\,.$$

III. Si fuerint $p$ et $r$ quantitates negativae, tum, nisi sit $qq > 4pr$, valor ipsius $y$ semper erit imaginarius. Quodsi autem fuerit $qq > 4pr$, expressio $p + qz + rzz$ in duos factores resolvi poterit, qui casus ad paragraphum praecedentem reducitur. Saepenumero autem commodius ad hanc formam reducitur

$$y = \sqrt{aa + (b + cz)(d + ez)}\,;$$

pro qua ad rationalitatem perducenda ponatur

$$y = a + (b + cz)x$$

eritque

$$d + ez = 2ax + bxx + cxxz\,,$$

unde fit

$$z = \tfrac{d-2ax-bxx}{cxx-e}$$

et

$$y = \frac{-ae+(cd-be)x-acxx}{cxx-e}.$$

Interdum commodius fieri potest reductio ad hanc formam

$$y = \sqrt{aazz+(b+cz)(d+ez)}.$$

Tum ponatur

$$y = az+(b+cz)x;$$

erit

$$d+ez = 2axz+bxx+cxxz$$

et

$$z = \frac{bxx-d}{e-2ax-cxx}$$

atque

$$y = \frac{-ad+(be-cd)x-abxx}{e-2ax-cxx}.$$

EXEMPLUM

Si habeatur ista ipsius $z$ functio irrationalis

$$y = \sqrt{(-1+3z-zz)},$$

quae, cum reduci queat ad hanc formam

$$y = \sqrt{(1-2+3z-zz)} = \sqrt{1-(1-z)(2-z)},$$

ponatur

$$y = 1-(1-z)x,$$

erit

$$-2+z = -2x+xx-xxz$$

et

$$z = \frac{2-2x+xx}{1+xx}$$

Deinde est

$$1-z = \frac{-1+2x}{1+xx}$$

et

$$y = 1-(1-z)x = \frac{1+x-xx}{1+xx}.$$

Atque hi sunt fere casus, quos Algebra indeterminata seu methodus *Diophantea* suppeditat, neque alios casus in his tractatis non comprehensos per substitutionem rationalem ad rationalitatem reducere licet. Quocirca ad alterum substitutionis usum monstrandum progredior.

52. *Si y eiusmodi fuerit functio ipsius z, ut sit invenire*

$$ay^{\alpha} + bz^{\beta} + cy^{\gamma} z^{\delta} = 0,$$

*invenire novam variabilem x, per quam valores ipsarum y et z explicite assignari queant.*

Quoniam resolutio aequationum generalis non habetur, ex aequatione proposita $ay^{\alpha} + bz^{\beta} + cy^{\gamma} z^{\delta} = 0$ neque $y$ per $z$ neque vicissim $z$ per $y$ exhiberi potest. Quo igitur huic incommodo remedium afferatur, ponatur

$$y = x^m z^n$$

eritque

$$ax^{\alpha m} z^{\alpha n} + bz^{\beta} + cx^{\gamma m} z^{\gamma n + \delta} = 0.$$

Determinetur nunc exponens $n$ ita, ut ex hac aequatione valor ipsius $z$ definiri queat, quod tribus modis praestari potest.
I. Sit
$an = \beta$ ideoque $n = \frac{\beta}{a}$ ;
erit aequatione per $z^{\alpha n} = z^{\beta}$ divisa

$$ax^{\alpha m} + b + cx^{\gamma m} z^{\gamma n - \beta + \delta} = 0,$$

unde oritur

$$z = \left( \frac{-ax^{\alpha m} - b}{cx^{\gamma m}} \right)^{\frac{1}{\gamma n - \beta + \delta}} \text{ sive } z = \left( \frac{-ax^{\alpha m} - b}{cx^{\gamma m}} \right)^{\frac{\alpha}{\beta\gamma - \alpha\beta + \alpha\delta}}$$

et

$$y = x^m \left( \frac{-ax^{\alpha m} - b}{cx^{\gamma m}} \right)^{\frac{\beta}{\beta n - \alpha\beta + \alpha\delta}}$$

II. Sit

$$\beta = \gamma n + \delta \text{ seu } n = \frac{\beta - \delta}{\gamma};$$

erit aequatione per $z^{\beta}$ divisa

$$ax^{\alpha m} z^{\alpha n - \beta} + b + cx^{\gamma m} = 0,$$

unde oritur

$$z = \left( \frac{-b - cx^{\gamma m}}{ax^{\alpha m}} \right)^{\frac{1}{\alpha n - \beta}} = \left( \frac{-b - cx^{\gamma m}}{ax^{\alpha m}} \right)^{\frac{\gamma}{\alpha\beta - \alpha\delta - \beta\gamma}}$$

atque

$$y = x^m \left(\frac{-b-cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\beta-\delta}{\alpha\beta-\alpha\delta-\beta\gamma}}$$

III. Sit

$$an = \gamma n + \delta \text{ seu } n = \frac{\delta}{\alpha-\gamma};$$

erit aequatione per $z^{\alpha n}$ divisa

$$ax^{\alpha m} + bz^{\beta-\alpha n} + cx^{\gamma m} = 0,$$

unde oritur

$$z = \left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{1}{\beta-\alpha n}} = \left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\alpha-\gamma}{\alpha\beta-\beta\gamma-\alpha\delta}}$$

atque

$$y = x^m \left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\delta}{\alpha\beta-\beta\gamma-\alpha\delta}}$$

Tribus igitur diversis modis erutae sunt functiones ipsius *x*, quae ipsis *z* et *y* sunt aequales. Praeterea vero pro *m* numerum pro lubitu substituere licet cyphra excepta sicque formulae ad commodissimam expressionem reduci poterunt.

EXEMPLUM

Exprimatur natura functionis *y* per hanc aequationem

$$y^3 + z^3 - cyz = 0$$

atque quaerantur functiones ipsius *x* ipsis *y* et *z* aequales.
Erit ergo

$$a = -1, \quad b = -1, \quad \alpha = 3, \quad \beta = 3, \quad \gamma = 1 \text{ et } \delta = 1.$$

Hinc primus modus dabit posito $m = 1$

$$z = \left(\frac{x^3+1}{cx}\right)^{-1} \text{ et } y = x\left(\frac{x^3+1}{cx}\right)^{-1}$$

sive

$$z = \frac{cx}{x^3+1} \text{ et } y = \frac{cxx}{1+x^3},$$

quarum expressionum utraque adeo est rationalis.

Secundus modus vero dabit hos valores

$$z=\left(\tfrac{cx-1}{x^3}\right)^{1:3} \text{ et } y=x\left(\tfrac{cx-1}{x^3}\right)^{2:3}$$

sive

$$z=\tfrac{1}{x}\sqrt[3]{(cx-1)} \text{ et } y=\tfrac{1}{x}\sqrt[3]{(cx-1)^2}\,.$$

Tertius modus ita rem expediet, ut sit

$$z=\left(cx-x^3\right)^{2:3} \text{ et } y=x\left(cx-x^3\right)^{1:3}.$$

53. *Hinc a posteriori intelligitur, cuiusmodi aequationes, quibus valor functionis y per z determinatur, hoc modo novam variabilem x introducendo resolvi queant.*

Ponamus enim resolutione iam instituta prodiisse has determinationes

$$z=\left(\tfrac{ax^\alpha+bx^\beta+cx^\gamma+\text{etc.}}{A+Bx^\mu+Cx^\nu+\text{etc.}}\right)^{p:r}$$

atque

$$y=x\left(\tfrac{ax^\alpha+bx^\beta+cx^\gamma+\text{etc.}}{A+Bx^\mu+Cx^\nu+\text{etc.}}\right)^{q:r}$$

eritque

$$y^p=x^pz^q$$

et hinc

$$x=yz^{-q:p}$$

Cum igitur sit

$$z^{r:p}=\tfrac{ax^\alpha+bx^\beta+cx^\gamma+\text{etc.}}{A+Bx^\mu+Cx^\nu+\text{etc.}},$$

si loco $x$ eius valorem $yz^{-q:p}$ substituamus, prodibit ista aequatio

$$z^{r:p}=\frac{ay^\alpha z^{-\alpha q:p}+by^\beta z^{-\beta q:p}+cy^\gamma z^{-\gamma q:p}+\text{etc.}}{A+By^\mu z^{-\mu q:p}+Cy^\nu z^{-\nu q:p}+\text{etc.}},$$

quae reducitur ad hanc

$$Az^{r:p}+By^\mu z^{(r-\mu q):p}+Cy^\nu z^{(r-\nu q):p}+\text{etc}=ay^\alpha z^{-\alpha q:p}+bx^\beta z^{-\beta q:p}+cx^\gamma z^{-\gamma q:p}+\text{etc.}$$

quae multiplicata per $z^{\alpha q:p}$ transibit in hanc

$$Az^{(\alpha q+v):p} + By^{\mu}z^{(\alpha q-\mu q+v):p} + Cy^{v}z^{(\alpha q-vq+v):p} + \text{etc.}$$
$$= ay^{\alpha} + by^{\beta}z^{(\alpha q-\beta q):p} + cy^{\gamma}z^{(\alpha q-\gamma q):p} + \text{etc.}$$

Ponatur

$$\frac{aq+r}{p} = m \text{ et } \frac{aq-\beta q}{p} = n;$$

fiet

$$p = \alpha - \beta,\ q = n \text{ et } r = am - \beta m - \alpha n$$

atque nascetur ista aequatio

$$Az^{m} + By^{\mu}z^{m\text{-}\mu n:(\alpha-\beta)} + Cy^{v}z^{m\text{-}vn:(\alpha-\beta)} + \text{etc.}$$
$$= ay^{\alpha} + by^{\beta}z^{n} + cy^{\gamma}z^{(\alpha-\gamma)n:(\alpha-\beta)} + \text{etc.},$$

quae propterea ita resolvetur, ut sit

$$z = \left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{v}+\text{etc.}}\right)^{\frac{\alpha-\beta}{\alpha m-\beta m-\alpha n}}$$

et

$$y = x\left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{v}+\text{etc.}}\right)^{\frac{n}{\alpha m-\beta m-\alpha n}}$$

Vel ponatur

$$\frac{\alpha q+r}{p} = m \quad \text{et} \quad \frac{\alpha q-\mu q+r}{p} = n;$$

erit

$$m-n = \frac{\mu q}{p} \quad \text{et} \quad \frac{q}{p} = \frac{m-n}{\mu} \text{ atque } \frac{r}{p} = m - \frac{\alpha m-\alpha n}{\mu}.$$

Hinc fit

$$p = \mu, \quad q = m-n \quad \text{et} \quad r = \mu m - \alpha m + \alpha n$$

atque haec aequatio resultabit

$$Az^{m} + By^{\mu}z^{n} + Cy^{v}z^{m\text{-}v(m-n):\mu} + \text{etc.}$$
$$= ay^{\alpha} + by^{\beta}z^{(\alpha-\beta)(m-n):\mu} + cy^{\gamma}z^{(\alpha-\gamma)(m-n):\mu} + \text{etc.},$$

quae ita resolvetur, ut sit

$$z = \left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{v}+\text{etc.}}\right)^{\frac{\mu}{\mu m-\alpha m+\alpha n}}$$

et

$$y = x\left(\frac{ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}}{A+Bx^{\mu}+Cx^{\nu}+\text{etc.}}\right)^{\frac{m-n}{\mu m-\alpha m+\alpha n}}$$

54. *Si y ita pendeat a z, ut sit*

$$ayy + byz + czz + dy + ez = 0\,,$$

*sequenti modo tam y quam z rationaliter* per *novam variabilem x exprimetur.*

Ponatur $y = xz$ *;* erit divisione per *z* facta

$$axxz + bxz + cz + dx + e = 0\,,$$

ex qua reperitur

$$z = \frac{-dx-e}{axx+bx+c}$$

et

$$y = \frac{-dxx-ex}{axx+bx+c}$$

At vero ad formam propositam reduci potest haec aequatio inter *y* et *z*

$$ayy + byz + czz + dy + ez + f = 0$$

diminuendo vel augendo utramque variabilem certa quadam quantitate constante, unde et haec aequatio per novam variabilem *x* rationaliter explicari potest.

55. *Si y ita pendeat a z, ut sit*

$$ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = 0\,,$$

*sequenti modo tam y quam z rationaliter per novam variabilem x exprimi poterit.*

Ponatur $y = xz$ et facta substitutione tota aequatio per *zz* dividi poterit; prodibit autem

$$ax^3z + bxxz + cxz + dz + exx + fx + g = 0\,.$$

Unde oritur

$$z = \frac{-cxx-fx-g}{ax^3+bxx+cx+d}\,,$$

ex quo erit

$$y = \frac{-ex^3-fxx-gx}{ax^3+bxx+cx+d}$$

Ex his casibus facile intelligitur, quemadmodum aequationes altiorum graduum, quibus *y* per *z* definitur, comparatae esse debeant, ut huiusmodi resolutio locum habere queat. Ceterum hi casus in superioribus formulis § 53 continentur, at, quia formulae generales non tam facile ad huiusmodi casus saepius occurrentes accommodantur, visum est horum aliquos seorsim evolvere.

56. *Si y ita pendeat a z, ut sit*

$$ayy + byz + czz = d\ ,$$

*hoc modo utraque quantitas y et z per novam variabilem x exprimetur.*

Ponatur $y = xz$ eritque

$$(axx + bx + c)zz = d$$

ideoque

$$z = \sqrt{\tfrac{d}{axx+bx+c}}$$

et

$$y = x\sqrt{\tfrac{d}{axx+bx+c}}.$$

Simili modo si fuerit

$$ay^3 + by^2z + cz^2 + dz^3 = ey + fz,$$

posito $y = xz$ tota aequatio per $z$ divisa dabit

$$\left(ax^3 + bxx + cz^2 + d\right)zz = ex + f\ ,$$

unde oritur

$$z = \sqrt{\tfrac{ex+f}{ax^3+bxx+cx+d}}$$

et

$$y = x\sqrt{\tfrac{ex+f}{ax^3+bxx+cx+d}}\ .$$

Hi autem casus aliique similes resolutiones admittentes comprehenduntur in sequente paragrapho.

57. *Si y ita pendeat a z, ut sit*

$$ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \text{etc.} = \alpha y^n + \beta y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \text{etc.},$$

*sequenti modo tam z quam y commode per novam variabilem x exprimetur.*

Sit $y = xz$ atque facta substitutione tota aequatio dividi poterit per $z^n$, siquidem exponens *m* sit maior quam *n*, eritque

$$\left(ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \text{etc.}\right)z^{m-n} = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.},$$

unde obtinebitur

$$z=\left(\frac{\alpha x^{n}+\beta x^{n-1}+\gamma x^{n-2}+\delta x^{n-3}+\text{etc.}}{ay^{m}+by^{m-1}z+cy^{m-2}z^{2}+dy^{m-3}z^{3}+\text{etc.}}\right)^{1:m-n}$$

et

$$y=x\left(\frac{\alpha x^{n}+\beta x^{n-1}+\gamma x^{n-2}+\delta x^{n-3}+\text{etc.}}{ax^{m}+bx^{m-1}+cx^{m-2}+dx^{m-3}+\text{etc.}}\right)^{1:m-n}$$

Haec scilicet resolutio locum habet, si in aequatione naturam functionis $y$ per $z$ exprimente duplex tantum ubique occurrit dimensionum ab $y$ et $z$ sumptarum numerus, uti in casu tractato in singulis terminis numerus dimensionum vel est $m$ vel $n$.

58. *Si in aequatione inter y et z triplicis generis dimensiones occurrant, quarum summa tantum superet mediam, quantum haec media infimam, ope resolutionis aequationis quadratae variabiles y et z per novam x exprimi poterunt.*

Si enim ponatur $y=xz$, divisione per minimam ipsius $z$ potestatem facta valor ipsius $z$ per $x$ ope extractionis radicis quadratae exhibebitur, id quod ex sequentibus exemplis erit manifestum.

EXEMPLUM 1

Sit

$$ay^{3}+byyz+cyzz+dz^{3}=2eyy+2fyz+2gzz+hy+iz$$

ac ponatur $y=xz$; erit divisione per $z$ facta

$$\left(ax^{3}+bxx+ex+d\right)zz=2\left(exx+fx+g\right)z+hx+i\ ,$$

ex qua sequens ipsius $z$ obtinebitur valor

$$z=\frac{exx+fx+g\pm\sqrt{\left(\left(exx+fx+g\right)^{2}+\left(ax^{3}+bxx+cx+d\right)\left(hx+i\right)\right)}}{ax^{3}+bxx+cx+d},$$

quo invento erit $y=xz$.

EXEMPLUM 2

Sit

$$y^5 = 2az^3 + by + cz$$

ac posito $y = xz$ erit

$$x^5z^4 = 2azz + bx + c$$

ex qua reperitur

$$zz = \frac{a \pm \sqrt{\left(aa + bx^6 + cx^5\right)}}{x^5}$$

et

$$z = \frac{\sqrt{\left(a \pm \sqrt{\left(aa + bx^6 + cx^5\right)}\right)}}{xx\sqrt{x}}$$

et

$$y = \frac{\sqrt{\left(a \pm \sqrt{\left(aa + bx^6 + cx^5\right)}\right)}}{x\sqrt{x}}.$$

EXEMPLUM 3

Sit

$$y^{10} = 2az^6 + byz^3 + cz^4;$$

in qua cum dimensiones sint 10, 7 et 4, ponatur $y = xz$ atque aequatio per $z^4$ divisa abibit in hanc

$$z^{10}z^6 = 2axz^3 + bx + c$$

seu

$$z^6 = \frac{2axz^3 + bx + c}{z^{10}}$$

unde invenitur

$$z^3 = \frac{ax \pm x\sqrt{\left(aa + bx^9 + cx^8\right)}}{x^{10}}$$

ideoque erit

$$z = \frac{\sqrt[3]{a \pm \sqrt{\left(aa + bx^9 + cx^8\right)}}}{x^3}$$

atque

$$y = \frac{\sqrt[3]{a \pm \sqrt{\left(aa + bx^9 + cx^8\right)}}}{x^2}$$

Ex quibus exemplis usus huiusmodi substitutionum abunde perspicitur.