CHAPTER II

THE TRANSFORMATION OF FUNCTIONS.

27**.** *Functions may be changed into other forms either by introducing another variable in place of the variable quantity, or by retaining the same variable quantity.*

But if the same quantity is kept, the function cannot change properly. But every transformation consists of the same function being expressed in another way, just as from algebra it is agreed that the same quantity is able to be expressed in several diverse forms. The transformations are of this kind, if in place of this function

$$2-3z+zz \text{ there is put } (1-z)(2-z),$$

or

$$(a+z)^3 \text{ in place of } a^3+3aaz+3azz+z^3,$$

or

$$\frac{a}{a-z}+\frac{a}{a+z} \text{ in place of } \frac{2aa}{aa-zz},$$

or

$$\sqrt{(1+zz)}+z \text{ in place of } \frac{1}{\sqrt{(1+zz)}-z};$$

which expressions, even if they differ in forms, yet actually are in agreement. But on many occasions one of these forms signifying the same is more suitable in effecting the proposition than the others, and on this account the most convenient that ought to be chosen.

The other kind of transformation, where another variable quantity *y* is introduced in place of the variable quantity *z*, which indeed may hold a relation to *z*, is said to be made by substitution ; and that thus it is agreed to be used, so that the proposed function may be expressed more succinctly and conveniently ; as, if this were the proposed function of *z*

$$a^4-4a^3z+6aazz-4az^3+z^4,$$

if *y* may be put in place of $a-z$, a much simpler function of *y* will be produced of *y*, namely the function y^4, and if this irrational function $\sqrt{(aa+zz)}$ of *z* may be had, if there is put $z=\frac{aa-yy}{2y}$, that function expressed by y becomes the rational $z=\frac{aa+yy}{2y}$.
But I will defer that method of transformation to the following chapter, and in this chapter we are going to explain that which proceeds without substitution.
[In this chapter, Euler introduces algebraic quantities analogous to numbers; thus, he talks about whole functions, fractional functions, etc.]

28. *On many occasions a whole function of z is resolved conveniently into its factors and thus may be transformed into a product.*

With this agreed upon, when a whole function is resolved into its factors, the nature of this is seen much easier ; for the case is noted at once, for which values of the function the value becomes $=0$. Thus this function of z

$$6-7z+z^3$$

is transformed into this product

$$(1-z)(2-z)(3+z),$$

from which at once it may be clear in which three cases the proposed function becomes $=0$, to wit if $z=1$ and $z=2$ and $z=-3$, which properties are not so easily understood from the form $6-7z+z^3$. Factors of this kind, in which only simple of the variable z occur, are called *simple* factors, so that they may be distinguished from *composite* factors, in the square or cube of z is involved, or some higher power. Therefore simple factors will be of the kind :

$$f+gz$$

the form of simple factors;

$$f+gz+hzz$$

the form of two factors;

$$f+gz+hzz+iz^3$$

the form of three factors, and thus henceforth. Moreover it is evident that the twofold [hence to be called here quadratic] function involves two simple factors, the threefold [or cubic] three simple factors, and thus henceforth. Hence a whole function of z, in which the exponent of the sum of the powers of z is $=n$, will contain n simple factors ; from which likewise, if which factors were either quadratic or cubic, etc., the number of factors would be known.

29.*Simple factors of any integral function Z of z are found, if the function Z is put equal to zero and from that equation all the roots of z itself are found ; for the single roots of z will give just as many simple factors of the function Z.*

For indeed if from the equation $Z=0$ there was a certain root $z=f$, $z-f$ will be a divisor and hence a factor of the function *Z;* therefore thus with all the roots of the equation $Z=0$ investigated, which shall be

$$z=f,\ z=g,\ z=h \text{ etc.},$$

the function Z may be resolved into its simple components and it will be transformed into the product

$$Z = (z - f)(z - g)(z - h) \text{ etc.};$$

where indeed it is to be noted, if the coefficient greatest power of z in Z were not $= +1$, then the product $(z - f)(z - g)$ etc. must in addition be multiplied by that coefficient. Thus, if it were

$$Z = Az^{n} + Bz^{n-1} + Cz^{n-2} + \text{ etc.},$$

it becomes

$$Z = A(z - f)(z - g)(z - h) \text{ etc.}$$

But if it were

$$Z = A + Bz + Cz^{2} + Dz^{3} + Ez^{4} + \text{etc.}$$

and the roots z of the equation $Z = 0$ found shall be f, g, h, i etc., it will be

$$Z = A\left(1 - \tfrac{z}{f}\right)\left(1 - \tfrac{z}{g}\right)\left(1 - \tfrac{z}{h}\right) \text{ etc.}$$

From these in turn it is understood, if a factor of the function Z were $z - f$ or $1 - \frac{z}{f}$, then the value of the function changes into nothing, if f is put in place of z. For indeed from the fact $z = f$, one factor $z - f$ or $1 - \frac{z}{f}$ of the function Z and thus must vanish from that function Z itself.

30. *Simple factors either will be real or imaginary and, if the function Z were to have imaginary values, the number of these will always be even.*

For since simple factors arise from the roots of the equation $Z = 0$, real roots present real factors and imaginary roots imaginary factors ; moreover in every equation the number of imaginary roots is even always, on which account the function Z either has no imaginary factors or two, four or six, etc. But if the function Z may have only two imaginary factors, the product of these will be real and will present a real quadratic factor. For P shall be equal to the product from all the real factors, and the product of two imaginary factors will be $= \frac{Z}{P}$ and hence will be real. In a similar manner if the function Z may have four, six, eight, etc. imaginary factors, the product of these will be real always, truly equal to the quotient, which arises, if the function Z is divided by the product of all the real factors.

31. *If Q were the real product from four simple imaginary factors, then likewise this product Q is able to be resolved into two real quadratic factors.*

For Q will have a form of this kind :

$$z^4 + Az^3 + Bz^2 + Cz + D;$$

which if it may be denied to be able to be resolved into two real factors, which will be resolvable by requiring two quadratic imaginary factors to be put in place, which will have a form of this kind:

$$zz - 2\left(p + q\sqrt{-1}\right)z + r + s\sqrt{-1}$$

and

$$zz - 2\left(p - q\sqrt{-1}\right)z + r - s\sqrt{-1};$$

for other imaginary forms cannot be conceived, the product of which becomes real, certainly $= z^4 + Az^3 + Bz^2 + Cz + D$. Moreover from these imaginary twofold factors the following four simple imaginary factors of Q emerge :

$$\text{I. } z - \left(p + q\sqrt{-1}\right) + \sqrt{\left(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1}\right)},$$

$$\text{II. } z - \left(p + q\sqrt{-1}\right) - \sqrt{\left(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1}\right)},$$

$$\text{III. } z - \left(p - q\sqrt{-1}\right) + \sqrt{\left(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1}\right)},$$

$$\text{IV. } z - \left(p - q\sqrt{-1}\right) - \sqrt{\left(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1}\right)}.$$

The first and the third of these may be multiplied in turn, on putting for the sake of brevity :

$$t = pp - qq - r \text{ and } u = 2pq - s$$

and of which the product of the factors produced will be

$$= zz - \left(2p - \sqrt{2t + 2\sqrt{(tt + uu)}}\right)z$$

$$+ pp + qq - p\sqrt{2t + 2\sqrt{(tt + uu)}} - q\sqrt{-2t + 2\sqrt{(tt + uu)}} + \sqrt{(tt + uu)},$$

which certainly is real. Moreover in a similar manner the product from the second and the fourth factors will be real, certainly

$$= zz - \left(2p + \sqrt{2t + 2\sqrt{(tt+uu)}}\right)z$$
$$+ pp + qq + p\sqrt{2t + 2\sqrt{(tt+uu)}} + q\sqrt{-2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)}.$$

On account of which the proposed product *Q*, which it was said could not be resolved into two quadratic real factors, nevertheless actually can be resolved into two real quadratic factors.

32. *If a whole function Z of z should have some simple imaginary factors, thus pairs are always able to be joined together, and the product of these becomes real.*

Because the number of imaginary roots always is even, that shall be $= 2n$, and indeed in the first place it is apparent that the product of all these imaginary roots is real. But if therefore only two imaginary roots are to be considered, the product of these certainly will be real ; but if four imaginary factors were had, then, as we have seen, the product of these can be resolved into two twofold [*i.e.* quadratic] real factors of the form $fzz + gz + h$. But though it is not allowed to extend the same to higher powers, to be demonstrated in the same manner, still it may be considered beyond doubt that the same property is in place in whatever number of factors taken together, thus so that always in place of the *2n* simple imaginary factors, they may to be lead into *n* real quadratic factors. Hence every whole function of *z* will be able to be resolved into real factors, either simple or quadratic. Because whatever shall be demonstrated without the maximum rigor, yet the truth of this in the following will be corroborated more, when functions of this kind

$$a + bz^n, \ \ a + bz^n + cz^{2n}, \ \ a + bz^n + cz^{2n} + dz^{3n} \text{ etc.}$$

are resolved actually into real quadratic factors of this kind.

33. *If a whole function Z adopts the value A on putting $z = a$ and on putting $z = b$ it may adopt the value B, then in place of z by putting intermediate values between a and b the function Z is able to take any intermediate values between A and B.*

Indeed since *Z* shall be a uniform function of *z*, whichever value may be attributed to *z* , the function *Z* also hence will take some real value. Therefore since *Z* in the first case $z = a$ obtains the value *A*, but the value *B* in the latter case $z = b$, it cannot cross over from *A* to *B* unless by passing through all the intermediate values. So that therefore if the equation $Z - A = 0$ shall have a real root and likewise $Z - B = 0$ supports a real root, then the equation $Z - C = 0$ also will have a real root, if indeed *C* may be contained between the values *A* and *B*. Hence if the expressions $Z - A$ and $Z - B$ may have a simple real factor, then some expression $Z - C$ will have a simple real factor, provided *C* may be contained between the values *A* and *B*.

34. *If the exponent of the greatest power of z in the whole function Z were the odd number* $2n+1$, *then that function Z will have at least a single real factor.*

It is evident that *Z* will have a form of this kind

$$z^{2n+1} + \alpha z^{2n} + \beta z^{2n-1} + \gamma z^{2n-2} + \text{etc.};$$

in which if there may be put $z = \infty$, because the values of the individual terms vanish in view of the first term, it becomes $Z = (\infty)^{2n+1} = \infty$ and thus $Z - \infty$ will have a single real factor, surely $z - \infty$. Moreover, if there is put $z = -\infty$, it becomes

$$Z = (-\infty)^{2n+1} = -\infty$$

and thus $Z + \infty$ will have a simple real factor $z + \infty$. Therefore since both $Z - \infty$ as well as $Z + \infty$ may have a simple real factor, it follows also that $Z - C$ is going to be a simple real factor, if indeed *C* shall be contained between the limits $+\infty$ and $-\infty$, that is, if *C* were some real number, either positive or negative. Hence on that account making $C = 0$ the function *Z* itself too will have a simple real factor $z - c$ and the magnitude *c* will be contained between the limits $+\infty$ and $-\infty$ and it will be either a positive or negative magnitude, or nothing.

35. *Therefore a whole function Z, in which the exponent of the greatest power of z is an odd number, either has a single real factor, or three, or five, or seven, etc.*

For since it may be shown that the function *Z* certainly has a single real factor $z - c$, we may put *Z* besides that to have a single [real] factor $z - d$ and the function *Z* may be divided by $(z - c)(z - d)$, in which the maximum power of *z* shall be z^{2n+1}; the maximum power of the quotient will be $= z^{2n-1}$, the exponent of which, since it shall be an odd number, indicates anew that a simple real factor of *Z* be given. Therefore if *Z* may have more than one simple real factor, it will have either three, or (because it is permitted to progress in the same manner) five, seven, etc. Clearly the number of simple real factors will be odd, and because the number of all the simple factors is $= 2n + 1$, the number of imaginary factors will be even.

36. *A whole function Z, in which the exponent of the maximum power of z is the even number* $2n$ *, will have either two simple real factors, or four, six, etc.*

We may put $2m+1$ to agree with the number of odd real factors Z ; if therefore the function Z is divided by the product of all these, the maximum power of the quotient will be $= z^{2n-2m-1}$ thus the exponent of which shall be an odd number; therefore the function Z besides certainly will have one simple real factor, from which the number of all the real factors at least will be $= 2m+2$ and thus even and the number of imaginary factors equally even. Therefore the simple imaginary factors of all whole functions are even in number, just as indeed we have stated now before.

37. *If in a whole function Z the exponent of the maximum power of z were an even number and the absolute or constant term were endowed with a – sign, then the function Z has at least two simple real factors.*

Therefore the function *Z,* for which this discussion is about, will have a form of this kind

$$z^{2n} \pm \alpha z^{2n-1} \pm \beta z^{2n-2} \pm \cdots \pm \nu z - A.$$

If now there may be put $z = \infty$, it becomes as we saw above, $Z = \infty$ and if there is put $z = 0$, it becomes $Z = -A$. Therefore $Z - \infty$ will have the real factor $z - \infty$ and $Z + A$ the factor $z - 0$; from which, since 0 is contained between the limits $-\infty$ and $+A$, it follows that $Z + 0$ has a simple real factor $z - c$, thus so that *c* may be contained within the limits 0 and ∞. Then, since on putting $z = -\infty$ there becomes $Z = \infty$ and thus $Z - \infty$ may have the factor $z + \infty$ and $Z + A$ the factor $z + 0$, it follows too that $Z + 0$ has the simple real factor $z + d$, thus so that *d* may be contained between the limits 0 and ∞; from which the proposition is agreed upon. Therefore from these it is seen, if *Z* were such a function, of the kind described here, the equation $Z = 0$ must have at least two real roots, the one positive and the other negative. Thus this equation

$$z^4 + \alpha z^3 + \beta z^2 + \gamma z - aa = 0$$

has two real roots, the one positive and the other negative.

38. *If in the fractional function the variable quantity z may have just as many or more dimensions in the numerator than in the denominator, then that function can be resolved into two parts, of which the one is a whole function, and the other a fraction, in the numerator of which the variable quantity z may have fewer dimensions than in the denominator.*

For if the exponent of the maximum power of *z* shall be less in the denominator than in the numerator, then the numerator is divided by the denominator in the customary manner, until in the quotient negative exponents of z itself are come upon; therefore in this place with the operation of division interrupted, and a quotient agreed upon from the whole part and with a fraction, in the numerator of which it will be of dimension of z itself will be less than in the denominator ; but here the quotient is equal to the proposed function. Thus, if this were the proposed fractional function :

$$\frac{1+z^4}{1+zz},$$

that may be resolved by division, thus :

$$\begin{array}{lll} zz+1) & z^4+1 & \left(zz-1+\frac{2}{1+zz}\right. \\ & \underline{z^4+zz} & \\ & \quad -zz+1 & \\ & \quad \underline{-zz-1} & \\ & \qquad +2 & \end{array}$$

and it will become

$$\frac{1+z^4}{zz+1} = zz-1+\frac{2}{1+zz}.$$

Fractional functions of this kind, in which the magnitude of the variable function z has as many or more dimensions in the numerator than in the denominator, by the similarity of arithmetic, can be called *spurious* fractions, or spurious fractional functions, from which they may be distinguished from *natural* fractional functions, in the numerator of which the variable quantity z has fewer dimensions than in the denominator. And thus a spurious fractional function will be able to be resolved into an whole function and a natural fractional function, and this resolution may be absolved by the operation of common division.

39. *If the denominator of a fractional function should have two factors relatively prime to each other, then that fractional function may be resolved into two fractions, the denominators of which shall be equal to these two factors respectively.*

Though this resolution applies equally to spurious and natural fractions, yet we will apply that chiefly to natural fractions. But with the denominator resolved into two relatively prime fractional functions of this kind, that function itself may be resolved into two other natural fractions, the denominators of which shall be equal respectively to these two factors, and this resolution, if the fractions are indeed natural, can be done in a single way ; the truth of this matter may be understood clearly from an example than by reasoning. Therefore this fractional function shall be proposed :

$$\frac{1-2z+3zz-4z^3}{1+4z^4};$$

the denominator $1+4z^4$ of this, since it shall be equal to this product

$$(1+2z+2zz)(1-2z+2zz),$$

may be resolved into the two fractions, the one denominator of which will be $1+2z+2zz$, the other $1-2z+2zz$; towards finding which, because they are natural fractions, these numerators are put in place, the former $=\alpha+\beta z$, the latter $=\gamma+\delta z$, and by hypothesis there will be

$$\frac{1-2z+3zz-4z^3}{1+4z^4}=\frac{\alpha+\beta z}{1+2z+2zz}+\frac{\gamma+\delta z}{1-2z+2zz} \;;$$

these two fractions can actually be added, and the numerator and denominator of the sum will be :

numerator	denominator
$+\alpha-2\alpha z+2\alpha zz$	
$+\beta z-\beta zz+2\beta z^3$	$1+4z^4$.
$+\gamma+2\gamma z+2\gamma zz$	
$+\delta z+2\delta zz+2\delta z^3$,	

Therefore since the denominator shall be equal to the denominator of the proposed fraction, the numerators also must be returned equal ; which on account of so many unknown letters $\alpha,\beta,\gamma,\delta,$ just as many equal terms are to be effected, and certainly that will be able to happen in a single way ; we obtain these four equations as a matter of course

$$\text{I.}\quad \alpha+\gamma=1,\qquad \text{III.}\ 2\alpha-2\beta+2\gamma+2\delta=3,$$
$$\text{II.}\ -2\alpha+\beta+2\gamma+\delta=-2,\qquad \text{IV.}\ 2\beta+2\delta=-4.$$

Hence on account of

$$\alpha+\gamma=1 \quad\text{and}\quad \beta+\delta=-2$$

the equations II. and III. will give

$$\alpha-\gamma=0 \quad\text{and}\quad \delta-\beta=\tfrac{1}{2},$$

from which is made

$$\alpha=\tfrac{1}{2},\ \gamma=\tfrac{1}{2},\ \beta=-\tfrac{5}{4},\ \delta=-\tfrac{3}{4},$$

and thus the proposed fraction

$$\frac{1-2z+3zz-4z^3}{1+4z^4}$$

is transformed into these two

$$\frac{\frac{1}{2}-\frac{5}{4}z}{1+2z+2zz}+\frac{\frac{1}{2}-\frac{3}{4}z}{1-2z+2zz}.$$

Moreover in a similar manner it may be seen that the resolution must always succeed, because always just as many unknown letters are introduced, as there is a need for eliciting the intended numerator. Truly from the common teaching of fractions it is understood that this resolution cannot succeed, unless these factors of the denominator were relatively prime.

40. *Therefore if a fractional function $\frac{M}{N}$ will be able to be resolved into so many simple fractions of the form $\frac{A}{p-qz}$, the denominator N will have just as many simple factors unequal to each other.*

Here the fractional function $\frac{M}{N}$ represents some natural fraction, thus so that *M* and *N* shall be whole functions of *z* and the greatest power of *z* in *M* shall be less than in *N*. So that therefore if the denominator *N* may be resolved into simple factors and these are unequal between themselves, the expression $\frac{M}{N}$ may be resolved into just as many, as there are simple factors contained in the denominator *N*, because each factor therefore will be changed into the denominator of a partial fraction. Therefore if $p-qz$ were a factor of *N* itself, this will be the denominator of a certain partial fraction and, since in the numerator of this fraction the number of the dimensions of *z* must be less than in the denominator $p-qz$, by necessity the numerator will be a constant quantity. Hence from each and very simple factor $p-qz$ of the denominator *N* a simple fraction $\frac{A}{p-qz}$ may arise, thus so that the sum of all these fractions $p-qz$ shall be equal to the fraction proposed $\frac{M}{N}$.

EXAMPLE

For the sake of an example this fractional function shall be $\frac{1+zz}{z-z^3}$.
Because the simple factors of the denominator are z, $1-z$ and $1+z$, this function may be resolved into these three simple fractions $\frac{A}{z}+\frac{B}{1-z}+\frac{C}{1+z}=\frac{1+zz}{z-z^3}$, where it is necessary to define the constant numerators *A*, *B* and *C*. These fractions may be reduced to the common denominator, which will be $z-z^3$, and the sum of the numerators will have to be equal to $1+zz$ itself, from which this equation arises

$$\begin{aligned} A+Bz-Azz&=1+zz=1+0z+zz, \\ +Cz+Bzz& \\ -Czz& \end{aligned}$$

which presents just as many comparisons, as there are unknown letters *A, B, C* ; evidently there will be

$$\text{I.}\quad A=1,$$
$$\text{II.}\ B+C=0,$$
$$\text{III.}-A+B-C=1.$$

Hence $B-C=2$, and again $A=1,\ B=1$ and $C=-1$.
Therefore the proposed function $\frac{1+zz}{z-z^3}$ is resolved into this form $\frac{1}{z}+\frac{1}{1-z}-\frac{1}{1+z}$.

Moreover it is understood in a similar manner, however many simple factors *N* may have unequal to each other, the fraction $\frac{M}{N}$ can always be resolved into just as many simple fractions. But if some factors were equal to each other, then the resolution must be put in place in another way to be explained later.

41. *Therefore since any simple factor of the denominator N supports a simple fraction for the resolution of the proposed function* $\frac{M}{N}$, *it is required to show, how from a simple factor of the denominator N with a known fraction, the corresponding simple fraction can be found.*

Let $p-qz$ be a simple factor of *N* itself, thus so that there becomes

$$N=(p-qz)S$$

and *S* a whole function of *z;* the fraction arising from the factor $p-qz$ is put

$$=\frac{A}{p-qz}$$

and let the fraction arising from the other factor of the denominator *S* be

$$=\frac{P}{S},$$

thus so that following § 39 the fraction becomes

$$\frac{M}{N}=\frac{A}{p-qz}+\frac{P}{S}=\frac{M}{(p-qz)S};$$

hence there will be

$$\frac{P}{S}=\frac{M-AS}{(p-qz)S};$$

since which fractions must agree, it is necessary that $M-AS$ shall be divisible by $p-qz$, because the whole function *P* shall be equal to the quotient itself. Truly when $p-qz$ becomes a divisor of $M-AS$, this expression disappears on putting $z=\frac{p}{q}$.
Therefore this constant value $\frac{p}{q}$ is put in place of *z* everywhere in *M* and *S;* there will be $M-AS=0$, from which there is made $A=\frac{M}{S}$, therefore in this manner the numerator *A*

of the fraction sought $\frac{A}{p-qz}$ is found ; and if from the individual simple factors of the denominator N, provided they shall not be equal to each other, simple fractions of this kind may be formed, then the sum of all the simple fractions of this kind will be equal to the proposed function $\frac{M}{N}$.

EXAMPLE

Thus, if in the preceding example

$$\frac{1+zz}{z-z^3}$$

where there is

$$M=1+zz \quad \text{and} \quad N=z-z^3,$$

z is taken for the simple factor, there will be $S=1-zz$, and the numerator of the simple fraction $\frac{A}{z}$ hence arising will be

$$A=\frac{1+zz}{1-zz}=1$$

on putting $z=0$, which value z obtains, if this simple factor z is put equal to zero. In a similar manner if the factor $1-z$ is taken for the factor of the denominator, so that there shall be $S=z+zz$, there will be $A=\frac{1+zz}{z+zz}$ on making $1-z=0$, from which there will be $A=1$, and from the factor $1-z$ the fraction $\frac{1}{1-z}$ arises.

Finally the third factor $1+z$ on account of $S=z-zz$ and $A=\frac{1+zz}{z+zz}$, on putting $1+z=0$ or $z=-1$ will give $A=-1$ and the simple fraction $=\frac{-1}{1+z}$.
Whereby by this rule it is found that $\frac{1+zz}{z-z^3}=\frac{1}{z}+\frac{1}{1-z}-\frac{1}{1+z}$ as before.

42. *A fractional function of this form* $\frac{P}{(p-qz)^n}$, *the numerator of which P does not involve a power of z as great as the denominator* $(p-qz)^n$, *can be changed into partial fractions of this kind*

$$\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\frac{D}{(p-qz)^{n-3}}+\cdots+\frac{K}{p-qz},$$

all the numerators of which shall be constant quantities.

Because the maximum power of z in P is less than z^n, it will be z^{n-1} and thus P will have a form of this kind : $\alpha+\beta z+\gamma z^2+\delta z^3+\cdots+\chi z^{n-1}$ with the number of terms present $=n$, for which the numerator must be equal to the sum of all the partial fractions, after the individual terms for the same denominator $\left(p-qz\right)^{n}$ were led through ; which numerator therefore will be

$$= A+B\left(p-qz\right)+C\left(p-qz\right)^{2}+ D\left(p-qz\right)^{3}+\cdots+K\left(p-qz\right)^{n-1}$$

The maximum power of *z* is as there, z^{n-1}, and just as many unknown letters *A, B, C, D, ... K* (the number of which is $=n$), as there are congruent terms to be returned. On which account the letters *A, B, C* etc. thus are able to be defined, so that the function becomes the natural fraction

$$\frac{A}{\left(p-qz\right)^{n}}+\frac{B}{\left(p-qz\right)^{n-1}}+\frac{C}{\left(p-qz\right)^{n-2}}+\frac{D}{\left(p-qz\right)^{n-3}}+\cdots+\frac{K}{p-qz},$$

But the discovery of these numerators soon will appear easier.

43. *If the denominator N of a fractional function* $\frac{M}{N}$ *may have the factor* $\left(p-qz\right)^{2}$, *the partial fractions arising from this factor may be found in the following manner.*

Partial fractions of this kind originate from the individual factors with simple denominators, which may not be had equal to each other, which has been shown before ; therefore now we may put the two factors to be equal to each other, or with these joined together, a factor of the common denominator *N* is $\left(p-qz\right)^{2}$. Therefore from this factor by the preceding paragraph, these two partial fractions may be produced

$$\frac{A}{\left(p-qz\right)^{2}}+\frac{B}{p-qz}.$$

But there shall be

$$N=\left(p-qz\right)^{2}S$$

and the fraction becomes

$$\frac{M}{N}=\frac{M}{\left(p-qz\right)^{2}S}=\frac{A}{\left(p-qz\right)^{2}}+\frac{B}{p-qz}+\frac{P}{S}$$

with $\frac{P}{S}$ denoting all the simple fractions taken jointly, arising from the factor S of the denominator *S*. Hence it will be

$$\frac{M}{S}=\frac{M-AS-B\left(p-qz\right)S}{\left(p-qz\right)^{2}S}$$

and

$$P = \frac{M - AS - B(p-qz)S}{(p-qz)^2} = \text{a whole function}.$$

Therefore

$$M - AS - B(p-qz)S$$

must be divisible by $(p-qz)^2$. First it shall be divisible by $p-qz$ and the whole expression $M - AS - B(p-qz)S$ will vanish on putting $p-qz=0$ or $z=\frac{p}{q}$; therefore $\frac{p}{q}$ is put everywhere in place of *z* and it becomes $M - AS = 0$ and thus $A=\frac{M}{S}$; clearly the fraction $\frac{M}{S}$ will give the constant value of *A*, if $\frac{p}{q}$ is put everywhere in place of *z*.

With this found, the quantity $M - AS - B(p-qz)S$ must also be divisible by $(p-qz)^2$ or $\frac{M-AS}{p-qz} - BS$ must be divisible again by $p-qz$.
Therefore with $z=\frac{p}{q}$ put in place everywhere the equation becomes

$$\frac{M-AS}{p-qz} = BS$$

and thus

$$B = \frac{M-AS}{(p-qz)S} = \frac{1}{p-qz}\left(\frac{M}{S} - A\right),$$

where it is to be noted, since $M - AS$ will be divisible by $p-qz$, this earlier division must be put in place, as $\frac{p}{q}$. may be substituted in place of *z*. Or putting

$$\frac{M-AS}{p-qz} = T$$

and there will be $B=\frac{T}{S}$ on putting $z=\frac{p}{q}$.

Therefore with the numerators *A* and *B* found, these partial fractions

$$\frac{A}{(p-qz)^2} + \frac{B}{p-qz}$$

have arisen from the denominator *N* with the factor $(p-qz)^2$.

EXAMPLE 1

This fractional function

$$\frac{1-zz}{zz(1+zz)};$$

shall be proposed, on account of the square factor *zz* of the denominator, there will be

$$S = 1 + zz \text{ et } M = 1 - zz.$$

$\frac{A}{zz} + \frac{B}{z}$ shall be the partial fractions arising from zz ; there will be $A = \frac{M}{S} = \frac{1-zz}{1+zz}$; and hence $A = 1$ on putting the factor $z = 0$.
Then there will be $M - AS = -2zz$, which divided by the simple factor z will give $T = -2z$ and hence

$$B = \frac{T}{S} = \frac{-2z}{1+zz}$$

on putting $z = 0$; from which there will be $B = 0$ and from the factor of the denominator zz this single partial fraction will arise $\frac{1}{zz}$.

EXAMPLE 2

Let this be the proposed fraction

$$\frac{z^3}{(1-z)^2(1+z^4)},$$

of which on account of the square factor of the denominator $(1-z)^2$ the partial fractions shall be

$$\frac{A}{(1-z)^2} + \frac{B}{1-z}.$$

Therefore there will be

$$M = z^3 \text{ and } S = 1 + z^4$$

and thus

$$A = \frac{M}{S} = \frac{z^3}{1+z^4};$$

putting $1 - z = 0$ or $z = 1$, from which arises $A = \frac{1}{2}$.
Therefore there will be produced $M - AS = z^3 - \frac{1}{2} - \frac{1}{2}z^4 = -\frac{1}{2} + z^3 - \frac{1}{2}z^4$, which divided by $1 - z$ gives

$$T = -\tfrac{1}{2} - \tfrac{1}{2}z - \tfrac{1}{2}zz + \tfrac{1}{2}z^3,$$

and thus

$$B = \frac{T}{S} = \frac{-1-z-zz+z^3}{2+2z^4}$$

on putting $z = 1$, thus so that there shall be $B = -\frac{1}{2}$.
Therefore the partial fractions sought are $\frac{1}{2(1-z)^2} - \frac{1}{2(1-z)}$.

44. *If the denominator N of the fractional function* $\frac{M}{N}$ *should have the factor* $(p-qz)^3$, *partial fractions of the following kind arise from this factor*

$$\frac{A}{(p-qz)^3}+\frac{B}{(p-qz)^2}+\frac{C}{p-qz}$$

will be found.

Putting $N=(p-qz)^3 S$ and the fraction shall be arising from the factor S, $=\frac{P}{S}$; there will be

$$P=\frac{M-AS-B(p-qz)S-C(p-qz)^2 S}{(p-qz)^3}=\text{a whole function.}$$

The numerator will be

$$M-AS-B(p-qz)S-C(p-qz)^2 S$$

before everything it must be divisible by $p-qz$, from which on putting $p-qz=0$ or $z=\frac{p}{q}$ that must vanish and thus there will be $M-AS=0$ and thus $A=\frac{M}{S}$ on putting $z=\frac{p}{q}$.

With this *A* found agreed on, $M-AS$ will be divisible by $p-qz$;therefore putting

$$\frac{M-AS}{p-qz}=T$$

and

$$T-BS-C(p-qz)S$$

now $(p-qz)^2$ will be divisible ; therefore it becomes $=0$ on putting $p-qz=0$, from which $B=\frac{T}{S}$ is produced on putting $z=\frac{p}{q}$.

But thus with *B* found, $T-BS$ will be divisible by $p-qz$. On that account, on putting $\frac{T-BS}{p-qz}=V$ it emerges that $V-CS$ will be divisible by $p-qz$,and therefore there will $V-CS=0$ on putting $p-qz=0$ and $C=\frac{V}{S}$ on putting $z=\frac{p}{q}$.

Therefore with the numerators *A*, *B*, *C* found, the partial fractions arising from the denominator *N* from the factor $(p-qz)^3$ will be:

$$\frac{A}{(p-qz)^3}+\frac{B}{(p-qz)^2}+\frac{C}{p-qz}.$$

EXAMPLE

This shall be the proposed fractional function

$$\frac{zz}{(1-z)^3(1+zz)},$$

from the cubic factor of this denominator $(1-z)^3$, these partial fractions are generated

$$\frac{A}{(1-z)^3}+\frac{B}{(1-z)^2}+\frac{C}{1-z}.$$

Therefore there will be

$$M = zz \quad \text{and} \quad S = 1+zz;$$

from which at first there becomes

$$A = \frac{zz}{1+zz}$$

on putting $1-z=0$ or $z=1$, from which $A=\frac{1}{2}$ is produced.
Now putting $T=\frac{M-AS}{1-z}$;
there will be

$$T=\frac{\frac{1}{2}zz-\frac{1}{2}}{1-z}=-\frac{1}{2}-\frac{1}{2}z$$

from which there comes about $B=\frac{-\frac{1}{2}-\frac{1}{2}z}{1+zz}$; on putting $z=1$, thus so that $B=-\frac{1}{2}$.

Again there is put

$$V=\frac{T-BS}{1-z}=\frac{T+\frac{1}{2}S}{1-z};$$

there will be

$$V=\frac{-\frac{1}{2}z+\frac{1}{2}zz}{1-z}=-\frac{1}{2}z;$$

from which there is made

$$C=\frac{V}{S}=\frac{-\frac{1}{2}z}{1+zz};$$

on putting $z=1$, thus so that there will be $C=-\frac{1}{4}$.
On account of which the partial fractions arising from the factor $(1-z)^3$ will be

$$\frac{1}{2(1-z)^3}-\frac{1}{2(1-z)^2}-\frac{1}{4(1-z)}.$$

45. *If the denominator N of a fractional function* $\frac{M}{N}$ *may have the factor* $(p-qz)^n$*, the partial fractions hence generated*

$$\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\cdots+\frac{K}{p-qz}$$

will be found in the following manner.

The denominator may be put $N=(p-qz)^n Z$ and it will be found by the reasoning put in place before, as follows, in the first place $A=\frac{M}{Z}$ on putting $z=\frac{p}{q}$.

Putting $P=\frac{M-AZ}{p-qz}$; in the second place $B=\frac{P}{Z}$ on putting $z=\frac{p}{q}$.

Putting $Q=\frac{P-BZ}{p-qz}$; in the third place $C=\frac{Q}{Z}$ on putting $z=\frac{p}{q}$.

Putting $R=\frac{Q-CZ}{p-qz}$; in the fourth place $D=\frac{R}{Z}$ on putting $z=\frac{p}{q}$.

Putting $S=\frac{R-DZ}{p-qz}$; in the fifth place $E=\frac{S}{Z}$ on putting $z=\frac{p}{q}$, etc.

Therefore, if the individual numerator constants *A*, *B*, *C*, *D* etc., may be defined in this manner, all the partial fractions will be found, which are produced from the factor $(p-qz)^n$ of the denominator *N*.

EXAMPLE

Let this fractional function be proposed

$$\frac{1+zz}{z^5(1+z^3)},$$

these partial fractions are produced from the factor z^5 of the denominator

$$\frac{A}{z^5}+\frac{B}{z^4}+\frac{C}{z^3}+\frac{D}{z^2}+\frac{E}{z}.$$

Towards finding the constant numerators of which there will be

$$M=1+zz,\quad Z=1+z^5,\ \text{and}\ \frac{p}{q}=0.$$

Therefore the following calculation may be undertaken.

Initially there is $A = \frac{M}{Z} = \frac{1+zz}{1+z^3}$ on putting $z = 0$, therefore $A = 1$.

Putting $P = \frac{M-AZ}{z} = \frac{zz-z^3}{z} = z - zz$, and there will be in the second case $B = \frac{P}{Z} = \frac{z-zz}{1+z^3}$ on putting $z = 0$, therefore $B = 0$.

Putting $Q = \frac{P-BZ}{p-qz} = \frac{z-zz}{z} = 1 - z$, becoming in the third place $C = \frac{Q}{Z} = \frac{1-z}{1+z^3}$ on putting $z = 0$, therefore $C = 1$.

Putting $R = \frac{Q-CZ}{z} = \frac{-z-z^3}{z} = -1 - zz$; becoming in the fourth place $D = \frac{R}{Z} = \frac{-1-zz}{1+z^3}$ on putting $z = 0$, from which $D = -1$.

Putting $S = \frac{R-DZ}{p-qz} = \frac{-zz+z^3}{z} = -zz + zz$; becoming in the fifth place $E = \frac{S}{Z} = \frac{-z+zz}{1+z^3}$ on putting $z = 0$, from which $E = 0$.

Concerning the partial fractions sought, these will be :

$$\frac{1}{z^5} + \frac{0}{z^4} + \frac{1}{z^3} - \frac{1}{z^2} + \frac{0}{z}.$$

45'. *Therefore whichever rational fractional function $\frac{M}{N}$ were proposed, that will be resolved in the following manner, and it will be transformed into the simplest form.*

All the simple factors of the denominator *N* may be sought, either real or imaginary, of those which shall not be equal to each other, they may be treated separately and each one elicited by a partial fraction by § 41. But if likewise a simple factor may occur twice or several times, these may be taken jointly and from the product of these, which will be a power of the form $(p - qz)^n$, convenient partial fractions may be sought by § 45. And in that manner since the partial fractions were elicited from individual simple factors of the denominator, then the sum of all those will be equal to the proposed function $\frac{M}{N}$, unless it was spurious, for if it were spurious, the whole part above must be extracted and added to these partial fractions found, so that the value of the function $\frac{M}{N}$ may appear expressed in the simplest form. But it is the same, whether the fractional parts may be sought either before the extraction of the whole part or after. For the same partial fractions appear from the individual factors of the denominator *N*, whether the numerator *M* may be used, or likewise to some multiple of the denominator *N* either increased or diminished ; because it will be clear how to consider the rules easily.

EXAMPLE

The value of the function is sought

$$\frac{1}{z^3(1-z)^2(1+z)}$$

expressed in the simplest form.

The first single value of the denominator is taken $1+z$, which gives $\frac{p}{q}=-1$; there will be $M=1$ and $Z=z^3-2z^4+z^5$.

Hence towards finding the fraction $\frac{A}{1+z}$ there will be

$$A=\frac{1}{z^3-2z^4+z^5}$$

on putting $z=-1$ and thus it becomes $A=-\frac{1}{4}$ and from the factor $1+z$ this partial fraction arises $-\frac{1}{4(1+z)}$.

Now the square factor is taken $(1-z)^2$, which gives

$$\frac{p}{q}=1,\ M=1 \text{ and } Z=z^3+z^4$$

Hence with the partial fractions in place arising

$$\frac{A}{(1-z)^2}+\frac{B}{1-z}$$

there will be $A=\frac{1}{z^3+z^4}$ on putting $z=1$, therefore $A=\frac{1}{2}$.

There becomes

$$P=\frac{M-\frac{1}{2}Z}{1-z}=\frac{1-\frac{1}{2}z^3-\frac{1}{2}z^4}{1-z}=1+z+zz+\frac{1}{2}z^3$$

and there becomes

$$B=\frac{P}{Z}=\frac{1+z+zz+\frac{1}{2}z^3}{z^3+z^4}$$

on putting $z=1$, therefore $B=\frac{7}{4}$ and the partial fractions sought

$$\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}$$

Finally the third cubic factor z^3 gives

$$\frac{p}{q}=1,\ M=1 \text{ and } Z=1-z-zz+z^3$$

Therefore with these partial fractions in place

$$\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}$$

in the first place there will be

$$A=\frac{M}{Z}=\frac{1}{1-z-zz+z^3}$$

on putting $z=0$, therefore

$$A=1.$$

There is put

$$P=\frac{M-Z}{z}=1+z-zz$$

there will be

$$B=\frac{P}{Z}$$

on putting $z=0$, therefore $B=1$.
Putting

$$Q=\frac{P-Z}{z}=2-zz;$$

there will be

$$C=\frac{Q}{Z}$$

on putting $z=0$, therefore $C=2$.

On that account the proposed function

$$\frac{1}{z^3(1-z)^2(1+z)}$$

is resolved into this form

$$\frac{1}{z^3}+\frac{1}{z^2}+\frac{2}{z}+\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}-\frac{1}{4(1+z)}.$$

Indeed it agrees with no integral part above, because the proposed fraction is not spurious.

CAPUT II

DE TRANSFORMATIONE FUNCTIONUM

27. *Functiones in alias formas transmutantur vel loco quantitatis variabilis aliam introducendo vel eandem quantitatem variabilem retinendo.*

Quodsi eadem quantitas variabilis servatur, functio proprie mutari non potest. Sed omnis transformatio consistit in alio modo eandem functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse. Huiusmodi transformationes sunt, si loco huius functionis $2-3z+zz$ ponatur $(1-z)(2-z)$,
vel

$$(a+z)^3 \text{ loco } a^3+3aaz+3azz+z^3,$$

vel

$$\frac{a}{a-z}+\frac{a}{a+z} \text{ loco } \frac{2aa}{aa-zz}$$

vel

$$\sqrt{(1+zz)}+z \text{ loco } \frac{1}{\sqrt{(1+zz)}-z};$$

,
quae expressiones, etsi forma differunt, tamen revera congruunt. Saepenumero autem harum plurium formarum idem significantium una aptior est ad propositum efficiendum quam reliquae et hanc ob rem formam commodissimam eligi oportet.

Alter transformationis modus, quo loco quantitatis variabilis z alia quantitas variabilis y introducitur, quae quidem ad z datam teneat relationem, per substitutionem fieri dicitur; hocque modo ita uti convenit, ut functio proposita succinctius et commodius exprimatur; uti, si ista proposita fuerit ipsius z functio

$$a^4-4a^3z+6aazz-4az^3+z^4,$$

si loco $a-z$ ponatur y, prodibit ista multo simplicior ipsius y functio y^4, et si habeatur haec functio irrationalis $\sqrt{(aa+zz)}$ ipsius z, si ponatur $z=\frac{aa-yy}{2y}$, ista functio per y expressa fiet rationalis $z=\frac{aa+yy}{2y}$.
Hunc autem transformationis modum in sequens caput differam hoc capite illum, qui sine substitutione procedit, expositurus.

28. *Functio integra ipsius z saepenumero commode in suos factores resolvitur sicque in productum transformatur.*

Quando functio integra hoc pacto in factores resolvitur, eius natura multo facilius perspicitur; casus enim statim innotescunt, quibus functionis valor fit $=0$. Sic haec ipsius z functio

$$6-7z+z^3$$

transformatur in hoc productum

$$(1-z)(2-z)(3+z),$$

ex quo statim liquet functionem propositam tribus casibus fieri $=0$, scilicet si $z=1$ et $z=2$ et $z=-3$, quae proprietates ex forma $6-7z+z^3$ non tam facile intelliguntur. Istiusmodi factores, in quibus variabilis z nulla [altior] occurrit potestas, vocantur factores *simplices*, ut distinguantur a factoribus *compositis*, in quibus ipsius z inest quadratum vel cubus vel alia potestas altior. Erit ergo in genere

$$f+gz$$

forma factorum simplicium,

$$f+gz+hzz$$

forma factorum duplicium,

$$f+gz+hzz+iz^3$$

forma factorum triplicium et ita porro. Perspicuum autem est factorem duplicem duos complecti factores simplices, factorem triplicem tres simplices et ita porro. Hinc functio ipsius z integra, in qua exponens summae potestatis ipsius z est $=n$, continebit n factores simplices; ex quo simul, si qui factores fuerint vel duplices vel triplices etc., numerus factorum cognoscetur.

29. *Factores simplices functionis cuiuscunque integrae Z ipsius z reperiuntur, si functio Z nihilo aequalis ponatur atque ex hac aequatione omnes ipsius z radices investigentur; singulae enim ipsius z radices dabunt totidem factores simplices functionis Z.*

Quodsi enim ex aequatione $Z=0$ fuerit quaepiam radix $z=f$, erit $z-f$ divisor ac proinde factor functionis Z; sic igitur investigandis omnibus radicibus aequationis $Z=0$, quae sint

$$z=f,\ z=g,\ z=h \text{ etc.},$$

functio Z resolvetur in suos factores simplices atque transformabitur in productum

$$Z = (z-f)(z-g)(z-h) \text{ etc.};$$

ubi quidem notandum est, si summae potestatis ipsius *z* in *Z* non fuerit coefficiens $=+1$, tum productum $(z-f)(z-g)$ etc. insuper per illum coefficientem multiplicari debere. Sic, si fuerit

$$Z = Az^n + Bz^{n-1} + Cz^{n-2} + \text{ etc.},$$

erit

$$Z = A(z-f)(z-g)(z-h) \text{ etc.}$$

At si fuerit

$$Z = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

atque aequationis $Z=0$ radices *z* repertae sint *f, g, h, i* etc., erit

$$Z = A\left(1-\tfrac{z}{f}\right)\left(1-\tfrac{z}{g}\right)\left(1-\tfrac{z}{h}\right) \text{ etc.}$$

Ex his autem vicissim intelligitur, si functionis *Z* factor fuerit $z-f$ seu $1-\tfrac{z}{f}$, tum valorem functionis in nihilum abire, si loco *z* ponatur *f*. Facto enim $z=f$, unus factor $z-f$ seu $1-\tfrac{z}{f}$ functionis *Z* ideoque ipsa functio *Z* evanescere debet.

30. *Factores simplices ergo erunt vel reales vel imaginarii et, si functio Z habeat factores imaginarios, eorum numerus semper erit par.*

Cum enim factores simplices nascantur ex radicibus aequationis $Z=0$, radices reales praebebunt factores reales et imaginariae imaginarios; in omni autem aequatione numerus radicum imaginariarum semper est par, quamobrem functio *Z* vel nullos habebit factores imaginarios vel duos vel quatuor vel sex etc. Quodsi functio *Z* duos tantum habeat factores imaginarios, eorum productum erit reale ideoque praebebit factorem duplicem realem. Sit enim $P=$ producto ex omnibus factoribus realibus, erit productum duorum factorum imaginariorum $=\tfrac{Z}{P}$ hincque reale. Simili modo si functio *Z* habeat quatuor vel sex vel octo etc. factores imaginarios, erit eorum productum semper reale, nempe aequale quoto, qui oritur, si functio *Z* dividatur per productum omnium factorum realium.

31. *Si fuerit Q productum reale ex quatuor factoribus simplicibus imaginariis, tum idem hoc productum Q resolvi poterit in duos factores duplices reales.*

Habebit enim *Q* eiusmodi formam

$$z^4 + Az^3 + Bz^2 + Cz + D;$$

quae si negetur in duos factores duplices reales resolvi posse, resolubilis erit statuenda in duos factores duplices imaginarios, qui huiusmodi formam habebunt

$$zz-2\left(p+q\sqrt{-1}\right)z+r+s\sqrt{-1}$$

et

$$zz-2\left(p-q\sqrt{-1}\right)z+r-s\sqrt{-1}\,;$$

aliae enim formae imaginariae concipi non possunt, quarum productum fiat reale, nempe $=z^4+Az^3+Bz^2+Cz+D$. Ex his autem factoribus imaginariis duplicibus sequentes emergent quatuor factores simplices imaginarii ipsius *Q:*

$$\text{I. } z-\left(p+q\sqrt{-1}\right)+\sqrt{\left(pp+2pq\sqrt{-1}-qq-r-s\sqrt{-1}\right)},$$

$$\text{II. } z-\left(p+q\sqrt{-1}\right)-\sqrt{\left(pp+2pq\sqrt{-1}-qq-r-s\sqrt{-1}\right)},$$

$$\text{III. } z-\left(p-q\sqrt{-1}\right)+\sqrt{\left(pp-2pq\sqrt{-1}-qq-r+s\sqrt{-1}\right)},$$

$$\text{IV. } z-\left(p-q\sqrt{-1}\right)-\sqrt{\left(pp-2pq\sqrt{-1}-qq-r+s\sqrt{-1}\right)}.$$

Horum factorum multiplicentur primus ac tertius in se invicem posito brevitatis gratia

$$t=pp-qq-r \text{ et } u=2pq-s$$

eritque horum factorum productum

$$=zz-\left(2p-\sqrt{2t+2\sqrt{(tt+uu)}}\right)z$$
$$+pp+qq-p\sqrt{2t+2\sqrt{(tt+uu)}}-q\sqrt{-2t+2\sqrt{(tt+uu)}}+\sqrt{(tt+uu)},$$

quod utique est reale. Simili autem modo productum ex factoribus secundo et quarto erit reale, nempe

$$=zz-\left(2p+\sqrt{2t+2\sqrt{(tt+uu)}}\right)z$$
$$+pp+qq+p\sqrt{2t+2\sqrt{(tt+uu)}}+q\sqrt{-2t+2\sqrt{(tt+uu)}}+\sqrt{(tt+uu)}.$$

Quocirca productum propositum *Q*, quod in duos factores duplices reales resolvi posse negabatur, nihilominus actu in duos factores duplices reales est resolutum.

32. *Si functio integra Z ipsius z quotcunque habeat factores simplices imaginarios, bini semper ita coniungi possunt, ut eorum productum fiat reale.*

Quoniam numerus radicum imaginariarum semper est par, sit is $=2n$ ac primo quidem patet productum harum radicum imaginariarum omnium esse reale. Quodsi ergo duae tantum radices imaginariae habeantur, erit earum productum utique reale; sin autem quatuor habeantur factores imaginarii, tum, uti vidimus, eorum productum resolvi potest in duos factores duplices reales formae $fzz+gz+h$. Quanquam autem eundem demonstrandi modum ad altiores potestates extendere non licet, tamen extra dubium videtur esse positum eandem proprietatem in quotcunque factores imaginarios competere, ita ut semper loco *2n* factorum simplicium imaginariorum induci queant *n* factores duplices reales. Hinc omnis functio integra ipsius *z* resolvi poterit in factores reales vel simplices vel duplices. Quod quamvis non summo rigore sit demonstratum, tamen eius veritas in sequentibus magis corroborabitur, ubi huius generis functiones

$$a+bz^n,\ \ a+bz^n+cz^{2n},\ \ a+bz^n+cz^{2n}+dz^{3n}\ \text{etc.}$$

actu in istiusmodi factores duplices reales resolventur.

33. *Si functio integra Z posito $z=a$ induat valorem A et posito $z=b$ induat valorem B, tum loco z valores medios inter a et b ponendo functio Z quosvis valores medios inter A et B accipere potest.*

Cum enim *Z* sit functio uniformis ipsius *z*, quicunque valor realis ipsi *z* tribuatur, functio quoque *Z* hinc valorem realem obtinebit. Cum igitur *Z* priore casu $z=a$ nanciscatur valorem *A*, posteriore casu $z=b$ autem valorem *B*, ab *A* ad *B* transire non poterit nisi per omnes valores medios transeundo. Quodsi ergo aequatio $Z-A=0$ habeat radicem realem simulque $Z-B=0$ radicem realem suppeditet, tum aequatio quoque $Z-C=0$ radicem habebit realem, siquidem *C* intra valores *A* et *B* contineatur. Hinc si expressiones $Z-A$ et $Z-B$ habeant factorem simplicem realem, tum expressio quaecunque $Z-C$ factorem simplicem habebit realem, dummodo *C* intra valores *A* et *B* contineatur.

34. *Si in functione integra Z exponens maximae ipsius z potestatis fuerit numerus impar $2n+1$, tum ea functio Z unicum ad minimum habebit factorem simplicem realem.*

Habebit scilicet *Z* huiusmodi formam

$$z^{2n+1}+\alpha z^{2n}+\beta z^{2n-1}+\gamma z^{2n-2}+\text{etc.};$$

in qua si ponatur $z=\infty$, quia valores singulorum terminorum prae prima evanescunt, fiet $Z = (\infty)^{2n+1} = \infty$ ideoque $Z-\infty$ factorem simplicem habebit realem, nempe $z-\infty$. Sin autem ponatur $z=-\infty$, fiet

$$Z = (-\infty)^{2n+1} = -\infty$$

ideoque habebit $Z+\infty$ factorem simplicem realem $z+\infty$. Cum igitur tam $Z-\infty$ quam $Z+\infty$ habeat factorem simplicem realem, sequitur etiam $Z-C$ habiturum esse factorem simplicem realem, siquidem C contineatur intra limites $+\infty$ et $-\infty$, hoc est, si C fuerit numerus realis quicunque, sive affirmativus sive negativus. Hanc ob rem facto $C = 0$ habebit quoque ipsa functio Z factorem simplicem realem $z-c$ atque quantitas c continebitur intra limites $+\infty$ et $-\infty$ eritque idcirco vel quantitas affirmativa vel negativa vel nihil.

35. *Functio igitur integra Z, in qua exponens maximae potestatis ipsius z est numerus impar, vel unum habebit factorem simplicem realem vel tres vel quinque vel septem etc.*

Cum enim demonstratum sit functionem Z certo unum habere factorem simplicem realem $z-c$, ponamus eam praeterea unum factorem habere $z-d$ atque dividatur functio Z, in qua maxima ipsius z potestas sit $= z^{2n-1}$, per $(z-c)(z-d)$; erit quoti maxima potestas $= z^{2n-1}$, cuius exponens, cum sit numerus impar, indicat denuo ipsius Z dari factorem simplicem realem. Si ergo Z plures uno habeat factores simplices reales, habebit vel tres vel (quoniam eodem modo progredi licet) quinque vel septem etc. Erit scilicet numerus factorum simplicium realium impar, et quia numerus omnium factorum simplicium est $= 2n+1$, erit numerus factorum imaginariorum par.

36. *Functio integra Z, in qua exponens maximae potestatis ipsius z est numerus par* $2n$, *vel duos habebit factores simplices reales vel quatuor vel sex vel etc.*

Ponamus ipsius Z constare factorum simplicium realium numerum imparem $2m+1$; si ergo per horum omnium productum dividatur functio Z, quoti maxima potestas erit $= z^{2n-2m-1}$ eiusque ideo exponens numerus impar; habebit ergo functio Z praeterea unum certo factorem simplicem realem, ex quo numerus omnium factorum simplicium realium ad minimum erit $= 2m+2$ ideoque par ac numerus factorum imaginariorum pariter par. Omnis ergo functionis integrae factores simplices imaginarii sunt numero pares, quemadmodum quidem iam ante statuimus.

37. *Si in functione integra Z exponens maximae potestatis ipsius z fuerit numerus par atque terminus absolutus seu constans signo – affectus, tum functio Z ad minimum duos habet factores simplices reales.*

Functio ergo *Z,* de qua hic sermo est, huiusmodi formam habebit

$$z^{2n} \pm \alpha z^{2n-1} \pm \beta z^{2n-2} \pm \cdots \pm \nu z - A.$$

Si iam ponatur $z = \infty$, fiet, uti supra vidimus, $Z = \infty$ atque, si ponatur $z = 0$, fiet $Z = -A$. Habebit ergo $Z - \infty$ factorem realem $z - \infty$ et $Z + A$ factorem $z - 0$; unde, cum 0 contineatur intra limites $-\infty$ et $+A$, sequitur $Z + 0$ habere factorem simplicem realem $z - c$, ita ut *c* contineatur intra limites 0 et ∞. Deinde, cum posito $z = -\infty$ fiat $Z = \infty$ ideoque $Z - \infty$ factorem habeat $z + \infty$ et $Z + A$ factorem $z + 0$, sequitur quoque $Z + 0$ factorem simplicem realem habere $z + d$, ita ut *d* intra limites 0 et ∞ contineatur; unde constat propositum. Ex his igitur perspicitur; si *Z* talis fuerit functio, qualis hic est descripta, aequationem $Z = 0$ duas ad minimum habere debere radices reales, alteram affirmativam, alteram negativam. Sic aequatio haec

$$z^4 + \alpha z^3 + \beta z^2 + \gamma z - aa = 0$$

duas habet radices reales, alteram affirmativam, alteram negativam.

38. *Si in functione fracta quantitas variabilis z tot vel plures habeat dimensiones in numeratore quam in denominatore, tum ista functio resolvi poterit in duas partes, quarum altera est functio integra, altera fracta, in cuius numeratore quantitas variabilis z pauciores habeat dimensiones quam in denominatore.*

Si enim exponens maximae potestatis ipsius *z* minor fuerit in denominatore quam in numeratore, tum numerator per denominatorem dividatur more solito, donec in quoto ad exponentes negativos ipsius *z* perveniatur; hoc ergo loco abrupta divisionis operatione quotus constabit ex parte integra atque fractione, in cuius numeratore minor erit dimensionum numerus ipsius *z* quam in denominatore; hic autem quotus functioni propositae est aequalis. Sic, si haec proposita fuerit functio fracta

$$\frac{1+z^4}{1+zz},$$

ea per divisionem, ita resolvetur:

$$zz+1)\quad z^4+1\quad \left(zz-1+\tfrac{2}{1+zz}\right.$$
$$\underline{z^4+zz}$$
$$-zz+1$$
$$\underline{-zz-1}$$
$$+2$$

eritque

$$\tfrac{1+z^4}{zz+1}=zz-1+\tfrac{2}{1+zz}.$$

Huiusmodi functiones fractae, in quibus quantitas variabilis z tot vel plures habet dimensiones in numeratore quam in denominatore, ad similitudinem Arithmeticae vocari possunt fractiones *spuriae* vel functiones fractae spuriae, quo distinguantur a functionibus fractis *genuinis,* in quarum numeratore quantitas variabilis z pauciores habet dimensiones quam in denominatore. Functio itaque fracta spuria resolvi poterit in functionem integram et functionem fractam genuinam haecque resolutio per vulgarem divisionis operationem absolvetur.

39. Si *denominator functionis fractae duos habeat factores inter se primos, tum ipsa functio fracta resolvetur in duas fractiones, quarum denominatores sint illis binis factoribus respective aequales.*

Quanquam haec resolutio ad functiones fractas spurias aeque pertinet atque ad genuinas, tamen eam ad genuinas potissimum accommodabimus. Resoluto autem denominatore huiusmodi functionis fractae in duos factores inter se primos, ipsa functio resolvetur in duas alias functiones fractas genuinas, quarum denominatores sint illis binis factoribus respective aequales, haecque resolutio, siquidem fractiones sint genuinae, unico modo fieri potest; cuius rei veritas ex exemplo clarius quam per ratiocinium perspicietur. Sit ergo proposita haec functio fracta

$$\tfrac{1-2z+3zz-4z^3}{1+4z^4};$$

cuius denominator $1+4z^4$ cum sit aequalis huic producto

$$(1+2z+2zz)(1-2z+2zz),$$

fractio proposita in duas fractiones resolvetur, quarum alterius denominator erit $1+2z+2zz$, alterius $1-2z+2zz$; ad quas inveniendas, quia sunt genuinae, statuantur numeratores illius $=\alpha+\beta z$, huius $=\gamma+\delta z$ eritque per hypothesin

$$\tfrac{1-2z+3zz-4z^3}{1+4z^4}=\tfrac{\alpha+\beta z}{1+2z+2zz}+\tfrac{\gamma+\delta z}{1-2z+2zz};$$

addantur actu hae duae fractiones eritque summae

$$\begin{array}{ll} \text{numerator} & \text{denominator} \\ +\ \alpha - 2\alpha z + 2\alpha zz & \\ \quad +\beta z \ - \ \beta zz + \ 2\beta z^3 & 1+4z^4. \\ +\ \gamma + 2\gamma z + \ 2\gamma zz & \\ \quad +\delta z \ \ + \ 2\delta zz + 2\delta z^3, & \end{array}$$

Cum ergo denominator aequalis sit denominatori fractionis propositae, numeratores quoque aequales reddi debent; quod ob tot litteras incognitas $\alpha, \beta, \gamma, \delta,$ quot sunt termini aequales efficiendi, utique fieri idque unico modo poterit; nanciscimur scilicet has quatuor aequationes

$$\text{I.} \quad \alpha + \gamma = 1, \qquad \text{III.}\ 2\alpha - 2\beta + 2\gamma + 2\delta = 3,$$
$$\text{II.} - 2\alpha + \beta + 2\gamma + \delta = -2, \qquad \text{IV.} 2\beta + 2\delta = -4.$$

Hinc ob

$$\alpha + \gamma = 1 \ \text{ et } \ \beta + \delta = -2$$

aequationes II. et III. dabunt

$$\alpha - \gamma = 0 \ \text{ et } \ \delta - \beta = \tfrac{1}{2},$$

ex quibus fit

$$\alpha = \tfrac{1}{2},\ \gamma = \tfrac{1}{2},\ \beta = -\tfrac{5}{4},\ \delta = -\tfrac{3}{4},$$

ideoque fractio proposita

$$\frac{1-2z+3zz-4z^3}{1+4z^4}$$

transformatur in has duas

$$\frac{\frac{1}{2}-\frac{5}{4}z}{1+2z+2zz} + \frac{\frac{1}{2}-\frac{3}{4}z}{1-2z+2zz}.$$

Simili autem modo facile perspicietur resolutionem semper succedere debere, quoniam semper tot litterae incognitae introducuntur, quot opus est ad numeratorem propositum eliciendum. Ex doctrina vero fractionum communi intelligitur hanc resolutionem succedere non posse, nisi isti denominatoris factores fuerint inter se primi.

40. *Functio igitur fracta $\frac{M}{N}$ in tot fractiones simplices formae resolvi $\frac{A}{p-qz}$ poterit, quot factores simplices habet denominator N inter se inaequales.*

Repraesentat hic fractio $\frac{M}{N}$ functionem quamcunque fractam genuinam, ita ut M et N sint functiones integrae ipsius z atque summa potestas ipsius z in M minor sit quam in N. Quodsi ergo denominator N in suos factores simplices resolvatur hique inter se fuerint inaequales, expressio $\frac{M}{N}$ in tot fractiones resolvetur, quot factores simplices in denominatore N continentur, propterea quod quisque factor abit in denominatorem fractionis partialis. Si ergo $p-qz$ fuerit factor ipsius N, is erit denominator fractionis cuiusdam partialis et, cum in numeratore huius fractionis numerus dimensionum ipsius z minor esse debeat quam in denominatore $p-qz$, numerator necessario erit quantitas constans. Hinc ex unoquoque factore simplici $p-qz$ denominatoris N nascetur fractio simplex $\frac{A}{p-qz}$, ita ut summa omnium harum fractionum $p-qz$ sit aequalis fractioni propositae $\frac{M}{N}$.

EXEMPLUM

Sit exempli causa proposita haec functio fracta

$$\frac{1+zz}{z-z^3}.$$

Quia factores simplices denominatoris sunt z, $1-z$ et $1+z$, ista functio resolvetur in has tres fractiones simplices

$$\frac{A}{z}+\frac{B}{1-z}+\frac{C}{1+z}=\frac{1+zz}{z-z^3},$$

ubi numeratores constantes A, B et C definire oportet. Reducantur hae fractiones ad communem denominatorem, qui erit $z-z^3$, atque numeratorum summa aequari debebit ipsi $1+zz$, unde ista aequatio oritur

$$\begin{aligned} A+Bz-Azz&=1+zz=1+0z+zz,\\ +Cz+Bzz&\\ -Czz&\end{aligned}$$

quae totidem comparationes praebet, quot sunt litterae incognitae A, B, C ;
erit scilicet

$$\begin{aligned} &\text{I.}\quad A=1,\\ &\text{II.}\quad B+C=0,\\ &\text{III.}-A+B-C=1.\end{aligned}$$

Hinc fit $B-C=2$,et porro $A=1$, $B=1$ et $C=-1$.
Functio ergo proposita

$$\frac{1+zz}{z-z^3}$$

resolvitur in hanc formam

$$\frac{1}{z}+\frac{1}{1-z}-\frac{1}{1+z}.$$

Simili autem modo intelligitur, quotcunque habuerit denominator *N* factores simplices inter se inaequales, semper fractionem $\frac{M}{N}$ in totidem fractiones simplices resolvi. Sin autem aliquot factores fuerint aequales inter se, tum alio modo post explicando resolutio institui debet.

41. *Cum igitur quilibet factor simplex denominatoris N suppeditet fractionem simplicem pro resolutione functionis propositae* $\frac{M}{N}$, *ostendendum est, quomodo ex factore simplici denominatoris N cognito fractio simplex respondens reperiatur.*

Sit $p-qz$ factor simplex ipsius *N*, ita ut sit

$$N=(p-qz)S$$

atque *S* functio integra ipsius *z;* ponatur fractio ex factore $p-qz$ orta

$$=\frac{A}{p-qz}$$

et sit fractio ex altero factore denominatoris *S* oriunda

$$=\frac{P}{S},$$

ita ut secundum § 39 futurum sit

$$\frac{M}{N}=\frac{A}{p-qz}+\frac{P}{\mathrm{S}}=\frac{M}{(p-qz)S};$$

hinc erit

$$\frac{P}{\mathrm{S}}=\frac{M-AS}{(p-qz)S};$$

quae fractiones cum congruere debeant, necesse est, ut $M-AS$ sit divisibile per $p-qz$, quoniam functio integra *P* ipsi quoto aequatur. Quando vero $p-qz$ divisor existit ipsius $M-AS$, haec expressio posito $z=\frac{p}{q}$ evanescit. Ponatur ergo ubique loco *z* hic valor constans $\frac{p}{q}$ in *M* et *S;* erit $M-AS=0$, ex quo fiet $A=\frac{M}{S}$, hocque ergo modo reperitur numerator *A* fractionis quaesitae $\frac{A}{p-qz}$*;* atque si ex singulis denominatoris *N* factoribus simplicibus, dummodo sint inter se inaequales, huiusmodi fractiones simplices formentur, harum fractionum simpli dum omnium summa erit aequalis functioni propositae $\frac{M}{N}$.

EXEMPLUM

Sic, si in exemplo praecedente

$$\frac{1+zz}{z-z^3}$$

ubi est

$$M=1+zz \text{ et } N=z-z^3,$$

sumatur z pro factore simplici, erit

$$S=1-zz$$

atque fractionis simplicis $\frac{A}{z}$ hinc ortae erit numerator

$$A=\frac{1+zz}{1-zz}=1$$

posito $z=0$, quem valorem z obtinet, si ipse hic factor simplex z nihilo aequalis ponatur. Simili modo si pro denominatoris factore sumatur $1-z$, ut sit

$$S=z+zz,$$

erit

$$A=\frac{1+zz}{z+zz}$$

facto $1-z=0$, unde erit $A=1$, et ex factore $1-z$ nascitur fractio $\frac{1}{1-z}$.

Tertius denique factor $1+z$ ob $S=z-zz$ et $A=\frac{1+zz}{z+zz}$, posito $1+z=0$ seu $z=-1$ dabit $A=-1$ et fractionem simplicem $=\frac{-1}{1+z}$.

Quare per hanc regulam reperitur

$$\frac{1+zz}{z-z^3}=\frac{1}{z}+\frac{1}{1-z}-\frac{1}{1+z}$$

ut ante.

42. *Functio fracta huius formae* $\frac{P}{(p-qz)^n}$, *cuius numerator P non tantam ipsius z potestatem involvit quantam denominator* $(p-qz)^n$, *transmutari potest in huiusmodi fractiones partiales*

$$\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\frac{D}{(p-qz)^{n-3}}+\cdots+\frac{K}{p-qz},$$

quarum omnium numeratores sint quantitates constantes.

Quoniam maxima potestas ipsius z in P minor est quam z^n, erit z^{n-1} ideoque P huiusmodi habebit formam

$\alpha+\beta z+\gamma z^2+\delta z^3+\cdots+\chi z^{n-1}$ existente terminorum numero $=n$, cui aequari debet numerator summae omnium fractionum partialium, postquam singulae ad eundem denominatorem $(p-qz)^n$ fuerint perductae; qui numerator propterea erit

$$=A+B(p-qz)+C(p-qz)^2+D(p-qz)^3+\cdots+K(p-qz)^{n-1}$$

Huius maxima ipsius z potestas est, ut ibi, z^{n-1} atque tot habentur litterae incognitae *A, B, C, D, ... K* (quarum numerus est $=n$), quot sunt termini congruentes reddendi. Quamobrem litterae *A, B, C* etc. ita definiri poterunt, ut fiat functio fracta genuina

$$\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\frac{D}{(p-qz)^{n-3}}+\cdots+\frac{K}{p-qz},$$

Ipsa autem horum numeratorum inventio mox facilis aperietur.

43. *Si functionis fractae $\frac{M}{N}$ denominator N factorem habeat $(p-qz)^2$, sequenti modo fractiones partiales ex hoc factore oriundae reperientur.*

Cuiusmodi fractiones partiales ex singulis factoribus denominatoris simplicibus, qui alios sibi aequales non habeant, oriantur, ante est ostensum; nunc igitur ponamus duos factores inter se esse aequales seu iis coniunctis denominatoris N factorem esse $(p-qz)^2$. Ex hoc ergo factore per paragraphum praecedentem duae nascentur fractiones partiales hae

$$\frac{A}{(p-qz)^2}+\frac{B}{p-qz}$$

Sit autem

$$N=(p-qz)^2 S$$

eritque

$$\frac{M}{N}=\frac{M}{(p-qz)^2 S}=\frac{A}{(p-qz)^2}+\frac{B}{p-qz}+\frac{P}{S}$$

denotante $\frac{P}{S}$ omnes fractiones simplices iunctim sumptas ex denominatoris factore *S* ortas. Hinc erit

$$\frac{M}{S}=\frac{M-AS-B(p-qz)S}{(p-qz)^2S}$$

et

$$P=\frac{M-AS-B(p-qz)S}{(p-qz)^2}=\text{functioni integrae}.$$

Debet ergo

$$M-AS-B(p-qz)S$$

divisibile esse per $(p-qz)^2$. Sit primum divisibile per $p-qz$ atque tota expressio $M-AS-B(p-qz)S$ evanescet posito $p-qz=0$ seu $z=\frac{p}{q}$; ponatur ergo ubique $\frac{p}{q}$ loco *z* eritque $M-AS=0$ ideoque $A=\frac{M}{S}$; scilicet fractio $\frac{M}{S}$, si loco *z* ubique ponatur $\frac{p}{q}$, dabit valorem ipsius *A* constantem.

Hoc invento quantitas $M-AS-B(p-qz)S$ etiam per $(p-qz)^2$ divisibilis esse debet seu $\frac{M-AS}{p-qz}-BS$ denuo per $p-qz$ divisibilis esse debet.
Posito ergo ubique $z=\frac{p}{q}$ erit

$$\frac{M-AS}{p-qz}=BS$$

ideoque

$$B=\frac{M-AS}{(p-qz)S}=\frac{1}{p-qz}\left(\frac{M}{S}-A\right),$$

ubi notandum est, cum $M-AS$ divisibile sit per $p-qz$, hanc divisionem prius institui debere, quam loco *z* substituatur $\frac{p}{q}$. Vel ponatur

$$\frac{M-AS}{p-qz}=T$$

eritque $B=\frac{T}{S}$ posito $z=\frac{p}{q}$.

Inventis ergo numeratoribus *A* et *B* erunt fractiones partiales ex denominatoris *N* factore $(p-qz)^2$ ortae hae $\frac{A}{(p-qz)^2}+\frac{B}{p-qz}$.

EXEMPLUM 1

Sit haec proposita functio fracta

$$\frac{1-zz}{zz(1+zz)};$$

erit ob denominatoris factorem quadratum *zz*

$$S = 1 + zz \text{ et } M = 1 - zz.$$

Sint fractiones partiales ex zz ortae $\frac{A}{zz} + \frac{B}{z}$; erit $A = \frac{M}{S} = \frac{1-zz}{1+zz}$
.
posito factore $z = 0$ hincque $A = 1$.
Tum erit $M - AS = -2zz$, quod divisum per factorem simplicem z dabit $T = -2z$ hincque

$$B = \frac{T}{S} = \frac{-2z}{1+zz}$$

posito $z = 0$; unde erit $B = 0$ atque ex factore denominatoris zz orietur unica haec fractio partialis $\frac{1}{zz}$.

EXEMPLUM 2

Sit haec proposita functio fracta

$$\frac{z^3}{(1-z)^2(1+z^4)},$$

cuius ob denomininatoris factorem quadratum $(1-z)^2$ fractiones partiales sint

$$\frac{A}{(1-z)^2} + \frac{B}{1-z}.$$

Erit ergo

$$M = z^3 \text{ et } S = 1 + z^4$$

ideoque

$$A = \frac{M}{S} = \frac{z^3}{1+z^4}$$

posito $1 - z = 0$ seu $z = 1$, unde fit $A = \frac{1}{2}$.
Prodibit ergo

$$M - AS = z^3 - \tfrac{1}{2} - \tfrac{1}{2}z^4 = -\tfrac{1}{2} + z^3 - \tfrac{1}{2}z^4,$$

quod per $1 - z$ divisum dat

$$T = -\tfrac{1}{2} - \tfrac{1}{2}z - \tfrac{1}{2}zz + \tfrac{1}{2}z^3,$$

ideoque

$$B = \frac{T}{S} = \frac{-1-z-zz+z^3}{2+2z^4}$$

posito $z = 1$, ita ut sit $B = -\frac{1}{2}$.
Fractiones ergo partiales quaesitae sunt $\frac{1}{2(1-z)^2} - \frac{1}{2(1-z)}$.

44. *Si functionis fractae* $\frac{M}{N}$ *denominator N factorem, habeat* $(p-qz)^3$, *sequenti modo fractiones partiales ex hoc factore oriundae*

$$\frac{A}{(p-qz)^3}+\frac{B}{(p-qz)^2}+\frac{C}{p-qz}$$

reperientur.

Ponatur $N=(p-qz)^3 S$ sitque fractio ex factore S orta $=\frac{P}{S}$; erit

$$P=\frac{M-AS-B(p-qz)S-C(p-qz)^2 S}{(p-qz)^3}=\text{functioni integrae.}$$

Numerator ergo

$$M-AS-B(p-qz)S-C(p-qz)^2 S$$

ante omnia divisibilis esse debet per $p-qz$, unde is posito $p-qz=0$ seu $z=\frac{p}{q}$ evanescere debet eritque adeo $M-AS=0$ ideoque $A=\frac{M}{S}$ posito $z=\frac{p}{q}$.

Invento hoc pacto A erit $M-AS$ divisibile per $p-qz$; ponatur ergo

$$\frac{M-AS}{p-qz}=T$$

atque

$$T-BS-C(p-qz)S$$

adhuc per $(p-qz)^2$ erit divisibile; fiet ergo $=0$ posito $p-qz=0$, ex quo prodit $B=\frac{T}{S}$ posito $z=\frac{p}{q}$.

Sic autem invento B erit $T-BS$ divisibile per $p-qz$. Hanc ob rem posito $\frac{T-BS}{p-qz}=V$ superest, ut $V-CS$ divisibile sit per $p-qz$, eritque ergo $V-CS=0$ posito $p-qz=0$ atque $C=\frac{V}{S}$ posito $z=\frac{p}{q}$.

Inventis ergo hoc modo numeratoribus A, B, C fractiones partiales ex denominatoris N factore $(p-qz)^3$ ortae erunt

$$\frac{A}{(p-qz)^3}+\frac{B}{(p-qz)^2}+\frac{C}{p-qz}.$$

EXEMPLUM

Sit proposita haec fracta functio

$$\frac{zz}{(1-z)^3(1+zz)},$$

ex cuius denominatoris factore cubico $(1-z)^3$ oriantur hae fractiones partiales

$$\frac{A}{(1-z)^3}+\frac{B}{(1-z)^2}+\frac{C}{1-z}.\ .$$

Erit ergo

$$M=zz \text{ et } S=1+zz;$$

unde fit primum

$$A=\frac{zz}{1+zz}$$

posito $1-z=0$ seu $z=1$, ex quo prodit $A=\frac{1}{2}$.
Iam ponatur $T=\frac{M-AS}{1-z}$;
erit

$$T=\frac{\frac{1}{2}zz-\frac{1}{2}}{1-z}=-\frac{1}{2}-\frac{1}{2}z$$

unde oritur $B=\frac{-\frac{1}{2}-\frac{1}{2}z}{1+zz}$ posito $z=1$, ita ut sit $B=-\frac{1}{2}$.

Ponatur porro

$$V=\frac{T-BS}{1-z}=\frac{T+\frac{1}{2}S}{1-z};$$

erit

$$V=\frac{-\frac{1}{2}z+\frac{1}{2}zz}{1-z}=-\frac{1}{2}z;$$

unde fit

$$C=\frac{V}{S}=\frac{-\frac{1}{2}z}{1+zz}$$

posito $z=1$, ita ut sit $C=-\frac{1}{4}$.
Quocirca fractiones partiales ex denominatoris factore $(1-z)^3$ ortae erunt

$$\frac{1}{2(1-z)^3}-\frac{1}{2(1-z)^2}-\frac{1}{4(1-z)}.$$

45. *Si functionis fractae* ; *denominator* $\frac{M}{N}$ *factorem habeat* $(p-qz)^n$, *fractiones partiales hinc ortae*

$$\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\cdots+\frac{K}{p-qz}$$

sequenti modo invenientur.

Ponatur denominator $N=(p-qz)^n Z$ atque ratiocinium ut ante instituendo reperietur, ut sequitur, primo $A=\frac{M}{Z}$ posito $z=\frac{p}{q}$.

Ponatur $P=\frac{M-AZ}{p-qz}$; secundo $B=\frac{P}{Z}$ posito $z=\frac{p}{q}$.

Ponatur $Q=\frac{P-BZ}{p-qz}$; tertio $C=\frac{Q}{Z}$ posito $z=\frac{p}{q}$.

Ponatur $R=\frac{Q-CZ}{p-qz}$; quarto $D=\frac{R}{Z}$ posito $z=\frac{p}{q}$.

Ponatur $S=\frac{R-DZ}{p-qz}$; quinto $E=\frac{S}{Z}$ posito $z=\frac{p}{q}$, etc.

Hoc ergo modo si definiantur singuli numeratores constantes *A*, *B*, *C*, *D* etc., invenientur omnes fractiones partiales, quae ex denominatoris *N* factore $(p-qz)^n$ nascuntur.

EXEMPLUM

Sit proposita ista functio fracta

$$\frac{1+zz}{z^5(1+z^3)},$$

ex cuius denominatoris factore z^5 nascantur hae fractiones partiales

$$\frac{A}{z^5}+\frac{B}{z^4}+\frac{C}{z^3}+\frac{D}{z^2}+\frac{E}{z}.$$

Ad quarum numeratores constantes inveniendos erit

$$M=1+zz \text{ atque } Z=1+z^5 \text{ et } \frac{p}{q}=0.$$

Sequens ergo calculus ineatur.

Primum est $A=\frac{M}{Z}=\frac{1+zz}{1+z^3}$ posito $z=0$, ergo $A=1$.

Ponatur $P=\frac{M-AZ}{z}=\frac{zz-z^3}{z}=z-zz$, eritque secundo $B=\frac{P}{Z}=\frac{z-zz}{1+z^3}$ posito $z=0$, ergo $B=0$.

Ponatur $Q=\frac{P-BZ}{p-qz}=\frac{z-zz}{z}=1-z$, eritque tertio $C=\frac{Q}{Z}=\frac{1-z}{1+z^3}$ posito $z=0$, ergo $C=1$.

Ponatur $R=\frac{Q-CZ}{z}=\frac{-z-z^3}{z}=-1-zz$; erit quarto $D=\frac{R}{Z}=\frac{-1-zz}{1+z^3}$ posito $z=0$, ex quo fit $D=-1$.

Ponatur $S=\frac{R-DZ}{p-qz}=\frac{-zz+z^3}{z}=-zz+zz$; erit quinto $E=\frac{S}{Z}=\frac{-z+zz}{1+z^3}$ posito $z=0$, unde fit $E=0$.

Quocirca fractiones partiales quaesitae erunt hae:

$$\frac{1}{z^5}+\frac{0}{z^4}+\frac{1}{z^3}-\frac{1}{z^2}+\frac{0}{z}.$$

45a. *Quaecunque ergo proposita fuerit functio rationalis fracta* $\frac{M}{N}$*, ea sequenti modo in partes resolvetur atque in formam simplicissimarn transmutabitur.*

Quaerantur denominatoris N omnes factores simplices sive reales sive imaginarii; quorum qui sibi pares non habeant, seorsim tractentur et ex unoquoque per § 41 fractio partialis eruatur. Quodsi idem factor simplex bis vel pluries occurrat, ii coniunctim sumantur atque ex eorum producto, quod erit potestas formae $(p-qz)^n$, quaerantur fractiones partiales convenientes per § 45. Hocque modo cum ex singulis factoribus simplicibus denominatoris erutae fuerint fractiones partiales, tum harum omnium aggregatum aequabitur functioni propositae $\frac{M}{N}$, nisi fuerit spuria; si enim fuerit spuria, pars integra insuper extrahi atque ad istas fractiones partiales inventas adiici debebit, quo prodeat valor functionis $\frac{M}{N}$ in forma simplicissima expressus. Perinde autem est, sive fractiones partiales ante extractionem partis integrae sive post quaerantur. Eaedem enim ex singulis denominatoris N factoribus prodeunt fractiones partiales, sive adhibeatur ipse numerator M sive idem quocunque denominatoris N multiplo vel auctus vel minutus; id quod regulas datas contemplanti facile patebit.

EXEMPLUM

Quaeratur valor functionis

$$\frac{1}{z^3(1-z)^2(1+z)}$$

in forma simplicissima expressus.

Sumatur primum factor denominatoris solitarius $1+z$, qui dat $\frac{p}{q}=-1$;

erit $M=1$ et $Z=z^3-2z^4+z^5$.

Hinc ad fractionem $\frac{A}{1+z}$ inveniendam erit

$$A=\frac{1}{z^3-2z^4+z^5}$$

posito $z=-1$ ideoque fit $A=-\frac{1}{4}$ atque ex factore $1+z$ oritur haec fractio partialis

$$-\frac{1}{4(1+z)}.$$

Iam sumatur factor quadratus $(1-z)^2$, qui dat

$$\tfrac{p}{q}=1,\ \ M=1 \ \text{ et } \ Z=z^3+z^4$$

Positis ergo fractionibus partialibus hinc ortis

$$\frac{A}{(1-z)^2}+\frac{B}{1-z}$$

erit

$$A=\frac{1}{z^3+z^4}$$

posito $z=1$, ergo $A=\frac{1}{2}$.
Fiat

$$P=\frac{M-\frac{1}{2}Z}{1-z}=\frac{1-\frac{1}{2}z^3-\frac{1}{2}z^4}{1-z}=1+z+zz+\tfrac{1}{2}z^3$$

eritque

$$B=\frac{P}{Z}=\frac{1+z+zz+\frac{1}{2}z^3}{z^3+z^4}$$

posito $z=1$, ergo $B=\frac{7}{4}$ et fractiones partiales quaesitae

$$\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}$$

Denique tertius factor cubicus z^3 dat

$$\tfrac{p}{q}=1,\ \ M=1 \ \text{ et } \ Z=1-z-zz+z^3$$

Positis ergo fractionibus partialibus his

$$\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}$$

erit primum

$$A=\frac{M}{Z}=\frac{1}{1-z-zz+z^3}$$

posito $z=0$, ergo

$$A=1.$$

Ponatur

$$P=\frac{M-Z}{z}=1+z-zz$$

erit

$$B=\frac{P}{Z}$$

posito $z=0$, ergo $B=1$.
Ponatur

$$Q=\frac{P-Z}{z}=2-zz$$

erit

$$C = \frac{Q}{Z}$$

posito $z = 0$, ergo $C = 2$.

Hanc ob rem functio proposita

$$\frac{1}{z^3(1-z)^2(1+z)}$$

in hanc formam resolvitur

$$\frac{1}{z^3} + \frac{1}{z^2} + \frac{2}{z} + \frac{1}{2(1-z)^2} + \frac{7}{4(1-z)} - \frac{1}{4(1+z)}.$$

Nulla. enim pars integra insuper accedit, quia fractio proposita non est spuria.