# INTRODUCTION
TO THE
# ANALYSIS OF THE INFINITES.

## FIRST BOOK.

## CHAPTER I

### ABOUT THE KINDS OF FUNCTIONS

1. *A constant quantity is known and continually keeps the same value.*

Quantities of this sort are numbers of any kind, which certainly maintain the same value continually, once they have been put in place; and if it is agreed to designate constant quantities by letters, the first letters of the alphabet $a$, $b$, $c$ etc are used. Indeed in common analysis, where only determined magnitudes are considered, these first letters of the alphabet are accustomed to denote the known magnitudes, truly the latter letters denoting unknown quantities ; but in higher analysis this distinction is not observed so much, since here some distinct magnitude may be considered especially, for which other constants and indeed other variables are put in place.

2. *A variable quantity is an indeterminate or universal quantity, which includes within itself all completely determined values.*

Therefore since all the values of the numbers to be determined are able to be expressed, the magnitude of a variable involves all the numbers of any kind whatever. Evidently to the extent that ideas of the appearances and kinds of variables are formed from notions of their indivisibility : thus a variable quantity is a kind, within which all the determined magnitudes may be contained. Moreover variable quantities of this kind are accustomed to be represented by the later letters of the alphabet $z$, $y$, $x$ etc.

3. *A variable quantity is determined, provided that some known value is attributed to that.*

Therefore a variable quantity can be determined in innumerable ways, since entirely all numbers may be substituted in place of that. Nor will all the significant magnitudes of the variable be exhausted, unless all the values of this to be determined should be substituted. Therefore the magnitude of a variable includes within itself evidently all the numbers, both positive and negative, both integers and fractions, both rational as well as irrational and transcending. So that zero and imaginary numbers too are not excluded from being significant magnitudes of variables.

4. *A function of a variable quantity is an analytical expression of some kind composed from that variable quantity and from constant numbers or magnitudes.*

Therefore every analytical expression, in which besides the variable quantity $z$ all the quantities composing that expression are constants, will be a function of $z$. Thus $a+3z$, $az-4zz$, $az+b\sqrt{(aa-zz)}$, $c^z$ etc. are functions of $z$.

5. *Therefore a function of a variable quantity will itself be a variable quantity.*

For since all the determined values are allowed to be substituted in place of a variable magnitude, hence a function may adopt innumerable determined values; nor any determined value may be excepted, that the function may not be able to adopt, since the magnitude of the variable may involve imaginary values also. Thus, even if this function

$$\sqrt{(9-zz)}$$

at no time is able to receive real numbers substituted in place of $z$ greater than the number three, yet imaginary values of $z$ may to be attributed, so that $5\sqrt{-1}$ cannot be assigned any determined value [of $z$] that cannot be elicited from the formula $\sqrt{(9-zz)}$. But sometimes only apparent functions occur, which no matter how the magnitude of the variable is changed, still maintain the same value all the time, such as

$$z^0,\ 1^z,\ \tfrac{aa-az}{a-z},$$

which, even if the appearance of the function is deceptive, yet actually are constant quantities.

6. *The particular characteristic of functions is put in place in the manner of composition, by which they are formed from a variable quantity and constant quantities.*

Therefore it depends on how the operations, by which the magnitudes are able to be put in place and to be mixed among themselves ; which operations are addition and subtraction, multiplication and division, by raising to powers and by the extraction of roots, by which also the resolution of equations is to be referred. Besides these operations, which are accustomed to be called algebraic, many other transcending functions are given, as exponentials, logarithmic and innumerable others, which the integral calculus provides.

Meanwhile certain kinds of functions are able to be noted, as the multiples

$$2z, 3z, \tfrac{3}{5}z, az, \text{ etc.}$$

and powers of $z$, as

$$z^2,\ z^3, z^{\frac{1}{2}}, z^{-1}\ \text{ etc.;}$$

which have been chosen to use from a single operation, thus the expressions, which arise from operations of any kind, are signified by the name function.

7. *Functions are divided into algebraic and transcending kinds ; the former are those which are composed from algebraic operations only; truly the latter, in which transcending operations are involved.*

Therefore multiples and powers of $z$ are algebraic functions of z, and all expressions entirely, which are formed by the aforementioned algebraic operations, an example of this kind is

$$\frac{a+bz^n-c\sqrt{(2z-z^2)}}{aaz-3bz^3}$$

But also algebraic functions are not able to be shown explicitly on many occasions ; $Z$ is a function of this kind of $z$, if it may be defined for example by an equation of this kind :

$$Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1.$$

[Thus, Euler is now dealing with multi–valued functions $Z$ of $z$.]
For although this equation is unable to be resolved, yet it is agreed that $Z$ be equal to some expression composed from the variable $z$ and constants, and on that account $Z$ to be a certain function of $z$. Moreover concerning transcending functions it is to be noted that these finally become transcending, not only if a transcending operation is present, but also it may affect the magnitude of the variable. For if transcending operations should only pertain to constant quantities, the function nevertheless is required to be considered as algebraic; just as if $c$ denotes the circumference of a circle, its radius shall be $=1$, and as $c$ shall be a transcending quantity, yet truly these expressions are algebraic functions of $z$ :

$$c+z,\quad cz^3,\quad 4z^c\quad \text{etc.}$$

Indeed there is the smallest moment of doubt, by which one may be moved from certainty, whether or not expressions of this kind $z^c$ may justly be counted as algebraic expressions; because they are powers of $z$ also, the exponents of which may be irrational numbers, such as $z^{\sqrt{2}}$, some are inclined to call the functions *interscending* rather than algebraic.

8. *Algebraic functions are subdivided into rationals and irrationals; functions are of the former kind, if the variable is not involved irrationally ; truly of the latter, in which the signs of the roots affect the magnitude of the variable.*

Therefore in rational functions all operations besides addition, subtraction, multiplication, division and raising to powers, the exponents of which shall be whole numbers, are not present ; therefore

$$a+z, \quad a-z, \quad az, \quad \tfrac{aa+zz}{a+z}, \quad az^3-bz^5 \text{ etc.}$$

will be rational functions of *z*. But expressions of this kind :

$$\sqrt{z}, \quad a+\sqrt{(aa-zz)}, \quad \sqrt[3]{(a-2z+zz)}, \quad \frac{aa-z\sqrt{(aa-zz)}}{a+z} \text{ etc.}$$

will be irrational functions of *z*.

*These may be separated conveniently into explicit and implicit functions.*

The functions are *explicit*, which have been set out through root signs, examples of this kind have been given just now. The functions are *implicit*, which truly are irrational functions, which have arisen from the resolution of equations. Thus *Z* will be an implicit function of irrational *z*, if it may be defined by an equation of this kind

$$Z^7 = azZ^2 - bz^5,$$

since the explicit value for *Z* cannot be shown with the root signs admitted, on that account by common algebra which has not yet risen to that level of perfection.

9. *Rational functions may be subdivided anew into whole and fractional functions.*

In the former, *z* anywhere has neither negative exponents nor expressions containing fractions, in which the denominators may have the variable quantity *z* present; from which it is understood the latter fractional functions to be fractions, in which denominators containing *z* or negative exponents of *z* occur. Therefore the general formula of these integral functions will be

$$a+bz+cz^2+dz^3+ez^4+fz^5+\text{etc.};$$

for no integral function of *z* is able to be devised, which is not contained in this expression. But all fractional functions, because they are able to collect several fractions into one in place, will be contained in this formula.

$$\frac{a+bz+cz^2+dz^3+ez^4+fz^5+\text{etc.}}{\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\xi z^5+\text{etc.}},$$

where it is required to note the constant quantities *a, b, c, d* etc., $\alpha,\beta,\gamma,\delta$ etc., which may be either positive or negative, whole or fractions, rational or irrational, or even transcending, do not change the nature of the functions.

10. *Then above all, the division of functions is to be considered into the uniform [i.e. one form or in modern terms single– valued function] and multiform kinds [i.e. many forms or multiple–valued functions].*

A function moreover is *uniform*, which, if some value be granted to the variable quantity *z*, itself may be attributed a single determined value also. But a function is *multiform*, which, for any single value determined substituted in place of the variable *z*, shows several determined values.

Therefore all the rational functions, either whole or fractional, are uniform functions, because expressions of this kind, whichever value may be granted to the variable quantity, only provide a single value. But all irrational functions are multiforms, because the root signs are ambiguous and therefore involve a twin value. But they are given also between transcending functions and both uniform and multiform functions ; why not indeed may there not be functions had of infinite forms, of this kind is the arc of a circle corresponding to the sine *z* ; for innumerable circular arcs are given which all may have the same sine.

Moreover these letters *P, Q, R, S, T* etc. will all denote single uniform functions of *z.*

11. *A biform function of z [i.e. one with two forms] is a function of such a kind, which bears twin values for some determined value of z.*

Square roots show functions of this kind, such as $\sqrt{(2z+zz)}$ *;* for whichever agreed value may be put in place for *z*, the expression $\sqrt{(2z+zz)}$ has a two–fold significance, either positive or negative. Truly generally *Z* will be a biform function of *z*, if it may be determined by the quadratic equation.

$$Z^2-PZ+Q,$$

if indeed *P* and *Q* were uniform functions of *z.* In as much as it will be

$$Z=\tfrac{1}{2}P\pm\sqrt{\left(\tfrac{1}{4}P^2-Q\right)};$$

from which it is apparent that for this determined value of *z* corresponds a double determined value of *Z*. But here it is to be noted that either each value of the function *z* is real or imaginary. Truly there will always be moreover, as agreed from the nature of the equation, the sum of two values of $Z = P$ and the product $= Q$.

12. *A function of z is tri–form [i.e. one of three forms, or of three kinds for a given z], which shows three determined values for some value of z.*

Functions of this kind are drawn from the resolution of an original cubic equation. For if *P, Q* and *R* were uniform functions and there were

$$Z^3 - PZ^2 + QZ - R = 0,$$

*Z* will be a tri–form function of *z*, because for whatever the determined value of *z*, three–fold values of *Z* prevail. The three values of *Z* for that one corresponding value of *z* either are all real or one will be real, while the two remaining are imaginary. Moreover it is agreed that the sum of the three values always to be $= P$, the sum of the factors from pairs of values to be $= Q$ and the product from all three to be $= R$.

13. *A function of z is of the quadri–form kind, which shows four determined values for some value of z.*

Functions of this kind arise from the resolution of biquadratic equations.
For if *P, Q, R* and *S* denote uniform functions of *z*, and the equation was

$$Z^4 - PZ^3 + QZ^2 - RZ + S = 0,$$

then *Z* will be a quadri–form function of *z*, because for that value of *z* there corresponds four–fold values of *Z*. Therefore of these values either all are real, or two are real and two imaginary, or all four will be imaginary. Moreover always the sum of these four values of *Z* is $= P$, the sum of the factors taken in pairs $= Q$, the sum of the factors taken in triples $= R$ and the product of all $= S$.

Moreover the account is prepared in a similar manner of functions of five forms and of the following.

14. *Therefore Z will be a multiform function of z, which for some value of z exhibits just as many values, as the number n contains units, if Z were defined by this equation :*

$$Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \text{etc.} = 0.$$

Where indeed it is noted that *n* is required to be a whole number always, so that it will be possible to decide, which multiform the function *Z* shall be of *z*, and the equation by which *Z* is defined may be reduced to rationality ; with which done the exponent of the greatest power of *Z* will indicate the number of values sought corresponding to the value of $z$ itself. Then also the letters $P, Q, R, S$ etc. required to be considered that should denote the uniform functions of $z$; indeed if some of these now should be multiform functions, then the function $Z$ may be going to offer many more values corresponding to a single value of $z$, than indeed the number of dimensions of $Z$ may indicate. But always, if which values of $Z$ were imaginary, the number of these will be even ; from which it is understood, if $n$ were an odd number, always at least one value of $Z$ to be real, but on the contrary it can happen, if the number $n$ were even, that in short no value of $Z$ shall be real.

15. *If Z were a multiform function of z of such a kind, that always it may have present only a single real value, then Z will imitate a uniform function of z and often will be able to be used in place of a uniform function.*

Functions of this kind will be :

$$\sqrt[3]{P}, \quad \sqrt[5]{P}, \quad \sqrt[7]{P}, \quad \text{etc.},$$

clearly which present only a single real value always with all the rest present imaginary, provided $P$ were a uniform function of $z$. Hence on this account an expression of this kind $P^{\frac{m}{n}}$, whenever $n$ should be an odd number, will be able to be counted with the uniform functions, whether $m$ were an even or odd number. But if $n$ were an even number, then $P^{\frac{m}{n}}$ either will have no real value or two ; from which expressions of this kind $P^{\frac{m}{n}}$ with the number $n$ present equal for the same pair justly can be added to the biform functions, if indeed the fraction $\frac{m}{n}$ were not reducible to smaller terms.

16. *If y were some function of z, then in turn z will be a function of y.*

Since indeed $y$ shall be a function of $z$, either uniform or multiform, an equation will be given, from which $y$ is defined through $z$ and constant quantities. Truly from the same equation in turn $z$ is able to be defined through $y$ and constants; from which, because $y$ is a variable quantity, $z$ will be equal to an expression composed from $y$ and with constants and thus it will be a function of $y$. Hence it will be apparent also, as $z$ will soon be a multiform function of $y$, and it can happen, that even if $y$ were a uniform function of $z$, yet $z$ may soon become a multiform function of $y$. Thus, if $y$ may be defined from this equation through $z$

$$y^3 = ayz - bzz,$$

each $y$ will be a tri–form function of $z$, truly on the contrary $z$ will be only a biform function of $y$.

17. *If y and x were functions of z, then y also will be a function of x and in turn x a function of y.*

Since indeed *y* shall be a function of *z,* also *z* will be a function of *y* and in a similar manner also *z* will be a function of *x.* On that account the function of *y* will be equal to the function of *x;* from which equation both *y* through *x* and in turn *x* through *y* will be able to be defined ; on account of which it is evident that *y* is a function of *x* and *x* a function of *y.* Indeed most often these functions are not able to be shown explicitly on account of the failure of algebra ; yet meanwhile nevertheless, as if all equations were able to be resolved, this reciprocation of the functions is observed. Moreover by the method treated in algebra, from pairs of equations given, of which the one contains *y* and *z,* truly the other *x* et *z,* by elimination of the quantity *z,* one equation will be formed expressing the relation between *x* and *y.*

18. *Finally certain kinds of singular functions are required to be noted ; thus an even function of z is one which gives the same value, whether for z there may be put the value* $+k$ *or* $-k$.

An even function of this kind of *z* will be *zz;* for there may be put $z=+k$ or $z=-k$, the expression *zz* will present the same value, in short $zz=+kk$. In a similar manner even functions of *z* will be these of the powers of *z* $z^4$, $z^6$, $z^8$ and generally all the powers $z^m$, if *m* were an even number, either *m* positive or negative. Why not also, since $z^{\frac{m}{n}}$ may be passed over as a uniform function of *z*, if *n* shall be an odd number, it is evident that $z^{\frac{m}{n}}$ becomes an even function of *z,* if *m* were an even number, *n* truly an odd number. Hence on this account expressions from powers of this kind composed in some manner will provide even functions of *z;* thus *Z* will be an even function of *z,* if it should be

$$Z=a+bz^2+cz^4+dz^6+\text{ etc.},$$

likewise if it were

$$Z=\frac{a+bz^2+cz^4+dz^6+\text{etc.}}{\alpha+\beta z^2+\gamma z^4+\delta z^6+\text{etc.}}.$$

And in a similar manner by introducing fractional exponents of *z*, *Z* will be an even function of *z,* if it were

$$Z=a+bz^{\frac{2}{3}}+cz^{\frac{2}{5}}+dz^{\frac{4}{7}}+\text{ etc.}$$

or

$$Z=a+bz^{\frac{2}{3}}+cz^{\frac{4}{3}}+dz^{\frac{2}{5}}+\text{ etc.}$$

or

$$Z = \frac{a+bz^{\frac{2}{7}}+cz^{\frac{4}{5}}+dz^{\frac{8}{3}}+\text{etc.}}{\alpha+\beta z^{\frac{2}{3}}+\gamma z^{\frac{2}{5}}+\delta z^{\frac{4}{7}}+\text{etc.}}.$$

Expressions of this kind, since all shall be uniform functions of *z*, can even be called uniform functions of *z*.

19. *A function is an even multiform of z ,which, even if it may show several determined values for some value of z, still proffers the same values, whether* $z = +k$ *or* $z = -k$ *is put in place.*

Let *Z* be an even multiform function of *z* of this kind ; because the nature of a multiform function is expressed by an equation between *Z* and *z*, in which *Z* may have just as many dimensions, as the various values may be multiply together, it is evident that *Z* becomes an even multiform function, if in the equation expressing the nature of *Z*, the magnitude of the variable *z* may have even dimensions everywhere. Thus, if it were

$$Z^2 = az^4 Z + bz^2$$

*Z* will be a an even biform of *z;* but if it were

$$Z^3 - az^2 Z^2 + bz^4 Z - cz^8 = 0,$$

*Z* will be an even tri–form function of *z ;* and generally, if *P*, *Q*, *R*, *S* etc. denote uniform even functions of *z*, *Z* will be an even biform function of *z*, if it shall be

$$Z^2 - PZ + Q = 0.$$

But *Z* will be an even tri–form function of *z*, if it shall be

$$Z^3 - PZ^2 + QZ - R = 0,$$

and thus henceforth.

20. *Therefore an even function of z, whether uniform or multiform, will be an expression of such a kind, assembled from the magnitude of the variable z and constants, in which the number of the dimensions of z will be even everywhere.*

Therefore functions of this kind besides the uniform, examples of which have been reported before, will be these expressions:

$$a+\sqrt{(bb-zz)},\ azz+\sqrt[3]{\left(a^6z^4-bz^2\right)},\ \text{likewise}\ az^{\frac{2}{3}}+\sqrt[3]{\left(z^2+\left(a^4-z^4\right)\right)}\ \text{etc.}$$

*From which it is apparent that even functions thus can be defined, so that they are said to be functions of zz.*

If indeed $y = zz$ is put in place and $Z$ becomes some function of $y$, by restoring everywhere $zz$ in place of $y$, $Z$ will be a function of $z$ of this kind, in which $z$ everywhere may have the dimensions of an even number. Yet these cases are required to be excepted, in which in the expression of $Z$, $\sqrt{y}$ occur and all the forms of this kind, which admit the root signs from the fact $y = zz$. For $y + \sqrt{ay}$ shall be some function of $y$, yet with $y = zz$ in place the same expression will not be an even function of $z$, since it becomes

$$y + \sqrt{ay} = zz + z\sqrt{a}.$$

Therefore with these cases excluded the final definition of even functions will be good and suitable for the formation of functions of this kind.

21. *An odd function of z is a function of such a kind, the value of which, if – z is put in place of z, will be made negative too.*

Therefore all the powers of $z$, of which the exponents are odd numbers, such as $z^1$, $z^2$, $z^5$, $z^7$ etc., will be functions of this kind, likewise $z^{-1}$, $z^{-3}$, $z^{-5}$ etc.; then truly also $z^{\frac{m}{n}}$ will be an odd function, if both the numbers $m$ and $n$ were odd numbers. Generally indeed every expression composed from powers of this kind will be an odd function of $z$; functions of this kind are :

$$az + bz^3,\ az + az^{-1},\ \text{likewise}\ z^{\frac{1}{3}} + az^{\frac{2}{5}} + bz^{-\frac{5}{3}}\ \text{etc.}$$

Moreover the nature and finding of these functions may be seen more easily from even functions.

22. *If an even function of z may be multiplied by z or by some odd function of the same, an odd function of z will be produced..*

Let $P$ be some even function of $z$, which on that account remains the same, if $-z$ may be put in place of $z$; and if therefore in the product $Pz$, $-z$ may be put in place of $z$, $-Pz$ will be produced, from which $Pz$ will be an odd function of $z$. Now let $P$ be some even function of z and $Q$ some odd function of $z$, and it is apparent from the definition, if $-z$ is put in place of $z$, the value of $P$ remains the same, but the value of $Q$ will change into its negative $-Q$, whereby the product $PQ$ on putting $-z$ in place of $z$ will be changed into $-PQ$, that is into its negative, and thus $PQ$ will be an odd function of $z$. Thus, since $a + \sqrt{(aa + zz)}$ shall be an even function, and $z^3$ an odd function of $z$, the product will be

$$az^3 + z^3\sqrt{(aa + zz)}$$

an odd function of *z*, and in a similar manner

$$z\times\frac{a+bzz}{\alpha+\beta zz}=\frac{az+bz^3}{\alpha+\beta zz}$$

will be an odd function of *z*.

From these truly it is understood also, if of the two functions *P* and *Q*, of which the one *P* is even, the other *Q* odd, the one may be divided by the other to become an odd function; therefore $\frac{P}{Q}$, and likewise $\frac{Q}{P}$, will be an odd function of *z*.

23. *If an odd function may be multiplied or divided by an odd function, what results will be an even function.*

*Q* and *S* shall be odd functions of *z*, thus so that on putting $-z$ in place of *z*, *Q* changes into $-Q$ and *S* into $-S$, and it is observed that the product *QS* as well as the quotient $\frac{Q}{S}$ retain the same value, even if for *z* there may be put $-z$, and thus each is an even function of *z*. And again it is evident that the square of odd functions of this kind are even functions, truly the cube an odd function, the biquadratic again an even function, and thus henceforth.

24. *If y were an odd function of z, in turn z will be an odd function of y.*

For since *y* shall be an odd function of *z*, if $-z$ may be put in place of *z*, *y* will change into $-y$. And therefore if *z* may be defined by *y*, it is necessary that, on putting $-y$ in place of *y*, also *z* will change into $-z$, and thus *z* will be an odd function of *y*. Thus, because on putting

$$y=z^3$$

*y* is an odd function of *z*, also from the equation

$$z^3=y \text{ or } z=y^{\frac{1}{3}}$$

*z* will be an odd function of *y*. And because, if it becomes

$$y=az+z^3$$

*y* is an odd function of *z*, in turn from the equation

$$bz^3+az=y$$

the value of *z* expressed by *y* will be an odd function of *y*.

25. *If the nature of the function y may be defined by an equation of such a kind, in the number of the dimensions of the individual terms, which y and z occupy jointly, each shall be either even or odd, then y will be an odd function of z.*

For if indeed in an equation of this kind everywhere $-z$ may be written in place of $z$ and likewise $-y$ in place of $y$, all the terms of the equation either remain the same or become negative, and in each case truly the equation will remain the same. From which it is apparent $-y$ is going to be determined in the same manner by $-z$, by which $+y$ is determined by $+z$, and on that account, if in place of $z$ there may be put $-z$, the value of $y$ will be changed into $-y$ or $y$ will be an odd function of $z$. Thus, if there were either

$$yy = ayz + bzz + c$$

or

$$y^3 + ayyz = byzz + cy + dz,$$

from each equation $y$ will be an odd function of $z$.

26. *If Z were a function of z and Y a function of y and Y may be defined in the same manner by the variable y and constants, by which Z is defined by the variable z and constants, then these functions Y and Z are called similar functions of the same y and z.*

Evidently if the equations were

$$Z = a + bz + cz^2 \text{ and } Y = a + by + cy^2,$$

$Z$ and $Y$ will be similar functions of $z$ and $y$ and in a similar manner with multiforms if there were

$$Z^3 = azzZ + b \text{ and } Y^3 = ayyY + b,$$

$Z$ and $Y$ will be similar functions of $z$ and $y$. Hence it follows, if $Y$ and $Z$ were similar functions of the same $y$ and $z$, then, if $y$ may be written in place of $z$, the function $Z$ will be changed into the function $Y$. It is usual to express this similarity in words, so that $Y$ is said to be such a function of $y$, as $Z$ shall be such a function of $z$. These expressions likewise occur, whether the variable quantities $z$ and $y$ may depend on each other or otherwise ; thus,

$$ay + by^3$$

is such a function of $y$;

$$a(y+n) + b(y+n)^3$$

will be such a function of $y + n$, evidently with $z = y + n$ arising; then,

$$\frac{a+bz+czz}{\alpha+\beta z+\gamma zz}$$

is such a function of *z*,

$$\frac{azz+bz+c}{\alpha zz+\beta z+\gamma}$$

will be such a function of $\frac{1}{z}$ on putting $y=\frac{1}{z}$. And from these the account of the similarity of the of the functions is observed most clearly, the use of which in higher analysis is most productive.

Moreover these are able to suffice in general concerning the nature of functions of one variable, since a fuller exposition is treated in the following application.

,

# INTRODUCTIO
IN
# ANALYSIN INFINITORUM.

## LLBER PRIMUS

### CAPUT I
### DE FUNCTIONIBUS IN GENERE

1. *Quantitas constans est quantitas determinata perpetuo eundem valorem servans.*

Eiusmodi quantitates sunt numeri cuiusvis generis, quippe qui eundem, quem semel obtinuerunt, valorem constanter conservant; atque si huiusmodi quantitates constantes per characteres indicare convenit, adhibentur litterae alphabeti initiales $a$, $b$, $c$ etc. In Analysi quidem communi, ubi tantum quantitates determinatae considerantur, hae litterae alphabeti priores quantitates cognitas denotare solent, posteriores vero quantitates incognitas; at in Analysi sublimiori hoc discrimen non tantopere spectatur, cum hic ad illud quantitatum discrimen praecipue respiciatur, quo aliae constantes, aliae vero variabiles statuuntur.

2. *Quantitas variabilis est quantitas indeterminata seu universalls, quae omnes omnino valores determinatos in se complectitur.*

Cum ergo omnes valores determinati numeris exprimi queant, quantitas variabilis omnes numeros cuiusvis generis involvit. Quemadmodum scilicit ex ideis individuorum formantur ideae specierum et generum, ita quantitas variabilis est genus, sub quo omnes quantitates determinatae continentur.

Huiusmodi autem quantitates variabiles per litteras alphabeti postremas $z$, $y$, $x$ etc. repraesentari solent.

3. *Quantitas variabilis determinatur, dum ei valor quicunque determinatus tribuitur.*

Quantitas ergo variabilis innumerabiibus modis determinari potest, cum omnes omnino numeros eius loco substituere liceat. Neque significatus quantitatis variabilis exhauritur, nisi omnes valores determinati eius loco fuerint substituti. Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales et transcendentes. Quin etiam cyphra et numeri imaginarii a significatu quantitatis variabilis non excluduntur.

4. *Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili et numeris seu quantitatibus constantibus.*

Omnis ergo expressio analytica, in qua praeter quantitatem variabilem $z$ omnes quantitates illam expressionem componentes sunt constantes, erit functio ipsius $z$. Sic $a+3z$, $az-4zz$, $az+b\sqrt{(aa-zz)}$, $c^z$ etc. sunt functiones ipsius $z$.

5. *Functio ergo quantitatis variabilis ipsa erit quantitas variabilis.*

Cum enim loco quantitatis variabilis omnes valores determinatos substituere liceat, hinc functio innumerabiles valores determinatos induet; neque ullus valor determinatus excipietur, quem functio induere nequeat, cum quantitas variabilis quoque valores imaginarios involvat. Sic, etsi haec functio

$$\sqrt{(9-zz)}$$

numeris realibus loco $z$ substituendis nunquam valorem ternario maiorem recipere potest, tamen ipsi $z$ valores imaginarios tribuendo, ut $5\sqrt{-1}$, nullus assignari poterit valor determinatus, quin ex formula $\sqrt{(9-zz)}$ elici queat. Occurrunt autem nonnunquam functiones tantum apparentes, quae, utcunque quantitas variabilis varietur, tamen usque eundem valorem retinent, ut

$$z^0,\ 1^z,\ \tfrac{aa-az}{a-z},$$

quae, etsi speciem functionis mentiuntur, tamen revera sunt quantitates constantes.

6. *Praecipuum functionum discrimen in modo compositionis, quo ex quantitate variabili et quantitatibus constantibus formantur, positum est.*

Pendet ergo ab operationibus, quibus quantitates inter se componi et permisceri possunt; quae operationes sunt additio et subtractio, multiplicatio et divisio, evectio ad potestates et radicum extractio, quo etiam resolutio aequationum est referenda. Praeter has operationes, quae algebraicae vocari solent, dantur complures aliae transcendentes, ut exponentiales, logarithmicae atque innumerabiles aliae, quas Calculus integralis suppeditat.

Interim species quaedam functionum notari possunt, ut multipla

$$2z, 3z, \tfrac{3}{5}z, az, \text{ etc.}$$

et potestates ipsius $z$, ut

$$z^2, z^3, z^{\frac{1}{2}}, z^{-1} \text{ etc.;}$$

quae uti ex unica operatione sunt desumptae, ita expressiones, quae ex operationibus quibuscunque nascuntur, functionum nomine insigniuntur.

7. *Functiones dividuntur in algebraicas et transcendentes; illae sunt, quae componuntur per operationes algebraicas solas; hae vero, in quibus operationes transcendentes insunt.*

Sunt ergo multipla ac potestates ipsius $z$ functiones algebraicae atque omnes omnino expressiones, quae per operationes algebraicas ante memoratas formantur, cuiusmodi est

$$\frac{a+bz^n-c\sqrt{(2z-z^2)}}{aaz-3bz^3}$$

Quin etiam functiones algebraicae saepenumero ne quidem explicite exhiberi possunt, cuiusmodi functio ipsius $z$ est $Z$, si definiatur per huiusmodi aequationem

$$Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1.$$

Quanquam enim haec aequatio resolvi nequit, tamen constat $Z$ aequari expressioni cuipiam ex variabili $z$ et constantibus compositae ac propterea fore $Z$ functionem quandam ipsius $z$. Ceterum de functionibus transcendentibus notandum est eas demum fore transcendentes, si operatio transcendens non solum ingrediatur, sed etiam quantitatem variabilem afficiat. Si enim operationes transcendentes tantum ad quantitates constantes pertineant, functio nihilominus algebraica est censenda; uti si $c$ denotet circumferentiam circuli, cuius radius sit $= 1$, erit utique $c$ quantitas transcendens, verumtamen hae expressiones

$$c+z, \quad cz^3, \quad 4z^c \quad \text{etc.}$$

erunt functiones algebraicae ipsius $z$. Parvi quidem est momenti dubium, quod a quibusdam movetur, utrum eiusmodi expressiones $z^c$ functionibus algebraicis annumerari iure possint necne; quin etiam potestates ipsius $z$, quarum exponentes sint numeri irrationales, uti $z^{\sqrt{2}}$, nonnulli maluerunt functiones *interscendentes* quam algebraicas appellare.

8. *Functiones algebraicae subdividuntur in rationales et irrationales; illae sunt,* si *quantitas variabilis in nulla irrationalitate involvitur; hae vero, in quibus signa· radicalia quantitatem variabilem afficiunt.*

In functionibus ergo rationaibus aliae operationes praeter additionem, subtractionem, multiplicationem, divisionem et evectionem ad potestates, quarum exponentes sint numeri integri, non insunt; erunt adeo

$$a+z,\quad a-z,\quad az,\quad \tfrac{aa+zz}{a+z},\quad az^3-bz^5 \text{ etc.}$$

functiones rationales ipsius *z.* At huiusmodi expressiones

$$\sqrt{z},\quad a+\sqrt{(aa-zz)},\quad \sqrt[3]{(a-2z+zz)},\quad \frac{aa-z\sqrt{(aa-zz)}}{a+z} \text{ etc.}$$

erunt functiones irrationales ipsius *z.*

*Hae commode distinguuntur in explicitas et implicitas.*

*Explicitae* sunt, quae per signa radicalia sunt evolutae, cuiusmodi exempla modo sunt data. *Implicitae* vero functiones irrationales sunt, quae ex resolutione aequationum ortum habent. Sic *Z* erit functio irrationalis implicita ipsius *z,* si per huiusmodi aequationem

$$Z^7=azZ^2-bz^5$$

definiatur, quoniam valorem explicitum pro *Z* admissis etiam signis radicalibus exhibere non licit, propterea quod Algebra communis nondum ad hunc perfectionis gradum est evecta.

9. *Functiones rationales denuo subdividuntur in integras et fractas.*

In illis neque *z* usquam habet exponentes negativos neque expressiones continent fractiones, in quarum denominatores quantitas variabilis *z* ingrediatur; unde intelligitur functiones fractas esse, in quibus denominatores *z* continentes vel exponentes negativi ipsius *z* occurrant. Functionum integrarum haec ergo erit formula generalis

$$a+bz+cz^2+dz^3+ez^4+fz^5+\text{etc.};$$

nulla enim functio ipsius *z* integra excogitari potest, quae non in hac expressione contineatur. Functiones autem fractae omnes, quia plures fractiones in unam cogi possunt, continebuntur in hac formula

$$\frac{a+bz+cz^2+dz^3+ez^4+fz^5+\text{etc.}}{\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\xi z^5+\text{etc.}},$$

ubi notandum est quantitates constantes *a, b, c, d* etc., $\alpha,\beta,\gamma,\delta$ etc., sive sint affirmativae sive negativae, sive integrae sive fractae, sive rationales sive irrationales sive etiam transcendentes, naturam functionum non mutare.

10*. Deinde potissimum tenenda est functionum divisio in uniformes ac multiformes.*

Functio autem *uniformis* est, quae, si quantitati variabili $z$ valor determinatus quicunque tribuatur, ipsa quoque unicum valorem determinatum obtineat. Functio autem *multiformis* est, quae pro unoquoque valore determinato in locum variabilis $z$ substituto plures valores determinatos exhibet.

Sunt igitur omnes functiones rationales, sive integrae sive fractae, functiones uniformes, quoniam eiusmodi expressiones, quicunque valor quantitati variabili tribuatur, non nisi unicum valorem praebent. Functiones autem irrationales omnes sunt multiformes, propterea quod signa radicalia sunt ambigua et geminum valorem involvunt. Dantur autem quoque inter functiones transcendentes et uniformes et multiformes; quin etiam habentur functiones infinitiformes, cuiusmodi est arcus circuli sinui $z$ respondens; dantur enim arcus circulares innumerabiles, qui omnes eundem habeant sinum.

Denotent autem hae litterae $P, Q, R, S, T$ etc. singulae functiones uniformes ipsius $z$.

11. *Functio biformis* ipsius $z$ *est eiusmodi functio, quae pro quovis ipsius* $z$ *valore determinato geminum valorem praebeat.*

Huiusmodi functiones radices quadratae exhibent, ut $\sqrt{(2z+zz)}$*;* quicunque enim valor pro $z$ statuatur, expressio $\sqrt{(2z+zz)}$ duplicem habet significatum, vel affirmativum vel negativum. Generatim vero $Z$ erit functio biformis ipsius $z$, si determinetur per aequationem quadraticam

$$Z^2-PZ+Q$$

si quidem $P$ et $Q$ fuerint functiones uniformes ipsius $z$. Erit namque

$$Z=\tfrac{1}{2}P\pm\sqrt{\left(\tfrac{1}{4}P^2-Q\right)};$$

ex quo patet cuique valori determinato ipsius $z$ duplicem valorem determinatum ipsius $Z$ respondere. Hic autem notandum est vel utrumque valorem functionis $z$ esse realem vel utrumque imaginarium. Tum vero erit semper, uti constat ex natura aequationum, binorum valorum ipsius $Z$ summa $=P$ ac productum $=Q$.

12*. Functio triformis ipsius z est, quae pro quovis ipsius z valore tres valores determinatos exhibet.*

Huiusmodi functiones ex resolutione aequationum cubicarum originem trahunt. Si enim fuerint *P*, *Q* et *R* functiones uniformes sitque

$$Z^3 - PZ^2 + QZ - R = 0,$$

erit *Z* functio triformis ipsius *z*, quia pro quolibet valore determinato ipsius *z* triplicem valorem obtinet. Tres isti ipsius *Z* valores unicuique valori ipsius *z* respondentes vel erunt omnes reales vel unicus erit realis, dum bini reliqui sunt imaginarii. Ceterum constat horum trium valorum summam perpetuo esse $= P$, summam factorum ex binis esse $= Q$ et productum ex omnibus tribus esse $= R$.

13*. Functio quadriformis ipsius z est, quae pro quovis ipsius z valore quatuor valores determinatos exhibet.*

Huiusmodi functiones ex resolutione aequationum biquadraticarum nascuntur. Quodsi enim *P*, *Q*, *R* et *S* denotent functiones uniformes ipsius *z* fueritque

$$Z^4 - PZ^3 + QZ^2 - RZ + S = 0,$$

erit *Z* functio quadriformis ipsius *z*, eo quod cuique valori ipsius *z* quadruplex valor ipsius *Z* respondet. Quatuor horum valorum ergo vel omnes erunt reales vel duo reales duoque imaginarii, vel omnes quatuor erunt imaginarii. Ceterum perpetuo summa horum quatuor valorum ipsius *Z* est $= P$, summa factorum ex binis $= Q$, summa factorum ex ternis $= R$ ac productum omnium $= S$.

Simili autem modo comparata est ratio functionum quinqueformium et sequentium.

14*. Erit ergo Z functio multiformis ipsius z, quae pro quovis valore ipsius z tot exhibet valores, quot numerus n continet unitates, si Z definiatur per hanc aequationem*

$$Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \text{etc.} = 0.$$

Ubi quidem notandum est *n* esse oportere numerum integrum atque perpetuo, ut diiudicari possit, quam multiformis sit functio *Z* ipsius *z*, aequatio, per quam *Z* definitur, reduci debet ad rationalitatum; quo facto exponens maximae potestatis ipsius *Z* indicabit quaesitum valorum numerum cuique ipsius *z* valori respondentium. Deinde quoque tenendum est litteras *P*, *Q*, *R*, *S* etc. denotare debere functiones uniformes ipsius *z*; si enim aliqua earum iam esset functio multiformis, tum functio *Z* multo plures praebitura esset valores unicuique valori ipsius $z$ respondentes, quam quidem numerus dimensionum ipsius $Z$ indicaret. Semper autem, si qui valores ipsius $Z$ fuerint imaginarii, eorum numerus erit par ; unde intelligitur, si fuerit $n$ numerus impar, perpetuo unum ad minimum valorem ipsius $Z$ fore realem, contra autem fieri posse, si numerus $n$ fuerit par, ut nullus prorsus valor ipsius $Z$ sit realis.

15. *Si Z eiusmodi fuerit functio multiformis ipsius z, ut perpetuo nonnisi unicum valorem exhibeat realem, tum Z functionem uniformem ipsius z mentietur ac plerumque loco functionis uniformis usurpari poterit.*

Eiusmodi functiones erunt

$$\sqrt[3]{P}, \quad \sqrt[5]{P}, \quad \sqrt[7]{P}, \quad \text{etc.},$$

quippe quae perpetuo nonnisi unicum valorem realem praebent reliquis omnibus existentibus imaginariis, dummodo $P$ fuerit functio uniformis ipsius $z$.Hanc ob rem huiusmodi expressio $P^{\frac{m}{n}}$, quoties $n$ fuerit numerus impar, functionibus uniformibus annumerari poterit, sive $m$ fuerit numerus par sive $m$ impar. Quodsi autem $n$ fuerit numerus par, tum $P^{\frac{m}{n}}$ vel nullum habebit valorem realem vel duos; ex quo eiusmodi expressiones $P^{\frac{m}{n}}$ existente $n$ numero pari eodem iure functionibus biformibus accenseri poterunt, siquidem fractio $\frac{m}{n}$ ad minores terminos non fuerit reducibilis.

16. *Si fuerit y functio quaecunque ipsius z, tum vicissim z erit functio ipsius y.*

Cum enim $y$ sit functio ipsius $z$, sive uniformis sive multiformis, dabitur aequatio, qua $y$ per $z$ et constantes quantitates definitur. Ex eadem vero aequatione vicissim $z$ per $y$ et constantes definiri poterit; unde, quoniam $y$ est quantitas variabilis, $z$ aequabitur expressioni ex $y$ et constantibus compositae eritque adeo functio ipsius $y$. Hinc quoque patebit, quam multiformis functio futura sit $z$ ipsius $y$, fierique potest, ut, etiamsi $y$ fuerit functio uniformis ipsius $z$, tamen $z$ futura sit functio multiformis ipsius $y$. Sic, si $y$ ex hac aequatione per $z$ definiatur

$$y^3 = ayz - bzz,$$

erit utique $y$ functio triformis ipsius $z$, contra vero $z$ functio tantum biformis ipsius $y$.

17. *Si fuerint y et x functiones ipsius z, erit quoque y functio ipsius x et vicissim x functio ipsius y*.

Cum enim sit *y* functio ipsius *z*, erit quoque *z* functio ipsius *y* similique modo erit etiam *z* functio ipsius *x*. Hanc ob rem functio ipsius *y* aequalis erit functioni ipsius *x;* ex qua aequatione et *y* per *x* et viceversa *x* per *y* definiri poterit; quocirca manifestum est esse *y* functionem ipsius *x* atque *x* functionem ipsius *y.* Saepissime quidem has functiones explicite exhibere non licit ob defectum Algebrae ; interim tamen nihilominus, quasi omnes aequationes resolvi possent, haec functionum reciprocatio perspicitur. Ceterum per methodum in Algebra traditam ex datis binis aequationibus, quarum altera continet *y* et *z*, altera vero *x* et *z*, per eliminationem quantitatis *z* formabitur una aequatio relationem inter *x* et *y* exprimens.

18. *Species denique quaedam functionum peculiares sunt notandae; sic functio par ipsius z est, quae eundem dat valorem, sive pro z ponatur valor determinatus + k sive – k.*

Huiusmodi ergo functio par ipsius *z* erit *zz;* sive enim ponatur $z=+k$ sive $z=-k$, eundem valorem praebebit expressio *zz*, nempe $zz=+kk$. Simili modo functiones pares ipsius *z* erunt hae ipsius *z* potestates $z^4$, $z^6$, $z^8$ et generatim omnis potestas $z^m$, si fuerit *m* numerus par, sive *m* affirmativus sive negativus. Quin etiam, cum $z^{\frac{m}{n}}$ mentiatur functionem ipsius *z* uniformem, si *n* sit numerus impar, perspicuum est $z^{\frac{m}{n}}$ fore functionem parem ipsius *z*, si *m* fuerit numerus par, *n* vero numerus impar. Hanc ob rem expressiones ex huiusmodi potestatibus utcunque compositae praebebunt functiones pares ipsius *z;* sic *Z* erit functio par ipsius *z*, si fuerit

$$Z=a+bz^2+cz^4+dz^6+\text{ etc.},$$

item si fuerit

$$Z=\frac{a+bz^2+cz^4+dz^6+\text{etc.}}{\alpha+\beta z^2+\gamma z^4+\delta z^6+\text{etc.}}.$$

Similique modo exponentes fractos ipsius *z* introducendo erit *Z* functio par ipsius *z*, si fuerit

$$Z=a+bz^{\frac{2}{3}}+cz^{\frac{2}{5}}+dz^{\frac{4}{7}}+\text{ etc.}$$

vel

$$Z=a+bz^{\frac{2}{3}}+cz^{\frac{4}{3}}+dz^{\frac{2}{5}}+\text{ etc.}$$

vel

$$Z=\frac{a+bz^{\frac{2}{7}}+cz^{\frac{4}{5}}+dz^{\frac{8}{3}}+\text{etc.}}{\alpha+\beta z^{\frac{2}{3}}+\gamma z^{\frac{2}{5}}+\delta z^{\frac{4}{7}}+\text{etc.}}.$$

Cuiusmodi expressiones, cum omnes sint functiones uniformes ipsius *z*, appellari poterunt functiones pares uniformes ipsius *z.*

19. *Functio multiformis par ipsius z est, quae, etiamsi pro quovis valore ipsius z plures exhibeat valores determinatos, tamen eosdem valores praebet, sive ponatur* $z=+k$ *sive* $z=-k$ .

Sit *Z* eiusmodi functio multiformis par ipsius *z;* quoniam natura functionis multiformis exprimitur per aequationem inter *Z* et *z*, in qua *Z* tot habeat dimensiones, quot varios valores complectatur, manifestum est *Z* fore functionem multiformem parem, si in aequatione naturam ipsius *Z* exprimente quantitas variabilis *z* ubique pares habeat dimensiones. Sic, si fuerit

$$Z^2 = az^4Z + bz^2$$

erit *Z* functio biformis par ipsius *z;* sin autem sit

$$Z^3 - az^2Z^2 + bz^4Z - cz^8 = 0,$$

erit *Z* functio triformis par ipsius *z;* atque generatim, si *P, Q, R, S* etc. denotent functiones uniformes pares ipsius *z*, erit *Z* functio biformis par ipsius *z*, si sit

$$Z^2 - PZ + Q = 0.$$

At *Z* erit functio triformis par ipsius *z*, si sit

$$Z^3 - PZ^2 + QZ - R = 0,$$

et ita porro.

20. *Functio ergo, sive uniformis sive multiformis, par ipsius z erit eiusmodi expressio ex quantitate variabili z et constantibus conflata, in qua ubique numerus dimensionum ipsius z sit par.*

Huiusmodi ergo functiones praeter uniformes, quarum exempla ante sunt allata, erunt hae expressiones

$$a+\sqrt{(bb-zz)},\ azz+\sqrt[3]{\left(a^6z^4-bz^2\right)},\ \text{item}\ az^{\frac{2}{3}}+\sqrt[3]{\left(z^2+\left(a^4-z^4\right)\right)}\ \text{etc.}$$

*Unde patet functiones pares ita definiri posse, ut dicantur esse functiones ipsius zz.*

Si enim ponatur $y = zz$ fueritque *Z* functio quaecunque ipsius *y*, restituto ubique *zz* loco *y* erit *Z* eiusmodi functio ipsius *z*, in qua *z* ubique parem habeat dimensionum numerum. Excipiendi tamen sunt ii casus, quibus in expressione ipsius *Z* occurrunt $\sqrt{y}$ ac

huiusmodi aliae formae, quae facto $y = zz$ signa radicalia amittunt. Quamvis enim sit $y + \sqrt{ay}$ functio ipsius *y* tamen posito $y = zz$ eadem expressio non erit functio par ipsius *z*, cum fiat $y + \sqrt{ay} = zz + z\sqrt{a}$.
Exclusis ergo his casibus definitio ultima functionum parium erit bona atque ad eiusmodi functiones formandas idonea.

21. *Functio impar ipsius z est eiusmodi functio, cuius valor, si loco z ponatur – z, fit quoque negativus.*

Huiusmodi functiones ergo impares erunt omnes potestates ipsius *z*, quarum exponentes sunt numeri impares, ut $z^1, z^2, z^5, z^7$ etc., item $z^{-1}, z^{-3}, z^{-5}$ etc.; tum vero etiam $z^{\frac{m}{n}}$ erit functio impar, si ambo numeri *m* et *n* fuerint numeri impares. Generatim vero omnis expressio ex huiusmodi potestatibus composita erit functio impar ipsius *z ;* cuiusmodi sunt

$$az + bz^3,\ az + az^{-1},\ \text{item}\ \mathrm{z}^{\frac{1}{3}} + az^{\frac{2}{5}} + bz^{-\frac{5}{3}}\ \text{etc.}$$

Harum autem functionum natura et inventio ex functionibus paribus facilius perspicietur.

22. *Si functio par ipsius z multiplicetur per z vel per eiusdem functionem imparem quamcunque, productum erit functio impar ipsius z.*

Sit *P* functio par ipsius *z*, quae idcirco manet eadem, si loco *z* ponatur – *z;* quodsi ergo in producto *Pz* ponatur – *z* loco *z*, prodibit – *Pz*, unde *Pz* erit functio impar ipsius *z*. Sit iam *P* functio par ipsius z et *Q* functio impar ipsius *z* atque ex definitione patet, si loco *z* ponatur – *z*, valorem ipsius *P* manere eundem, at valorem ipsius *Q* abire in sui negativum – *Q*, quare productum *PQ* posito – *z* loco *z* abibit in –*PQ*, hoc est in sui negativum, eritque ideo *PQ* functio impar ipsius z. Sic, cum sit $a + \sqrt{(aa + zz)}$ functio par et $z^3$ functio impar ipsius *z*, erit productum

$$az^3 + z^3\sqrt{(aa + zz)}$$

functio impar ipsius *z* similique modo

$$z \times \frac{a+bzz}{\alpha+\beta zz} = \frac{az+bz^3}{\alpha+\beta zz}$$

functio impar ipsius *z*.

Ex his vero etiam intelligitur, si duarum functionum *P* et *Q*, quarum altera *P* est par, altera *Q* impar, altera per alteram dividatur, quotum fore functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ functio impar ipsius *z*.

23. *Si functio impar per functionem imparem vel multiplicetur vel dividatur, quod resultat erit functio par.*

Sint $Q$ et $S$ functiones impares ipsius z, ita ut posito $-z$ loco $z$ $Q$ abeat in $-Q$ et $S$ in $-S$, atque perspicuum est tam productum $QS$ quam quotum $\frac{Q}{S}$ eundem valorem retinere, etiamsi pro $z$ ponatur $-z$, ideoque esse utrumque functionem parem ipsius $z$. Manifestum itaque porro est cuiusque functionis imparis quadratum esse functionem parem, cubum vero functionem imparem, biquadratum iterum functionem parem atque ita porro.

24. *Si fuerit y functio impar ipsius z, erit vicissim z functio impar ipsius y.*

Cum enim sit $y$ functio impar ipsius $z$, si ponatur $-z$ loco $z$, abibit $y$ in $-y$. Quodsi ergo $z$ per $y$ definiatur, necesse est, ut posito $-y$ loco $y$ quoque $z$ abeat in $-z$, eritque ideo $z$ functio impar ipsius $y$. Sic, quia posito

$$y = z^3$$

est $y$ functio impar ipsius $z$, erit quoque ex aequatione

$$z^3 = y \text{ seu } z = y^{\frac{1}{3}}$$

$z$ functio impar ipsius $y$. Et quia, si fuerit

$$y = az + z^3$$

est $y$ functio impar ipsius $z$, erit vicissim ex aequatione

$$bz^3 + az = y$$

valor ipsius $z$ per $y$ expressus functio impar ipsius $y$.

25. *Si natura functionis y per eiusmodi aequationem definiatur, in cuius singulis terminis numerus dimensionum, quas y et z occupant coniunctim, sit vel par ubique vel impar, tum erit y functio impar ipsius z.*

Quodsi enim in eiusmodi aequatione ubique loco $z$ scribatur $-z$ simulque $-y$ loco $y$, omnes aequationis termini vel manebunt iidem vel fient negativi, utroque vero casu aequatio manebit eadem. Unde patet $-y$ eodem modo per $-z$ determinatum iri, quo $+y$ per $+z$ determinatur, et hanc ob rem, si loco $z$ ponatur $-z$, valor ipsius $y$ abibit in $-y$ seu $y$ erit functio impar ipsius $z$. Sic, si fuerit vel

$$yy = ayz + bzz + c$$

vel

$$y^3 + ayyz = byzz + cy + dz,$$

ex utraque aequatione *y* erit functio impar ipsius *z*.

26. Si *Z fuerit functio ipsius z et Y functio ipsius y atque Y eodem modo definiatur per variabilem y et constantes, quo Z definitur per variabilem z et constantes, tum hae functiones Y et Z vocantur functiones similes ipsarum y et z.*

Si scilicet fuerit

$$Z = a + bz + cz^2 \text{ et } Y = a + by + cy^2,$$

erunt *Z* et *Y* functiones similes ipsarum *z* et *y* similique modo in multiformibus, si fuerit

$$Z^3 = azzZ + b \text{ et } Y^3 = ayyY + b,$$

erunt *Z* et *Y* functiones similes ipsarum *z* et *y*. Hinc sequitur, si *Y* et *Z* fuerint huiusmodi functiones similes ipsarum *y* et *z*, tum, si loco *z* scribatur *y*, functionem *Z* abituram esse in functionem *Y*. Solet haec similitudo etiam hoc modo verbis exprimi, ut *Y* talis functio dicatur ipsius *y*, qualis functio sit *Z* ipsius *z*. Hae locutiones perinde occurrent, sive quantitates variabiles *z* et *y* a se invicem pendeant sive secus; sic, qualis functio est

$$ay + by^3$$

ipsius *y*, talis functio erit

$$a(y+n) + b(y+n)^3$$

ipsius $y + n$, existente scilicet $z = y + n$; tum, qualis functio est

$$\frac{a+bz+czz}{\alpha+\beta z+\gamma zz}$$

ipsius *z*, talis functio erit

$$\frac{azz+bz+c}{\alpha zz+\beta z+\gamma}$$

ipsius $\frac{1}{z}$ posito $y = \frac{1}{z}$. Atque ex his luculenter perspicitur ratio similitudinis functionum, cuius per universam Analysin sublimiorem uberrimus est usus.

Ceterum haec in genere de natura functionum unius variabilis sufficere possunt, cum plenior expositio in applicatione sequente tradatur.

,