CHAPTER XVIII

CONTINUED FRACTIONS

356. Because I have talked so much in the preceding chapters about constructing both infinite series as well as products from an infinite number of factors, it is seen to be fitting, if I may add also a certain other kind of infinite expression, which may contain continued fractions or divisions. For although at present this kind of expansion has been little developed, yet we may have no doubt, why at some time there should not be the fullest use made from this in the analysis of infinite quantities. Indeed I have produced some examples of this kind, in which this expectation is returned more than probable (see E71 & E123). Truly at first this speculation is not to be distained for offering help to arithmetic itself and common algebra, which I have decided to set up and indicate briefly in this chapter.

357. Moreover I call a continued fraction a fraction of this kind, the denominator of which is agreed to be made from a while number with a fraction, whose denominator afresh is the sum of an integer and a fraction, which shall be prepared in a like manner, thus effected either it may progress indefinitely or stop somewhere. Therefore the following expression will be a continued fraction of this kind

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\text{ etc.}}}}}} \quad \text{or} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{ etc.}}}}}}$$

in the first form of which all the numerators of the fractions are one, as we will consider here mainly, truly in the other the numerators are any numbers.

358. Therefore the first form of continued fractions is required to be explained, just as the meaning of these may be able to be found in the accustomed manner. Which so that it may be found more easily, we shall progress by breaking these fractions into steps, the first into the first, then into the second, after into the third, and thus again with a fraction ; which done it will become apparent

$$a+\frac{1}{b} = \frac{ab+1}{b},$$

$$a+\cfrac{1}{b+\cfrac{1}{c}} = \frac{abc+a+c}{bc+1},$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d},$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+b+c}{bcde+be+de+bc+1}$$

etc.

359. If the law is not easily seen, by which in these ordinary fractions, following which the numerator and denominator may be composed from the letters *a, b, c, d* etc., yet it will be noticed at once, in what manner any fraction may be formed from the previous. For any numerator is the sum formed from the final numerator multiplied by the new letter and from the simple penultimate numerator; and the same law is observed in the denominator. Therefore with the letters *a, b, c, d* etc. written in order, the fractions will be formed readily in this manner from these :

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+a+c}{bc+1}, & \frac{abcd+ab+ad+cd+1}{bcd+b+d} & \text{etc.,} \end{array}$$

where any numerator is found, if the preceding final numerator may be multiplied by the letter indicated above, and the penultimate numerator is added to the product ; which same law prevails for the denominator. But so that this law may be allowed to be used from the start, the fraction $\frac{1}{0}$ is to be placed first, which even if it may not arise from a continued fraction, yet makes the law of the progression more clear. Moreover any fraction shows the value of the continued fraction as far as to that letter continued inclusively, which overhangs the antecedent.

360. In a similar manner the form of the continued fraction

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{ etc.}}}}}}$$

will give the following values, just as it is stopped in different and in other different places

$$a = a,$$

$$a+\frac{\alpha}{b} = \frac{ab+\alpha}{b},$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} = \frac{abc+\beta a+\alpha c}{bc+\beta},$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b},$$

etc.

each of the fractions will be found from the two preceding fractions in the following manner :

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+\alpha}{b}, & \frac{abc+\beta a+\alpha c}{bc+\beta}, & \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} & \text{etc.} \\ \alpha & \beta & \gamma & \delta & \varepsilon & \end{array}$$

361. It is evident the indices *a, b, c, d* etc. may be inscribed above, but below the indices $\alpha, \beta, \gamma, \delta$ etc. are inscribed. The first fraction $\frac{1}{0}$ may be put in place again, the following $\frac{a}{1}$. Then any of the following may be formed, if the final preceding numerator may be multiplied by the index written above, truly the penultimate numerator may be multiplied by the letter written below, and both products may be added ; the sum will be the numerator of the following fraction. In a similar manner its denominator will be the sum formed from the final denominator multiplied by the index written above and from the penultimate denominator multiplied by the index written below. Truly any fraction found in this manner will provide the value of the continued fraction as far as that denominator, which has been written for the preceding fraction, continued inclusively.

362. Therefore if these fractions are continued to that point, as long as the continued fraction may supply the indices, then the final fraction will give the true value of the continued fraction. Truly the preceding fractions continually approach closer to the true value and thus through which the final fraction may suggest an extremely close sufficient approximation. Indeed for the true value of the continued fraction we may put

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\text{ etc.}}}}}$$

to be $=x$ and it is clear the first fraction $\frac{1}{0}$ to be greater than *x*, the second truly $\frac{a}{1}$ will be less than *x*, the third $a+\frac{\alpha}{b}$ again truly will be a greater value, the fourth anew less, and thus again these fractions alternately will be greater and less than *x*. Moreover again it is clear any proper fraction approaches closer to the true value *x* than any preceding that, from which with this agreed upon the approximate value of *x* may be found most quickly and most conveniently, even if the continued fraction may progress to infinity, as long as the numerators $\alpha, \beta, \gamma, \delta$ etc. may not increase very much: moreover if all these numerators were unity, then the approximation is liable to no inconvenience.

363. So that the account of this approximation to the true fraction may be understood better, we will consider the differences of the fractions found.
And indeed with the first term omitted $\frac{1}{0}$, the difference between the second and third terms is

$$=\frac{\alpha}{b},$$

the fourth taken from the third leaves

$$\frac{\alpha\beta}{b(bc+\beta)},$$

the fourth subtracted from the fifth leaves

$$\frac{\alpha\beta\gamma}{(bc+\beta)\ (bcd+\beta d+\gamma b)}$$

etc. Hence the value of the continued fraction may be expressed by the accustomed series of terms in this manner, so that there shall be

$$x=a+\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)\ (bcd+\beta d+\gamma b)}-\text{ etc.},$$

which series ends just as often as the continued fraction does not progress to infinity.

364. Therefore we have found the manner in which some continued fraction may be changed into a series of terms of which the signs alternate, if indeed the first letter *a* may vanish. For if the continued fraction were

$$x=\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{ etc.}}}}}}$$

for that there will be the series, in the manner we have found,

$$x=\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)\ (bcd+\beta d+\gamma b)}$$

$$-\frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)\ (bcde+\beta de+\gamma be+\delta bc+\beta\delta)}\ +\text{ etc.}$$

From which, if $\alpha, \beta, \gamma, \delta$ etc. were non increasing numbers, such as all ones, and *a, b, c, d* etc. all positive whole numbers, then the value of the continued fraction may be expressed by a series of strongly convergent terms.

365. From these properly considered, some series of alternating terms in turn may be converted into a continued fraction or the continued fraction to be found, the value of which shall be equal to the sum of the proposed series. For let this series be proposed :

$$x=A-B+C-D+E-F+\text{etc.};$$

there will be, from the individual terms compared with the series arising from the continued fraction :

$$A=\frac{\alpha}{b}, \qquad \text{from which} \qquad \alpha = Ab,$$

$$\frac{B}{A}=\frac{\beta}{bc+\beta}, \qquad \beta=\frac{Bbc}{A-B},$$

$$\frac{C}{B}=\frac{\gamma b}{bcd+\beta d+\gamma b}, \qquad \gamma=\frac{Cd\left(bc+\beta\right)}{b\left(B-C\right)},$$

$$\frac{D}{C}=\frac{\delta\left(bc+\beta\right)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}, \qquad \delta=\frac{Ce\left(bcd+\beta d+\gamma b\right)}{\left(bc+\beta\right)\left(C-D\right)}$$

etc., etc.

But since $\beta=\frac{Bbc}{A-B}$, there will be

$$bc+\beta=\frac{Abc}{A-B},$$

from which

$$\gamma=\frac{ACcd}{\left(A-B\right)\left(B-C\right)}.$$

From which

$$bcd+\beta d+\gamma b=\left(bc+\beta\right)d+\gamma b=\frac{Abcd}{\left(A-B\right)}+\frac{ACbcd}{\left(A-B\right)\left(B-C\right)}=\frac{ABbcd}{\left(A-B\right)\left(B-C\right)},$$

from which there will be

$$\frac{bcd+\beta d+\gamma b}{bc+\beta}=\frac{Bd}{B-C}$$

and

$$\delta=\frac{BDde}{\left(B-C\right)\left(C-D\right)}.$$

In a similar manner there will be found

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

and thus henceforth.

366. So that this law may appear clearer, we may put

$$\begin{aligned}
&P = b,\\
&Q = bc + \beta,\\
&R = bcd + \beta d + \gamma b,\\
&S = bcde + \beta de + \gamma be + \delta bc + \beta\delta,\\
&T = bcdef + \text{ etc.},\\
&V = bcdefg + \text{ etc.}\\
&\qquad\text{etc.};
\end{aligned}$$

from the law of these expressions there will be

$$\begin{aligned}
&Q = Pc + \beta,\\
&R = Qd + \gamma P,\\
&S = Re + \delta Q,\\
&T = Sf + \varepsilon R,\\
&V = Tg + \zeta S\\
&\qquad\text{etc.}
\end{aligned}$$

Since therefore, using these letters, the series becomes

$$x = \frac{\alpha}{P} - \frac{\alpha\beta}{PQ} + \frac{\alpha\beta\gamma}{QR} - \frac{\alpha\beta\gamma\delta}{RS} + \frac{\alpha\beta\gamma\delta\varepsilon}{ST} - \text{etc.}$$

367. Therefore because we may put

$$x = A - B + C - D + E - \mathrm{F} + \text{ etc.},$$

there will be

$$A=\frac{\alpha}{P},\qquad \alpha=AP,$$

$$\frac{B}{A}=\frac{\beta}{Q},\qquad \beta=\frac{BQ}{A},$$

$$\frac{C}{B}=\frac{\gamma P}{R},\qquad \gamma=\frac{CR}{BP},$$

$$\frac{D}{C}=\frac{\delta Q}{S},\qquad \delta=\frac{DS}{CQ},$$

$$\frac{E}{D}=\frac{\varepsilon R}{T},\qquad \varepsilon=\frac{ET}{DR}$$

etc.

Again truly from the differences taken there will be had

$$A-B=\frac{\alpha\left(Q-\beta\right)}{PQ}=\frac{\alpha c}{Q}=\frac{APc}{Q},$$

$$B-C=\frac{\alpha\beta\left(R-\gamma P\right)}{PQR}=\frac{\alpha\beta d}{PR}=\frac{BQd}{R},$$

$$C-D=\frac{\alpha\beta\gamma\left(S-\delta Q\right)}{QRS}=\frac{\alpha\beta\gamma e}{QS}=\frac{CRe}{S},$$

$$D-E=\frac{\alpha\beta\gamma\delta\left(T-\varepsilon R\right)}{RST}=\frac{\alpha\beta\gamma\delta f}{RT}=\frac{DSf}{T}$$

etc.

If the two therefore may be multiplied together, there becomes

$$(A-B)(B-C)=ABcd\frac{P}{R} \text{ and } \frac{R}{P}=\frac{ABcd}{(A-B)(B-C)},$$

$$(B-C)(C-D)=BCde\frac{Q}{S} \text{ and } \frac{S}{Q}=\frac{BCde}{(B-C)(C-D)},$$

$$(C-D)(D-E)=CDef\frac{R}{T} \text{ and } \frac{T}{R}=\frac{CDef}{(C-D)(D-E)}$$

etc.

From which, since there shall be $P=b,\ Q=\frac{\alpha c}{A-B}=\frac{\alpha bc}{A-B}$, there will be

$$\alpha = Ab,$$

$$\beta=\frac{Bbc}{A-B},$$

$$\gamma=\frac{ACcd}{(A-B)(B-C)},$$

$$\delta=\frac{BDde}{(B-C)(C-D)},$$

$$\varepsilon=\frac{CEef}{(C-D)(D-E)}$$

etc.

368. Therefore with the values of the numbers $\alpha, \beta, \gamma, \delta$ etc. found, the denominators *b*, *c*, *d*, *e* etc. are left to our choice ; thus moreover it is convenient to assume these themselves to be whole numbers, since then integral values may be shown for $\alpha, \beta, \gamma, \delta$ etc. Truly this depends also on the nature of the numbers *A*, *B*, *C* etc., whether they shall be whole or fractions. We may put whole numbers in place and the question is satisfied by putting in place

$$\begin{array}{ll} b=1, \text{ from which} & \alpha = A, \\ c = A-B, & \beta = B, \\ d = B-C, & \gamma = AC, \\ e = C-D, & \delta = BD, \\ f = D-E, & \varepsilon = CE \\ \text{etc.,} & \text{etc.} \end{array}$$

On account of which, if there were

$$x = A - B + C - D + E - F + \text{ etc.},$$

the same value of x can be expressed thus by a continued fraction, so that there shall be

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \text{etc.}}}}}}$$

369. But if all the terms of the series shall be fractional numbers, thus so that there becomes

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \text{etc.},$$

the following values will be had for $\alpha, \beta, \gamma, \delta$ etc.

$$\alpha = \frac{b}{A},$$

$$\beta = \frac{Abc}{B-A},$$

$$\gamma = \frac{B^2cd}{(B-A)(C-B)},$$

$$\delta = \frac{C^2de}{(C-B)(D-C)},$$

$$\varepsilon = \frac{D^2ef}{(D-C)(E-D)}$$

etc.

Therefore there may be put, as follows,

$$\begin{array}{lcl} b = A, & \text{from which} & \alpha = 1, \\ c = B-A, & & \beta = AA, \\ d = C-B, & & \gamma = BB, \\ e = D-C, & & \delta = CC, \\ \text{etc.,} & & \text{etc.} \end{array}$$

and the continued fraction will be

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \text{etc.}}}}}$$

EXAMPLE 1

This infinite series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.}$$

shall be transformed into a continued fraction.

Therefore there will be

$$A=1,\ B=2,\ C=3,\ D=4 \text{ etc.},$$

and since the value of the proposed series shall be $=l2$, there will be

$$l2=\cfrac{1}{1+\cfrac{1}{1+\cfrac{4}{1+\cfrac{9}{1+\cfrac{16}{1+\cfrac{25}{1+\text{etc.}}}}}}}$$

EXAMPLE 2

This infinite series may be transformed

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.},$$

into a continued fraction, where π denotes the periphery of the circle, of which the diameter $=1$.

With the numbers 1, 3, 5, 7 etc. substituted in place of *A, B, C, D* etc. there may be generated

$$\frac{\pi}{4}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}}$$

and hence with the fraction inverted there will be

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}},$$

which is the expression that Brouncker first proposed for the quadrature of the circle.

EXAMPLE 3

Let this infinite series be proposed

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \text{ etc.},$$

which on account of

$$A = m,\ B = m+n,\ C = m+2n \quad \text{etc.}$$

may be changed into this continued fraction

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \text{ etc.}}}}}}$$

from which by inverting , there becomes

$$\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \text{ etc.}}}}}$$

EXAMPLE 4

Since above (§ 178) we have found

$$\frac{\pi \cos.\frac{m\pi}{n}}{\pi \sin.\frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \text{etc.},$$

for the continued fraction there will be

$$A = m,\ B = n-m,\ C = n+m,\ D = 2n-m \ \text{ etc.},$$

from which the expression becomes :

$$\frac{\pi\cos.\frac{m\pi}{n}}{\pi\sin.\frac{m\pi}{n}}=\cfrac{1}{m+\cfrac{mm}{n-2m+\cfrac{(n-m)^2}{2m+\cfrac{(n+m)^2}{n-2m+\cfrac{(2n-m)^2}{2m+\cfrac{(2n+m)^2}{n-2m+\text{etc.}}}}}}}$$

370. If the series proposed may be progressing by continued factors, so that it shall be

$$x=\frac{1}{A}-\frac{1}{AB}+\frac{1}{ABC}-\frac{1}{ABCD}+\frac{1}{ABCDE}-\text{ etc.},$$

then the following determinations will be produced :

$$\alpha=\frac{b}{A}-,$$
$$\beta=\frac{bc}{B-1},$$
$$\gamma=\frac{Bcd}{(B-1)(C-1)},$$
$$\delta=\frac{Cde}{(C-1)(D-1)},$$
$$\varepsilon=\frac{Def}{(D-1)(E-1)}$$
etc.

Therefore it becomes, as follows,

$$\begin{array}{lcl} b=A, & \text{from which} & \alpha=1, \\ c=B-1, & & \beta=A, \\ d=C-1, & & \gamma=B, \\ e=D-1, & & \delta=C, \\ f=E-1, & & \varepsilon=D \\ \text{etc.,} & & \text{etc.,} \end{array}$$

from which consequently there becomes

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \text{etc.}}}}}}$$

EXAMPLE 1

Because on putting in place the number e, of which the logarithm is $= 1$, we have found above to be

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} - \text{ etc.}$$

or

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} - \frac{1}{1\cdot 2\cdot 3\cdot 4} + \text{ etc.},$$

this series will be converted into a continued fraction by putting

$$A = 1,\ B = 2,\ C = 3,\ D = 4 \ \text{ etc.};$$

with which done therefore there will be had

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{ etc.}}}}}}}$$

from which with the initial asymmetry removed, it becomes

$$\frac{1}{e-1}=\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\text{ etc.}}}}}}$$

EXAMPLE 2

Also we have found the cosine of the arc, which is taken equal to the radius, to be

$$=1-\frac{1}{2}+\frac{1}{2\cdot 12}-\frac{1}{2\cdot 12\cdot 30}+\frac{1}{2\cdot 12\cdot 30\cdot 56}-\text{ etc.}$$

Therefore if there becomes

$$A=1,\ B=2,\ C=12,\ D=30,\ E=56\quad \text{etc.}$$

and the cosine of the arc, which is equal to the radius, may be put $=x$, there will be

$$x=\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{11+\cfrac{12}{29+\cfrac{30}{55\ +\text{etc.}}}}}}$$

or

$$\frac{1}{x}-1=\cfrac{1}{1+\cfrac{2}{11+\cfrac{12}{29+\cfrac{30}{55+\text{etc.}}}}}$$

[*i.e.* x is the angle of one radian.]

371. Let the above series be taken jointly with the above geometric series, evidently

$$x=A-Bz+Cz^2-Dz^3+Ez^4-Fz^5+\text{ etc.};$$

there will be

$$\alpha = Ab,$$
$$\beta = \frac{Bbcz}{A - Bz},$$
$$\gamma = \frac{ACcdz}{(A - Bz)(B - Cz)},$$
$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)},$$
$$\varepsilon = \frac{CEefz}{(C - Dz)(D - Ez)}$$
etc.

Now there may be put

$$\begin{array}{lcl} b = 1, & \text{will become} & \alpha = A, \\ c = A - Bz, & & \beta = Bz, \\ d = B - Cz, & & \gamma = ACz \\ e = C - Dz, & & \delta = BDz \\ \text{etc.}, & & \text{etc.}, \end{array}$$

from which there is made :

$$x = \cfrac{A}{1 + \cfrac{Bz}{A - Bz + \cfrac{ACz}{B - Cz + \cfrac{BDz}{C - Dz + \text{ etc.}}}}}$$

372. But so that we may resolve this more general matter, we may put in place

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} - \frac{Dy^3}{Oz^3} + \frac{Ey^4}{Pz^4} - \text{etc.}$$

and there becomes from the comparison put in place

$$\alpha=\frac{Ab}{L},$$
$$\beta=\frac{BLbcy}{AMz-BLy},$$
$$\gamma=\frac{ACM^{2}cdyz}{(AMz-BLy)(BNz-CMy)},$$
$$\delta=\frac{BDN^{2}deyz}{(BNz-CMy)\ (COz-DNy)}$$
etc,

The values *b*, *c*, *d* etc. may be put in place in the following manner :

$$\begin{array}{lcl} b=L, & \text{becomes} & \alpha=A, \\ c=AMz-BLy, & & \beta=BLLy, \\ d=BNz-CMy, & & \gamma=ACM^{2}yz, \\ e=COz-DNy, & & \delta=BDN^{2}yz, \\ f=DPz-EOy, & & \varepsilon=CEO^{2}yz \\ \text{etc.}; & & \text{etc.}, \end{array}$$

from which the proposed series will be expressed by the following continued fraction

$$x=\cfrac{A}{L+\cfrac{BLLy}{AMz-Bly+\cfrac{ACMMyz}{BNz-CMy+\cfrac{BDNNyz}{COz-DNy+\text{etc.}}}}}$$

373. Finally a proposed series of this kind may be had

$$x=\frac{A}{L}-\frac{ABy}{LMz}+\frac{ABCy^{2}}{LMNz^{2}}-\frac{ABCDy^{3}}{LMNOz^{3}}+\text{etc.}$$

and the following values will be produced

$$\alpha = \frac{Ab}{L},$$
$$\beta = \frac{Bbcy}{Mz - By},$$
$$\gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)},$$
$$\delta = \frac{DNdeyz}{(Nz - Cy)\ (Oz - Dy)},$$
$$\varepsilon = \frac{EOefyz}{(Oz - Dy)\ (Pz - Ey)}$$
$$\text{etc},$$

According to the integral values being found, therefore

$$\begin{array}{lcl} b = Lz, & \text{will become} & a = Az, \\ c = Mz - By, & & \beta = BLyz, \\ d = Nz - Cy, & & \gamma = CMyz \\ e = Oz - Dy, & & \delta = DNyz, \\ f = Pz - Ey, & & \varepsilon = EOyz \\ \text{etc.;} & & \text{etc.,} \end{array}$$

from which the value of the series will be expressed thus, so that it becomes

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \text{etc.}}}}}$$

or, so that the law of the progression is made clear at once, it will become

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy\ \text{etc.}}}}$$

374. Innumerable continued fractions can be found in this manner progressing to infinity, the true value of which may be able to be shown. Since indeed from the above treatment the infinite series, the sums of which may be agreed on, are able to be adapted to this situation, each one can be transformed into a continued fraction, the value of which therefore is equal to the sum of that series. The examples, which now have been brought forwards, are sufficient to show this use. But yet there may be a wish by which one can uncover a method, by means of which, having proposed some continued fraction, the value of this could be found at once. For although a continued fraction can be transformed into an infinite series, the sum of which may be able to be investigated by known methods, yet most such series become so much more complicated, so that the sum of these, even if it shall be simple enough, scarcely will be able to be found, if indeed at all.

375. But so that continued fractions of this kind may appear to be given clearer, the value of which may be able to be assigned readily, even if from the infinite series, into which they are converted, certainly may allow nothing to be deduced, we will consider this continued fraction

$$x=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{ etc.}}}}}$$

all the denominators of which are equal to each other. For if hence in this manner set out above we may put

$$\begin{array}{ccccccc} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70} \text{ etc.,} \end{array}$$

this series will be produce

$$x=0+\frac{1}{2}-\frac{1}{2\cdot 5}+\frac{1}{5\cdot 12}-\frac{1}{12\cdot 29}+\frac{1}{29\cdot 70}-\text{ etc.}$$

or, if each two terms are joined together, it will become

$$x=\frac{2}{1\cdot 5}+\frac{2}{5\cdot 29}+\frac{2}{29\cdot 169}+\text{etc.}$$

or

$$x=\frac{1}{2}-\frac{2}{2\cdot 12}-\frac{2}{12\cdot 70}-\text{etc.}$$

Further, as there may be

$$x=\frac{1}{4}-\frac{1}{2\cdot 2\cdot 5}+\frac{1}{2\cdot 5\cdot 12}-\frac{1}{2\cdot 12\cdot 29}+\text{etc.}$$
$$+\frac{1}{4}-\frac{1}{2\cdot 2\cdot 5}+\frac{1}{2\cdot 5\cdot 12}-\frac{1}{12\cdot 12\cdot 29}\ +\text{etc.},$$

there becomes

$$x=\frac{1}{4}+\frac{1}{1\cdot 5}-\frac{1}{2\cdot 12}+\frac{1}{5\cdot 29}-\frac{1}{12\cdot 70}+\ \text{etc.};$$

which series even if they may be strongly convergent, yet their true sum is unable to be deduced from their form.

376. But for continued fractions of this kind, in which all the denominators either are equal or are returned the same, thus so that fraction, if it may be cut off at some term from the beginning, at that point shall be equal to the total, the sums of these to be investigated may be found in an easy manner. For in the proposed example since there shall be :

$$x=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ \text{etc.}}}}}$$

there will be

$$x=\frac{1}{2+x}$$

and thus

$$xx+2x=1$$

and

$$x+1=\sqrt{2}$$

thus so that the value of this continued fraction shall be

$$=\sqrt{2}-1$$

Indeed fractions elicited before from the continued fraction approach closer to the true value continually and so quickly, that scarcely a more efficient way may be found for expressing that irrational value by rational numbers.

For $\sqrt{2}-1$ is so close to $\frac{29}{70}$, that the error shall be unnoticed ; and as the root being extracted shall be

$$\sqrt{2}-1=0,41421356237$$

and

$$\frac{29}{70}=0,41428571428,$$

thus so that the error may stand only at parts per hundred thousand.

377. Therefore just as continued fractions supply the way most conveniently to approximating the value $\sqrt{2}$, thus from the same source a way is uncovered for investigating the roots of other numbers approximately. Towards this end, we may put

$$x=\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\text{etc.}}}}}},$$

there will be

$$x=\frac{1}{a+x}$$

and

$$xx+ax=1,$$

from which there becomes

$$x=-\frac{1}{2}a+\sqrt{(1+\frac{1}{4}aa)}=\frac{\sqrt{(aa+4)}-a}{2}.$$

Therefore this continued fraction will act to provide the value of the square root by finding it from the number $aa+4$. And therefore by substituting the successive numbers 1, 2, 3, 4 etc. in place of a, hence $\sqrt{5}$, $\sqrt{2}$, $\sqrt{13}$, $\sqrt{5}$, $\sqrt{29}$, $\sqrt{10}$, $\sqrt{53}$ etc. will be found, clearly on reducing these roots to their simplest form. Hence there will be

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{3}, & \frac{3}{5}, & \frac{5}{8} \end{array} \quad \text{etc.} = \frac{\sqrt{5}-1}{2},$$

$$\begin{array}{cccccc} 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70} \end{array} \quad \text{etc.} = \sqrt{2}-1,$$

$$\begin{array}{cccccc} 3 & 3 & 3 & 3 & 3 & 3 \\ \frac{0}{1}, & \frac{1}{3}, & \frac{3}{10}, & \frac{10}{33}, & \frac{33}{109}, & \frac{109}{360} \end{array} \quad \text{etc.} = \frac{\sqrt{13}-3}{2},$$

$$\begin{array}{cccccc} 4 & 4 & 4 & 4 & 4 & 4 \\ \frac{0}{1}, & \frac{1}{4}, & \frac{4}{17}, & \frac{17}{72}, & \frac{72}{305}, & \frac{305}{1292} \end{array} \quad \text{etc.} = \sqrt{5}-2$$

etc.

Moreover it is to be observed from that the approximation is more prompt, as the number *a* becomes greater. Thus in the final example, there will be

$$\sqrt{5} = 2\frac{305}{1292},$$

so that the error shall be less than $\frac{1}{1292 \cdot 5473}$, where 5473 is the denominator of the following fraction $\frac{1292}{5473}$.

378. Indeed the roots of other numbers are unable to be shown in this way, unless they shall be the sum of two squares. Therefore so that this approximation may be extended to other numbers, we may put

$$x=\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\text{etc.}}}}}}}$$

There will be

$$x=\cfrac{1}{a+\cfrac{1}{b+x}}=\frac{b+x}{ab+1+ax}$$

and thus

$$axx+abx=b$$

and

$$x=-\frac{1}{2}b\pm\sqrt{\left(\frac{1}{4}bb+\frac{b}{a}\right)}=\frac{-ab+\sqrt{(aabb+4ab)}}{2a}.$$

From which now the roots of all the roots may be found. For the sake of an example let $a=2,\ b=7$; there will be

$$x=\frac{-14+\sqrt{(14\cdot 18)}}{4}=\frac{-7+3\sqrt{7}}{2}.$$

But the following fractions will show the value of x approximately

$$\begin{array}{cccccc} 2 & 7 & 2 & 7 & 2 & 7 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510} \text{ etc.} \end{array}$$

Therefore there will be approximately

$$\frac{-7+3\sqrt{7}}{2}=\frac{239}{510}$$

and

$$\sqrt{7}=\frac{2024}{765}=2{,}645\,751\,63;$$

but actually

$$\sqrt{7} = 2,645\,751\,31\,;$$

thus so that the error shall be less than $\frac{33}{100\,000\,000}$.

379. Moreover we may progress further by putting

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \text{etc.}}}}}}}}$$

There becomes

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + x}}} = \cfrac{1}{a + \cfrac{c + x}{bx + bc + 1}} = \frac{bx + bc + 1}{(ab + 1)x + abc + a + c},$$

from which

$$(ab + 1)xx + (abc + a - b + c)x = bc + 1$$

and

$$x = \frac{-abc - a + b - c + \sqrt{\left((abc + a + b + c)^2 + 4\right)}}{2(ab + 1)}$$

where the quantity put under the sign for the root again is the sum of two squares ; therefore neither does this form serve for roots extracted from other numbers, unless numbers of the first form may now be substituted. In a similar manner if the four letters *a*, *b*, *c*, *d* may continually stand for the denominators of continued fractions, then these do not have more use than the second, which will contain only two letters, and thus henceforth.

380. Therefore since continued fractions may be able to be used so effectively for the extraction of square roots, likewise they may serve in resolving quadratic equations ; which indeed is evident from that calculation itself, while x may be determined from the affected equation. Moreover the root of each quadratic equation will be able to be expressed easily by a continued fraction in this manner. Let this equation be proposed

$$xx = ax + b\,;$$

from which since there shall be

$$x = a + \frac{b}{x}\ ,$$

the same value x found now may be substituted in the final place and there will be

$$x = a + \cfrac{b}{a + \cfrac{b}{x}}$$

therefore by preceding in a similar manner there will be the continued fraction

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \text{etc.}}}}$$

but which, since the numerators b shall not be ones, so not able to be used so conveniently.

381. But so that the use in arithmetic may be shown, first it is to be observed every ordinary fraction can be changed into a continued fraction. For the fraction may be proposed

$$x = \frac{A}{B},$$

in which there shall be $A > B$; A may be divided by B and the quotient shall be $= a$ and the remainder C ; then the preceding divisor B may be divided by this remainder C and the quotient b and remainder D may be produced, by which again the preceding divisor C may be divided ; and thus this operation, which is commonly accustomed to be used for finding the greatest common divisor of the numbers A and B , in the following manner:

$$
\begin{array}{l}
B)\underline{A}(a \\
\quad C)\underline{B}(b \\
\qquad D)\underline{C}(c \\
\qquad\quad E)\underline{D}(d \\
\qquad\qquad F \text{ etc.}
\end{array}
$$

And there will be, from the nature of the division

$$
\begin{array}{llll}
A = \alpha B + C, & \text{from which} & \dfrac{A}{B} = a + \dfrac{C}{B}, & \\
B = bC + D, & & \dfrac{B}{C} = b + \dfrac{D}{C}, & \dfrac{C}{B} = \dfrac{1}{b + \dfrac{D}{C}}, \\
C = cD + E, & & \dfrac{C}{D} = c + \dfrac{E}{D}, & \dfrac{D}{C} = \dfrac{1}{c + \dfrac{E}{D}}, \\
D = dE + F & & \dfrac{D}{E} = d + \dfrac{F}{E}, & \dfrac{E}{D} = \dfrac{1}{d + \dfrac{F}{E}} \\
\text{etc.,} & & \text{etc.} &
\end{array}
$$

Hence by substituting the following values into the preceding, there will be

$$
x = \frac{A}{B} = a + \frac{C}{B} = a + \frac{1}{b + \dfrac{D}{C}} = a + \frac{1}{b + \dfrac{1}{c + \dfrac{E}{D}}}
$$

from which finally x may be expressed through the following mixed quotients found *a, b, c, d* etc., so that there shall be

$$
x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \text{etc.}}}}}}}
$$

EXAMPLE 1

Let this fraction be proposed $\frac{1461}{59}$, which will be changed into a continued fraction in the following manner, all the numerators of which are ones.

Evidently the same operation may be put in place, from which the maximum common divisor of the numbers 59 and 1461 is accustomed to be found :

```
59) 1461 (24
    118
     281
     236
        45) 59 (1
            45
            14) 45 (3
                42
                 3) 14 (4
                    12
                     2)3(1
                       2
                       1)2(2
                         2
                         0
```

Hence from these quotients there is made

$$\frac{1461}{59}=24+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{2}}}}}$$

EXAMPLE 2

Decimal fractions also can be changed in the same manner.
For let there be proposed

$$\sqrt{2} = 1,41421356 = \frac{141\,421\,356}{100\,000\,000},$$

from which this operation may be put in place :

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418328	1010160	2
86752	209164	

etc.

Now it is apparent from this operation that all the denominators are 2 and thus there shall be

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\text{etc.}}}}}}$$

now the ratio of this expression is apparent from above.

EXAMPLE 3

Truly in the first place also the number e is worthy of attention here , the logarithm of which is $=1$, which is

$$e = 2,718281828459 .$$

From which there arises

$$\frac{e-1}{2}=0,8591409142295,$$

which decimal fraction, if it may be examined in the above manner, will give the following quotients

8591409142295	10000000000000	1
451545146224	8591409142295	6
139863996071	1408590857704	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	
	etc.	

If this calculation at this stage may be continued further with a more exact assumed value of *e*, then these quotients will be produced :

1, 6, 10, 14, 18, 22, 26, 30, 34 etc.,

which constitute an arithmetic progression taken from the start, from which it is apparent :

$$\frac{e-1}{2}=\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\text{etc.}}}}}}}},$$

an account of this fraction can be given from the infinitesimal calculus.

382. Therefore since fractions may be able to be elicited from expressions of this kind, which as the quickest may be reduced to the true value of the expression, this method can be used for expressing decimal fractions through ordinary fractions, which approach close to these. Also one will be able, if some fraction were proposed, the numerator and denominator of which shall be very large numbers, to find fractions from small constant numbers, which even if the proposed numbers shall not be exactly equal, yet from these they shall be different minimally. Hence a problem treated by Wallis long ago can be resolved easily, in which fractions are expressed by smaller numbers, which so closely approach the value of some fraction proposed with larger numbers, as often as they can

with smaller numbers. But our fractions arise in this method approach so closely to the value of continued fractions, from which they may be elicited, so that no constants may be given with greater numbers, which do approach closer.

EXAMPLE 1

The ratio of the diameter to the periphery may be expressed by numbers so small, so that it may not be shown more accurately, unless greater numbers may be used.

If the known decimal fraction

$$3{,}1415926535 \text{ etc.}$$

may be expanded out in the way shown by continued division, the following quotients will be found

$$3,\ 7,\ 15,\ 1,\ 292,\ 1,\ 1 \text{ etc.},$$

from which the following fractions will be formed

$$\frac{1}{0},\quad \frac{3}{1},\quad \frac{22}{7},\quad \frac{333}{106},\quad \frac{355}{113},\quad \frac{103993}{33102} \quad \text{etc.}$$

The second fraction thus has been shown the diameter to the periphery to be as 1: 3 nor surely will it be able to be given more accurately by larger numbers. The third fraction gives the *Archimedes* ratio 7: 22, but the fifth of *Metius*, which approaches so close to the true value, that the error shall be less than the $\frac{1}{113\cdot 33102}$ th part. These remaining fractions alternately are greater of less than the true value.

EXAMPLE 2

The ratio of the day to the mean solar year may be expressed approximately in the smallest numbers.

Since the year shall be itself $365^d\ 5^h\ 48'\ 55''$, in one year there will be contained a fraction of $365\frac{20935}{68640}$ days. Therefore, it is necessary only, that this fraction may be expanded out, which will give the following quotients 4, 7, 1, 6, 1, 2, 2, 4, from which the fractions themselves may be elicited

$$\frac{0}{1},\quad \frac{1}{4},\quad \frac{7}{29},\quad \frac{8}{33},\quad \frac{55}{227},\quad \frac{63}{260},\quad \frac{181}{747} \quad \text{etc.}$$

Therefore the hours with the minutes and seconds which are present in the above 365 days, make around four years and one day, from which the Julian calendar has its origin. But more exact 33 years gives 8 days, or 747 years gives 181 days; from which it follows in four hundred years 97 days are left over. Whereby, since in this interval of the Julian calendar 100 days may be inserted, [the Pope] Gregory changed this in a duration of four centuries composed of three years and a leap year in common.

End of Book I.

CAPUT XVIII

DE FRACTIONIBUS CONTINUIS

356. Quoniam in praecedentibus capitibus plura cum de seriebus infinitis tum de productis ex infinitis factoribus conflatis disserui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est excultum, tamen non dubitamus, quin ex eo amplissimus usus in analysin infinitorum aliquando sit redundaturus. Exhibui enim iam aliquoties eiusmodi specimina, quibus haec expectatio non parum probabilis redditur. Imprimis vero ad ipsam arithmeticam et algebram communem non contemnenda subsidia affert ista speculatio, quae hoc capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco eiusmodi fractionem, cuius denominator constat ex numero integro cum fractione, cuius denominator denuo est aggregatum ex integro et fractione, quae porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Huiusmodi ergo fractio continua erit sequens expressio

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\text{ etc.}}}}}} \quad \text{vel} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{ etc.}}}}}}$$

in quarum forma priori omnes fractionum numeratores sunt unitates, quam potissimum hic contemplabor, in altera vero forma sunt numeratores numeri quicunque.

358. Exposita ergo fractionum harum continuarum forma primum videndum est, quemadmodum earum significatio consueto more expressa inveniri queat. Quae ut facilius inveniri possit, progrediamur per gradus abrumpendo illas fractiones primo in prima, tum in secunda, post in tertia et ita porro fractione; quo facto patebit fore

$$a+\frac{1}{b} = \frac{ab+1}{b},$$

$$a+\cfrac{1}{b+\cfrac{1}{c}} = \frac{abc+a+c}{bc+1},$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d},$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+b+c}{bcde+be+de+bc+1}$$

etc.

359. Etsi in his fractionibus ordinariis non facile lex, secundum quam numerator ac denominator ex litteris *a, b, c, d* etc. componantur, perspicitur, tamen attendenti statim patebit, quemadmodum quaelibet fractio ex praecedentibus formari queat. Quilibet enim numerator est aggregatum ex numeratore ultimo per novam litteram multiplicato et ex numeratore penultimo simplici; eademque lex in denominatoribus observatur. Scriptis ergo ordine litteris *a, b, c, d* etc. ex iis fractiones inventae facile formabuntur hoc modo

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+a+c}{bc+1}, & \frac{abcd+ab+ad+cd+1}{bcd+b+d} \end{array} \text{ etc.}$$

ubi quilibet numerator invenitur, si praecedentium ultimus per indicem supra scriptum multiplicetur atque ad productum antepenultimus addatur; quae eadem lex pro denominatoribus valet. Quo autem hac lege ab ipso initio uti liceat, praefixi fractionem $\frac{1}{0}$, quae, etiamsi e fractione continua non oriatur, tamen progressionis legem clariorem efficit. Quaelibet autem fractio exhibet valorem fractionis continuae usque ad eam litteram, quae antecedenti imminet, inclusive continuatae.

360. Simili modo altera fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{ etc.}}}}}}$$

dabit, prout aliis aliisque locis abrumpitur, sequentes valores

$$a = a,$$
$$a+\frac{\alpha}{b} = \frac{ab+\alpha}{b},$$
$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} = \frac{abc+\beta a+\alpha c}{bc+\beta},$$
$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b},$$
etc.

quarum fractionum quaeque ex binis praecedentibus sequentem in modum invenietur:

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+\alpha}{b}, & \frac{abc+\beta a+\alpha c}{bc+\beta}, & \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} & \text{etc.} \\ \alpha & \beta & \gamma & \delta & \varepsilon & \end{array}$$

361. Fractionibus scilicet formandis supra inscribantur indices *a, b, c, d* etc., infra autem subscribantur indices $\alpha, \beta, \gamma, \delta$ etc. Prima fractio iterum constituatur $\frac{1}{0}$, secunda $\frac{a}{1}$. Tum sequentium quaevis formabitur, si antecedentium ultimae numerator per indicem supra scriptum, penultimae vero numerator per indicem infra scriptum multiplicetur et ambo producta addantur; aggregatum erit numerator fractionis sequentis. Simili modo eius denominator erit aggregatum ex ultimo denominatore per indicem supra scriptum et ex penultimo denominatore per indicem infra scriptum multiplicatis. Quaelibet vero fractio hoc modo inventa praebebit valorem fractionis continuae ad eum usque denominatorem, qui fractioni antecedenti est inscriptus, continuatae inclusive.

362. Quodsi ergo hae fractiones eousque continuentur, quoad fractio continua indices suppeditet, tum ultima fractio verum dabit valorem fractionis continuae. Praecedentes fractiones vero continuo propius ad hunc valorem accedent ideoque perquam idoneam appropinquationem suggerent. Ponamus enim verum valorem fractionis continuae

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\text{ etc.}}}}}$$

esse $=x$ atque manifestum est fractionem primam $\frac{1}{0}$ esse maiorem quam *x*, secunda vero $\frac{a}{1}$ minor erit quam *x*, tertia $a+\frac{\alpha}{b}$; iterum vero valore erit maior, quarta denuo minor, atque ita porro hae fractiones alternatim erunt maiores et minores quam *x*. Porro autem perspicuum est quamlibet fractionem propius accedere ad verum valorem *x* quam ulla praecedentium, unde hoc pacto citissime et commodissime valor ipsius *x* proxime obtinetur, etiamsi fractio continua in infinitum progrediatur, dummodo numeratores $\alpha, \beta, \gamma, \delta$ etc. non nimis crescant: sin autem omnes isti numeratores fuerint unitates, tum appropinquatio nulli incommodo est obnoxia.

363. Quo ratio huius appropinquationis ad verum fractionis continuae valorem melius percipiatur, consideremus fractionum inventarum differentiae.
Ac prima quidem $\frac{1}{0}$ praetermissa differentia inter secundam ac tertiam est

$$=\frac{\alpha}{b},$$

quarta a tertia subtracta relinquit

$$=\frac{\alpha\beta}{b(bc+\beta)},$$

quarta a quinta subtracta relinquit

$$=\frac{\alpha\beta\gamma}{(bc+\beta)\ (bcd+\beta d+\gamma b)}$$

etc. Hinc exprimetur valor fractionis continuae per seriem terminorum consuetam hoc modo, ut sit

$$x=a+\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)\ (bcd+\beta d+\gamma b)}-\text{ etc.}$$

quae series toties abrumpitur, quoties fractio continua non in infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quamcunque in seriem terminorum, quorum signa alternantur, convertendi, siquidem prima littera *a* evanescat. Si enim fuerit

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{ etc.}}}}}}$$

erit per ea, quae modo invenimus,

$$x=a+\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)\ (bcd+\beta d+\gamma b)}$$

$$-\frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)\ (bcde+\beta de+\gamma be+\delta bc+\beta\delta)}+\text{ etc.}$$

Unde, si $\alpha, \beta, \gamma, \delta$ etc. fuerint numeri non crescentes, uti omnes unitates, denominatores vero *a, b, c, d* etc. numeri integri quicunque affirmativi, valor fractionis continuae exprimetur per seriem terminorum maxime convergentem.

365. His probe consideratis poterit vicissim series quaecunque terminorum alternantium in fractionem continuam converti seu fractio continua inveniri, cuius valor aequalis sit summae seriei propositae. Sit enim proposita haec series

$$x=A-B+C-D+E-F+\text{etc.};$$

erit singulis terminis cum serie ex fractione continua orta comparandis

$$A=\frac{\alpha}{b}, \qquad \text{unde fit} \quad \alpha=Ab,$$

$$\frac{B}{A}=\frac{\beta}{bc+\beta}, \qquad \beta=\frac{Bbc}{A-B},$$

$$\frac{C}{B}=\frac{\gamma b}{bcd+\beta d+\gamma b}, \qquad \gamma=\frac{Cd(bc+\beta)}{b(B-C)},$$

$$\frac{D}{C}=\frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}, \qquad \delta=\frac{Ce(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$$

etc., etc.

At cum $\beta = \frac{Bbc}{A-B}$, erit

$$bc + \beta = \frac{Abc}{A-B},$$

unde

$$\gamma = \frac{ACcd}{(A-B)(B-C)}.$$

Porro fit

$$bcd + \beta d + \gamma b = (bc+\beta)d + \gamma b = \frac{Abcd}{(A-B)} + \frac{ACbcd}{(A-B)(B-C)} = \frac{ABbcd}{(A-B)(B-C)},$$

unde erit

$$\frac{bcd + \beta d + \gamma b}{bc+\beta} = \frac{Bd}{B-C}$$

et

$$\delta = \frac{BDde}{(B-C)(C-D)}.$$

Simili modo reperietur

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

et ita porro.

366. Quo ista lex clarius appareat, ponamus esse

$$\begin{aligned}
&P = b,\\
&Q = bc + \beta,\\
&R = bcd + \beta d + \gamma b,\\
&S = bcde + \beta de + \gamma be + \delta bc + \beta\delta,\\
&T = bcdef + \text{ etc.},\\
&V = bcdefg + \text{ etc.}\\
&\qquad\qquad \text{etc.};
\end{aligned}$$

erit ex lege harum expressionum

$$Q = Pc + \beta,$$
$$R = Qd + \gamma P,$$
$$S = Re + \delta Q,$$
$$T = Sf + \varepsilon R,$$
$$V = Tg + \zeta S$$
etc.

Cum igitur his adhibendis litteris fit

$$x = \frac{\alpha}{P} - \frac{\alpha\beta}{PQ} + \frac{\alpha\beta\gamma}{QR} - \frac{\alpha\beta\gamma\delta}{RS} + \frac{\alpha\beta\gamma\delta\varepsilon}{ST} - \text{etc.}$$

367. Quoniam ergo ponimus esse

$$x = A - B + C - D + E - \mathrm{F} + \text{ etc.},$$

erit

$$A = \frac{\alpha}{P}, \qquad \alpha = AP,$$

$$\frac{B}{A} = \frac{\beta}{Q}, \qquad \beta = \frac{BQ}{A},$$

$$\frac{C}{B} = \frac{\gamma P}{R}, \qquad \gamma = \frac{CR}{BP},$$

$$\frac{D}{C} = \frac{\delta Q}{S}, \qquad \delta = \frac{DS}{CQ},$$

$$\frac{E}{D} = \frac{\varepsilon R}{T}, \qquad \varepsilon = \frac{ET}{DR}$$

etc.

Porro vero differentiis sumendis habebitur

$$A-B=\frac{\alpha(Q-\beta)}{PQ}=\frac{\alpha c}{Q}=\frac{APc}{Q},$$

$$B-C=\frac{\alpha\beta(R-\gamma P)}{PQR}=\frac{\alpha\beta d}{PR}=\frac{BQd}{R},$$

$$C-D=\frac{\alpha\beta\gamma(S-\delta Q)}{QRS}=\frac{\alpha\beta\gamma e}{QS}=\frac{CRe}{S},$$

$$D-E=\frac{\alpha\beta\gamma\delta(T-\varepsilon R)}{RST}=\frac{\alpha\beta\gamma\delta f}{RT}=\frac{DSf}{T}$$

etc.

Si bini igitur in se invicem ducantur, fiet

$$(A-B)(B-C)=ABcd\frac{P}{R} \text{ et } \frac{R}{P}=\frac{ABcd}{(A-B)(B-C)},$$

$$(B-C)(C-D)=BCde\frac{Q}{S} \text{ et } \frac{S}{Q}=\frac{BCde}{(B-C)(C-D)},$$

$$(C-D)(D-E)=CDef\frac{R}{T} \text{ et } \frac{T}{R}=\frac{CDef}{(C-D)(D-E)}$$

etc.

Unde, cum sit $P=b,\ Q=\frac{\alpha c}{A-B}=\frac{\alpha bc}{A-B}$ erit

$$\alpha = Ab,$$

$$\beta = \frac{Bbc}{A-B},$$

$$\gamma = \frac{ACcd}{(A-B)(B-C)},$$

$$\delta = \frac{BDde}{(B-C)(C-D)},$$

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

etc.

368. Inventis ergo valoribus numeratorum $\alpha, \beta, \gamma, \delta$ etc. denominatores *b, c, d, e* etc. arbitrio nostro relinquuntur; ita autem eos assumi convenit, ut cum ipsi sint numeri integri, tum valores integros pro $\alpha, \beta, \gamma, \delta$ etc. exhibeant. Hoc vero pendet quoque a natura numerorum *A*, *B*, *C* etc., utrum sint integri an fracti. Ponamus esse numeros integros atque quaesito satisfiet statuendo

$$\begin{array}{lll} b = 1, & \text{unde fit} & \alpha = A, \\ c = A-B, & & \beta = B, \\ d = B-C, & & \gamma = AC, \\ e = C-D, & & \delta = BD, \\ f = D-E, & & \varepsilon = CE \\ \text{etc.,} & & \text{etc.} \end{array}$$

Quocirca si fuerit

$$x = A - B + C - D + E - F + \text{ etc.},$$

idem ipsius *x* valor per fractionem continuam ita exprimi poterit, ut sit

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \text{etc.}}}}}}$$

369. Sin autem omnes termini seriei sint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \text{etc.},$$

habebuntur pro $\alpha, \beta, \gamma, \delta$ etc. sequentes valores

$$\alpha = \frac{b}{A},$$

$$\beta = \frac{Abc}{B - A},$$

$$\gamma = \frac{B^2cd}{(B - A)(C - B)},$$

$$\delta = \frac{C^2de}{(C - B)(D - C)},$$

$$\varepsilon = \frac{D^2ef}{(D - C)(E - D)}$$

etc.

Ponatur ergo, ut sequitur,

$$\begin{array}{lll} b = A, & \text{unde fit} & \alpha = 1, \\ c = B - A, & & \beta = AA \\ d = C - B, & & \gamma = BB \\ e = D - C, & & \delta = CC \\ \text{etc.,} & & \text{etc.} \end{array}$$

eritque per fractionem continuam

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \text{etc.}}}}}$$

EXEMPLUM 1

Transformetur haec series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.}$$

in fractionem continuam.

Erit ergo

$$A = 1,\ B = 2,\ C = 3,\ D = 4\ \text{ etc.},$$

atque cum seriei propositae valor sit $= l2$, erit

$$l2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cfrac{25}{1 + \text{etc.}}}}}}}$$

EXEMPLUM 2

Transformetur haec series infinita

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.},$$

ubi π denotat peripheriam circuli, cuius diameter $= 1$, in fractionem continuam.

Substitutis loco *A, B, C, D* etc. numeris 1, 3, 5, 7 etc. orietur

$$\frac{\pi}{4}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}}$$

hincque invertendo fractionem erit

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

quae est expressio, quam BROUNCKERUS primum pro quadratura circuli protulit,

EXEMPLUM 3

Sit proposita ista series infinita

$$x=\frac{1}{m}-\frac{1}{m+n}+\frac{1}{m+2n}-\frac{1}{m+3n}+\text{ etc.},$$

quae ob

$$A=m,\ B=m+n,\ C=m+2n \quad \text{etc.},$$

in hanc fractionem continuam mutatur

$$x=\cfrac{1}{m+\cfrac{mm}{n+\cfrac{(m+n)^2}{n+\cfrac{(m+2n)^2}{n+\cfrac{(m+3n)^2}{n+\text{ etc.}}}}}}$$

ex qua fit invertendo

$$\frac{1}{x}-m=\cfrac{mm}{n+\cfrac{(m+n)^2}{n+\cfrac{(m+2n)^2}{n+\cfrac{(m+3n)^2}{n+\text{ etc.}}}}}$$

EXEMPLUM 4

Quoniam supra (§ 178) invenimus esse

$$\frac{\pi\cos.\frac{m\pi}{n}}{\pi\sin.\frac{m\pi}{n}}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\text{etc.},$$

erit pro fractione continuanda

$$A=m,\ B=n-m,\ C=n+m,\ D=2n-m\ \text{ etc.},$$

unde fiet

$$\frac{\pi\cos.\frac{m\pi}{n}}{\pi\sin.\frac{m\pi}{n}}=\cfrac{1}{m+\cfrac{mm}{n-2m+\cfrac{(n-m)^2}{2m+\cfrac{(n+m)^2}{n-2m+\cfrac{(2n-m)^2}{2m+\cfrac{(2n+m)^2}{n-2m+\text{etc.}}}}}}}$$

370. Si series proposita per continuos factores progrediatur, ut sit

$$x=\frac{1}{A}-\frac{1}{AB}+\frac{1}{ABC}-\frac{1}{ABCD}+\frac{1}{ABCDE}-\text{ etc.},$$

tum prodibunt sequentes determinationes

$$\alpha = \frac{b}{A} -,$$
$$\beta = \frac{bc}{B-1},$$
$$\gamma = \frac{Bcd}{(B-1)(C-1)},$$
$$\delta = \frac{Cde}{(C-1)(D-1)},$$
$$\varepsilon = \frac{Def}{(D-1)(E-1)}$$
$$\text{etc.}$$

Fiat ergo, ut sequitur,

$$\begin{array}{lcl} b = A, & \text{unde fit} & \alpha = 1, \\ c = B-1, & & \beta = A, \\ d = C-1, & & \gamma = B, \\ e = D-1, & & \delta = C, \\ f = E-1, & & \varepsilon = D \\ \text{etc.}, & & \text{etc.}, \end{array}$$

unde consequenter fiet

$$x = \cfrac{1}{A + \cfrac{A}{B-1 + \cfrac{B}{\mathrm{C}-1 + \cfrac{C}{D-1 + \cfrac{D}{E-1+\text{etc.}}}}}}$$

EXEMPLUM 1

Quoniam posito e numero, cuius logarithmus est = 1, supra invenimus esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} - \text{ etc.}$$

seu

$$1-\frac{1}{e}=\frac{1}{1}-\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}-\frac{1}{1\cdot 2\cdot 3\cdot 4}+\text{ etc.},$$

haec series in fractionem continuam convertetur ponendo

$$A=1,\ B=2,\ C=3,\ D=4\ \text{ etc.};$$

quo ergo facto habebitur

$$1-\frac{1}{e}=\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\text{ etc.}}}}}}}$$

unde asymmetria initio reiecta erit

$$\frac{1}{e-1}=\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\text{ etc.}}}}}}$$

EXEMPLUM 2

Invenimus quoque arcus, qui radio aequalis sumitur, cosinum esse

$$=1-\frac{1}{2}+\frac{1}{2\cdot 12}-\frac{1}{2\cdot 12\cdot 30}+\frac{1}{2\cdot 12\cdot 30\cdot 56}-\text{ etc.}$$

Si ergo fiat

$$A=1,\ B=2,\ C=12,\ D=30,\ E=56\quad\text{etc.}$$

atque cosinus arcus, qui radio aequatur, ponatur $=x$, erit

$$x = \cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{11+\cfrac{12}{29+\cfrac{30}{55\ +\text{etc.}}}}}}$$

seu

$$\frac{1}{x}-1 = \cfrac{1}{1+\cfrac{2}{11+\cfrac{12}{29+\cfrac{30}{55+\text{etc.}}}}}$$

371. Sit series insuper cum geometrica coniuncta, scilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \text{ etc.};$$

erit

$$\alpha = Ab,$$
$$\beta = \frac{Bbcz}{A-Bz},$$
$$\gamma = \frac{ACcdz}{(A-Bz)(B-Cz)},$$
$$\delta = \frac{BDdez}{(B-Cz)(C-Dz)},$$
$$\varepsilon = \frac{CEefz}{(C-Dz)(D-Ez)}$$
$$\text{etc.}$$

Ponatur nunc

$$\begin{array}{lll} b=1, & \text{erit} & \alpha = A, \\ c = A-Bz, & & \beta = Bz, \\ d = B-Cz, & & \gamma = ACz \\ e = C-Dz, & & \delta = BDz \\ \text{etc.,} & & \text{etc.,} \end{array}$$

unde fiet

$$x=\cfrac{A}{1+\cfrac{Bz}{A-Bz+\cfrac{ACz}{B-Cz+\cfrac{BDz}{C-Dz+\text{ etc.}}}}}$$

372. Quo autem hoc negotium generalius absolvamus, ponamus esse

$$x=\frac{A}{L}-\frac{By}{Mz}+\frac{Cy^2}{Nz^2}-\frac{Dy^3}{Oz^3}+\frac{Ey^4}{Pz^4}-\text{etc.}$$

fietque comparatione instituta

$$\alpha=\frac{Ab}{L},$$
$$\beta=\frac{BLbcy}{AMz-BLy},$$
$$\gamma=\frac{ACM^2cdyz}{(AMz-BLy)(BNz-CMy)},$$
$$\delta=\frac{BDN^2deyz}{(BNz-CMy)\ (COz-DNy)}$$
etc,

Statuantur valores *b*, *c*, *d* etc. sequenti modo

$$\begin{array}{lcl}
b=L, & \text{erit} & \alpha=A,\\
c=AMz-BLy, & & \beta=BLLy,\\
d=BNz-CMy, & & \gamma=ACM^2yz,\\
e=COz-DNy, & & \delta=BDN^2yz,\\
f=DPz-EOy, & & \varepsilon=CEO^2yz\\
\text{etc.}; & & \text{etc.},
\end{array}$$

unde series proposita per sequentem fractionem continuam exprimetur

$$x=\cfrac{A}{L+\cfrac{BLLy}{AMz-Bly+\cfrac{ACMMyz}{BNz-CMy+\cfrac{BDNNyz}{COz-DNy+\text{etc.}}}}}$$

373. Habeat denique series proposita huiusmodi formam

$$x=\frac{A}{L}-\frac{ABy}{LMz}+\frac{ABCy^2}{LMNz^2}-\frac{ABCDy^3}{LMNOz^3}+\text{etc.}$$

atque sequentes valores prodibunt

$$\alpha=\frac{Ab}{L},$$
$$\beta=\frac{Bbcy}{Mz-By},$$
$$\gamma=\frac{CMcdyz}{(Mz-By)(Nz-Cy)},$$
$$\delta=\frac{DNdeyz}{(Nz-Cy)\,(Oz-Dy)},$$
$$\varepsilon=\frac{EOefyz}{(Oz-Dy)\,(Pz-Ey)}$$
$$\text{etc,}$$

Ad valores ergo integros inveniendos fiat

$$\begin{array}{lll} b=Lz, & \text{erit} & a=Az, \\ c=Mz-By, & & \beta=BLyz, \\ d=Nz-Cy, & & \gamma=CMyz \\ e=Oz-Dy, & & \delta=DNyz, \\ f=Pz-Ey, & & \varepsilon=EOyz \\ \text{etc.;} & & \text{etc.,} \end{array}$$

unde valor seriei propositae ita exprimetur, ut sit

$$x=\cfrac{Az}{Lz+\cfrac{BLyz}{Mz-By+\cfrac{CMyz}{Nz-Cy+\cfrac{DNyz}{Oz-Dy+\text{etc.}}}}}$$

vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy \text{ etc.}}}}$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuae in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim ex supra traditis infinitae series, quarum summae constent, ad hoc negotium accommodari queant, unaquaeque transformari poterit in fractionem continuam, cuius adeo valor summae illius seriei est aequalis. Exempla, quae iam hic sunt allata, sufficiunt ad hunc usum ostendendum. Verumtamen optandum esset, ut methodus detegeretur, cuius beneficio, si proposita fuerit fractio continua quaecunque, eius valor immediate inveniri posset. Quanquam enim fractio continua transmutari potest in seriem infinitam, cuius summa per methodos cognitas investigari queat, tamen plerumque istae series tantopere fiunt intricatae, ut earum summa, etiamsi sit satis simplex, vix ac ne vix quidem obtineri possit.

375. Quo autem clarius perspiciatur dari eiusmodi fractiones continuas, quarum valor aliunde facile assignari queat, etiamsi ex seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, consideremus hanc fractionem continuam

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \text{ etc.}}}}}$$

cuius omnes denominatores sunt inter se aequales. Si enim hinc modo supra exposito fractiones formemus

$$\begin{array}{ccccccc} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70} \text{ etc.,} \end{array}$$

oritur haec series

$$x = 0 + \frac{1}{2} - \frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 12} - \frac{1}{12 \cdot 29} + \frac{1}{29 \cdot 70} - \text{ etc.}$$

vel, si bini termini coniungantur, erit

$$x = \frac{2}{1\cdot 5} + \frac{2}{5\cdot 29} + \frac{2}{29\cdot 169} + \text{etc.}$$

vel

$$x = \frac{1}{2} - \frac{2}{2\cdot 12} - \frac{2}{12\cdot 70} - \text{etc.}$$

Quin etiam, cum sit

$$\begin{aligned} x &= \frac{1}{4} - \frac{1}{2\cdot 2\cdot 5} + \frac{1}{2\cdot 5\cdot 12} - \frac{1}{2\cdot 12\cdot 29} + \text{etc.} \\ &+ \frac{1}{4} - \frac{1}{2\cdot 2\cdot 5} + \frac{1}{2\cdot 5\cdot 12} - \frac{1}{12\cdot 12\cdot 29} + \text{etc.}, \end{aligned}$$

erit

$$x = \frac{1}{4} + \frac{1}{1\cdot 5} - \frac{1}{2\cdot 12} + \frac{1}{5\cdot 29} - \frac{1}{12\cdot 70} + \text{ etc.};$$

quae series etiamsi vehementer convergant, tamen vera earum summa ex earum forma colligi nequit.

376. Pro huiusmodi autem fractionibus continuis, in quibus denominatores omnes vel sunt aequales vel iidem revertuntur, ita ut ea fractio, si ab initio aliquot terminis truncetur, toti adhuc sit aequalis, facilis habetur modus earum summas explorandi. In exemplo enim proposito icum sit

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \text{ etc.}}}}}$$

erit

$$x = \frac{1}{2 + x}$$

ideoque

$$xx + 2x = 1$$

et

$$x + 1 = \sqrt{2}$$

ita ut valor huius fractionis continuae sit

$$=\sqrt{2}-1$$

Fractiones vero ex fractione continua ante erutae continuo propius ad hunc valorem accedunt idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum inveniri queat.

Est enim $\sqrt{2}-1$ tam prope $\frac{29}{70}$, ut error sit insensibilis; namque radicem extrahendo erit

$$\sqrt{2}-1=0,41421356237$$

atque

$$\frac{29}{70}=0,41428571428,$$

ita ut error tantum in partibus centesimis millesimis consistat.

377. Quemadmodum ergo fractiones continuae commodissimum suppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem facillima via aperitur ad radices aliorum numerorum proxime investigandas. Ponamus hunc in finem

$$x=\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\text{etc.}}}}}}$$

erit

$$x=\frac{1}{a+x}$$

et

$$xx+ax=1,$$

unde fit

$$x=-\frac{1}{2}a+\sqrt{(1+\frac{1}{4}aa)}=\frac{\sqrt{(aa+4)}-a}{2}.$$

Haec ergo fractio continua inserviet valori radicis quadratae ex numero $aa+4$ inveniendo. Hincque adeo substituendo loco *a* successive numeros 1, 2, 3, 4 etc. reperientur $\sqrt{5},\ \sqrt{2},\ \sqrt{13},\ \sqrt{5},\ \sqrt{29},\ \sqrt{10},\ \sqrt{53}$ etc., perductis scilicet his radicibus ad formam simplicissimam. Erit ergo

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{3}, & \frac{3}{5}, & \frac{5}{8} \end{array} \quad \text{etc.} = \frac{\sqrt{5}-1}{2},$$

$$\begin{array}{cccccc} 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70} \end{array} \quad \text{etc.} = \sqrt{2}-1,$$

$$\begin{array}{cccccc} 3 & 3 & 3 & 3 & 3 & 3 \\ \frac{0}{1}, & \frac{1}{3}, & \frac{3}{10}, & \frac{10}{33}, & \frac{33}{109}, & \frac{109}{360} \end{array} \quad \text{etc.} = \frac{\sqrt{13}-3}{2},$$

$$\begin{array}{cccccc} 4 & 4 & 4 & 4 & 4 & 4 \\ \frac{0}{1}, & \frac{1}{4}, & \frac{4}{17}, & \frac{17}{72}, & \frac{72}{305}, & \frac{305}{1292} \end{array} \quad \text{etc.} = \sqrt{5}-2$$

etc.

Notandum autem eo promptiorem esse approximationem, quo maior fuerit numerus *a*. Sic in ultimo exemplo erit

$$\sqrt{5} = 2\frac{305}{1292},$$

ut error minor sit quam $\frac{1}{1292\cdot 5473}$ ubi 5473 est denominator sequentis fractionis $\frac{1292}{5473}$.

378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur haec approximatio ad alios numeros extendatur, ponamus esse

$$x=\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\text{etc.}}}}}}}$$

Erit

$$x=\cfrac{1}{a+\cfrac{1}{b+x}}=\frac{b+x}{ab+1+ax}$$

ideoque

$$axx+abx=b$$

et

$$x=-\frac{1}{2}b\pm\sqrt{\left(\frac{1}{4}bb+\frac{b}{a}\right)}\;=\frac{-ab+\sqrt{(aabb+4ab)}}{2a}.$$

Unde iam omnium numerorum radices inveniri poterunt. Sit verbi gratia $a=2,\ b=7$; erit

$$x=\frac{-14+\sqrt{(14\cdot 18)}}{4}=\frac{-7+3\sqrt{7}}{2}.$$

At valorem ipsius x proxime exhibebunt sequentes fractiones

$$\begin{array}{cccccc} 2 & 7 & 2 & 7 & 2 & 7 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510} \text{ etc.} \end{array}$$

Erit ergo proxime

$$\frac{-7\ +3\sqrt{7}}{2}=\frac{239}{510}$$

et

$$\sqrt{7}=\frac{2024}{765}=2{,}645\,751\,63\,;$$

at revera est

$$\sqrt{7}=2{,}645\,751\,31\,;$$

ita ut error minor sit quam $\frac{33}{100\,000\,000}$.

379. Progrediamur autem ulterius ponendo

$$x=\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{a+\text{etc.}}}}}}}}$$

Erit

$$x=\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+x}}}=\cfrac{1}{a+\cfrac{c+x}{bx+bc+1}}=\frac{bx+bc+1}{(ab+1)x+abc+a+c},$$

unde

$$(ab+1)xx+(abc+a-b+c)x=bc+1$$

atque

$$x=\frac{-abc-a+b-c+\sqrt{\left((abc+a+b+c)^2+4\right)}}{2(ab+1)}$$

ubi quantitas post signum radicale posita iterum est summa duorum quadratorum; neque ergo haec forma radicibus ex allis numeris extrahendis inservit, nisi ad quos prima forma iam suffecerat, Simili modo si quatuor litterae *a, b, c, d* continuo repetitae denominatores fractionis continuae constituant, tum ea plus non inserviet quam secunda, quae duas tantum litteras continebat, et ita porro.

380. Cum igitur fractiones continuae tam utiliter ad extractionem radicis quadratae adhiberi queant, simul inservient aequationibus quadratis resolvendis; quod quidem ex

ipso calculo est manifestum, dum x per aequationem quadraticam affectam determinatur. Potest autem vicissim facile cuiusque aequationis quadratae radix per fractionem continuam hoc modo exprimi. Sit proposita ista aequatio

$$xx = ax + b;$$

ex qua cum sit

$$x = a + \frac{b}{x},$$

substituatur in ultimo termino loco x valor idem iam inventus eritque

$$x = a + \cfrac{b}{a + \cfrac{b}{x}}$$

simili ergo modo procedendo erit per fractionem continuam infinitam

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \text{etc.}}}}$$

quae autem, cum numeratores b non sint unitates, non tam commode adhiberi potest.

381. Ut autem usus in arithmetica ostendatur, primum notandum est omnem fractionem ordinariam in fractionem continuam converti posse. Sit enim proposita fractio

$$x = \frac{A}{B},$$

in qua sit $A > B$; dividatur A per B sitque quotus $= a$ et residuum C; tum per hoc residuum C dividatur praecedens divisor B prodeatque quotus b et relinquatur residuum D, per quod denuo praecedens divisor C dividatur; sicque haec operatio, quae vulgo ad maximum communem divisorem numerorum A et B investigandum usurpari solet, continuetur, donec ipsa finiatur, sequenti modo:

$$\begin{array}{l} B)\underline{A}(a \\ \quad C)\underline{B}(b \\ \qquad D)\underline{C}(c \\ \qquad\quad E)\underline{D}(d \\ \qquad\qquad F \;\text{etc.} \end{array}$$

Eritque per naturam divisionis

$$A=\alpha B+C, \quad \text{unde} \quad \frac{A}{B}=a+\frac{C}{B},$$

$$B=bC+D, \quad \frac{B}{C}=b+\frac{D}{C}, \quad \frac{C}{B}=\frac{1}{b+\frac{D}{C}},$$

$$C=cD+E, \quad \frac{C}{D}=c+\frac{E}{D}, \quad \frac{D}{C}=\frac{1}{c+\frac{E}{D}},$$

$$D=dE+F \quad \frac{D}{E}=d+\frac{F}{E}, \quad \frac{E}{D}=\frac{1}{d+\frac{F}{E}}$$

etc., etc.

Hinc sequentes valores in praecedentibus substituendo erit

$$x=\frac{A}{B}=a+\frac{C}{B}=a+\frac{1}{b+\frac{D}{C}}=a+\frac{1}{b+\frac{1}{c+\frac{E}{D}}}$$

unde tandem x per meros quotos inventos a, b, c, d etc. sequentem in modum exprimetur, ut sit

$$x=\frac{1}{a+\frac{1}{b+\frac{1}{c+\frac{1}{d+\frac{1}{e+\frac{1}{f+\text{etc.}}}}}}}$$

EXEMPLUM 1

Sit proposita ista fractio $\frac{1461}{59}$, quae sequenti modo in fractionem continuam transmutabitur, cuius omnes numeratores erunt unitates.

Instituatur scilicet eadem operatio, qua maximus communis divisor numerorum 59 et 1461 quaeri solet:

59) 1461 (24
118
281
236
45) 59 (1
45
14) 45 (3
42
3) 14 (4
12
2)3(1
2
1)2(2
2
0

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$$

EXEMPLUM 2

Fractiones quoque decimales eodem modo transmutari poterunt.
Sit enim proposita

$$\sqrt{2} = 1,41421356 = \frac{141\,421\,356}{100\,000\,000},$$

unde haec operatio instituatur

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418328	1010160	2
86752	209164	

etc.

Ex qua operatione iam patet omnes denominatores esse 2 atque adeo esse

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\text{etc.}}}}}}$$

cuius expressionis ratio iam ex superioribus patet.

EXEMPLUM 3

Imprimis vero etiam hic attentione dignus est numerus e, cuius logarithmus est $=1$, qui est

$$e = 2,718281828459.$$

Unde oritur

$$\frac{e-1}{2} = 0,8591409142295,$$

quae fractio decimalis, si superiori modo tractetur, dabit quotes sequentes

8591409142295	10000000000000	1
451545146224	8591409142295	6
139863996071	1408590857704	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	
	etc.	

Si iste calculus exactius adhuc assumpto valore ipsius e ulterius continuetur, tum prodibunt isti quoti

$$1, 6, 10, 14, 18, 22, 26, 30, 34 \text{ etc.},$$

qui dempto primo progressionem arithmeticam constituunt, unde patet fore

$$\frac{e-1}{2}=\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\text{etc.}}}}}}}}$$

cuius fractionis ratio ex Calculo infinitesimali dari potest.

382. Cum igitur ex huiusmodi expressionibus fractiones erui queant, quae quam citissime ad verum valorem expressionis deducant, haec methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quae ad ipsas proxime accedant, exprimendas. Quin etiam, si fractio fuerit proposita, cuius numerator et denominator sint numeri valde magni, fractiones ex minoribus numeris constantes inveniri poterunt, quae, etiamsi propositae non sint penitus aequales, tamen ab ea quam minime discrepent. Hincque problema a WALLISIO olim tractatum facile resolvi potest, quo quaeruntur fractiones minoribus numeris expressae, quae tam prope exhauriant valorem fractionis cuiuspiam in numeris maioribus propositae, quantum fieri poterit numeris non maioribus. Fractiones autem nostra hac methodo ortae tam prope ad valorem fractionis continuae, ex qua eliciuntur, accedunt, ut nullae numeris non maioribus constantes dentur, quae propius accedant.

EXEMPLUM 1

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nisi numeri maiores adhibeantur.

Si fractio decimalis cognita

$$3,1415926535 \text{ etc.}$$

modo exposito per divisionem continuam evolvatur, reperientur sequentes quoti

$$3,\ 7,\ 15,\ 1,\ 292,\ 1,\ 1 \text{ etc.,}$$

ex quibus sequentes fractiones formabuntur

$$\frac{1}{0},\quad \frac{3}{1},\quad \frac{22}{7},\quad \frac{333}{106},\quad \frac{355}{113},\quad \frac{103993}{33102}\quad \text{etc.}$$

Secunda fractio iam ostendit esse diametrum ad peripheriam ut 1: 3 neque certe numeris non maioribus accuratius dari poterit. Tertia fractio dat rationem ARCHIMEDEAM 7: 22, at quinta METIANAM, quae ad verum tam prope accedit, ut error minor sit parte $\frac{1}{113\cdot 33102}$. Ceterum hae fractiones alternatim vero sunt maiores minoresque.

EXEMPLUM 2

Exprimatur ratio diei ad annum solarem medium in numeris minimis proxime.

Cum annus iste sit $365^d\ 5^h\ 48'\,55''$, continebit in fractione annus unus $365\frac{20935}{68640}$ dies. Tantum ergo opus est, ut haec fractio evolvatur, quae dabit sequentes quotos 4, 7, 1, 6, 1, 2, 2, 4, unde istae eliciuntur fractiones

$$\frac{0}{1},\quad \frac{1}{4},\quad \frac{7}{29},\quad \frac{8}{33},\quad \frac{55}{227},\quad \frac{63}{260},\quad \frac{181}{747}\quad \text{etc.}$$

Horae ergo cum minutis primis et secundis, quae supra 365 dies adsunt, quatuor annis unum diem circiter faciunt, unde calendarium JULIANUM originem habet. Exactius autem 33 annis 8 dies implentur vel 747 annis 181 dies; unde sequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium JULIANUM inserat 100 dies, GREGORIANUM quaternis seculis tres annos bissextiles in communes convertit.

FINIS TOMI PRIMI.