CHAPTER XVII

CONCERNING THE USE OF RECURRING SERIES IN INVESTIGATING THE ROOTS OF EQUATIONS

332. The most celebrated DANIEL BERNOULLI has indicated a good use of recurring series in investigating the roots of equations of any order, in Vol. III of the *Comment. Acad. Petropol.*, where he has shown how the values of the true roots of each algebraic equation, of whatever dimensions it should be, are able to be found approximately with the help of recurring series. Which discovery affords the maximum utility in innumerable cases, that I have decided to explain here carefully, so that it may be understood in which cases it may be able to be used. Yet sometimes more than the expected happens, so that no root of the equation may be able to be found by the aid of this method. On which account, so that the strength of this method may be seen more clearly, we will consider the whole fundamentals from the properties of recurring series, which it depends on.

333. Because all recurring series arise from the expansion of certain rational fractions, the this fraction shall be

$$=\frac{a+bz+cz^2+dz^3+ez^4+\text{etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{ etc.}},$$

from which the following series arises

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.},$$

the coefficients of which , *A, B, C, D* etc., thus may be determined, so that there shall be

$$\begin{aligned}
&A=a,\\
&B=\alpha A+b,\\
&C=\alpha B+\beta A+c,\\
&D=\alpha C+\beta B+\gamma A+d,\\
&E=\alpha D+\beta C+\gamma B+\delta A+\text{etc.}
\end{aligned}$$

Moreover the general term or the coefficient of the power z^n found from the resolution of the proposed fraction into simple fractions, the denominators of which shall be the factors of the denominators

$$1-\alpha z-\beta zz-\gamma z^3-\text{etc.},$$

as has been shown in Ch. XIII.

334. But the form of the general term chiefly depends on the nature of the simple factors of the denominator, whether they shall be real or imaginary, and whether they shall not be equal to each other or whether two or several shall be equal. So that we may run through which various cases, in the first place we may put all the factors of the denominator to be simple both to be real as well as unequal to each other. Therefore all the simple factors for the denominator

$$(1-pz)(1-qz)(1-rz)(1-sz) \text{ etc.},$$

from which the proposed fraction may be resolved into the simple fractions

$$\frac{\mathfrak{A}}{1-pz}+\frac{\mathfrak{B}}{1-qz}+\frac{\mathfrak{C}}{1-rz}+\frac{\mathfrak{D}}{1-sz}+ \text{ etc.}$$

With which known, the general term of the recurring series will be, see § 215,

$$=z^n\left(\mathfrak{A}p^n+\mathfrak{B}q^n+\mathfrak{C}r^n+\mathfrak{D}s^n+ \text{ etc.}\right);$$

which we may set $=Pz^n$; clearly P shall be the coefficient of the power z^n and Q, R etc. of the following so that the series becomes

$$A+Bz+Cz^2+Dz^3+\cdots+Pz^n+Qz^{n+1}+Rz^{n+2}+ \text{ etc.}$$

335. Now we may put n to be a very large number, or the series to be continued to many more terms. Because the powers of unequal numbers therefore become more unequal, by which the greater they are made, so great will be the differences in the powers $\mathfrak{A}p^n, \mathfrak{B}q^n, \mathfrak{C}r^n$, etc., so that that, which arises from the greatest of the numbers p, q, r etc., will surpass the rest in magnitude and before which the rest definitely vanish, if n were an infinitely great number. Therefore since the numbers p, q, r etc. shall be unequal to each other, we may put p to be the maximum among these. And therefore, if n shall be an infinite number, there becomes

$$P=\mathfrak{A}p^n;$$

but if n were an exceedingly large number, only approximately will there be $P=\mathfrak{A}p^n$. Truly in a similar manner there will be

$$Q=\mathfrak{A}p^{n+1}$$

And thus

$$\frac{Q}{P} = p\,.$$

From which it is apparent, if the recurring series were now produced, the coefficient of each term divided by the preceding will be approximately the value of the maximum letter *p* shown.

336. Therefore if the denominator in the proposed fraction

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{ etc.}}$$

may have all the factors real and unequal to each other, thus from the recurring series arising a single factor will be able to be known, evidently this one $1 - pz$, in which the letter *p* has the greatest value of all. And nor are the coefficients of the numbers *a, b, c, d* etc. introduced into the calculation here, as whatever of these may be put in place, yet finally the same true value of the letter *p* is found. Then indeed at last the value of *p* will become known, when the series may be continued to infinity ; yet meanwhile, if now more terms of this were formed, from that a closer value of *p* will become known, from which the number of terms may become greater, and from that the more that letter *p* may exceed the remaining letters *q, r, s* etc. Therefore truly there is the matter whether this maximum letter *p* should be governed by a $+$ or a $-$ sign, because the powers of this will increase equally.

337. Now in so far as this investigation is clear enough, it will be able to be applied to find the root of some algebraic equation. For from the known factors of the denominator

$$1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \text{etc}\,.$$

the roots of this equation may be assigned readily

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.} = 0\,,$$

thus so that, if $1 - pz$ were a factor, one the root of this equation shall become $z = \frac{1}{p}$.
Therefore since the maximum number *p* may be found from the recurring series, from the same place the minimum root of the equation

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.} = 0$$

will be found. Or if there may be put $z = \frac{1}{x}$ so that this equation may be produced

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0,$$

with the aid of the same method the maximum root $x = p$ of this equation may be elicited.

338. Therefore if this equation may be proposed

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0,$$

which may have all the roots real and unequal to each other, the maximum root of this may be found in the following way. The fraction

$$\frac{a + bz + cz^2 + dz^3 + ez^4 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

may be formed from the coefficients of this equation. And hence the recurring series may be formed by assuming an arbitrary number or, which amounts to the same, by assuming the initial terms as you wish. Which shall become

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{ etc.}$$

and the fraction $\frac{Q}{P}$ will give a largest value of the root x for the proposed equation closer to that, for which the number n shall be a greater number.

EXAMPLE 1

This equation shall be proposed

$$xx - 3x - 1 = 0,$$

of which it is required to find the maximum root.

The fraction may be formed :

$$\frac{a + bz}{1 - 3z - zz},$$

from which on putting 1, 2 for the two first terms this recurring series may arise

1, 2, 7, 23, 76, 251, 829, 2738 etc.

Therefore

$$\frac{2738}{829}$$

will be approximately equal to the maximum root of the proposed equation. But the value of this fraction expressed in decimal parts is

$$3{,}3027744;$$

the true maximum root of the equation is

$$= \tfrac{3+\sqrt{13}}{2} = 3{,}3027756\,,$$

which exceeds the root found by one part in a million only. Moreover it is to be noted the fractions $\frac{Q}{P}$ to be alternately greater and less than the true root.

[i.e. from § 333, we have $\alpha = 3$ and $\beta = 1$; The coefficients *A, B, C*, etc. can be replaced in general by the suffixed terms, a notation that was not yet in use :

$$\text{i.e. } s_{n+1} = 3s_n + s_{n-1}\,; \quad s_1 = 1\,; s_2 = 2\,; \text{for } n \geq 3. \text{ Thus,}$$

$$A = a = 1\,; B = \alpha A + b = 3 - 1 = 2\,; C = \alpha B + \beta A = 6 + 1 = 7\,; D = \alpha C + \beta B = 21 + 2 = 23 \text{ etc.]}$$

EXAMPLE 2

This equation shall be proposed

$$3x - 4x^3 = \tfrac{1}{2}\,,$$

the roots of which show the sines of three arcs, of which the sine of the triple arc is $\frac{1}{2}$.
With the equation adapted to this form

$$0 = 1 - 6x * + 8x^3$$

[The * sign indicates that a power is missing.]
of this the smallest root may be sought, so that we remain with whole numbers, so that it will be no need for $\frac{1}{x}$ to be put in place of x. Therefore this fraction may be formed

$$\frac{a+bx+cxx}{1-6x*+8x^3}$$

with which there may be taken as it pleases for the three initial terms 0, 0, 1, because in this way the calculation will be carried out the most easily, this recurring series may arise with the powers of *z* omitted, because there is a need for the coefficients only,

$$0,\ 0,\ 1,\ 6,\ 36,\ 208,\ 1200,\ 6912,\ 39808,\ 229248\,.$$

Therefore an approximation to the minimum root will be

$$\frac{39808}{229248}=\frac{311}{1791}=0{,}1736460\,,$$

which has to be the sine of the angle 10°; but this from tables is 0,1736482, which exceeds the root found by the part $\frac{22}{10\,000\,000}$.

But this same root can be found more easily by putting $x=\frac{1}{2}y$, so that the equation may be produced

$$1-3y*+y^3=0\,,$$

from which treated in a like manner, the series arises

$$0,\ 0,\ 1,\ 3,\ 9,\ 26,\ 75,\ 216,\ 622,\ 1791,\ 5157\text{ etc}\,.$$

Therefore the approximate smallest root will be

$$y=\frac{1791}{5157}=\frac{199}{573}=0{,}3472949\,,$$

from which there becomes

$$x=\frac{1}{2}y=0{,}1736475\,,$$

which approaches three times closer than the preceding.

[Here, and elsewhere, we have used the corrected arithmetical values from earlier editions.]

EXAMPLE 3

If the maximum root is desired from the same proposed equation

$$0=1-6x*+8x^3\,,$$

on putting $x = \frac{y}{2}$ this will become $y^3 * -3y + 1 = 0.$

The maximum root of this equation may be found from the recurring series, of which the scale of the relation is 0, 3, −1, for which therefore with the three initial terms taken as you wish,

$$1,\ 1,\ 1,\ 2,\ 2,\ 5,\ 4,\ 13,\ 7,\ 35,\ 8,\ 98, -\ 11\ \text{ etc .;}$$

$$[\text{i.e. } s_{n+1} = 0s_n + 3s_{n-1} - s_{n-2}\ ;\quad s_1 = 1\,; s_2 = 1\,; s_3 = 1\,; \text{for } n \geq 4.]$$

in which series it is indicated, since one comes upon negative terms, that the maximum root is to be negative; for there is

$$x = -\sin.70° = -0,9396926\ .$$

Whereby a reason for this can be had from the initial terms in this manner :

$$1, -2\ , +4, -7, +14, -25, +49, -89, +172, -316, +605 \text{ etc .,}$$

from which there will be

$$y = \tfrac{-605}{316} \text{ and } x = \tfrac{-\ 605}{632} = -0,957\ ,$$

which differs markedly from the truth.

339. The reason for this disagreement is mainly, that the roots of the proposed equation shall be :

$$\sin.10°,\ \sin.50° \text{ and } -\sin.70°\ ,$$

[since the angles shown $\times 3$ each have the same sine;]

of which the two greatest roots differ little from each other, so that with the powers, to which we have continued the series, the second root sin. 50° at this stage may maintain an unusual ratio to the maximum root and thus may not vanish before that root. Hence which also depends on the jump in value, as the other values found become exceedingly large and exceedingly small. Thus on taking

$$y = \tfrac{-\ 316}{172}$$

there becomes

$$x = \tfrac{-158}{172} = \tfrac{-79}{86} = -0,919\ .$$

because now the powers of the maximum root alternately become positive and negative, alternately too the powers of the second root are added and taken away ; on account of which, so that this discrepancy shall become unnoticed, the series must be continued much further.

340. Truly another remedy can be brought to this inconvenience by changing the equation with the aid of a suitable substitution into another form, of which the roots themselves shall not be so close. Thus if in the equation

$$0 = 1 - 6x + 8x^3,$$

the roots of which are $-\sin.70°$, $+\sin.50°$, $+\sin.10°$, there may be put $x = y - 1$, and the roots of the equation

$$0 = 8y^3 - 24yy + 18y - 1$$

will be $1 - \sin.\ 70°$, $1 + \sin.\ 50°$, $1 + \sin.\ 10°$ and thus the minimum root of this will be $1 - \sin.70°$, since still this sin. 70° shall be the maximum root of the preceding equation, and $1 + \sin.50°$ now is the maximum root, since sin. 50° was before the middle root. And in this manner some root by substitution can become changed in a maximum or minimum root of the new equation, and thus will be able to be found by the same method treated here. Therefore because in this example the root $1 - \sin.70°$ is much less than the two remaining, also it will easily become known approximately by the recurring series. [Thus, by examining a related function of the initial function translated horizontally, a maximum root may be chosen further apart from the other roots.]

EXAMPLE 4

To find the minimum root of the equation

$$0 = 8y^3 - 24yy + 18y - 1,$$

which taken from unity will leave the sine of the angle 70°.
There may be put $y = \frac{1}{2}z$, so that there shall be

$$0 = z^3 - 6zz + 9z - 1,$$

the smallest root of which shall be found by a recurring series, of which the scale of the relation is 9, − 6, +1; but for the maximum root to be found the scale of the relation must be taken to be 6, − 9, +1. Therefore for the minimum this series will be formed [i.e. $s_{n+1} = 9s_n - 6s_{n-1} + s_{n-2}$; $s_1 = 1$; $s_2 = 1$; $s_3 = 1$; for $n \geq 4$.]

$$1,\ 1,\ 1,\ 4,\ 31,\ 256,\ 2122,\ 17593,\ 145861 \text{ etc}.$$

Therefore it will be approximately

$$z = \tfrac{17593}{145861} = 0{,}12061483$$

and

$$y = 0{,}06030741$$

And

$$\sin.70° = 1 - y = 0{,}93969258,$$

which does not disagree with the truth except in the final figure. Therefore from this example it is understood, how much use a suitable transformation of the equation with the help of a substitution brings to finding the root, and which with this agreed upon, the method treated not only places bounds on the maximum and minimum roots, but also may be able to show all the roots.

341. Therefore with the approximate root of some equation thus now known ; so that for example the number k may differ minimally from some root, there may be put, $x - k = y$ or $x = y + k$ and in this manner an equation will be produced, the smallest root of which will be $= x - k$ *;* which therefore if it may be sought by a recurring series, so that it may be easily done, because this root will be much smaller than the rest, if to that k may be added, the true root of x will be found for the proposed equation. Truly this artifice is so commonly known that its use may be retained, even if the equation contains imaginary roots.

342. Moreover in the first place without this artifice a root cannot be known, for which another equal root is given, but with the contrary sign. Clearly, if an equation of which the maximum root is p*,* may have the same root $-p$*,* then, even if the recurring series will be continued to infinity, yet this root p on no account will be obtained. Let there be, so that we may illustrate by an example, the proposed equation

$$x^3 - x^2 - 5x + 5 = 0,$$

of which the maximum root is $\sqrt{5}$, besides which truly the root $-\sqrt{5}$ is present also. Therefore if we may use the method prescribed before for finding the maximum root, and we may form the recurring series from the scale of the relation 1, +5, − 5, this will be

$$1,\ 2,\ 3,\ 8,\ 13,\ 38,\ 63,\ 188,\ 313,\ 938,\ 1563 \text{ etc}.,$$

[i.e. $s_{n+1} = s_n + 5s_{n-1} - 5s_{n-2}$; $s_0 = 0\,; s_1 = 1\,; s_2 = 2\,; s_3 = 3\,;$for $n \geq 4$.]

where it is not reaching any constant ratio.

[Otherwise from § 333, we recall that

$$\frac{a+ bz+cz^2+dz^3+ ez^4+\text{etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{ etc.}} = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

In this case we have, in order to find the maximum root, to use the equation :

$$1 - x - 5x^2 + 5x^3 = 0\,;$$

$$\frac{a+bz+cz^2}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{ etc.}} = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc., and thus}$$

$\alpha = 1$, $\beta = 5$ and, $\gamma = -5$; it follows that

$A = a = 1\,; B = \alpha A + b = 2\,; C = 3 = 7 + c \therefore c = -4\,; D = 3 + 10 - 5 = 8\,; E = 8 + 15 - 10 = 13$, etc.]

Truly the alternating terms of the series adopt an equal ratio ; of which if any one may be divided by the preceding, the square of the maximum root is found ; for thus there is approximately

$$5 \; = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}\,.$$

Therefore as many times as the alternate terms themselves only give a constant ratio, so the square of the of the root sought will be obtained. But the root $x = \sqrt{5}$ itself is found by putting $x = y + 2$, from which there becomes

$$1 - 3y - 5yy - y^3 = 0\,,$$

the minimum root of which is found from the series

$$1,\; 1,\; 1,\; 9,\; 33,\; 145,\; 609,\; 2585,\; 10945 \text{ etc.}:$$

for it is approximately

$$= \frac{2585}{10945} = 0{,}2361\,;$$

but 2,2361 is approximately $=\sqrt{5}$, which is the maximum root of the equation.

343. Although the numerator of the fraction, from which the recurring series may be formed, depends on our choice, yet a most suitable arrangement of that is put in place, so that the value may be quickly shown approximately. For since with the assumed factors of the denominator as above (§334) the general term of the recurring series shall be

$$=z^n\left(\mathfrak{A}p^n+\mathfrak{B}q^n+\mathfrak{C}r^n+\text{ etc.}\right),$$

these coefficients $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ may be determined by the numerator of the fraction, so that $\mathfrak{A}$ may obtain either a large or a small value ; in the first case the maximum root p is found quickly, in the latter case truly slowly. Also one can also take a numerator, so that in short $\mathfrak{A}$ vanishes, in which case, even if the series may be continued indefinitely, yet it will on no account reach the maximum root p. But this comes about, if the numerator is taken thus, so that it may have the same numerator $1-pz$ itself; thus indeed it may be completely removed from the calculation. Thus if the equation may be proposed

$$x^3-6xx+10x-3=0,$$

of which the maximum root is $=3$, and thence the fraction is formed

$$\frac{1-3z}{1-6z+10z^2-3z^3}$$

so that the scale of the recurring series shall be 6, −10, + 3, and the series will be

$$1,\ 3,\ 8,\ 21,\ 55,\ 144,\ 377\ \text{ etc.},$$

$$[\text{i.e. } s_{n+1}=6s_n-10s_{n-1}+3s_{n-2}\ ;\quad s_0=0\,;s_1=1\,;s_2=3\,;\text{for } n\geq 2\,]$$

the terms of which do not converge at once to the ratio 1 : 3. For the same series arises from the fraction

$$\frac{1}{1-3z+zz}$$

and therefore the maximum root of the equation

$$x^2-3x+1=0$$

is found.

344. One can thus also assume a numerator, so that some root of the equation may be found by a recurring series, which can be done, if the numerator were a product from all the factors of the denominator except that, to which the root corresponds, as we wish. Thus if in the former example the numerator may be taken $1-3z+zz$, the fraction

$$\frac{1-3z+zz}{1-6z+10z^2-3z^3}$$

will give this recurring series

$$1,\ 3,\ 9,\ 27,\ 81,\ 243\ \text{ etc.},$$

which, since it shall be geometric, will show at once the root $x=3$. For that fraction is equal to this simple fraction

$$\frac{1}{1-3z}$$

Hence it is apparent, if the initial terms, which for argument's sake it is allowed to assume, thus may be taken, so that they make a geometric progression, the exponent of which is equal to one of the roots of the equation, then the whole recurring series becomes geometric and thus that same root can be shown, even if it shall be neither a maximum nor a minimum.

345. Therefore since, while we search for either a maximum or a minimum root, besides the root expected by us another root may be shown by the recurring series, a numerator of this kind must be selected, which shall have no common factor with the denominator, which comes about if one may be taken for the numerator, from which the first term of the series will be $=1$, and from which alone all the following are defined by the scale of the relation. And in this way certainly a root of the equation, always either a maximum or minimum, may be elicited just as it was proposed. Thus for the proposed equation

$$y^3 * -3y+1=0,$$

of which the maximum root may be desired, from the scale of the relation 0, +3, −1 by beginning from unity the following recurring series arises

$$1,\ -0,\ +3,\ -1,\ +9,\ -6,\ +28,\ -27,\ +90,\ -109,\ +297,\ -417,$$
$$+1000,\ -1548,\ +3417,\ -5644 \text{ etc.},$$

$$[\text{i.e. } s_{n+1}=3s_{n-1}-s_{n-2}\,;\quad s_0=0\,;s_1=1\,;s_2=0\,;\text{for } n\geq 3.]$$

which will converge to a constant ratio and will show the maximum root to be negative and approximately

$$y = \frac{-5644}{3417} = -1,651741,$$

which ought to be $= -1,8793852$. [Euler made a slip in his calculation, in that the last term was given as −6544, rather than the corrected value −5644 used here in the *O.O.* corrected edition, and from the earlier French translation of J. B. Labey in 1796. Thus, the series would need to be carried further for convergence.] But the above ratio has been taken [§ 330], because it approaches the true value so slowly, so that therefore the other root shall not be much less than the maximum and likewise it shall be positive.

346. With these considered properly, which since in general as well as according to the examples we have brought to advise us, the great usefulness of this method will be seen for finding the roots of equations more clearly. Truly the artifices, by which operation may be able to be drawn together and with that to be returned more promptly, have been indicated too in a satisfactory manner, thus so that nothing further may be required to be added, except the cases in which the equation has equal or imaginary roots, remaining to be examined. Therefore we may put the denominator of the fraction

$$\frac{a+bz+cz^2+dz^3 \text{ +etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{etc.}}$$

to have the factor $(1-pz)^2$, with the remaining factors present $1-qz$, $1-rz$ etc. Therefore the general term of the recurring series hence generated will be

$$= z^n\left((n+1)\mathfrak{A}p^n+\mathfrak{B}p^n+\mathfrak{C}q^n+\text{ etc.}\right);$$

[Recall that the separation of a denominator of the form

$$\frac{1}{(1-zp)^2(1-zq)(1-zr)......}$$

into factors including the square term introduces partial fractions of the form :

$$\frac{\mathfrak{A}}{(1-zp)^2}+\frac{\mathfrak{B}}{(1-zp)}+\frac{\mathfrak{C}}{(1-zq)}.....;$$

these can be expanded in series, the n^{th} term of which has the form

$$z^n\left((n+1)\mathfrak{A}p^n+\mathfrak{B}p^n+\mathfrak{C}q^n+\text{ etc.}\right).]$$

which value of this kind shall soon be arrived at, if n were an exceedingly great number, two cases are to be distinguished, the one in which p is a number greater than the remaining numbers q, r etc., and the other, in which p does not provide the maximum root. In the first case, for which p likewise is the maximum root, on account of the coefficient $n+1$ the remaining terms $\mathfrak{B}p^n, \mathfrak{C}q^n$ etc. besides do not vanish so quickly from that term ; but if q were $> p$, then also the term $(n+1)\mathfrak{A}p^n$ will vanish more slowly besides $\mathfrak{C}q^n$ and thus the investigation certainly will avoid the trouble.

EXAMPLE 1

Let the proposed equation be

$$x^3 - 3xx + 4 = 0\,,$$

of which the maximum root 2 occurs twice.

Therefore this maximum root may be sought in the manner set out before by the expansion of the fraction,
[i.e. the fraction $\frac{1}{1-3z*+4z^3}$ is expanded out, according to the iterations of the sequence

$$s_{n+1} = 3s_n - 4s_{n-2}\,; \quad s_0 = 0\,; s_1 = 1\,; s_2 = 3\,; \text{for } n \geq 3.\,]$$

which will give that recurring series

$$1,\ 3,\ 9,\ 23,\ 57,\ 135,\ 313,\ 711,\ 1593 \text{ etc.},$$

where indeed some term divided by the preceding gives a quotient greater than two [for the root]. The ratio of which is readily apparent from the general term. For with the terms $\mathfrak{C}q^n$ etc. rejected from that, the corresponding term of the power z^n will be

$$= (n+1)\mathfrak{A}p^n + \mathfrak{B}p^n\,,$$

and the following

$$= (n+2)\mathfrak{A}p^{n+1} + \mathfrak{B}p^{n+1}\,,$$

which divided by the first gives

$$\frac{(n+2)\mathfrak{A}+\mathfrak{B}}{(n+1)\mathfrak{A}+\mathfrak{B}}\,p > p\,,$$

unless n now increases to infinity.

EXAMPLE 2

Now the equation shall be proposed :

$$x^3 - xx - 5x - 3 = 0$$

of which the maximum root is $= 3$, and the remaining two roots are $= -1$.
The maximum root is sought with the aid of recurring series, the scale of the relation of which is $1, \ +5, \ +3$; from which there is generated :

$$1, \ 1, \ 6, \ 14, \ 47, \ 135, \ 412, \ 1228 \text{ etc.},$$

$$[\text{i.e. } s_{n+1} = s_n + 5s_{n-1} + 3s_{n-2}\,; \quad s_0 = 0\,; s_1 = 1\,; s_2 = 1\,; \text{for } n \geq 3.]$$

which thus quickly shows the value 3, because the powers of the smaller root -1, even if it may be multiplied by $n+1$, yet still will vanish compared with the powers of 3.

EXAMPLE 3

But if the equation may be put in place

$$x^3 + x^2 - 8x - 12 = 0,$$

the roots of which are $3, \ -2, \ -2$, the maximum itself will be produced much more slowly.
For this series will arise

$$1, \ -1, \ 9, \ -5, \ 65, \ 3, \ 457, \ 347, \ 3345, \ 4915 \text{ etc.},$$

which still must be continued at length, before the root thence arising will appear to be $= 3$.

347. In a similar manner if three factors were equal, thus so that the factor of one denominator shall be $(1 - pz)^3$, the remaining factors $1 - qz, \ 1 - rz$ etc., then the general term of the recurring series will be

$$= z^n \left(\frac{(n+1)(n+2)}{1 \cdot 2} \mathfrak{A} p^n + (n+1) \mathfrak{B} p^n + \mathfrak{C} p^n + \mathfrak{D} q^n + \mathfrak{E} r^n + \text{ etc.} \right).$$

Therefore if p were the maximum root and n were a number so great, that the powers

q^n, r^n etc. may vanish besides p^n, then from the recurring series the root will be generated :

$$\frac{\frac{1}{2}(n+2)(n+3)\mathfrak{A}+(n+2)\mathfrak{B}+\mathfrak{C}}{\frac{1}{2}(n+1)(n+2)\mathfrak{A}+(n+1)\mathfrak{B}+\mathfrak{C}}p,$$

which, unless n shall be a great number and as if infinite, will not indicate the true value of p. But the value of this root will be

$$= p+\frac{(n+2)\mathfrak{A}+\mathfrak{B}}{\frac{1}{2}(n+1)(n+2)\mathfrak{A}+(n+1)\mathfrak{B}+\mathfrak{C}}p$$

But if moreover p were not the maximum root, then at this point the discovery of the maximum may be much more hindered ; from which it follows the equations, which contain equal roots, are much more difficult to be resolved by this method, than if all the roots were unequal to each other.

348. We may now see, in what manner a recurring series continued to infinity must be prepared, when the denominator of the fraction has imaginary factors.
Let the real factors of the denominator of the fraction

$$\frac{a+bz+cz^2+dz^3+\text{ etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{ etc.}}$$

be

$$1-qz,\ 1-rz\ \text{ etc.}$$

and from above the factor

$$1-2pz\cos.\varphi+ppzz$$

containing two simple imaginary factors. But if therefore the recurring series arising from that fraction were

$$A+Bz+Cz^2+Dz^3+\cdots+Pz^n+Qz^{n+1}+\text{ etc.},$$

the coefficient P, by that which we have established above (see §218), will be

$$=\frac{\mathfrak{A}\sin.(n+1)\varphi+\mathfrak{B}\sin.n\varphi}{\sin.\varphi}p^n+\mathfrak{C}q^n+\mathfrak{D}r^n+\text{ etc.}$$

Therefore if the number p were less than one of the others q, r etc., thus so that the maximum root of the equation

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{ etc.} = 0$$

shall be real, then that will be found equally by a recurring series, and as if no imaginary roots were to be present.

349. Therefore the discovery of the maximum real root will not be disturbed by imaginary roots, if these thus were prepared, so that of the two, which make a real factor, the product shall not be greater than the square of the maximum root. But if two imaginary roots of the same kind shall be present, so that the product of these either is equal or thus exceeds the square of the maximum real root, then the investigation set out before will tell us nothing, because the power p^n therefore on no account vanishes before a similar power of the maximum root, even if the series may be continued to infinity. It has been considered to add here examples of this illustrating the cause.

EXAMPLE 1

Let the proposed equation be

$$x^3 - 2x - 4 = 0,$$

of which it may be required to investigate the maximum root.

This equation is resolved into two factors

$$(x-2)(xx+2x+2);$$

from which it has one real root 2 and two imaginary roots, of which the product is 2, less than the square of the real root. That on account of the manner so far treated will be able to be known. Therefore the recurring series may be formed from the scale of the relation $0,\ +2,\ +4$, which will be

$$1,\ 0,\ 2,\ 4,\ 4,\ 16,\ 24,\ 48,\ 112,\ 192,\ 416,\ 832 \text{ etc.},$$

from which the real root 2 can become known clearly enough.

EXAMPLE 2

Let the proposed equation be

$$x^3 - 4xx + 8x - 8 = 0,$$

of which the one real root is 2, the product of the two imaginary roots truly will be $= 4$ and thus equal to the square of the real root 2.

Therefore we may seek the root by a recurring series; but so that it may be able to come about more easily, we may put $x = 2y$, so that we may have

$$y^3 - 2yy + 2y - 1 = 0,$$

from which the recurring series may be formed

$$1,\ 2,\ 2,\ 1,\ 0,\ 0,\ 1,\ 2,\ 2,\ 1,\ 0,\ 0,\ 1,\ 2,\ 2,\ 1\ \text{ etc.};$$

in which since the same terms may be returned perpetually, from that nothing other can be deduced, unless the maximum root either is not real or imaginary roots are to be given, the product of which shall be equal to or exceed the square of the real root.

EXAMPLE 3

Now let the proposed equation be

$$x^3 - 3xx + 4x - 2 = 0,$$

the real root of which is 1, truly the product of the imaginary roots $= 2$.

Therefore from the scale of the relation $3,\ -4,\ +2$ the series

$$1,\ 3,\ 5,\ 5,\ 1,\ -7,\ -15,\ -15,\ +1,\ 33,\ 65,\ 65,\ 1\ \text{etc.};$$

will be formed, in which since the terms are made positive and negative in this way and that, the real root 1 cannot be found from that in any way. Truly revolutions of this kind always show the root that the series must provide to be imaginary ; for here the imaginary roots are with a greater power than of the real root 1.

350. Therefore let *pp* be greater than the square of any real roots in the general fraction produced of two imaginary roots, thus so that the remaining powers q^n, r^n etc. may vanish before p^n, if *n* shall be an infinite number. Therefore in this case let there become

$$P = \frac{\mathfrak{A}\sin.(n+1)\varphi + \mathfrak{B}\sin.n\varphi}{\sin.\varphi}p^n$$

and

$$Q = \frac{\mathfrak{A}\sin.(n+2)\varphi + \mathfrak{B}\sin.(n+1)\varphi}{\sin.\varphi}p^{n+1}$$

and thus

$$\frac{Q}{P} = \frac{\mathfrak{A}\sin.(n+2)\varphi + \mathfrak{B}\sin.(n+1)\varphi}{\mathfrak{A}\sin.(n+1)\varphi + \mathfrak{B}\sin.n\varphi}p.$$

Which expression at no time adopts a constant value, even if *n* shall be an infinite number. Indeed the sines of angles perpetually keep changing especially, thus so that soon they shall be positive, soon negative.

351. Yet meanwhile if the following fractions $\frac{R}{Q}$, $\frac{S}{R}$ may be taken similar in the same manner and thus the letters $\mathfrak{A}$ and $\mathfrak{B}$ may be eliminated, the same number *n* may emerge from the calculation ; for [see § 352 following for a detailed account of this calculation.]

$$Ppp + R = 2Qp\cos.\varphi,$$

will be found, from which there becomes

$$\cos.\varphi = \frac{Ppp+R}{2Qp};$$

indeed similarly there will be

$$\cos.\varphi = \frac{Qpp+S}{2Rp},$$

from the comparison of which two values there shall become

$$p = \sqrt{\frac{RR-QS}{QQ-PR}}$$

and

$$\cos.\varphi = \frac{QR-PS}{2\sqrt{(Q^2-PR)(R^2-QS)}}.$$

On which account if the recurring series now were continued to that point, so that before p^n the powers of the remaining roots vanish, then in this manner the factor of the trinomial $1-2pz\cos.\varphi+ppzz$ will be able to be found.

352. Because this calculation may be difficult to create with enough training, I shall set it out here as a whole. From the value of $\frac{Q}{P}$ found there arises

$$\mathfrak{A}\, Pp\sin.(n+2)\varphi+\mathfrak{B}\, Pp\sin.(n+1)\varphi=\mathfrak{A}Q\sin.(n+1)\varphi+\mathfrak{B}Q\sin.n\varphi,$$

from which there becomes

$$\frac{\mathfrak{A}}{\mathfrak{B}}=\frac{Q\sin.n\varphi-Pp\sin.(n+1)\varphi}{Pp\sin.(n+2)\varphi-Q\sin.(n+1)\varphi}.$$

There will be in an equal ratio

$$\frac{\mathfrak{A}}{\mathfrak{B}}=\frac{R\sin.(n+1)\varphi-Qp\sin.(n+2)\varphi}{Qp\sin.(n+3)\varphi-R\sin.(n+2)\varphi}.$$

From these two values equated there becomes

$$\begin{aligned}0&=QQp\sin.n\varphi\sin.(n+3)\varphi-QR\sin.n\varphi\sin.(n+2)\varphi\\&-PQpp\sin.(n+1)\varphi\sin.(n+3)\varphi-QQp\sin.(n+1)\varphi\sin.(n+2)\varphi\\&+QR\sin.(n+1)\varphi\sin.(n+1)\varphi+PQpp\sin.(n+2)\varphi\sin.(n+2)\varphi.\end{aligned}$$

But since there shall be

$$\sin.a\sin.b=\frac{1}{2}\cos.(a-b)-\frac{1}{2}\cos.(a+b),$$

the equation becomes

$$0=\frac{1}{2}QQp(\cos.3\varphi-\cos.\varphi)+\frac{1}{2}QR(1-\cos.2\varphi)+\frac{1}{2}PQpp(1-\cos.2\varphi),$$

which divided by $\frac{1}{2}Q$ gives

$$(Ppp+R)(1-\cos.2\varphi)=Qp(\cos.\varphi-\cos.3\varphi).$$

But there is

$$\cos.\varphi=\cos.2\varphi\cos.\varphi+\sin.2\varphi\sin.\varphi$$

and

$$\cos.3\varphi = \cos.2\varphi \cos.\varphi - \sin.2\varphi \sin.\varphi ,$$

from which

$$\cos.\varphi - \cos.3\varphi = 2 \sin.2\varphi \sin.\varphi = 4 \sin.\varphi^2 \cos.\varphi ,$$

and

$$1 - \cos.2\varphi = 2 \sin.\varphi^2 ,$$

from which there will be

$$Ppp + R = 2Qp\cos.\varphi$$

both

$$\cos.\varphi = \frac{Ppp+R}{2Qp}$$

and

$$\cos.\varphi = \frac{Qpp+S}{2Rp}$$

from which the above values will be produced, clearly

$$p = \sqrt{\frac{RR-QS}{QQ-PR}}$$

and

$$\cos.\varphi = \frac{QR-PS}{2\sqrt{(Q^2-PR)(RR-QS)}} .$$

353. If the denominator of the fraction, from which the recurring series may be formed, may have several trinomial factors equal to each other, then from the general form of the terms seen given above it will be apparent that the finding of the roots becomes much more uncertain. Yet meanwhile if some one real root were found approximately, then the transformations of the equation will elicit always the value the same root much closer. For x may be put equal to that value now found $+y$ and the smallest root for y may be sought of the new equation, which added to that value will provide the true value of x.

EXAMPLE

This equation shall be proposed :

$$x^3 - 3xx + 5x - 4 = 0 ;$$

one root of which agreed from that to be almost equal to 1, because on putting $x=1$ it produces

$$x^3-3xx+5x-4=-1.$$

Therefore there may be put $x=1+y$ and there becomes

$$1-2y-y^3=0,$$

from which for finding the minimum root this recurring series may will be formed, of which the scale of the relation will be 2, 0, +1, which will be

$$1,\ 2,\ 4,\ 9,\ 20,\ 44,\ 97,\ 214,\ 472,\ 1041,\ 2296\ \text{ etc.},$$

from which the minimum root of y will be approximately

$$\tfrac{1041}{2296}=0,453397,$$

thus so that there shall be

$$x=1,453397,$$

which value so near can scarcely be obtained with equal ease by any other method.

354. But if moreover some recurring series finally may converge closely to a geometric progression, then from the law of the progression it will at once be known easily, the root of which equation shall become the amount, which arises from the division of one term by the preceding . Let

$$P,\ Q,\ R,\ S,\ T\quad \text{etc}.$$

be the terms of a recurring series now a great length from a distant beginning, thus so that it may be combined with a geometric progression, and there shall become

$$T=\alpha S+\beta R+\gamma Q+\delta P$$

or from the scale of the relation $a,\ +\beta,\ +\gamma,\ +\delta$. The value of the fraction may be put in place $\frac{Q}{P}=x$; there will be

$$\tfrac{R}{P}=xx,\quad \tfrac{S}{P}=x^3 \text{ and } \tfrac{T}{P}=x^4,$$

which by substitution in the above equation will give

$$x^4 = \alpha x^3 + \beta x^2 + \gamma x + \delta ,$$

from which it is apparent the quotient $\frac{Q}{P}$ finally provides one root of the equation found. Truly this and the preceding method indicate, and indeed it shows as well, that the fraction $\frac{Q}{P}$ gives the maximum root of the equation.

355. Also this method of finding the roots is often useful in practice, if the equation shall be infinite. Towards showing which the equation shall be proposed :

$$\frac{1}{2} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \text{etc.},$$

the smallest root of which z shall show the arc of 30° or the sixth part of the semi circumference of a circle. Therefore the equation may be changed into this form

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \text{etc.} = 0.$$

Hence a recurring series therefore may be formed, the scale of which is infinite, evidently

$$2,\ 0,\ -\tfrac{1}{3},\ 0,\ +\tfrac{1}{60},\ 0,\ -\tfrac{1}{2520},\ 0\ \text{ etc.},$$

and there will be the recurring series

$$1,\ 2,\ 4,\ \frac{23}{3},\ \frac{44}{3},\ \frac{1681}{60},\ \frac{2408}{45}\ \text{ etc.}:$$

and therefore there will be approximately

$$z = \frac{1681 \cdot 45}{2408 \cdot 60} = \frac{1681 \cdot 3}{2408 \cdot 4} = \frac{5043}{9632} = 0{,}52356.$$

But from the known proportion of the circumference to the diameter it must become $z = 0{,}523598$, and thus the root found differs only by the part $\frac{3}{100\,000}$ from the true value. But this comes from the convenient use in this equation, because all the roots of this shall be real and the others differ far enough from the minimum. Which condition since it may have the rarest place in infinite equations, hence there are few cases where this method can be used for their resolution.

CAPUT XVII

DE USU SERIERUM RECURRENTIUM IN RADICIBUS AEQUATIONUM INDAGANDIS

332. Indicavit Vir Celeb. DANIEL BERNOULLI insignem usum serierum recurrentium in investigandis radicibus aequationum cuiusvis gradus in Comment. Acad. Petropol. Tomo III, ubi ostendit, quemadmodum cuiusque aequationis algebraicae, quotcunque fuerit dimensionum, valores radicum veris proximi ope serierum recurrentium assignari queant. Quae inventio cum saepenumero maximum afferat utilitatem, eam hic diligentius explicare constitui, ut intelligatur, quibus casibus adhiberi possit. Interdum enim praeter expectationem evenit, ut nulla aequationis radix ope huius methodi cognosci queat. Quocirca, ut vis huius methodi clarius perspiciatur, ex proprietatibus serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis series recurrens ex evolutione cuiusdam fractionis rationalis oritur, sit ista fractio

$$=\frac{a+bz+cz^2+dz^3+ez^4+\text{etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{etc.}},$$

unde oriatur sequens series recurrens

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.},$$

cuius ceofficientes *A, B, C, D* etc. ita determinantur, ut sit

$$\begin{aligned}
&A=a,\\
&B=\alpha A+b,\\
&C=\alpha B+\beta A+c,\\
&D=\alpha C+\beta B+\gamma A+d,\\
&E=\alpha D+\beta C+\gamma B+\delta A+\text{etc.}
\end{aligned}$$

Terminus autem generalis seu coefficiens potestatis z^n invenitur ex resolutione fractionis propositae in fractiones simplices, quarum denominatores sint factores denominatoris

$$1-\alpha z-\beta zz-\gamma z^3-\text{etc.},$$

uti cap. XIII est ostensum.

334. Forma autem termini generalis potissimum pendet ab indole factorum simplicium denominatoris, utrum sint reales an imaginarii, et utrum sint inter se inaequales an eorum bini pluresve aequales. Quos varios casus ut ordine percurramus, ponamus primum omnes denominatoris factores simplices cum reales esse tum inter se inaequales. Sint ergo factores simplices denominatoris omnes

$$(1-pz)(1-qz)(1-rz)(1-sz) \text{ etc.},$$

ex quibus fractio proposita in sequentes fractiones simplices resolvatur

$$\frac{\mathfrak{A}}{1-pz}+\frac{\mathfrak{B}}{1-qz}+\frac{\mathfrak{C}}{1-rz}+\frac{\mathfrak{D}}{1-sz}+\text{ etc.}$$

Quibus cognitis erit seriei recurrentis terminus generalis

$$=z^n\left(\mathfrak{A}p^n+\mathfrak{B}q^n+\mathfrak{C}r^n+\mathfrak{D}s^n+\text{ etc.}\right);$$

quem statuamus $=Pz^n$ sit scilicet *P* coefficiens potestatis z^n sequentiumque *Q, R* etc., ita ut series recurrens fiat

$$A+Bz+Cz^2+Dz^3+\cdots+Pz^n+Qz^{n+1}+Rz^{n+2}+\text{ etc.}$$

335. Ponamus iam *n* esse numerum maximum seu seriem recurrentem ad plurimos terminos esse continuatam. Quoniam numerorum inaequalium potestates eo magis fiunt inaequales, quo fuerint altiores, tanta erit diversitas in potestatibus $\mathfrak{A}p^n, \mathfrak{B}q^n, \mathfrak{C}r^n$, etc., ut ea, quae oritur ex maximo numerorum *p, q, r* etc., reliquas magnitudine longe superet prae eaque reliquae penitus evanescant, si *n* fuerit numerus plane infinite magnus. Cum igitur numeri *p, q, r* etc. sint inter se inaequales, ponamus inter eos *p* esse maximum. Ac propterea, si *n* sit numerus infinitus, fiet

$$P=\mathfrak{A}p^n;$$

sin autem *n* sit numerus vehementer magnus, erit tantum proxime $P=\mathfrak{A}p^n$.
Simili vero modo erit

$$Q=\mathfrak{A}p^{n+1}$$

Ideoque

$$\frac{Q}{P}=p.$$

Unde patet, si series recurrens iam longe fuerit producta, coefficientem cuiusque termini per praecedentem divisum proxime esse exhibiturum valorem maximae litterae *p*.

336. Si igitur in fractione proposita

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

denominator habeat omnes factores simplices reales et inter se inaequales, ex serie recurrente inde orta cognosci poterit unus factor simplex, is scilicet $1 - pz$, in quo littera *p* omnium maximum habet valorem. Neque in hoc negotio coefficientes numeratoris *a, b, c, d* etc. in computum ingrediuntur, sed quicunque ii statuantur, tamen denique idem verus valor litterae maximae *p* invenitur. Verus quidem valor ipsius *p* tum demum innotescit, quando series in infinitum fuerit continuata; interim tamen, si iam plures eius termini fuerint formati, eo propius valor ipsius *p* cognoscetur, quo maior fuerit terminorum numerus et quo magis littera ista *p* excedat reliquas *q, r, s* etc. Perinde vero est, utrum haec maxima littera *p* fuerit signo + an signo − affecta, quoniam eius potestates aeque increscunt.

337. Quemadmodum nunc haec investigatio ad inventionem radicum aequationis cuiusvis algebraicae accommodari possit, satis est perspicuum. Ex factoribus enim denominatoris

$$1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \text{etc}.$$

cognitis facile assignantur radices aequationis huius

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.} = 0,$$

ita ut, si factor fuerit $1 - pz$, huius aequationis radix una futura sit $z = \frac{1}{p}$.
Cum igitur ex serie recurrente reperiatur maximus numerus *p*, indidem obtinebitur minima radix aequationis

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.} = 0.$$

Vel si ponatur $z = \frac{1}{x}$ ut prodeat haec aequatio

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0,$$

eiusdem methodi ope eruitur maxima huius aequationis radix $x = p$.

338. Si igitur proponatur aequatio haec

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0,$$

quae omnes radices habeat reales et inter se inaequales, harum radicum maxima sequenti modo reperietur. Formetur ex coefficientibus huius aequationis fractio

$$\frac{a + bz + cz^2 + dz^3 + ez^4 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

Hincque formetur series recurrens assumendo pro arbitrio numeratorem seu, quod eodem redit, assumendo pro lubitu terminos initiales. Quae sit

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{ etc.}$$

dabitque fractio $\frac{Q}{P}$ valorem radicis maximae x pro aequatione proposita eo propius, quo maior fuerit numerus n.

EXEMPLUM 1

Sit proposita ista aequatio

$$xx - 3x - 1 = 0,$$

cuius maximam radicem inveniri oporteat.
Formetur fractio

$$\frac{a + bz}{1 - 3z - zz},$$

unde positis duobus primis terminis 1, 2 orietur ista series recurrens

$$1, 2, 7, 23, 76, 251, 829, 2738 \text{ etc.}$$

Erit ergo

$$\frac{2738}{829}$$

proxime aequalis radici aequationis propositae maximae. Valor autem huius fractionis in partibus decimalibus expressus est

$$3{,}3027744;$$

aequationis vero radix maxima est

$$=\frac{3+\sqrt{13}}{2}=3,3027756,$$

quae inventam superat tantum una parte millionesima. Ceterum notandum est fractiones $\frac{Q}{P}$ alternatim vera radice esse maiores et minores.

EXEMPLUM 2

Proposita sit ista aequatio

$$3x-4x^3=\tfrac{1}{2},$$

cuius radices exhibent sinus trium arcuum, quorum triplorum sinus est $\frac{1}{2}$.
Aequatione perducta ad hanc formam

$$0=1-6x*+8x^3$$

quaeratur huius, ut in numeris integris maneamus, radix minima, ita ut non opus sit pro x ponere $\frac{1}{x}$. Formetur ergo haec fractio

$$\frac{a+bx+cxx}{1-6x*+8x^3}$$

ex qua sumendis pro lubitu tribus terminis initialibus 0, 0, 1, quia hoc modo calculus facillime expeditur, orietur haec series recurrens omittendis potestatibus ipsius z, quia tantum coefficientibus opus est,

$$0,\ 0,\ 1,\ 6,\ 36,\ 208,\ 1200,\ 6912,\ 39808,\ 229248.$$

Erit ergo proxime aequationis radix minima

$$\frac{39808}{229248}=\frac{311}{1791}=0,1736460,$$

quae propterea esse deberet sinus anguli 10°; hic autem ex tabulis est 0,1736482, qui superat radicem inventam parte $\frac{22}{10\,000\,000}$.

Facilius autem haec eadem radix inveniri potest ponendo $x=\frac{1}{2}y$, ut prodeat aequatio

$$1 - 3y * + y^3 = 0,$$

ex qua simili modo tractata oritur series

$$0, \ 0, \ 1, \ 3, \ 9, \ 26, \ 75, \ 216, \ 622, \ 1791, \ 5157 \text{ etc}.$$

Erit ergo proxime aequationis radix minima

$$y = \tfrac{1791}{5157} = \tfrac{199}{573} = 0{,}3472949,$$

unde fit

$$x = \tfrac{1}{2} y = 0{,}1736475,$$

qui valor fere ter propius accedit quam praecedens,

EXEMPLUM 3

Si desideretur eiusdem aequationis propositae

$$0 = 1 - 6x * + 8x^3$$

radix maxima, ponatur $x = \frac{y}{2}$ eritque $y^3 * - 3y + 1 = 0.$
Cuius aequationis radix maxima reperietur per seriem recurrentem, cuius scala relationis est 0, 3, −1, unde ergo oritur sumptis tribus terminis initialibus pro arbitrio

$$1, \ 1, \ 1, \ 2, \ 2, \ 5, \ 4, \ 13, \ 7, \ 35, \ 8, \ 98, - \ 11 \ \text{ etc}.;$$

in qua serie cum ad terminos negativos perveniatur, id indicio est maximam radicem esse negativam; est enim

$$x = -\sin.70^\circ = -0{,}9396926.$$

Quare huius ratio in terminis initialibus est habenda hoc modo

$$1, - \ 2, +4, -7, +14, -25, +49, -89, +172, -316, +605 \text{ etc}.,$$

ex qua erit

$$y = \tfrac{-605}{316} \text{ et } x = \tfrac{-605}{632} = -0{,}957,$$

quae a veritate vehementer abludit.

339. Ratio huius dissensus potissimum est, quod aequationis propositae radices sint .

$$\sin.10°,\ \sin.50° \text{ et } -\ \sin.70°,$$

quarum binae maximae tam parum a se invicem discrepant, ut in potestatibus, ad quas seriem continuavimus, secunda radix sin. 50° adhuc notabilem teneat rationem ad radicem maximam ideoque prae ea non evanescat. Hinc que etiam saltus pendet, quod alternatim valores inventi fiant nimis magni et nimis parvi. Sic sumendo

$$y = \frac{-316}{172}$$

Fit

$$x = \frac{-158}{172} = \frac{-79}{86} = -0,919\ .$$

Nam quoniam potestates radicis maximae alternatim fiunt affirmativae et negativae, alternatim quoque potestates secundae radicis adduntur et tolluntur; quamobrem, quo haec discrepantia fiat insensibilis, series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommode afferri potest transmutando aequationem ope idoneae substitutionis in aliam formam, cuius radices sibi non amplius sint tam vicinae, Sic si in aequatione

$$0 = 1 - 6x + 8x^3,$$

cuius radices sunt − sin.70°, + sin.50°, + sin.10°, ponatur $x = y - 1$, aequationis

$$0 = 8y^3 - 24yy + 18y - 1$$

radices erunt $1 - \sin.\ 70°,\ 1 + \sin.\ 50°,\ 1 + \sin.\ 10°$ ideoque eius radix minima erit $1 - \sin.70°$, cum tamen haec sin. 70° esset radix maxima aequationis praecedentis, atque $1 + \sin.50°$ nunc est radix maxima, cum sin. 50° ante esset media. Atque hoc modo quaevis radix per substitutionem in maximam minimamve radicem novae aequationis transmutari ideoque per methodum hic traditam inveniri poterit. Quia praeterea in hoc exemplo radix $1 - \sin.70°$ multo minor est quam binae reliquae, etiam facile per seriem recurrentem proxime cognoscetur.

EXEMPLUM 4

Invenire radicem minimam aequationis

$$0 = 8y^3 - 24yy + 18y - 1,$$

quae ab unit ate subtracta relinquet sinum anguli 70°.
Ponatur $y=\frac{1}{2}z$, ut sit

$$0=z^3-6zz+9z-1,$$

cuius radix minima invenietur per seriem recurrentem, cuius scala relationis est $9,-6,+1$; pro radice autem maxima invenienda scala relationis sumi deberet $6,-9,+1$. Pro minima ergo formetur haec series

$$1,\ 1,\ 1,\ 4,\ 31,\ 256,\ 2122,\ 17593,\ 145861 \text{ etc}.$$

Erit ergo proxime

$$z=\frac{17593}{145861}=0,12061483$$

et

$$y=0,06030741$$

Atque

$$\sin.70°=1-y=0,93969258,$$

quae a veritate ne in ultima quidem figura discrepant. Ex hoc ergo exemplo intelligitur, quantam utilitatem idonea transformatio aequationis ope substitutionis ad inventionem radicum afferat et quod hoc pacta methodus tradita non solum ad maximas minimasve radices adstringatur, sed etiam omnes radices exhibere queat.

341. Cognita ergo iam quacunque aequationis propositae radice proxime ita; ut verbi gratia numerus *k* quam minime a quapiam radice differat, ponatur $x-k=y$ seu $x=y+k$ hocque modo prodibit aequatio, cuius radix minima erit $=x-k$; quae igitur si per series recurrentes indagetur, quod facillime fiet, quia haec radix multo minor erit quam ceterae, si ea ad *k* addatur, habebitur radix vera ipsius *x* pro aequatione proposita. Hoc vero artificium tam late patet, ut, etiamsi aequatio contineat radices imaginarias, usum suum retineat.

342. Imprimis autem sine hoc artificio radix cognosci nequit, cui datur alia aequalis, sed signo contrario affecta. Scilicet, si aequatio, cuius maxima radix *p,* eadem radicem habeat $-p$, tum, etiamsi series recurrens in infinitum continuetur, tamen radix haec *p* nunquam obtinebitur. Sit, ut hoc exemplo illustremus, proposita aequatio

$$x^3-x^2-5x+5=0,$$

cuius maxima radix est $\sqrt{5}$, praeter quam vero inest quoque $-\sqrt{5}$. Si igitur modo ante praescripto pro radice maxima invenienda utamur atque seriem recurrentem formemus ex scala relationis 1, +5, − 5, erit haec

$$1,\ 2,\ 3,\ 8,\ 13,\ 38,\ 63,\ 188,\ 313,\ 938,\ 1563 \text{ etc.},$$

ubi ad nullam rationem constantem pervenitur. Termini vero alterni rationem aequabilem induunt; quorum si quisque per praecedentem dividatur, reperietur quadratum maximae radicis; sic enim est proximo

$$5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}.$$

Quoties ergo termini tantum alterni sese ad rationem constantem componunt, toties quadratum radicis quaesitae proxime obtinetur. Ipsa autem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$, unde fit

$$1 - 3y - 5yy - y^3 = 0,$$

cuius radix minima cognoscetur ex serie

$$1,\ 1,\ 1,\ 9,\ 33,\ 145,\ 609,\ 2585,\ 10945 \text{ etc.}:$$

erit enim proxime

$$= \frac{2585}{10945} = 0,2361;;$$

at 2,2361 est proxime $= \sqrt{5}$, quae est radix maxima aequationis.

343. Quanquam numerator fractionis, ex qua series recurrens formatur, a nostro arbitrio pendet, tamen idonea eius constitutio plurimum confert, ut valor radicis cito vero proxime exhibeatur. Cum enim assumptis ut supra factoribus denominatoris (§ 334) sit terminus generalis seriei recurrentis

$$= z^n \left(\mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n + \text{ etc.}\right),$$

isti coefficients $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ per numeratorem fractionis determinantur, unde fieri potest, ut $\mathfrak{A}$ sive magnum sive isti coefficients parvum valorem obtineat; priori casu radix maxima p cito reperitur, posteriore vero tarde. Quin etiam numerator ita accipi potest, ut $\mathfrak{A}$ prorsus evanescat, quo casu, etiamsi series in infinitum continuetur, tamen nunquam

radicem maximam p praebebit. Hoc autem evenit, si numerator ita accipiatur, ut ipse eundem habeat factorem $1-pz$;
sic enim ex computo penitus tolletur. Sic si proponatur aequatio

$$x^3-6xx+10x-3=0,$$

cuius maxima radix est $=3$, indeque formetur fractio

$$\frac{1-3z}{1-6z+10z^2-3z^3}$$

ut seriei recurrentis sit scala relationis 6, −10, + 3,

$$1,\ 3,\ 8,\ 21,\ 55,\ 144,\ 377\ \text{etc.},$$

cuius termini prorsus non convergunt ad rationem 1 : 3. Eadem enim series oritur ex fractione

$$\frac{1}{1-3z+zz}$$

ac propterea maximam radicem aequationis

$$x^2-3x+1=0$$

exhibet.

344. Quin etiam numerator ita assumi potest, ut per seriem recurrentem quaevis radix aequationis reperiatur, quod fiet, si numerator fuerit productum ex omnibus factoribus denominatoris praeter eum, cui respondet radix, quam velimus. Sic si in priori exemplo sumatur numerator $1-3z+zz$, fractio

$$\frac{1-3z+zz}{1-6z+10z^2-3z^3}$$

dabit hanc seriem recurrentem

$$1,\ 3,\ 9,\ 27,\ 81,\ 243\ \text{etc.},$$

quae, cum sit geometrica, statim monstrat radicem $x=3$. Fractio enim illa aequalis est huic simplici

$$\frac{1}{1-3z}$$

Hinc apparet, si termini initiales, quos pro lubitu assumere licet, ita accipiantur, ut progressionem geometricam constituant, cuius exponens aequetur uni radici aequationis, tum totam seriem recurrentem fore geometricam ideoque eam ipsam radicem esse exhibituram, etiamsi neque sit maxima neque minima.

345. Ne igitur, dum quaerimus radicem vel maximam vel minimam, praeter expectationem nobis alia radix per seriem recurrentem exhibeatur, eiusmodi numerator debet eligi, qui cum denominatore nullum factorem habeat communem, quod fiet, si pro numeratore unitas accipiatur, unde terminus primus seriei erit $=1$, ex quo solo secundum scalam relationis sequentes omnes definiantur. Hocque modo semper certe radix aequationis vel maxima vel minima, prout fuerit propositum, eruetur. Sic proposita aequatione

$$y^3 * -3y+1=0,$$

cuius radix maxima desideratur, ex scala relationis 0, +3, −1 incipiendo ab unitate sequens oritur series recurrens

$$1,\ -0,\ +3,\ -1,\ +9,\ -\ 6,\ +28,\ -27,\ +90,\ -109,\ +297,\ -417,\\ +1000,\ -1548,\ +3417,\ -5644 \text{ etc.},$$

quae ad rationem constantem convergit ostenditque radicem maximam esse negativam atque proxime

$$y=\tfrac{-5644}{3417}=-1,651741,$$

quae esse debebat $=-1,8793852$. Ratio autem supra [§ 330] est allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo sit minor maxima simulque sit affirmativa.

346. His probe perpensis, quae cum in genere tum ad exempla allata monuimus, summa utilitas huius methodi ad investigandas aequationum radices luculenter perspicietur. Artificia vero, quibus operatio contrahi eoque promptior reddi queat, satis quoque sunt indicata, ita ut nihil insuper addendum esset, nisi casus, quibus aequatio vel radices habet aequales vel imaginarias, evolvendi superessent. Ponamus ergo denominatorem fractionis

$$\frac{a+bz+cz^2+dz^3\ +\text{etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{etc.}}$$

habere factorem $(1-pz)^2$ reliquis factoribus existentibus $1-qz,\quad 1-rz$ etc.
Seriei ergo recurrentis hinc natae terminus generalis erit

$$= z^n\left((n+1)\mathfrak{A}p^n + \mathfrak{B}p^n + \mathfrak{C}q^n + \text{ etc.}\right);$$

quae cuiusmodi valorem sit adeptura, si n fuerit numerus vehementer magnus, duo casus sunt distinguendi, alter, quo p est numerus maior reliquis q, r etc., alter, quo p non praebet radicem maximum. Casu priori, quo p simul est radix maxima, ob coefficientem $n+1$ reliqui termini $\mathfrak{B}p^n, \mathfrak{C}q^n$ etc. non tam cito prae eo evanescent quam ante; sin autem q fuerit $> p$, tum quoque tarde terminus $(n+1)\mathfrak{A}p^n$ prae $\mathfrak{C}q^n$ evanescet ideoque investigatio radicis maximae admodum evadet molesta.

EXEMPLUM 1

Sit proposita aequatio

$$x^3 - 3xx + 4 = 0,$$

cuius maxima radix 2 bis occurrit.

Quaeratur ergo maxima radix haec modo ante exposito per evolutionem fractionis quae dabit hanc seriem recurrentem

$$1, \ 3, \ 9, \ 23, \ 57, \ 135, \ 313, \ 711, \ 1593 \text{ etc.},$$

ubi quidem quivis terminus per praecedentem divisus dat quotum binario maiorem. Cuius ratio ex termino generali facillime patet. Reiectis enim in eo terminis $\mathfrak{C}q^n$ etc. erit terminus potestati z^n respondens

$$= (n+1)\mathfrak{A}p^n + \mathfrak{B}p^n,$$

Sequens

$$= (n+2)\mathfrak{A}p^{n+1} + \mathfrak{B}p^{n+1},$$

qui per illum divisus dat

$$\frac{(n+2)\mathfrak{A}+\mathfrak{B}}{(n+1)\mathfrak{A}+\mathfrak{B}}p > p,$$

nisi n iam in infinitum excreverit.

EXEMPLUM 2

Sit iam proposita aequatio

$$x^3 - xx - 5x - 3 = 0$$

cuius maxima radix $=3$, reliquae duae aequales $=-1$.
Quaeratur maxima radix ope seriei recurrentis, cuius scala relationis est $1,\ +5,\ +3$; unde oritur

$$1,\ 1,\ 6,\ 14,\ 47,\ 135,\ 412,\ 1228 \text{ etc.},$$

quae ideo satis cito valorem 3 exhibet, quod potestates minoris radicis -1, etiamsi multiplicentur per $n+1$, tamen mox prae potestatibus ipsius 3 evanescant.

EXEMPLUM 3

Sin autem proponeretur aequatio

$$x^3 + x^2 - 8x - 12 = 0,$$

cuius radices sunt $3,\ -2,\ -2$, multo tardius maxima sese prodet.

Orietur enim haec series

$$1,\ -1,\ 9,\ -5,\ 65,\ 3,\ 457,\ 347,\ 3345,\ 4915 \text{ etc.},$$

quae adhuc longissime continuari, deberet, antequam pateret radicem inde oriundam esse $=3$.

347. Simili modo si tres factores essent aequales, ita ut denominatoris factor unus esset $(1-pz)^3$, reliqui $1-qz,\ 1-rz$ etc., seriei recurrentis terminus generalis erit

$$= z^n \left(\frac{(n+1)(n+2)}{1\cdot 2} \mathfrak{A}p^n + (n+1)\mathfrak{B}p^n + \mathfrak{C}p^n + \mathfrak{D}q^n + \mathfrak{E}r^n + \text{ etc.} \right).$$

Si ergo *p* fuerit maxima radix atque *n* fuerit numerus tantus, ut potestates $q^n,\ r^n$ etc. prae p^n evanescant, tum ex serie recurrente orietur radix

$$\frac{\frac{1}{2}(n+2)(n+3)\mathfrak{A} + (n+2)\mathfrak{B} + \mathfrak{C}}{\frac{1}{2}(n+1)(n+2)\mathfrak{A} + (n+1)\mathfrak{B} + \mathfrak{C}}\, p,$$

quae, nisi sit *n* numerus maximus et quasi infinitus, verum ipsius *p* valorem indicabit. Erit autem iste radicis valor

$$= p + \frac{(n+2)\mathfrak{A}+\mathfrak{B}}{\frac{1}{2}(n+1)(n+2)\mathfrak{A}+(n+1)\mathfrak{B}+\mathfrak{C}}\, p$$

Quodsi autem p non fuerit radix maxima, tum inventio maximae multo magis adhuc impedietur; unde sequitur aequationes, quae contineant radices aequales, hac methodo per series recurrentes multo difficilius resolvi, quam si omnes radices essent inter se inaequales.

348. Videamus nunc, quomodo series recurrens in infinitum continuata debeat esse comparata, quando denominator fractionis habet factores imaginarios.
Sint igitur fractionis

$$\frac{a+ bz+cz^2+dz^3+\text{ etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{ etc.}}$$

factores denominatoris reales

$$1-qz,\ 1-rz \text{ etc.}$$

insuperque factor trinomialis

$$1-2pz\cos.\varphi+ppzz$$

continens duos factores simplices imaginarios. Quodsi ergo series recurrens ex illa fractione orta fuerit

$$A+Bz+Cz^2+Dz^3+\cdots+Pz^n+Qz^{n+1}+\text{ etc.},$$

erit per ea, quae supra exposuimus, coefficiens P

$$=\frac{\mathfrak{A}\sin.(n+1)\varphi+\mathfrak{B}\sin.n\varphi}{\sin.\varphi}p^n+\mathfrak{C}q^n+\mathfrak{D}r^n+\text{ etc.}$$

Si igitur numerus p minor fuerit quam unus ceterorum q, r etc., ita ut maxima radix aequationis

$$x^m-\alpha x^{m-1}-\beta x^{m-2}-\gamma x^{m-3}-\text{ etc.}=0$$

sit realis, tum ea per series recurrentes aeque reperietur, ac si nullae radices inessent imaginariae.

349. Inventio ergo maximae radicis realis per radices imaginarias non perturbabitur, si hae ita fuerint comparatae, ut binarum, quae factorem realem componunt, productum non sit maius quadrato radicis maximae. Sin autem binae eiusmodi insint radices imaginariae, ut earum productum adaequet vel adeo superet quadratum maximae radicis realis, tum investigatio ante exposita nihil declarabit, propterea quod potestas p^n prae simili potestate radicis maximae nunquam evanescit, etiamsi series in infinitum continuetur. Cuius exempla illustrationis causa hic adiicere visum est.

EXEMPLUM 1

Sit proposita aequatio

$$x^3-2x-4=0,$$

cuius radicem maximam investigari oporteat.

Resolvitur haec aequatio in duos factores

$$(x-2)(xx+2x+2);$$

unde unam habet radicem realem 2 et duas reliquas imaginarias, quarum productum est 2, minus quam quadratum radicis realis. Quamobrem ea per modum hactenus traditum cognosci poterit. Formetur ergo series recurrens ex scala relationis $0,\ +2,\ +4$, quae erit

$$1,\ 0,\ 2,\ 4,\ 4,\ 16,\ 24,\ 48,\ 112,\ 192,\ 416,\ 832 \text{ etc.},$$

unde satis luculenter radix realis 2 cognosci potest.

EXEMPLUM 2

Proposita sit aequatio

$$x^3-4xx+8x-8=0,$$

cuius radix una realis est 2, binarum imaginariarum productum vera $=4$ ideoque aequale quadrato radicis realis 2.

Quaeramus ergo radicem per seriem recurrentem; quod quo facilius fieri queat, ponamus $x=2y$, ut habeatur

$$y^3-2yy+2y-1=0,$$

unde formetur series recurrens

$$1,\ 2,\ 2,\ 1,\ 0,\ 0,\ 1,\ 2,\ 2,\ 1,\ 0,\ 0,\ 1,\ 2,\ 2,\ 1 \text{ etc.};$$

in qua cum iidem termini perpetuo revertantur, nihil inde aliud colligi potest, nisi radicem maximam vel non esse realem vel dari imaginarias, quarum productum aequale sit aut superet quadratum radicis realis.

EXEMPLUM 3

Sit iam proposita aequatio

$$x^3 - 3xx + 4x - 2 = 0,$$

cuius radix realis est 1, imaginariarum vero productum $= 2$.

Formetur ergo ex scala relationis $3, \ -4, \ +2$ series

$$1, \ 3, \ 5, \ 5, \ 1, \ -7, \ -15, \ -15, \ +1, \ 33, \ 65, \ 65, \ 1 \text{ etc.};$$

in qua cum termini modo fiant affirmativi modo negativi, radix realis 1 inde nullo modo cognosci poterit. Huiusmodi vero revolutiones semper ostendunt radicem, quam series praebere debebat, esse imaginariam; hic enim radices imaginariae potestate sunt maiores quam realis 1.

350. Sit igitur in fractione generali productum binarum radicum imaginariarum *pp* maius quam ullius radicis realis quadratum, ita ut prae p^n reliquae potestates q^n, r^n etc. evanescant, si *n* sit numerus infinitus. Hoc ergo casu fiet

$$P = \frac{\mathfrak{A} \sin.(n+1)\varphi + \mathfrak{B} \sin. n\varphi}{\sin. \varphi} p^n$$

et

$$Q = \frac{\mathfrak{A} \sin.(n+2)\varphi + \mathfrak{B} \sin. (n+1)\varphi}{\sin. \varphi} p^{n+1}$$

ideoque

$$\frac{Q}{P} = \frac{\mathfrak{A} \sin.(n+2)\varphi + \mathfrak{B} \sin. (n+1)\varphi}{\mathfrak{A} \sin.(n+1)\varphi + \mathfrak{B} \sin. n\varphi} p.$$

Quae expressio nunquam valorem constantem induet, etiamsi *n* sit numerus infinitus. Sinus enim angulorum perpetuo maxime manent mutabiles, ita ut mox sint affirmativi mox negativi.

351. Interim tamen si fractiones sequentes $\frac{R}{Q}$, $\frac{S}{R}$ simili modo sumantur indeque litterae $\mathfrak{A}$ et $\mathfrak{B}$ eliminentur, simul numerus *n* ex calculo egredietur ; reperietur enim

$$Ppp + R = 2Qp\cos.\varphi,$$

unde fit

$$\cos.\varphi = \frac{Ppp+R}{2Qp};$$

similiter vero erit

$$\cos.\varphi = \frac{Qpp+S}{2Rp}$$

ex quorum duorum valorum comparatione fit

$$p = \sqrt{\frac{RR-QS}{QQ-PR}}$$

atque

$$\cos.\varphi = \frac{QR-PS}{2\sqrt{(Q^2-PR)(R^2-QS)}}.$$

Quamobrem si series recurrens iam eousque fuerit continuata, ut prae p^n reliquarum radicum potestates evanescant, tum hoc modo factor trinomialis $1-2pz\cos.\varphi + ppzz$ poterit inveniri.

352. Quoniam iste calculus non satis exercitatis molestiam creare posset, eum totum hic apponam. Ex valore ipsius $\frac{Q}{P}$ invento oritur

$$\mathfrak{A}\, Pp\sin.(n+2)\varphi + \mathfrak{B}\, Pp\sin.(n+1)\varphi = \mathfrak{A}Q\sin.(n+1)\varphi + \mathfrak{B}Q\sin.n\varphi,$$

unde fit

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{Q\sin.n\varphi - Pp\sin.(n+1)\varphi}{Pp\sin.(n+2)\varphi - Q\sin.(n+1)\varphi}.$$

Pari ratione erit

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{R\sin.(n+1)\varphi - Qp\sin.(n+2)\varphi}{Qp\sin.(n+3)\varphi - R\sin.(n+2)\varphi}.$$

Aequatis his duobus valoribus fiet

$$\begin{aligned}0 = {} & QQp\sin.n\varphi\sin.(n+3)\varphi - QR\sin.n\varphi\sin.(n+2)\varphi \\ & -PQpp\sin.(n+1)\varphi\sin.(n+3)\varphi - QQp\sin.(n+1)\varphi\sin.(n+2)\varphi \\ & + QR\sin.(n+1)\varphi\sin.(n+1)\varphi + PQpp\sin.(n+2)\varphi\sin.(n+2)\varphi.\end{aligned}$$

Cum autem sit

$$\sin.a \sin.b = \frac{1}{2}\cos.(a-b) - \frac{1}{2}\cos.(a+b)\,,$$

fiet

$$0 = \frac{1}{2}QQp(\cos.3\varphi - \cos.\varphi) + \frac{1}{2}QR(1-\cos.2\varphi) + \frac{1}{2}PQpp(1-\cos.2\varphi)\,,$$

quae per $\frac{1}{2}Q$ divisa dat

$$(Ppp+R)(1-\cos.2\varphi) = Qp(\cos.\varphi - \cos.3\varphi)\,.$$

At est

$$\cos.\varphi = \cos.2\varphi\cos.\varphi + \sin.2\varphi\sin.\varphi$$

et

$$\cos.3\varphi = \cos.2\varphi\cos.\varphi - \sin.2\varphi\sin.\varphi\,,$$

unde

$$\cos.\varphi - \cos.3\varphi = 2\ \sin.2\varphi\ \sin.\varphi = 4\ \sin.\varphi^2\cos.\varphi\,,$$

et

$$1-\cos.2\varphi = 2\ \sin.\varphi^2\,,$$

ex quo erit

$$Ppp + R = 2Qp\cos.\varphi$$

et

$$\cos.\varphi = \frac{Ppp+R}{2Qp}$$

atque

$$\cos.\varphi = \frac{Qpp+S}{2Rp}$$

unde superiores valores prodeunt, scilicet

$$p = \sqrt{\frac{RR-QS}{QQ-PR}}$$

et

$$\cos.\varphi = \frac{QR-PS}{2\sqrt{(Q^2-PR)(RR-QS)}}.$$

353. Si denominator fractionis, ex qua series recurrens formatur, plures habeat factores trinomiales inter se aequales, tum spectata forma termini generalis supra data patebit inventionem radicum multo magis fieri incertam. Interim tamen si una quaecunque radix realis iam proxime fuerit detecta, tum aequationis transformations semper valor eiusdem radicis multo propior eruetur. Ponatur enim x aequalis valori illi iam detecto + y atque novae aequationis quaeratur minima radix pro y, quae addita ad illum valorem praebebit verum ipsius x valorem.

EXEMPLUM

Sit proposita ista aequatio

$$x^3-3xx+5x-4=0\,;$$

cuius unam radicem fere esse $=1$ inde constat, quod posita $x=1$ prodit

$$x^3-3xx+5x-4=-1.$$

Ponatur ergo $x=1+y$ fietque

$$1-2y-y^3=0,$$

unde pro radice minima invenienda formetur series recurrens, cuius scala relationis 2, 0, +1, quae erit

$$1,\ 2,\ 4,\ 9,\ 20,\ 44,\ 97,\ 214,\ 472,\ 1041,\ 2296\ \text{etc.},$$

unde radix minima ipsius y erit proxime

$$\tfrac{1041}{2296}=0,453397,$$

ita ut sit

$$x=1,453397,$$

qui valor tam prope vix alia methodo aeque facile obtineri poterit.

354. Quodsi autem series quaecunque recurrens tandem tam prope ad progressionem geometricam convergat, tum ex ipsa lege progressionis statim facile cognosci poterit,

cuiusnam aequationis radix sit futura quotus, qui ex divisione unius termini per praecedentem oritur. Sint

$$P,\ Q,\ R,\ S,\ T \quad \text{etc}.$$

termini seriei recurrentis a principio iam longissime remoti, ita ut cum progressione geometrica confundantur, sitque

$$T=\alpha S+\beta R+\gamma Q+\delta P$$

seu scala relationis $a,\ +\beta,\ +\gamma,\ +\delta$. Ponatur valor fractionis $\frac{Q}{P}=x$; erit

$$\tfrac{R}{P}=xx,\quad \tfrac{S}{P}=x^3 \ \text{ et } \ \tfrac{T}{P}=x^4,$$

qui in superiori aequatione substituti dabunt

$$x^4=\alpha x^3+\beta x^2+\gamma x+\delta\,,$$

unde patet quotum $\frac{Q}{P}$ tandem praebere radicem unam aequationis inventae. Hoc vero et praecedens methodus indicat, praeterea vero docet fractionem $\frac{Q}{P}$ dare maximam aequationis radicem.

355. Potest quoque haec methodus investigandarum radicum saepenumero utiliter adhiberi, si aequatio sit infinita. Ad quod ostendendum proposita sit aequatio

$$\frac{1}{2}=z-\frac{z^3}{6}+\frac{z^5}{120}-\frac{z^7}{5040}+\text{etc.},$$

cuius radix minutia z exhibet arcum 30° seu semiperipheriae circuli sextantem. Perducatur ergo aequatio ad hanc formam

$$1-2z+\frac{z^3}{3}-\frac{z^5}{60}+\frac{z^7}{2520}-\text{etc.}=0.$$

Hinc ergo formetur series recurrens, cuius scala relationis est infinita, scilicet

$$2,\ 0,\ -\tfrac{1}{3},\ 0,\ +\tfrac{1}{60},\ 0,\ -\tfrac{1}{2520},\ 0\ \text{ etc.},$$

eritque series recurrens

$$1,\ 2,\ 4,\ \frac{23}{3},\ \frac{44}{3},\ \frac{1681}{60},\ \frac{2408}{45}\ \text{ etc.}:$$

erit ergo proxime

$$z = \frac{1681 \cdot 45}{2408 \cdot 60} = \frac{1681 \cdot 3}{2408 \cdot 4} = \frac{5043}{9632} = 0,52356.$$

At ex proportione peripheriae ad diametrum cognita debebat esse $z = 0,523598$, ita ut radix inventa tantum parte $\frac{3}{100\,000}$ a vero discrepet. Hoc autem in hac aequatione commode usu venit, quod eius omnes radices sint reales atque a minima reliquae satis notabiliter discrepent. Quae conditio cum rarissime in aequationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.