## CHAPTER XVI

# CONCERNING THE PARTITION OF NUMBERS

297. This expression shall be proposed

$$\left(1+x^{\alpha}z\right)\left(1+x^{\beta}z\right)\left(1+x^{\gamma}z\right)\left(1+x^{\delta}z\right)\left(1+x^{\varepsilon}z\right) \text{ etc.};$$

we may inquire what kind of form it adopts on expanding out by multiplication.
We may put

$$1+Pz+Qz^{2}+Rz^{3}+Sz^{4}+\text{etc.}$$

to emerge, and it is evident that $P$ becomes the sum of the powers

$$x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\varepsilon}+\text{etc.}$$

In the next place, $Q$ is the sum of the factors from two different powers, or $Q$ will be the sum of several powers of $x$, the exponents of which are the sums of two different terms of this series

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \text{etc.}$$

In a similar manner, $R$ will be the sum of the powers of $x$, the exponents of which are the sums of three different terms. And $S$ will be the sum of the powers of $x$, the exponents of which are the sums of four diverse terms of the same series $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc., and thus henceforth.

298. These individual powers of $x$, which are present in the values of the letters $P$, $Q$, $R$, $S$ etc., have one for the coefficient, if indeed the exponents of these are able to be formed in a single way from $\alpha,\ \beta,\ \gamma,\ \delta$ etc.; but if the exponent of the same power may be a sum in several manners of two, three or more terms of the series $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc., then the power also will have that coefficient, which is present as often as one is included within itself. Thus if $Nx^{n}$ may be found in the value of $Q$, from this evidence the number $n$ is the sum of two different terms of the series $\alpha,\ \beta,\ \gamma,\ \delta$ etc., in $N$ different manners.

And if in the expansion of the proposed factors the term $Nx^{n}z^{m}$ may occur, the coefficient $N$ of this will indicate, in how many different ways the number $n$ may be able to be the sum of $m$ different terms of the series $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta$ etc.

299. But if the proposed product

$$\left(1+x^{\alpha}z\right)\left(1+x^{\beta}z\right)\left(1+x^{\gamma}z\right)\left(1+x^{\delta}z\right)\text{ etc.}$$

therefore truly may be expanded out by multiplication, from the resulting expression it will be apparent at once, in how many different ways a given number will be able to be the sum of just as many different terms of the series $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ etc., as many as desired from which. Clearly, if it may be sought, in how many different ways a number *n* may be the sum of *m* different terms of that series, then in the expression expanded out the term $x^n z^m$ must be sought, and the coefficient of this term will indicate the number sought.

300. So that this may be made more apparent, this shall be the agreed product proposed from infinite factors which expanded out by actual multiplication gives

$$\begin{aligned}
1 &+z\left(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\text{etc.}\right)\\
&+z^2\left(x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+4x^{10}+5x^{11}+\text{etc.}\right)\\
&+z^3\left(x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+8x^{13}+10x^{14}+\text{etc.}\right)\\
&+z^4\left(x^{10}+x^{11}+2x^{12}+3x^{13}+5x^{14}+6x^{15}+9x^{16}+11x^{17}+15x^{18}+\text{etc.}\right)\\
&+z^5\left(x^{15}+x^{16}+2x^{17}+3x^{18}+5x^{19}+7x^{20}+10x^{21}+13x^{22}+18x^{23}+\text{etc.}\right)\\
&+z^6\left(x^{21}+x^{22}+2x^{23}+3x^{24}+5x^{25}+7x^{26}+11x^{27}+14x^{28}+20x^{29}+\text{etc.}\right)\\
&+z^7\left(x^{28}+x^{29}+2x^{30}+3x^{31}+5x^{32}+7x^{33}+11x^{34}+15x^{35}+21x^{36}+\text{etc.}\right)\\
&+z^8\left(x^{36}+x^{37}+2x^{38}+3x^{39}+5x^{40}+7x^{41}+11x^{42}+15x^{43}+22x^{44}+\text{etc.}\right)\\
&\qquad\text{etc.}
\end{aligned}$$

Therefore from these series it is possible to define at once, in how many different ways a proposed number may be able to be generated from the given different terms of this series

1, 2, 3, 4, 5, 6, 7, 8 etc.

Thus if it may be wished to know, in how many different ways the number 35 can be the sum of seven different terms of the series 1, 2, 3, 4, 5, 6, 7, 8 etc., the power $x^{35}$ may be

sought in the series multiplying $z^7$, the coefficient of which 15 will indicate that the proposed number 35 can be the sum of the seven terms of the series
1, 2, 3, 4, 5, 6, 7, 8, etc., in fifteen different ways.

301. But if moreover there may be put $z = 1$ and similar powers of $x$ may be joined together in as single sum or, which returns the same, if this infinite expression may be expanded out :

$$(1+x)\left(1+x^2\right)\left(1+x^3\right)\left(1+x^4\right)\left(1+x^5\right)\left(1+x^6\right) \text{ etc.,}$$

with which done this series is generated :

$$x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+\text{etc.,}$$

where whatever coefficient indicates, in how many different ways the exponent of the powers of $x$ joined together may emerge by addition from the different powers of the series 1, 2, 3, 4, 5, 6, 7 etc. Thus it is apparent that the number 8 is produced in six different ways by the addition of the different numbers, which are

$$\begin{array}{l|l} 8=8 & 8=5+3 \\ 8=7+1 & 8=5+2+1 \\ 8=6+2 & 8=4+3+1 \end{array}$$

where it is to be observed that likewise the number itself proposed must be calculated, because the number of terms is not defined and thus unity cannot be excluded from that. [*i.e.* the term $x^0$ .]

302. Hence therefore it is understood, in what manner each number may be produced by the addition of different numbers. But this condition which considers the numbers different can be laid aside, if we may transpose these factors into a denominator. Therefore let this expression be proposed

$$\frac{1}{\left(1-x^\alpha z\right)\left(1-x^\beta z\right)\left(1-x^\gamma z\right)\left(1-x^\delta z\right)\left(1-x^\varepsilon z\right) \text{ etc.}},$$

which expanded out by division will give

$$1+Pz+Qz^2+Rz^3+Sz^4+\text{etc.}$$

And it is clear that $P$ becomes the sum of the powers of $x$, the exponents of which may be held in this series

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \text{etc.}$$

Then *Q* will be the sum of the powers of *x*, the exponents of which shall be the sums of two terms of this series, either of the same or different. Then *R* will be the sum of powers of *x,* the exponents of which may be generated from the addition of three terms of that series, and *S* the sum of the powers, the exponents of which may be formed from the addition of four terms contained in that series, and hence so forth.

303. Therefore if the whole expression may be set out by the individual terms and like terms may be joined together, it may be understood, in how many different ways a proposed number *n* may be able to be produced by the addition of *m* terms, either of the same or different terms of the series

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \text{etc.}$$

Evidently the term $x^n z^m$ may be sought in the expression expanded, the coefficient of which shall be *N*, thus so that the whole term shall be $= Nx^n z^m$, and which will indicate the coefficient *N*, in as many ways as the number *n* may be produced by the addition of the terms *m* contained in the series $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc. Therefore this question may be resolved in agreement with the first of a similar nature, that we have considered before.

304. We may apply this to an especially noteworthy case and this expression shall be proposed

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\ \text{etc.}},$$

which expanded out by division will give

$$\begin{aligned}1 &+ z\left(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \text{etc.}\right)\\ &+ z^2\left(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + 5x^{10} + \text{etc.}\right)\\ &+ z^3\left(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + \text{etc.}\right)\\ &+ z^4\left(x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10} + 11x^{11} + 15x^{12} + \text{etc.}\right)\\ &+ z^5\left(x^5 + x^6 + 2x^7 + 3x^8 + 5x^9 + 7x^{10} + 10x^{11} + 13x^{12} + 18x^{13} + \text{etc.}\right)\\ &+ z^6\left(x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 7x^{11} + 11x^{12} + 14x^{13} + 20x^{14} + \text{etc.}\right)\\ &+ z^7\left(x^7 + x^8 + 2x^9 + 3x^{10} + 5x^{11} + 7x^{12} + 11x^{13} + 15x^{14} + 21x^{15} + \text{etc.}\right)\\ &+ z^8\left(x^8 + x^9 + 2x^{10} + 3x^{11} + 5x^{12} + 7x^{13} + 11x^{14} + 15x^{15} + 22x^{16} + \text{etc.}\right)\\ &\qquad\text{etc.}\end{aligned}$$

Therefore from these series it is possible to define at once, in how many different ways a proposed number may be able to be produced by addition from a given number of the terms of this series 1, 2, 3, 4, 5, 6, 7 etc. Thus if, the number of ways is sought in which the number 13 can be generated by adding five whole numbers, then the term $x^{13}z^5$ must be examined, the coefficient 18 of which indicates the number of ways the proposed number 13 can arise from the addition of five numbers.

305. If there may be put $z=1$ and like powers of $x$ may be expressed jointly, this expression

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\text{ etc.}}$$

may be expanded into this series

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \text{ etc.};$$

in which any coefficient indicates, in how many ways the exponent from the powers joined together by addition may be produced from whole numbers, be they either equal or unequal. Evidently from the term $11x^6$ the number 6 can be produced in eleven ways by the addition of whole numbers, which are :

| | |
|---|---|
| $6=6$ | $6=3+1+1+1$ |
| $6=5+1$ | $6=2+2+2$ |
| $6=4+2$ | $6=2+2+1+1$ |
| $6=4+1+1$ | $6=2+1+1+1+1$ |
| $6=3+3$ | $6=1+1+1+1+1+1$ |
| $6=3+2+1$ | |

where it must also be observed the proposed number itself, since it shall be contained in the proposed series of numbers 1, 2, 3, 4, 5, 6 etc., to be present in a single way.

306. We must inquire more carefully from these set out in general, the manner in which this multitude of compositions are to be found. And indeed in the first place we may consider that composition from whole numbers, in which only different numbers are allowed, as we have mentioned at first. Therefore let this expression be proposed towards this end :

$$(1+xz)(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z) \text{ etc.},$$

which expanded out and provides the following powers of $z$ in order

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\text{etc.},$$

where a method is desired whereby these functions *P*, *Q*, *R*, *S*, *T* etc. of *x* functions can be conveniently found; indeed with this agreed on, it will provide satisfaction to the proposed question in a most convenient manner.

307. Moreover it is apparent, if *xz* may be put in place of *z*, for

$$(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)\text{etc.}=\tfrac{Z}{1+xz}$$

to appear. Therefore on putting *xz* in place of $z$ a value was produced, which was *Z*, which will be changed into $\tfrac{Z}{1+xz}$ ; and thus, since there shall be

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\text{etc.},$$

there will be

$$\tfrac{Z}{1+xz}=1+Pxz+Qx^2z^2+Rx^3z^3+Sx^4z^4+\text{etc.}$$

Therefore the equation actually may be multiplied by $1+xz$ and there will be produced

$$Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \text{etc.}$$
$$+ \quad xz + Px^2z^2 + Qx^3z^3 + Rx^4z^4 + \text{etc.},$$

which value of $Z$ compared with the superior will give

$$P = \frac{x}{1-x}, \quad Q = \frac{Px^2}{1-x^2}, \quad R = \frac{Qx^3}{1-x^3}, \quad S = \frac{Rx^4}{1-x^4} \quad \text{etc.}$$

Therefore the following values will be found for *P, Q, R, S* etc. :

$$P = \frac{x}{1-x},$$
$$Q = \frac{x^3}{(1-x)(1-x^2)},$$
$$R = \frac{x^6}{(1-x)(1-x^2)(1-x^3)},$$
$$S = \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)},$$
$$T = \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$
etc.

308. Therefore thus we are able to show each and every power of $x$ of the series separately, from which it is possible to define, in how many ways the proposed number will be able to be defined from the given number of the whole partition by addition. Moreover it is evident again that these individual series are recurring, because $x$ is generated from the expansion of the fractional function. In the first place clearly the expression

$$P = \frac{x}{1-x}$$

gives the geometric series

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \text{etc.},$$

from which indeed any number is evident once in the continued series of whole numbers.

309. The second expression

$$Q = \frac{x^3}{(1-x)(1-x^2)}$$

gives this series

$$x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \text{etc.},$$

in which any coefficient of a term indicates, in how many ways the exponent of $x$ will be able to be separated into two unequal parts. Thus the term $4x^9$ indicates that the number 9 is able to be cut into two unequal parts in four ways. But if we may divide this series by $x^3$, it will produce the series, that provides this fraction

$$\frac{1}{(1-x)(1-x^2)},$$

which will be

$$1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+\text{etc.},$$

the general term of which becomes $=Nx^n$ ; and from the origins of this series it is known that the coefficient $N$ indicates, in how many ways the exponent $n$ may be able to be generated from the numbers 1 and 2 by addition. Therefore since the general term of the first series shall be $=Nx^{n+3}$, such a theorem hence is deduced :

*In as many ways as the number n can be produced by addition from the numbers* 1 *and* 2, *the number* $n+3$ *can be cut into two unequal parts in just many ways.*

310. The third expression

$$\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$

expanded out into a series will give

$$x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+8x^{13}+\text{etc.},$$

in which the coefficient of any term shows, in how many different ways the exponent of the power $x$ will be able to be added together divided up in three unequal parts. But if the fraction

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

may be expanded out, it will produce this series

$$1+x+2x^2+3x^3+4x^4+5x^5+7x^6+8x^7+\text{etc.};$$

the general term of which, if it may be put $=Nx^n$, the coefficient $N$ will indicate in how many different ways the number $n$ may be produced from the numbers 1, 2, 3 by addition.

Therefore since the general term of the first series shall be $Nx^{n+6}$, hence such a theorem follows :

*In as many different ways as the number n can be produced from the numbers* 1, 2, 3 *by addition, the number* $n+6$ *will be able to be cut into three unequal parts in just as many ways.*

311. The fourth expression

$$\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

expanded out into a recurring series will give

$$x^{10}+x^{11}+2x^{12}+3x^{13}+5x^{14}+6x^{15}+9x^{16}+11x^{17}+\text{ etc.},$$

in which the coefficient of any term will indicate, in how many different ways the exponent of the power *x* shall be able to be taken together in four unequal separated parts. But if moreover this expression

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

may be expanded out, the above series will be produced divided by $x^{10}$, clearly

$$1+x+2x^2+3x^3+5x^4+6x^5+9x^6+11x^7+\text{ etc.},$$

the general term of which we may put $=Nx^n$; and hence it will be apparent that the coefficient *N* indicates, in how many different ways the number *n* will be able to arise by addition from these four numbers 1, 2, 3, 4. Therefore since the general term of the first series shall become $=Nx^{n+10}$, this theorem is deduced :

*In as many different ways as the number n is able to be produced by addition from the numbers* 1, 2, 3, 4, *the number* $n+10$ *will be able to be cut into four unequal parts in just as many ways.*

312. Generallly therefore, if this expression

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

may be expanded out into a series of which the general term were $= Nx^n$, the coefficient $N$ will show, in how many different ways the number $n$ will be able to be produced by addition from these numbers 1, 2, 3, 4, . . . $m$. But if moreover this expression

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

may be expanded out into a series, the general term of which will be $= Nx^{n+\frac{m(m+1)}{2}}$ and this coefficient $N$ indicates, in how many different ways the number $n+\frac{m(m+1)}{1\cdot 2}$ may be able to be cut into $m$ unequal parts, from which this theorem follows :

*In as many different ways as the number n is able to be produced by addition from the numbers* 1, 2, 3, 4, ... *m, the number* $n+\frac{m(m+1)}{1\cdot 2}$ *will be able to be cut into m unequal parts in just as many ways.*

313. From the partition of the numbers put into unequal parts we can consider also the partition into parts, where the equality of the parts is not excluded ; which partition has its origin from this expression

$$Z=\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}}.$$

We may put in place the expansion by division to produce

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\text{etc.}$$

Moreover it is clear, if in place of $z$ we may put $xz$, to give rise to

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}}=(1-xz)Z.$$

Therefore from the expansion made in series from the same change made

$$(1-xz)Z=1+Pxz+Qx^2z^2+Rx^3z^3+Sx^4z^4+\text{etc.}$$

Therefore the above series equally may be multiplied by $(1-xz)$ and there will be

$$\begin{aligned}(1-xz)Z = 1 + Pz + \quad & Qz^2 + \quad Rz^3 + \quad Sz^4 + \text{etc.}\\ - xz - Pxz^2 & - Qxz^3 - Rxz^4 - \text{etc.}\end{aligned}$$

Therefore with a comparison made there will be generated

$$P = \tfrac{x}{1-x}, \quad Q = \tfrac{Px}{1-x^2}, \quad R = \tfrac{Qx}{1-x^3}, \quad S = \tfrac{Rx}{1-x^4} \text{ etc.},$$

from which the following values arise for *P, Q, R, S* etc. :

$$P = \tfrac{x}{1-x},$$
$$Q = \frac{x^2}{(1-x)(1-x^2)},$$
$$R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)},$$
$$S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$
etc.

314. These expressions do not differ from the above in any way, except that the numerators here have smaller exponents than in the preceding case. And on account of this result the series which arise from the expansion, generally will agree generally in the account of the coefficients, which now conveniently is seen from the comparison of §300 and §304, now truly the method of this finally is understood. Hence therefore generally similar theorems will follow, which are :

*In as many ways as the number n can be produced by addition from the numbers* 1, 2, $n+2$ *can be separated into two parts in just as many ways.*

*In as many ways as the number n can be produced by addition from the numbers* 1, 2, 3, *the number* $n+3$ *can be separated into three parts in just as many ways.*

*In as many ways as the number n can be produced by addition from the numbers* 1, 2, 3, 4, *the number* $n+4$ *can be separated into three parts in just as many ways.*

And generally this theorem will be had :

*In as many ways as the number n can be produced by addition from the numbers* 1, 2, 3, ... *m, the number* $n+m$ *can be separated into m parts in just as many ways.*

315. Therefore whether it may be sought, in how many ways a given number *m* may be able to be separated into unequal parts, or whether it may be separated into *m* parts with equality excluded, each question will be resolved, if it may be known, in how many ways each number may be able to be produced by addition from the numbers 1, 2, 3, 4, . . . *m*, just as this will be apparent from the following theorems, which have been derived from the above :

*The number n is able to be separated into as many unequal parts m, as the number of ways the number* $n-\frac{m(m+1)}{2}$ *can be produced by addition from the numbers* 1, 2, 3, 4,...*m*.

*The number n can be separated into m parts in as many ways, either equal or unequal, as the number of ways the number* $n-m$ *can be produced by addition from the numbers* 1, 2, 3, ... *m*.

Hence again these theorems follow:

*The number n can be cut in just as many ways into m unequal parts, as the number of ways* $n-\frac{m(m+1)}{2}$ *may be separated into m parts, either equal or unequal.*

*The number n can be cut into m parts in just as many ways, either equal or unequal, as the number of ways the number* $n+\frac{m(m+1)}{2}$ *can be separated into m unequal parts.*

316. Moreover by the formation of recurring series one can find, in how many different ways the given number *n* may be produced by addition from the numbers 1, 2, 3, .... *m*. For towards finding this series the fraction

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

must be expanded out and the recurring series will have to be continued as far as to the term $Nx^n$, the coefficient of which *N* will indicate, in how many ways the number *n* may be able to be produced from the numbers 1, 2, 3, 4, . . . *m*. But truly this manner of solving will have more than a little difficulty, if the numbers *m* and *n* shall be moderately large ; for the scale of the relation, [*i.e.* the recurring relation ; here and elsewhere Euler uses the word *scala* meaning ladder, indicating joining steps.] that provides the denominator expanded out by multiplication, depends on a number of terms, from which it will be laborious to continue the series to several terms.

317. But this inquiry will be less tiresome, if the simpler cases may be set out first ; for from these it will be easy to progress to more composite cases. Let the general term of the series, which originates from this fraction,

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

be $= Nx^n$ ; but the general term shall be $Mx^n$, arising from this form of series

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)},$$

where the coefficient $M$ will indicate, in how many different ways the number $n-m$ will be able to be produced by addition from the numbers 1, 2, 3, ... $m$. The latter expression may be taken from the former and

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^{m-1})}$$

will remain, and it is clear the general term of this series arising will become $(N-M)x^n$ ; because the coefficient $N-M$ will indicate, in how many different ways the number $n$ will be able to be produced by addition from the numbers $1, 2, 3, \ldots. m-1$.

318. Hence therefore we come upon the following rule:

*Let L be the number of ways, in which the number n can be produced by addition from the numbers* $1, 2, 3, \ldots m-1$,
*let M be the number of ways, in which the number* $n-m$ *can be produced by addition from the numbers* 1, 2, 3, ... *m*,
*and N shall be the number of ways, in which the number n can be produced by addition from the numbers* 1, 2, 3, ... *m*;
*with these in place there will be, as we have seen,*

$$L = N - M$$

*and thus*

$$N = L + M .$$

But if therefore now we have found, in how many different ways the numbers $n$ and $n-m$ are able to be produced by addition, that from the numbers $1, 2, 3, \ldots m-1$, truly this from the numbers 1, 2, 3, ... *m*, hence we know on adding, in how many different ways the number *n* by addition may be produced from the numbers 1, 2, 3, ... *m*. With the aid of these theorems from the simpler cases, which offer nothing of difficulty, one can progress by continuation to the more composite, and in this way the table here adjoined had been computed, the use of which thus may be considered :

If it may be sought, in how many different ways the number 50 shall be separated into 7 unequal parts, in the first vertical column the number $50-\frac{7\cdot 8}{2}=22$ is taken, but on the horizontal uppermost row the roman numeral VII is taken; and the number put in the angle position 522 will indicate the number of ways sought. But if it is sought, in how many ways the number 50 may be separated into 7 parts, either equal or unequal, in the first vertical column the number $50-7=43$ may be taken, to which the number sought 8946 will correspond in the seventh column.

TABLE.

| *n* | I | II | III | IV | V | VI | VII | VIII | IX | X | XI |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 3 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 5 | 1 | 3 | 5 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 6 | 1 | 4 | 7 | 9 | 10 | 11 | 11 | 11 | 11 | 11 | 11 |
| 7 | 1 | 4 | 8 | 11 | 13 | 14 | 15 | 15 | 15 | 15 | 15 |
| 8 | 1 | 5 | 10 | 15 | 18 | 20 | 21 | 22 | 22 | 22 | 22 |
| 9 | 1 | 5 | 12 | 18 | 23 | 26 | 28 | 29 | 30 | 30 | 30 |
| 10 | 1 | 6 | 14 | 23 | 30 | 35 | 38 | 40 | 41 | 42 | 42 |
| 11 | 1 | 6 | 16 | 27 | 37 | 44 | 49 | 52 | 54 | 55 | 56 |
| 12 | 1 | 7 | 19 | 34 | 47 | 58 | 65 | 70 | 73 | 75 | 76 |
| 13 | 1 | 7 | 21 | 39 | 57 | 71 | 82 | 89 | 94 | 97 | 99 |
| 14 | 1 | 8 | 24 | 47 | 70 | 90 | 105 | 116 | 123 | 128 | 131 |
| 15 | 1 | 8 | 27 | 54 | 84 | 110 | 131 | 146 | 157 | 164 | 169 |
| 16 | 1 | 9 | 30 | 64 | 101 | 136 | 164 | 186 | 201 | 212 | 219 |
| 17 | 1 | 9 | 33 | 72 | 119 | 163 | 201 | 230 | 252 | 267 | 278 |
| 18 | 1 | 10 | 37 | 84 | 141 | 199 | 248 | 288 | 318 | 340 | 355 |
| 19 | 1 | 10 | 40 | 94 | 164 | 235 | 300 | 352 | 393 | 423 | 445 |
| 20 | 1 | 11 | 44 | 108 | 192 | 282 | 364 | 434 | 488 | 530 | 560 |
| 21 | 1 | 11 | 48 | 120 | 221 | 331 | 436 | 525 | 598 | 653 | 695 |
| 22 | 1 | 12 | 52 | 136 | 255 | 391 | 522 | 638 | 732 | 807 | 863 |
| 23 | 1 | 12 | 56 | 150 | 291 | 454 | 618 | 764 | 887 | 984 | 1060 |
| 24 | 1 | 13 | 61 | 169 | 333 | 532 | 733 | 919 | 1076 | 1204 | 1303 |
| 25 | 1 | 13 | 65 | 185 | 377 | 612 | 860 | 1090 | 1291 | 1455 | 1586 |
| 26 | 1 | 14 | 70 | 206 | 427 | 709 | 1009 | 1297 | 1549 | 1761 | 1930 |
| 27 | 1 | 14 | 75 | 225 | 480 | 811 | 1175 | 1527 | 1845 | 2112 | 2331 |
| 28 | 1 | 15 | 80 | 249 | 540 | 931 | 1367 | 1801 | 2194 | 2534 | 2812 |
| 29 | 1 | 15 | 85 | 270 | 603 | 1057 | 1579 | 2104 | 2592 | 3015 | 3370 |
| 30 | 1 | 16 | 91 | 297 | 674 | 1206 | 1824 | 2462 | 3060 | 3590 | 4035 |
| 31 | 1 | 16 | 96 | 321 | 748 | 1360 | 2093 | 2857 | 3589 | 4242 | 4802 |
| 32 | 1 | 17 | 102 | 351 | 831 | 1540 | 2400 | 3319 | 4206 | 5013 | 5708 |
| 33 | 1 | 17 | 108 | 378 | 918 | 1729 | 2738 | 3828 | 4904 | 5888 | 6751 |
| 34 | 1 | 18 | 114 | 411 | 1014 | 1945 | 3120 | 4417 | 5708 | 6912 | 7972 |
| 35 | 1 | 18 | 120 | 441 | 1115 | 2172 | 3539 | 5066 | 6615 | 8070 | 9373 |
| 36 | 1 | 19 | 127 | 478 | 1226 | 2432 | 4011 | 5812 | 7657 | 9418 | 11004 |
| 37 | 1 | 19 | 133 | 511 | 1342 | 2702 | 4526 | 6630 | 8824 | 10936 | 12866 |
| 38 | 1 | 20 | 140 | 551 | 1469 | 3009 | 5102 | 7564 | 10156 | 12690 | 15021 |
| 39 | 1 | 20 | 147 | 588 | 1602 | 3331 | 5731 | 8588 | 11648 | 14663 | 17475 |
| 40 | 1 | 21 | 154 | 632 | 1747 | 3692 | 6430 | 9749 | 13338 | 16928 | 20298 |
| 41 | 1 | 21 | 161 | 672 | 1898 | 4070 | 7190 | 11018 | 15224 | 19466 | 23501 |
| 42 | 1 | 22 | 169 | 720 | 2062 | 4494 | 8033 | 12450 | 17354 | 22367 | 27169 |
| 43 | 1 | 22 | 176 | 764 | 2233 | 4935 | 8946 | 14012 | 19720 | 25608 | 31316 |
| 44 | 1 | 23 | 184 | 816 | 2418 | 5427 | 9953 | 15765 | 22380 | 29292 | 36043 |
| 45 | 1 | 23 | 192 | 864 | 2611 | 5942 | 11044 | 17674 | 25331 | 33401 | 41373 |
| 46 | 1 | 24 | 200 | 920 | 2818 | 6510 | 12241 | 19805 | 28629 | 38047 | 47420 |
| 47 | 1 | 24 | 208 | 972 | 3034 | 7104 | 13534 | 22122 | 32278 | 43214 | 54218 |
| 48 | 1 | 25 | 217 | 1033 | 3266 | 7760 | 14950 | 24699 | 36347 | 49037 | 61903 |
| 49 | 1 | 25 | 225 | 1089 | 3507 | 8442 | 16475 | 27493 | 40831 | 55494 | 70515 |

| *n* | I | II | III | IV | V | VI | VII | VIII | IX | X | XI |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 1 | 26 | 234 | 1154 | 3765 | 9192 | 18138 | 30588 | 45812 | 62740 | 80215 |
| 51 | 1 | 26 | 243 | 1215 | 4033 | 9975 | 19928 | 33940 | 51294 | 70760 | 91058 |
| 52 | 1 | 27 | 252 | 1285 | 4319 | 10829 | 21873 | 37638 | 57358 | 79725 | 103226 |
| 53 | 1 | 27 | 261 | 1350 | 4616 | 11720 | 23961 | 41635 | 64015 | 89623 | 116792 |
| 54 | 1 | 28 | 271 | 1425 | 4932 | 12692 | 26226 | 46031 | 71362 | 100654 | 131970 |
| 55 | 1 | 28 | 280 | 1495 | 5260 | 13702 | 28652 | 50774 | 79403 | 112804 | 148847 |
| 56 | 1 | 29 | 290 | 1575 | 5608 | 14800 | 31275 | 55974 | 88252 | 126299 | 167672 |
| 57 | 1 | 29 | 300 | 1650 | 5969 | 15944 | 34082 | 61575 | 97922 | 141136 | 188556 |
| 58 | 1 | 30 | 310 | 1735 | 6351 | 17180 | 37108 | 67696 | 108527 | 157564 | 211782 |
| 59 | 1 | 30 | 320 | 1815 | 6747 | 18467 | 40340 | 74280 | 120092 | 175586 | 237489 |
| 60 | 1 | 31 | 331 | 1906 | 7166 | 19858 | 43819 | 81457 | 132751 | 195491 | 266006 |
| 61 | 1 | 31 | 341 | 1991 | 7599 | 21301 | 47527 | 89162 | 146520 | 217280 | 297495 |
| 62 | 1 | 32 | 352 | 2087 | 8056 | 22856 | 51508 | 97539 | 161554 | 241279 | 332337 |
| 63 | 1 | 32 | 363 | 2178 | 8529 | 24473 | 55748 | 106522 | 177884 | 267507 | 370733 |
| 64 | 1 | 33 | 374 | 2280 | 9027 | 26207 | 60289 | 116263 | 195666 | 296320 | 413112 |
| 65 | 1 | 33 | 385 | 2376 | 9542 | 28009 | 65117 | 126692 | 214944 | 327748 | 459718 |
| 66 | 1 | 34 | 397 | 2484 | 10083 | 29941 | 70281 | 137977 | 235899 | 362198 | 511045 |
| 67 | 1 | 34 | 408 | 2586 | 10642 | 31943 | 75762 | 150042 | 258569 | 399705 | 567377 |
| 68 | 1 | 35 | 420 | 2700 | 11229 | 34085 | 81612 | 163069 | 283161 | 440725 | 629281 |
| 69 | 1 | 35 | 432 | 2808 | 11835 | 36308 | 87816 | 176978 | 309729 | 485315 | 697097 |

319. The vertical series of this table, although they are recurring, still have a remarkable connection with the natural numbers, with the triangular, pyramidal and with the following series, so that it will be worth the effort to set out a few. Because indeed from the fraction

$$\frac{1}{(1-x)(1-xx)}$$

the series arises

$$1+x+2x^2+2x^3+3x^4+3x^5+\text{etc}.\ [\text{II}]$$

and hence from the fraction

$$\frac{x}{(1-x)(1-xx)}$$

this series arises

$$x+x^2+2x^3+2x^4+3x^5+3x^6+\text{etc.},\ [\text{II}']$$

if these two series may be added, this series is generated :

$$1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+\text{etc.},\ [\text{III}]$$

which arises through division from the fraction

$$\frac{1+x}{(1-x)(1-xx)}=\frac{1}{(1-x)^2};$$

from which it is apparent the numerical terms of the latter series constitute the series of natural numbers [i.e. the coefficients of the powers from $[\text{III}] =$ the coefficients of the same powers of $[\text{II}] + [\text{II}']$ ]. Hence from the second series of the table by adding the two terms the series of natural numbers is come upon on putting $x = 1$:

$$1+1+2+2+3+3+4+4+5+5+6+6+\text{etc.}$$
$$1+2+3+4+5+6+7+8+9+10+11+12+\text{ etc.}$$

Therefore, in turn, the above series is found from the series of natural numbers by subtracting each of the terms of the above series from the following term of the lower series. [i.e. $[\text{II}] = [\text{III}] - [\text{II}']$ ]

320. The third vertical series arises from the fraction

$$\frac{1}{(1-x)(1-xx)(1-x^3)}.$$

But since there shall be

$$\frac{1}{(1-x)^3} = \frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)}$$

it is evident, if at first three terms of that series may be added, then two terms of this new series may be added anew, which must produce triangular numbers; as that will become apparent from the following scheme:

$$1+1+2+3+4+5+7+8+10+12+14+16+19+\text{etc.}$$
$$1+2+4+6+9+12+16+20+25+30+36+42+49+\text{etc.}$$
$$1+3+6+10+15+21+28+36+45+55+66+78+91+\text{etc.}$$

Moreover, it will be apparent in turn, how the upper series may be elicited from the series of triangular numbers.
[The triangular numbers are of course $1, 3, 6, 10, 15, \ldots\binom{n+1}{2}\ldots\ldots$ ]

321. In a similar manner, because the fourth series is generated from the fraction

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)},$$

there will be

$$\frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)}=\frac{1}{(1-x)^4}.$$

If in the fourth series, the first four terms may be added together, then in the resulting series the terms are added three at a time, and then in that two at a time, a series of pyramidal numbers will be produced, as we can see in the following calculation :

$$\begin{array}{l}
1+1+2+3+5+6+9+11+15+18+23+27+\text{etc.}\\
1+2+4+7+11+16+23+31+41+53+67+83+\text{etc.}\\
1+3+7+13+22+34+50+70+95+125+161+203+\text{etc.}\\
1+4+10+20+35+56+84+120+165+220+286+364+\text{etc.}
\end{array}$$

Moreover in a similar manner the series of the fifth order may be reduced to pyramidal numbers of the second order, the sixth to the third order, and thus henceforth.

322. Therefore in turn from these numbers formed these series themselves, which occur in the table, are able to be formed by operations, which will be at once elucidated.

$$\begin{array}{ll}
1+2+3+4+5+6+7+8+9+10+\text{etc.} & \\
1+1+2+2+3+3+4+4+5+5+\text{etc.} & \text{II}
\end{array}$$

$$\begin{array}{ll}
1+3+6+10+15+21+28+36+45+55+\text{etc.} & \\
1+2+4+6+9+12+16+20+25+30+\text{etc.} & \text{III}\\
1+1+2+3+4+5+7+8+10+12+\text{etc.} &
\end{array}$$

$$\begin{array}{ll}
1+4+10+20+35+56+84+120+165+220+\text{etc.} & \\
1+3+7+13+22+34+50+70+95+125+\text{etc.} & \\
1+2+4+7+11+16+23+31+41+53+\text{etc.} & \text{IV}\\
1+1+2+3+5+6+9+11+15+18+\text{etc.} &
\end{array}$$

$$\begin{array}{ll}
1+5+15+35+70+126+210+330+495+715+\text{etc.} & \\
1+4+11+24+46+80+130+200+295+420+\text{etc.} & \\
1+3+7+14+25+41+64+95+136+189+\text{etc.} & \text{V}\\
1+2+4+7+12+18+27+38+53+71+\text{etc.} & \\
1+1+2+3+5+7+10+13+18+23+\text{etc.} &
\end{array}$$

etc.

In these orders [of series], the first series are the numbers formed, from which each of the terms of the second series may be formed, by subtraction a second order term from the following term of the first order. Then two terms of the third series taken together may be subtracted from the following term of the second, and thus the third series is generated. And in this manner by subtracting further the sum of three terms, four and thus henceforth of terms from the following term of the above series, the remaining series will be formed, finally the series will arrive at that which begins from $1+1+2+\text{etc}$., and this will be the series shown in the table.

323. All the vertical series of the table begin similarly and continually have more common terms ; from which it is understood at infinity these series become congruent among themselves. But the series will be produced, which is generated from this fraction

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\text{ etc.}};$$

which since it shall be recurring, the first denominator must be examined, so that hence the scale of the relation may be obtained. But if moreover the factors of the denominator may be multiplied together continually, it will produce

$$1-x-x^2+x^5+x^7-x^{12}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+\text{etc.};$$

which series if it may be considered carefully, does not depend on other powers of $x$ to be present, except for the exponents of those contained in this formula $\frac{3nn\pm n}{2}$, and if $n$ shall be an odd number, the powers will be negative, but positive if $n$ were an even number.

324. Therefore since the scale of the relation shall be

$$+1,\ +1,\ 0,\ 0,\ -1,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0,\ +1,\ 0,\ 0,\ +1,\ 0,\ 0\ \text{etc.},$$

the recurring series arising from the expansion of the fraction

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\text{ etc.}}$$

will be this :

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 56x^{11}$$
$$+ 77x^{12} + 10x^{13} + 135x^{14} + 176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20}$$
$$+ 792x^{21} + 1002x^{22} + 1255x^{25} + 1575x^{24} + \text{ etc.}$$

Therefore in this series each coefficient indicates, in how many different ways the exponent of *x* may be able to be generated from whole numbers. Thus the number 7 can arise in fifteen ways by addition :

| | | |
|---|---|---|
| $7 = 7$ | $7 = 4 + 2 + 1$ | $7 = 3 + 1 + 1 + 1 + 1$ |
| $7 = 6 + 1$ | $7 = 4 + 1 + 1 + 1$ | $7 = 2 + 2 + 2 + 1$ |
| $7 = 5 + 2$ | $7 = 3 + 3 + 1$ | $7 = 2 + 2 + 1 + 1 + 1$ |
| $7 = 5 + 1 + 1$ | $7 = 3 + 2 + 2$ | $7 = 2 + 1 + 1 + 1 + 1 + 1$ |
| $7 = 4 + 3$ | $7 = 3 + 2 + 1 + 1$ | $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$ |

325. But if moreover this product

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6)\text{etc.}$$

may be expanded out, it will produce the following series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + 10x^{10} + \text{etc.},$$

in which each coefficient indicates, in how many different ways the exponent of *x* may be able to arise through the addition of unequal numbers. Thus the number 9 can be formed in eight different ways from unequal numbers:

| | |
|---|---|
| $9 = 9$ | $9 = 6 + 2 + 1$ |
| $9 = 8 + 1$ | $9 = 5 + 4$ |
| $9 = 7 + 2$ | $9 = 5 + 3 + 1$ |
| $9 = 6 + 3$ | $9 = 4 + 3 + 2$ |

[It is clear from these tables, that for the second generating function, the lesser powers of *x* contributing to $x^n$ are different *per se*; however, in the first generating function, which is inverted, the different factors can contribute more than once to $x^n$ on expansion ; this is not yet a proof of the proposition, but suggestive of one. ]

326. So that we may put in place a comparison between these forms, let

$$P=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\text{ etc.}$$

and

$$Q=(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\text{ etc.};$$

$$PQ=(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\text{ etc.};$$

which factors, since all shall be contained in *P*, *P* may be divided by *PQ;* there will be

$$\tfrac{1}{Q}=(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\text{ etc.};$$

and thus

$$Q=\frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\text{ etc.}};$$

which fraction, if it may be expanded out, will produce a series, in which each coefficient will indicate, in how many different ways the exponent of *x* will be able to be produced from odd numbers. Therefore since this expression shall be equal to that, which we have considered in the preceding paragraph, hence this theorem follows :

*A given number will be able to be formed by addition in as many ways with all the whole numbers unequal to each other, in just as many ways as the same number can be formed by addition from odd numbers only, either equal or unequal.*

327. Therefore since, since as we have seen before, there shall be

$$P=1-x-x^2+x^5+x^7-x^{12}+x^{22}+x^{26}-x^{35}-x^{40}+\text{etc.},$$

on writing *xx* in place of *x* there will be

$$PQ=1-x^2-x^4+x^{10}+x^{14}-x^{24}-x^{30}+x^{44}+x^{52}-\text{etc.}$$

On account of which by dividing this by the former equality there will be

$$Q=\frac{1-x^2-x^4+x^{10}+x^{14}-x^{24}-x^{30}+\text{etc.}}{1-x-x^2+x^5+x^7-x^{12}+x^{22}+x^{26}-\text{etc.}};$$

therefore *Q* will be equally recurring and this series is generated from the series $\frac{1}{P}$ on multipling by

$$1-x^2-x^4+x^{10}+x^{14}-x^{24}-\text{etc.}$$

Evidently, since for § 324 there shall be

$$\tfrac{1}{P} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{etc.},$$

if that may be multiplied by

$$1 - x^2 - x^4 + x^{10} + x^{14} - \text{etc.},$$

it becomes

$$\begin{array}{l} 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{etc.} \\ \quad - x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \text{etc.} \\ \quad\quad - x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \text{etc.} \end{array}$$

or

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \text{etc.} = Q.$$

Hence therefore, if the formation of numbers through the addition of numbers, either of equal or unequal, may be agreed upon, the formation of numbers through the addition of unequal numbers may be deduced, and hence again the formation of numbers through the addition of unequal numbers only.

328. Certain memorable cases remain in this catagory, the expansion of which will not be without some usefulness for inquiring into the nature of numbers. Certainly this expression may be considered

$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})(1 + x^{32}) \text{ etc.},$$

in which the exponents of $x$ are progressing in the square ratio. If this expression is expanded out, this series indeed may be found

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.},$$

because truly there can be doubt, whether this series may progress to infinity by this geometric law, hence we will investigate this series itself. Therefore let

$$P = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16}) \text{ etc.}$$

and the series arising from expansion is put to be

$$P = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \eta x^7 + \theta x^8 + \text{etc.}$$

But it is apparent, if $xx$ is written in place of $x$, then the product is produced

$$(1+xx)\left(1+x^4\right)\left(1+x^8\right)\left(1+x^{16}\right)\left(1+x^{32}\right)\text{ etc.}=\tfrac{P}{1+x}.$$

Therefore with the substitution made in the same series there will be

$$\tfrac{P}{1+x}=1+\alpha x^2+\beta x^4+\gamma x^6+\delta x^8+\varepsilon x^{10}+\zeta x^{12}+\text{etc.}$$

Therefore it will be multiplied by $1+x$ and there will be

$$P=1+x+\alpha x^2+\alpha x^3+\beta x^4+\beta x^5+\gamma x^6+\gamma x^7+\delta x^8+\delta x^9+\text{etc.};$$

which value may be compared with the above value of *P*, and there will be found

$$\alpha=1,\ \beta=\alpha,\ \gamma=\alpha,\ \delta=\beta,\ \varepsilon=\beta,\ \zeta=\gamma,\ \eta=\gamma\text{ etc.};$$

therefore all the coefficients $=1$ and thus the proposed product *P* expanded out will give the geometric series

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\text{ etc}.$$

329. Therefore since here all the individual powers of *x* occur once, the form of the product

$$\left(1+x\right)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)\left(1+x^{16}\right)\left(1+x^{32}\right)\text{ etc.},$$

follows all the whole numbers from the different terms of the squared geometric progression

$$1,\ 2,\ 4,\ 8,\ 16,\ 32\text{ etc}.$$

able to be formed, and thus in a single way.
Note that this property is to be considered in practice. For if weights of $1,\ 2,\ 4,\ 8,\ 16,\ 32\text{ etc}.$ pounds [*lb*] may be had, from these weights alone all the load will be able to be weighed, unless parts of weighs are required. Thus from these ten weights, evidently

1 *lb*,2 *lb*,4 *lb*,8 *lb*,16 *lb*,32 *lb*,64 *lb*,128 *lb*,256 *lb*,512 *lb* ,

all weights as far as to 1024 *lb* are able to be weighed, and if one weight of 1024 *lb* may be added, will be sufficient for all the weights as far as to 2048 *lb* requiring to be weighed.

330.But in practice above it is customary to be weighing with fewer weights, which evidently may be progressing in the triplicate geometric ratio, clearly of $1,\ 3,\ 9,\ 27,\ 81\text{ etc.}$ pounds, equally all weights are able to be weighed, unless there shall be a need for fractional weights. But in this practice the weights must not placed only in one pan of the balance, but in both, as the necessity arises. This practice depends on this basis, as from the terms of the triple geometric progression 1, 3, 9, 27, 81 etc., from the differences always taken through addition and subtraction all the numbers are able to be produced completely ; clearly there will be:

| | | |
|---|---|---|
| $1=1$ | $5=9-3-1$ | $9=9$ |
| $2=3-1$ | $6=9-3$ | $10=9+1$ |
| $3=3$ | $7=9-3+1$ | $11=9+3-1$ |
| $4=3+1$ | $8=9-1$ | $12=9+3$ |
| | etc. | |

331. Towards showing this truth I may consider this infinite product

$$\left(x^{-1}+1+x^{1}\right)\left(x^{-3}+1+x^{3}\right)\left(x^{-9}+1+x^{9}\right)\left(x^{-27}+1+x^{27}\right) \text{ etc.} = P,$$

which expanded out will not give other powers of $x$, unless the exponents of which shall be formed from the numbers 1, 3, 9, 27, 81 etc., either by adding or subtracting. Thus, I shall examine whether truly all the individual powers shall be produced and that once. Let

$$P = \text{etc.} + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^{1} + \beta x^{2} + \gamma x^{3} + \delta x^{4} + \varepsilon x^{5} + \text{etc .}$$

Truly it is clear, if $x^3$ may be written in place of $x$, then

$$\frac{P}{x^{-1}+1+x^{1}} = \text{etc.} + bx^{-6} + ax^{-3} + 1 + \alpha x^{3} + \beta x^{6} + \gamma x^{9} + \text{etc.}$$

is produced. Hence therefore

$$P = \text{etc.} + ax^{-4} + ax^{-3} + ax^{-1} + x^{-1} + 1 + x + \alpha x^{2} + \alpha x^{3} + \alpha x^{4} + \beta x^{5} + \beta x^{6} + \beta x^{7} + \text{etc.},$$

will be found, which expression compared with the assumed will give

$$\alpha = 1,\ \beta = \alpha,\ \gamma = \alpha,\ \delta = \alpha,\ \varepsilon = \beta,\ \zeta = \beta \ \text{ etc..}$$

and

$$a = 1,\ b = a,\ c = a,\ d = a,\ e = b \ \text{ etc..}$$

And thus hence there becomes

$$\begin{aligned} P = 1 + x &+ x^{2} + x^{3} + x^{4} + x^{5} + x^{6} + x^{7} + \text{ etc.} \\ + x^{-1} &+ x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \text{etc.,} \end{aligned}$$

from which it is apparent all the powers of $x$, both positive as well as negative, occur here and thus all the numbers from the terms of the triple geometric progression, either by addition or by subtraction, are able to be formed, and any single number only in one way.

## CAPUT XVI

# DE PARTITIONE NUMERORUM

297. Proposita sit ista expressio

$$\left(1+x^{\alpha}z\right)\left(1+x^{\beta}z\right)\left(1+x^{\gamma}z\right)\left(1+x^{\delta}z\right)\left(1+x^{\varepsilon}z\right) \text{ etc.};$$

quae cuiusmodi induat formam, si per multiplicationem evolvatur, inquiramus.
Ponamus prodire

$$1+Pz+Qz^2+Rz^3+Sz^4+\text{etc.}$$

atque manifestum est $P$ fore summam potestatum

$$x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\varepsilon}+\text{etc.}$$

Deinde $Q$ est summa factorum ex binis potestatibus diversis seu $Q$ erit aggregatum plurium potestatum ipsius $x$, quarum exponentes sunt summae duorum terminorum diversorum huius seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \text{ etc.}$$

Simili modo $R$ erit aggregatum potestatum ipsius $x$, quarum exponentes sunt summae trium terminorum diversorum. Atque $S$ erit aggregatum potestatum ipsius $x$, quarum exponentes sunt summae quatuor terminorum diversorum eiusdem seriei $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc., et ita porro.

298. Singulae hae potestates ipsius $x$, quae in valoribus litterarum $P, Q, R, S$ etc. insunt, unitatem pro coefficiente habebunt, siquidem earum exponentes unico modo ex $\alpha,\ \beta,\ \gamma,\ \delta$ etc. formari queant; sin autem eiusdem potestatis exponens pluribus modis possit esse summa duorum, trium pluriumve terminorum seriei $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc., tum etiam potestas illa coefficientem habebit, qui unitatem toties in se complectatur. Sic si in valore ipsius $Q$ reperiatur $Nx^n$, indicio hoc erit numerum $n$ esse $N$ diversis modis summam duorum terminorum diversorum seriei $\alpha,\ \beta,\ \gamma,\ \delta$ etc. Atque si in evolutione factorum propositorum occurrat terminus $Nx^n z^m$, eius coefficiens $N$ indicabit, quot variis modis numerus $n$ possit esse summa $m$ terminorum diversorum seriei $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta$ etc.

299. Quodsi ergo productum propositum

$$\left(1+x^{\alpha}z\right)\left(1+x^{\beta}z\right)\left(1+x^{\gamma}z\right)\left(1+x^{\delta}z\right) \text{ etc.}$$

per multiplicationem veram evolvatur, ex expressione resultante statim apparebit, quot variis modis datus numerus possit esse summa tot terminorum diversorum seriei $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ etc., quot quis voluerit. Scilicet, si quaeratur, quot variis modis numerus *n* possit esse summa *m* terminorum illius seriei diversorum, in expressione evoluta quaeri debet terminus $x^n z^m$ eiusque coefficiens indicabit numuerum quaesitum.

300. Quo haec fiant planiora, sit propositum hoc productum ex factoribus constans infinitis quod per multiplicationem actualem evolutum dat

$$\begin{aligned}
1 &+z\left(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\text{etc.}\right)\\
&+z^2\left(x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+4x^{10}+5x^{11}+\text{etc.}\right)\\
&+z^3\left(x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+8x^{13}+10x^{14}+\text{etc.}\right)\\
&+z^4\left(x^{10}+x^{11}+2x^{12}+3x^{13}+5x^{14}+6x^{15}+9x^{16}+11x^{17}+15x^{18}+\text{etc.}\right)\\
&+z^5\left(x^{15}+x^{16}+2x^{17}+3x^{18}+5x^{19}+7x^{20}+10x^{21}+13x^{22}+18x^{23}+\text{etc.}\right)\\
&+z^6\left(x^{21}+x^{22}+2x^{23}+3x^{24}+5x^{25}+7x^{26}+11x^{27}+14x^{28}+20x^{29}+\text{etc.}\right)\\
&+z^7\left(x^{28}+x^{29}+2x^{30}+3x^{31}+5x^{32}+7x^{33}+11x^{34}+15x^{35}+21x^{36}+\text{etc.}\right)\\
&+z^8\left(x^{36}+x^{37}+2x^{38}+3x^{39}+5x^{40}+7x^{41}+11x^{42}+15x^{43}+22x^{44}+\text{etc.}\right)\\
&\qquad\text{etc.}
\end{aligned}$$

Ex his ergo seriebus statim definire licet, quot variis modis propositus numerus ex dato terminorum diversorum huius seriei

$$1, 2, 3, 4, 5, 6, 7, 8 \text{ etc.}$$

numero oriri queat. Sic si quaeratur, quot variis modis numerus 35 possit esse summa septem terminorum diversorum seriei 1, 2, 3, 4, 5, 6, 7, 8 etc., quaeratur in serie $z^7$ multiplicante potestas $x^{35}$ eiusque coefficiens 15 indicabit numerum propositum 35 quindecim, variis modis esse summam septem terminorum seriei 1, 2, 3, 4, 5, 6, 7, 8 etc.

301. Quodsi autem ponatur $z=1$ et similes potestates ipsius $x$ in unam summam coniiciantur seu, quod eodem redit, si evolvatur haec expressio infinita

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) \text{ etc.},$$

quo facto orietur haec series

$$x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+\text{etc.},$$

ubi quivis coefficiens indicat, quot variis modis exponens potestatis ipsius $x$ coniunctae ex terminis diversis seriei 1, 2, 3, 4, 5, 6, 7 etc. per additionem emergere possit. Sic apparet numerum 8 sex modis per additionem diversorum numerorum produci, qui sunt

| | |
|---|---|
| 8=8 | 8=5+3 |
| 8=7+1 | 8=5+2+1 |
| 8=6+2 | 8=4+3+1 |

ubi notandum est numerum propositum ipsum simul computari debere, quia numerus terminorum non definitur ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quisque numerus per additionem diversorum numerorum producatur. Conditio autem diversitatis omittetur, si factores illos in denominatorem transponamus. Sit igitur proposita haec expressio

$$\frac{1}{(1-x^\alpha z)(1-x^\beta z)(1-x^\gamma z)(1-x^\delta z)(1-x^\varepsilon z) \text{ etc.}},$$

quae per divisionem evoluta det

$$1+Pz+Qz^2+Rz^3+Sz^4+\text{etc.}$$

Atque manifestum est fore $P$ aggregatum potestatum ipsius $x$, quarum exponentes contineantur in hac serie

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \text{etc.}$$

Deinde $Q$ erit aggregatum potestatum ipsius $x$, quarum exponentes sint summae duorum terminorum huius seriei, sive eorundem sive diversorum. Tum erit $R$ summa potestatum ipsius $x$, quarum exponentes ex additione trium terminorum illius seriei oriantur, et $S$ summa potestatum, quarum exponentes ex additione quatuor terminorum in illa serie contentorum formantur, et ita porro.

303. Si igitur tota expressio per singulos terminos explicetur et termini similes coniunctim exprimantur, intelligetur, quot variis modis propositus numerus $n$ per additionem $m$ terminorum, sive diversorum sive non diversorum, seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \text{etc.}$$

produci queat. Quaeratur scilicet in expressione evoluta terminus $x^n z^m$ eiusque coefficiens, qui sit $N$, ita ut totus terminus sit $= Nx^n z^m$, atque coefficiens $N$ indicabit, quot variis modis numerus $n$ per additionem $m$ terminorum in serie $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc. contentorum produci queat. Hoc igitur pacto quaestio priori, quam ante sumus contemplati, similis resolvetur.

304. Accommodemus haec ad casum inprimis notatu dignum sitque proposita haec expressio

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}},$$

quae per divisionem evoluta dabit

$$\begin{aligned}
1 &+ z\left(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \text{etc.}\right)\\
&+ z^2\left(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + 5x^{10} + \text{etc.}\right)\\
&+ z^3\left(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + \text{etc.}\right)\\
&+ z^4\left(x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10} + 11x^{11} + 15x^{12} + \text{etc.}\right)\\
&+ z^5\left(x^5 + x^6 + 2x^7 + 3x^8 + 5x^9 + 7x^{10} + 10x^{11} + 13x^{12} + 18x^{13} + \text{etc.}\right)\\
&+ z^6\left(x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 7x^{11} + 11x^{12} + 14x^{13} + 20x^{14} + \text{etc.}\right)\\
&+ z^7\left(x^7 + x^8 + 2x^9 + 3x^{10} + 5x^{11} + 7x^{12} + 11x^{13} + 15x^{14} + 21x^{15} + \text{etc.}\right)\\
&+ z^8\left(x^8 + x^9 + 2x^{10} + 3x^{11} + 5x^{12} + 7x^{13} + 11x^{14} + 15x^{15} + 22x^{16} + \text{etc.}\right)\\
&\qquad\qquad\text{etc.}
\end{aligned}$$

Ex his ergo seriebus statim definire licet, quot variis modis propositus numerus per additionem ex dato terminorum huius seriei 1, 2, 3, 4, 5, 6, 7 etc. numero produci queat. Sic si quaeratur, quot variis modis numerus 13 oriri possit per additionem quinque numerorum integrorum, spectari debebit terminus $x^{13}z^5$, cuius coefficiens 13 indicat numerum propositum 13 ex quinque numerorum additione octodecim modis oriri posse.

305. Si ponatur $z=1$ atque similes potestates ipsius $x$ coniunctim exprimantur, haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\text{ etc.}}$$

evolvetur in hanc seriem

$$1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\text{ etc.};$$

in qua quilibet coefficiens indicat, quot variis modis exponens potestatis adiunctae per additionem produci queat ex numeris integris, sive aequalibus sive inaequalibus. Scilicet ex termino $11x^6$ cognoscitur numerum 6 undecim modis per additionem numerorum integrorum produci posse, qui sunt

$$\begin{array}{l|l} 6=6 & 6=3+1+1+1 \\ 6=5+1 & 6=2+2+2 \\ 6=4+2 & 6=2+2+1+1 \\ 6=4+1+1 & 6=2+1+1+1+1 \\ 6=3+3 & 6=1+1+1+1+1+1 \\ 6=3+2+1 & \end{array}$$

ubi quoque notari debet ipsum numerum propositum, cum in serie numerorum 1, 2, 3, 4, 5, 6 etc. proposita contineatur, unum modum praebere.

306. His in genere expositis diligentius inquiramus in modum hanc compositionum multitudinem inveniendi. Ac primo quidem consideremus eam ex numeris integris compositionem, in qua numeri tantum diversi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem proposita haec expressio

$$(1+xz)(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)\text{ etc.},$$

quae evoluta et secundum potestates ipsius $z$ digesta praebeat

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\text{etc.},$$

ubi methodus desideratur has ipsius $x$ functiones $P$, $Q$, $R$, $S$, $T$ etc. expedite inveniendi; hoc enim pacto quaestioni propositae convenientissime satisfiet.

307. Patet autem, si loco $z$ ponatur $xz$, prodire

$$\left(1+x^2z\right)\left(1+x^3z\right)\left(1+x^4z\right)\left(1+x^5z\right)\text{etc.}=\tfrac{Z}{1+xz}.$$

Ergo posito $xz$ loco $z$ valor producti, qui erat $Z$, abibit in $\tfrac{Z}{1+xz}$ ; sicque, cum sit

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\text{etc.},$$

erit

$$\tfrac{Z}{1+xz}=1+Pxz+Qx^2z^2+Rx^3z^3+Sx^4z^4+\text{etc.}$$

Multiplicetur ergo actu per $1+xz$ atque prodibit

$$\begin{aligned}Z=1+Pxz+Qx^2z^2+Rx^3z^3+Sx^4z^4+\text{etc.}\\ +\quad xz+Px^2z^2+Qx^3z^3+Rx^4z^4+\text{etc.},\end{aligned}$$

qui valor ipsius $Z$ cum superiori comparatus dabit

$$P=\tfrac{x}{1-x},\quad Q=\tfrac{Px^2}{1-x^2},\quad R=\tfrac{Qx^3}{1-x^3},\quad S=\tfrac{Rx^4}{1-x^4}\quad \text{etc.}$$

Sequentes ergo pro $P$, $Q$, $R$, $S$ etc. obtinentur valores:

$$P=\tfrac{x}{1-x},$$
$$Q=\frac{x^3}{(1-x)\left(1-x^2\right)},$$
$$R=\frac{x^6}{(1-x)\left(1-x^2\right)\left(1-x^3\right)},$$
$$S=\frac{x^{10}}{(1-x)\left(1-x^2\right)\left(1-x^3\right)\left(1-x^4\right)},$$
$$T=\frac{x^{15}}{(1-x)\left(1-x^2\right)\left(1-x^3\right)\left(1-x^4\right)\left(1-x^5\right)}$$
etc.

308. Sic igitur seorsim unamquamque seriem potestatum ipsius $x$ exhibere possumus, ex qua definire licet, quot variis modis propositus numerus ex dato partium integrarum numero per additionem formari possit. Manifestum autem porro est has singulas series

esse recurrentes, quia ex evolutione functionis fractae ipsius $x$ nascuntur. Prima scilicet expressio

$$P = \frac{x}{1-x}$$

dat seriem geometricam

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \text{etc.},$$

ex qua quidem manifestum est quemvis numerum semel in serie numerorum integrorum contineri.

309. Expressio secunda

$$Q = \frac{x^3}{(1-x)(1-x^2)}$$

dat hanc seriem

$$x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \text{etc.},$$

in qua cuiusvis termini coefficiens indicat, quot modis exponens ipsius $x$ in duas partes inaequales dispertiri possit. Sic terminus $4x^9$ indicat numerum 9 quatuor modis in duas partes inaequales secari posse. Quodsi hanc seriem per $x^3$ dividamus, prodibit series, quam praebet ista fractio

$$\frac{1}{(1-x)(1-x^2)},$$

quae erit

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \text{etc.},$$

cuius terminus generalis sit $= Nx^n$; atque ex genesi huius seriei intelligitur coefficientem $N$ indicare, quot variis modis exponens $n$ ex numeris 1 et 2 per additionem nasci queat. Cum igitur prioris seriei terminus generalis sit $= Nx^{n+3}$, deducitur hinc istud theorema:

*Quot variis modis numerus n per additionem ex numeris* 1 *et* 2 *produci potest, totidem variis modis numerus* $n+3$ *in duas partes inaequales secari poterit.*

310. Expressio tertia

$$\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$

in seriem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \text{etc.},$$

in qua cuiusvis termini coefficiens indicat, quot variis modis exponens potestatis $x$ adiunctae in tres partes inaequales dispertiri possit. Quodsi autem haec fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

evolvatur, prodibit haec series

$$1+x+2x^2+3x^3+4x^4+5x^5+7x^6+8x^7+\text{etc.};$$

cuius terminus generalis si ponatur $=Nx^n$, coefficiens $N$ indicabit, quot variis modis numerus $n$ ex numeris 1, 2, 3 per additionem produci possit. Cum igitur prioris seriei terminus generalis sit $Nx^{n+6}$, sequetur hinc istud theorema:

*Quot variis modis numerus n per additionem ex numeris* 1, 2, 3 *produci potest, totidem variis modis numerus* $n+6$ *in tres partes inaequales secari poterit.*

311. Expressio quarta

$$\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

in seriem recurrentem evoluta dabit

$$x^{10}+x^{11}+2x^{12}+3x^{13}+5x^{14}+6x^{15}+9x^{16}+11x^{17}+\text{ etc.},$$

in qua cuiusvis termini coefficiens indicabit, quot variis modis exponens potestatis $x$ adiunctae in quatuor partes inaequales dispertiri possit. Quodsi autem haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

evolvatur, prodibit superior series per $x^{10}$ divisa, nempe

$$1+x+2x^2+3x^3+5x^4+6x^5+9x^6+11x^7+\text{ etc.},$$

cuius terminum generalem ponamus $=Nx^n$; atque hinc patebit coefficientem $N$ indicare, quot variis modis numerus $n$ per additionem oriri possit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris seriei terminus generalis futurus sit $=Nx^{n+10}$, deducitur hoc theorema:

*Quot variis modis numerus n per additionem produci potest ex numeris* 1, 2, 3, 4, *totidem variis modis numerus* $n+10$ *in quatuor partes inaequales secari poterit.*

312. Generaliter ergo, si haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

in seriem evolvatur eiusque terminus generalis fuerit $= Nx^n$, coefficiens $N$ indicabit, quot variis modis numerus $n$ per additionem produci possit ex his numeris 1, 2, 3, 4, . . . $m$. Quodsi autem haec expressio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

in seriem evolvatur, erit eius terminus generalis $= Nx^{n+\frac{m(m+1)}{2}}$ atque hic coefficients $N$ indicat,quot variis modis numerus $n+\frac{m(m+1)}{1\cdot 2}$ in $m$ partes inaequales secari possit, unde hoc habetur theorema:

*Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, ... m, totidem modis numerus* $n+\frac{m(m+1)}{1\cdot 2}$ *in m partes inaequales secari potent.*

313. Ex posita partitione numerorum in partes inaequales perpendamus quoque partitionem in partes, ubi aequalitas partium non excluditur; quae partitio ex hac expressione originem habet

$$Z=\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}}.$$

Ponamus evolutione per divisionem instituta prodire

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\text{etc.}$$

Perspicuum autem est, si loco $z$ ponatur $xz$, prodire

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}}=(1-xz)Z.$$

Facta ergo in serie evoluta eadem mutatione fiet

$$(1-xz)Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \text{etc.}$$

Multiplicetur ergo superior series pariter per $(1-xz)$ eritque

$$\begin{aligned}(1-xz)Z = 1 + Pz + {} & Qz^2 + Rz^3 + Sz^4 + \text{etc.}\\ -xz - {} & Pxz^2 - Qxz^3 - Rxz^4 - \text{etc.}\end{aligned}$$

Comparatione ergo instituta orietur

$$P = \tfrac{x}{1-x}, \quad Q = \tfrac{Px}{1-x^2}, \quad R = \tfrac{Qx}{1-x^3}, \quad S = \tfrac{Rx}{1-x^4} \text{ etc.},$$

unde pro *P, Q, R, S* etc. sequentes valores proveniunt:

$$P = \tfrac{x}{1-x},$$
$$Q = \frac{x^2}{(1-x)(1-x^2)},$$
$$R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)},$$
$$S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$
$$\text{etc.}$$

314. Expressiones istae a superioribus aliter non discrepant, nisi quod numeratores hic minores habeant exponentes quam casu praecedente. Atque hanc ob rem series, quae per evolutionem nascuntur, ratione coefficientium omnino convenient, quae convenientia iam ex comparatione § 300 et § 304 perspicitur, nunc vero demum eius ratio intelligitur. Hinc ergo omnino similia theoremata eonsequentur, quae sunt:

*Quot variis modis numerus n per additionem produci potest ex numeris* 1, 2, *totidem modis numerus* $n+2$ *in duas partes dispertiri poterit.*
*Quot variis modis numerus n per additionem produci potest ex numeris* 1, 2, 3, *totidem modis numerus* $n+3$ *in tres partes dispertiri poterit.*
*Quot variis modis numerus n per additionem produci potest ex numeris* 1, 2, 3, 4, *totidem modis numerus* $n+4$ *in quatuor partes dispertiri poterit.*

Atque generaliter habebitur hoc theorema:
*Quot variis modis numerus n per additionem produci potest ex numeris* 1, 2, 3, ... *m*, *totidem· modis numerus* $n+m$ *in m partes dispertiri poterit.*

315. Sive ergo quaeratur, quot modis datus numerus in *m* partes inaequales, sive in *m* partes aequalibus non exclusis dispertiri possit, utraque quaestio resolvetur, si cognoscatur, quot modis quisque numerus per additionem produci possit ex numeris 1, 2, 3, 4, . . . *m,* quemadmodum hoc patebit ex sequentibus theorematis, quae ex superioribus sunt derivata:

*Numerus n tot modis in m partes inaequales dispertiri potest, quot modis numerus* $n-\frac{m(m+1)}{2}$ *per additionem produci potest ex numeris* 1, 2, 3, 4, ... *m.*

*Numerus n tot modis in m partes, sive aequales sive inaequales, dispertiri potest, quot modis numerus* $n-m$ *per additionem produci potest ex numeris* 1, 2, 3, ... *m.*

Hinc porro sequuntur haec theoremata:

*Numerus n totidem modis in m partes inaequales secari potest, quot modis numerus* $n-\frac{m(m+1)}{2}$ *in m partes, sive aequales sive inaequales, dispertitur.*

*Numerus n totidem modis in m partes, sive inaequales sive aequales, secari potest, quot modis numerus* $n+\frac{m(m+1)}{2}$ *in m partes inaequales dispertirit potest.*

316. Per formationem autem serierum recurrentium inveniri poterit, quot variis modis datus numerus *n* per additionem produci possit ex numeris 1, 2, 3, .... *m.* Ad hoc enim inveniendum evolvi debebit fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

atque series recurrens continuari debebit usque ad terminum $Nx^n$, cuius coefficiens *N* indicabit, quot modis numerus *n* per additionem produci possit ex numeris 1, 2, 3, 4, . . . *m.* At vero hic solvendi modus non parum habebit difficultatis, si numeri *m* et *n* sint modice magni; scala enim relationis, quam praebet denominator per multiplicationem evolutus, ex pluribus terminis constat, unde operosum erit seriem ad plures terminos continuare.

317. Haec autem disquisitio minus erit molesta, si casus simpliciores primum expediantur; ex his enim facile erit ad casus magis compositos progredi. Sit seriei, quae ex hac fractione oritur

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

terminus generalis $= Nx^n$; at seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

ortae terminus generalis sit $Mx^n$, ubi coefficiens $M$ indicabit, quot variis modis numerus $n-m$ per additionem produci possit ex numeris 1, 2, 3, ... *m*. Subtrahatur posterior expressio a priori ac remanebit

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^{m-1})}$$

atque manifestum est seriei hinc ortae terminum generalem futurum esse $(N-M)x^n$; quare coefficiens $N-M$ indicabit, quot variis modis numerus *n* per additionem produci possit ex numeris 1, 2, 3, ... *m* - 1.

318. Hinc ergo sequentem regulam nanciscimur:
*Sit L numerus modorum, quibus numerus n per additionem produci potest ex numeris* $1, 2, 3, \ldots m-1$, *sit M numerus modorum, quibus numerus* $n-m$ *per additionem produci potest ex numeris* 1, 2, 3, ... *m, sitque N numerus modorum, quibus numerus n per additionem produci potest ex numeris* 1, 2, 3, ... *m*; *his positis erit, ut vidimus,*

$$L = N - M$$

*ideoque*

$$N = L + M .$$

Quodsi ergo iam invenerimus, quot variis modis numeri *n* et $n-m$ per additionem produci queant, ille ex numeris $1, 2, 3, \ldots m-1$, hic vero ex numeris 1, 2, 3, ... *m*, hinc addenda cognoscemus, quot variis modis numerus *n* per additionem produci queat ex numeris 1, 2, 3, ... *m*. Ope huius theorematis a casibus simplicioribus, qui nihil habent difficultatis, continuo ad magis compositos progredi licebit hocque modo tabula hic annexa est computata, cuius usus ita se habet:
Si quaeratur, quot variis modis numerus 50 in 7 partes inaequales dispertiri possit, sumatur in prima columna verticali numerus $50-\frac{7\cdot 8}{2}=22$, in horizontali autem suprema numerus romanus VII; atque numerus in angulo positus 522 indicabit modorum numerum quaesitum. Sin autem quaeratur, quot variis modis numerus 50 in 7 partes, sive aequales sive inaequales, dispertiri possit, in prima columna verticali sumatur numerus $50-7=43$, cui in columna septima respondebit numerus quaesitus 8946.

TABULA.

| *n* | I | II | III | IV | V | VI | VII | VIII | IX | X | XI |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 3 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 5 | 1 | 3 | 5 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 6 | 1 | 4 | 7 | 9 | 10 | 11 | 11 | 11 | 11 | 11 | 11 |
| 7 | 1 | 4 | 8 | 11 | 13 | 14 | 15 | 15 | 15 | 15 | 15 |
| 8 | 1 | 5 | 10 | 15 | 18 | 20 | 21 | 22 | 22 | 22 | 22 |
| 9 | 1 | 5 | 12 | 18 | 23 | 26 | 28 | 29 | 30 | 30 | 30 |
| 10 | 1 | 6 | 14 | 23 | 30 | 35 | 38 | 40 | 41 | 42 | 42 |
| 11 | 1 | 6 | 16 | 27 | 37 | 44 | 49 | 52 | 54 | 55 | 56 |
| 12 | 1 | 7 | 19 | 34 | 47 | 58 | 65 | 70 | 73 | 75 | 76 |
| 13 | 1 | 7 | 21 | 39 | 57 | 71 | 82 | 89 | 94 | 97 | 99 |
| 14 | 1 | 8 | 24 | 47 | 70 | 90 | 105 | 116 | 123 | 128 | 131 |
| 15 | 1 | 8 | 27 | 54 | 84 | 110 | 131 | 146 | 157 | 164 | 169 |
| 16 | 1 | 9 | 30 | 64 | 101 | 136 | 164 | 186 | 201 | 212 | 219 |
| 17 | 1 | 9 | 33 | 72 | 119 | 163 | 201 | 230 | 252 | 267 | 278 |
| 18 | 1 | 10 | 37 | 84 | 141 | 199 | 248 | 288 | 318 | 340 | 355 |
| 19 | 1 | 10 | 40 | 94 | 164 | 235 | 300 | 352 | 393 | 423 | 445 |
| 20 | 1 | 11 | 44 | 108 | 192 | 282 | 364 | 434 | 488 | 530 | 560 |
| 21 | 1 | 11 | 48 | 120 | 221 | 331 | 436 | 525 | 598 | 653 | 695 |
| 22 | 1 | 12 | 52 | 136 | 255 | 391 | 522 | 638 | 732 | 807 | 863 |
| 23 | 1 | 12 | 56 | 150 | 291 | 454 | 618 | 764 | 887 | 984 | 1060 |
| 24 | 1 | 13 | 61 | 169 | 333 | 532 | 733 | 919 | 1076 | 1204 | 1303 |
| 25 | 1 | 13 | 65 | 185 | 377 | 612 | 860 | 1090 | 1291 | 1455 | 1586 |
| 26 | 1 | 14 | 70 | 206 | 427 | 709 | 1009 | 1297 | 1549 | 1761 | 1930 |
| 27 | 1 | 14 | 75 | 225 | 480 | 811 | 1175 | 1527 | 1845 | 2112 | 2331 |
| 28 | 1 | 15 | 80 | 249 | 540 | 931 | 1367 | 1801 | 2194 | 2534 | 2812 |
| 29 | 1 | 15 | 85 | 270 | 603 | 1057 | 1579 | 2104 | 2592 | 3015 | 3370 |
| 30 | 1 | 16 | 91 | 297 | 674 | 1206 | 1824 | 2462 | 3060 | 3590 | 4035 |
| 31 | 1 | 16 | 96 | 321 | 748 | 1360 | 2093 | 2857 | 3589 | 4242 | 4802 |
| 32 | 1 | 17 | 102 | 351 | 831 | 1540 | 2400 | 3319 | 4206 | 5013 | 5708 |
| 33 | 1 | 17 | 108 | 378 | 918 | 1729 | 2738 | 3828 | 4904 | 5888 | 6751 |
| 34 | 1 | 18 | 114 | 411 | 1014 | 1945 | 3120 | 4417 | 5708 | 6912 | 7972 |
| 35 | 1 | 18 | 120 | 441 | 1115 | 2172 | 3539 | 5066 | 6615 | 8070 | 9373 |
| 36 | 1 | 19 | 127 | 478 | 1226 | 2432 | 4011 | 5812 | 7657 | 9418 | 11004 |
| 37 | 1 | 19 | 133 | 511 | 1342 | 2702 | 4526 | 6630 | 8824 | 10936 | 12866 |
| 38 | 1 | 20 | 140 | 551 | 1469 | 3009 | 5102 | 7564 | 10156 | 12690 | 15021 |
| 39 | 1 | 20 | 147 | 588 | 1602 | 3331 | 5731 | 8588 | 11648 | 14663 | 17475 |
| 40 | 1 | 21 | 154 | 632 | 1747 | 3692 | 6430 | 9749 | 13338 | 16928 | 20298 |
| 41 | 1 | 21 | 161 | 672 | 1898 | 4070 | 7190 | 11018 | 15224 | 19466 | 23501 |
| 42 | 1 | 22 | 169 | 720 | 2062 | 4494 | 8033 | 12450 | 17354 | 22367 | 27169 |
| 43 | 1 | 22 | 176 | 764 | 2233 | 4935 | 8946 | 14012 | 19720 | 25608 | 31316 |
| 44 | 1 | 23 | 184 | 816 | 2418 | 5427 | 9953 | 15765 | 22380 | 29292 | 36043 |
| 45 | 1 | 23 | 192 | 864 | 2611 | 5942 | 11044 | 17674 | 25331 | 33401 | 41373 |
| 46 | 1 | 24 | 200 | 920 | 2818 | 6510 | 12241 | 19805 | 28629 | 38047 | 47420 |
| 47 | 1 | 24 | 208 | 972 | 3034 | 7104 | 13534 | 22122 | 32278 | 43214 | 54218 |
| 48 | 1 | 25 | 217 | 1033 | 3266 | 7760 | 14950 | 24699 | 36347 | 49037 | 61903 |
| 49 | 1 | 25 | 225 | 1089 | 3507 | 8442 | 16475 | 27493 | 40831 | 55494 | 70515 |
| 50 | 1 | 26 | 234 | 1154 | 3765 | 9192 | 18138 | 30588 | 45812 | 62740 | 80215 |
| 51 | 1 | 26 | 243 | 1215 | 4033 | 9975 | 19928 | 33940 | 51294 | 70760 | 91058 |
| 52 | 1 | 27 | 252 | 1285 | 4319 | 10829 | 21873 | 37638 | 57358 | 79725 | 103226 |
| 53 | 1 | 27 | 261 | 1350 | 4616 | 11720 | 23961 | 41635 | 64015 | 89623 | 116792 |
| 54 | 1 | 28 | 271 | 1425 | 4932 | 12692 | 26226 | 46031 | 71362 | 100654 | 131970 |
| 55 | 1 | 28 | 280 | 1495 | 5260 | 13702 | 28652 | 50774 | 79403 | 112804 | 148847 |
| 56 | 1 | 29 | 290 | 1575 | 5608 | 14800 | 31275 | 55974 | 88252 | 126299 | 167672 |
| 57 | 1 | 29 | 300 | 1650 | 5969 | 15944 | 34082 | 61575 | 97922 | 141136 | 188556 |
| 58 | 1 | 30 | 310 | 1735 | 6351 | 17180 | 37108 | 67696 | 108527 | 157564 | 211782 |
| 59 | 1 | 30 | 320 | 1815 | 6747 | 18467 | 40340 | 74280 | 120092 | 175586 | 237489 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 60 | 1 | 31 | 331 | 1906 | 7166 | 19858 | 43819 | 81457 | 132751 | 195491 | 266006 |
| 61 | 1 | 31 | 341 | 1991 | 7599 | 21301 | 47527 | 89162 | 146520 | 217280 | 297495 |
| 62 | 1 | 32 | 352 | 2087 | 8056 | 22856 | 51508 | 97539 | 161554 | 241279 | 332337 |
| 63 | 1 | 32 | 363 | 2178 | 8529 | 24473 | 55748 | 106522 | 177884 | 267507 | 370733 |
| 64 | 1 | 33 | 374 | 2280 | 9027 | 26207 | 60289 | 116263 | 195666 | 296320 | 413112 |
| 65 | 1 | 33 | 385 | 2376 | 9542 | 28009 | 65117 | 126692 | 214944 | 327748 | 459718 |
| 66 | 1 | 34 | 397 | 2484 | 10083 | 29941 | 70281 | 137977 | 235899 | 362198 | 511045 |
| 67 | 1 | 34 | 408 | 2586 | 10642 | 31943 | 75762 | 150042 | 258569 | 399705 | 567377 |
| 68 | 1 | 35 | 420 | 2700 | 11229 | 34085 | 81612 | 163069 | 283161 | 440725 | 629281 |
| 69 | 1 | 35 | 432 | 2808 | 11835 | 36308 | 87816 | 176978 | 309729 | 485315 | 697097 |

319. Series huius tabulae verticales, etsi sunt recurrentes, tamen ingentem habent connexionem cum numeris naturalibus, trigonalibus, pyramidalibus et sequentibus, quam paucis exponere operae pretium erit. Quoniam enim ex fractione

$$\frac{1}{(1-x)(1-xx)}$$

oritur series

$$1+x+2x^2+2x^3+3x^4+3x^5+\text{etc}.$$

ac proinde ex fractione

$$\frac{x}{(1-x)(1-xx)}$$

haec

$$x+x^2+2x^3+2x^4+3x^5+3x^6+\text{etc.},$$

si duae hae series addantur, nascitur ista

$$1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+\text{etc.},$$

quae per divisionem oritur ex fractione

$$\frac{1+x}{(1-x)(1-xx)}=\frac{1}{(1-x)^2};$$

unde patet serie postremae terminos numericos seriem numerorum naturalium constituere. Hinc ex serie tabulae secunda addendo binos terminos proveniet series numerorum naturalium posito $x=1$:

$$1+\ 1\ +2+\ 2+\ 3+\ 3+\ 4\ +4+\ 5\ +\ \ 5+\ \ 6\ +\ \ 6\ +\text{etc.}$$
$$1+2+3+4+5+6+7+8+9+10+11+12+\ \text{etc.}$$

Vicissim ergo ex serie numerorum naturalium superior invenitur subtrahendo quemque terminum seriei superioris a termino inferioris sequente.

320. Series verticalis tertia oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)}.$$

Cum autem sit

$$\frac{1}{(1-x)^3}=\frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)}$$

manifestum est, si primo seriei illius terni termini addantur, tum bini huius novae seriei denuo addantur, prodire debere numeros trigonales; id quod ex schemate sequente apparebit:

$$\begin{array}{l}
1+1+2+3+4+5+7+8+10+12+14+16+19+\text{etc.}\\
1+2+4+6+9+12+16+20+25+30+36+42+49+\text{etc.}\\
1+3+6+10+15+21+28+36+45+55+66+78+91+\text{etc.}
\end{array}$$

Vicissim autem apparet, quomodo ex serie trigonalium erui debeat series superior.

321. Simili modo, quia series quarta oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)},$$

erit

$$\frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)}=\frac{1}{(1-x)^4}.$$

Si in serie quarta primum quaterni termini addantur, tum in serie resultante terni, denique in hac bini, prodibit series numerorum pyramidalium, uti ex sequenti calculo videre licet:

$$\begin{array}{l}
1+1+2+3+5+6+9+11+15+18+23+27+\text{etc.}\\
1+2+4+7+11+16+23+31+41+53+67+83+\text{etc.}\\
1+3+7+13+22+34+50+70+95+125+161+203+\text{etc.}\\
1+4+10+20+35+56+84+120+165+220+286+364+\text{etc.}
\end{array}$$

Simili autem modo series quinta deducet ad numeros pyramidales secundi ordinis, sexta ad tertii ordinis, et ita porro.

322. Vicissim igitur ex numeris figuratis illae ipsae series, quae in tabula occurrunt, formari poterunt per operationes, quae ex inspectione calculi sequentis sponte elucebunt.

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + etc.
1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + etc. II

1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 +etc.
1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + etc. III
1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + etc.

1 + 4 + 10 + 20 + 35 +56 + 84 + 120+ 165 + 220 + etc.
1 + 3 + 7 +13 + 22 +34 + 50 + 70 + 95 + 125 + etc.
1 + 2 + 4 + 7 + 11 +16 + 23 + 31 + 41 + 53 + etc. IV
1+ 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + etc.

1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + 495 + 715 + etc.
1 + 4 + 11 + 24 + 46 + 80 + 130 + 200 + 295 + 420 + etc.
1 + 3 + 7 + 14 + 25 + 41 + 64 + 95 + 136 + 189 + etc. V
1 + 2 + 4 + 7 + 12 + 18 + 27 + 38 + 53 + 71 + etc.
1 + 1 + 2 + 3 + 5 + 7 + 10 + 13 + 18 + 23 + etc.
etc.

In his ordinibus primae series sunt numeri figurati, unde subtrahendo quemvis terminum seriei secundae a termino primae sequente formatur series secunda. Tum seriei tertiae bini termini coniunctim subtrahantur a termino sequente seriei secundae sicque oritur series tertia. Hocque modo subtrahendo ulterius summam trium, quatuor et ita porro terminorum a termino superioris seriei sequente formabuntur reliquae series, donec perveniatur ad seriem, quae incipit ab $1+1+2+\text{etc}$ ., haecque erit series in tabula exhibita.

323. Series verticales tabulae omnes similiter incipiunt continuoque plures habent terminos communes; ex quo intelligitur in infinitum has series inter se fore congruentes. Prodibit autem series, quae oritur ex hac fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\text{ etc.}};$$

quae cum sit recurrens, primum denominator spectari debet, ut hinc scala relationis habeatur. Quodsi autem factores denominatoris continuo in se multiplicentur, prodibit

$$1-x-x^2+x^5+x^7-x^{12}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+\text{etc.};$$

quae series si attentius consideretur, aliae potestates ipsius $x$ adesse non deprehenduntur, nisi quarum exponentes contineantur in hac formula $\frac{3nn\pm n}{2}$, atque si $n$ sit numerus impar, potestates erunt negativae, affirmativae autem, si $n$ fuerit numerus par.

324. Cum igitur scala relationis sit

$$+1,\ +1,\ 0,\ 0,\ -1,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0,\ +1,\ 0,\ 0,\ +1,\ 0,\ 0 \text{ etc.},$$

series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\text{ etc.}}$$

oriunda erit haec

$$1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+30x^9+42x^{10}+56x^{11}$$
$$+77x^{12}+10x^{13}+135x^{14}+176x^{15}+231x^{16}+297x^{17}+385x^{18}+490x^{19}+627x^{20}$$
$$+792x^{21}+1002x^{22}+1255x^{25}+1575x^{24}+\text{ etc.}$$

In hac ergo serie coefficiens quisque indicat, quot variis modis exponens ipsius $x$ per additionem ex numeris integris oriri queat. Sic numerus 7 quindecim modis per additionem oriri potest:

| | | |
|---|---|---|
| $7=7$ | $7=4+2+1$ | $7=3+1+1+1+1$ |
| $7=6+1$ | $7=4+1+1+1$ | $7=2+2+2+1$ |
| $7=5+2$ | $7=3+3+1$ | $7=2+2+1+1+1$ |
| $7=5+1+1$ | $7=3+2+2$ | $7=2+1+1+1+1+1$ |
| $7=4+3$ | $7=3+2+1+1$ | $7=1+1+1+1+1+1+1$ |

325. Quodsi autem hoc productum

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\text{etc.}$$

evolvatur, sequens prodibit series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\text{etc.},$$

in qua quisque coefficiens indicat, quot variis modis exponens ipsius $x$ per additionem numerorum inaequalium oriri possit. Sic numerus 9 octo variis modis per additionem ex numeris inaequalibus formari potest:

| | |
|---|---|
| $9=9$ | $9=6+2+1$ |
| $9=8+1$ | $9=5+4$ |
| $9=7+2$ | $9=5+3+1$ |
| $9=6+3$ | $9=4+3+2$ |

326. Ut comparationem inter has formas instituamus, sit

$$P=(1-x)\left(1-x^2\right)\left(1-x^3\right)\left(1-x^4\right)\left(1-x^5\right)\left(1-x^6\right) \text{ etc.}$$

et

$$Q=(1+x)\left(1+x^2\right)\left(1+x^3\right)\left(1+x^4\right)\left(1+x^5\right)\left(1+x^6\right) \text{ etc.};$$

$$PQ=\left(1-x^2\right)\left(1-x^4\right)\left(1-x^6\right)\left(1-x^8\right)\left(1-x^{10}\right)\left(1-x^{12}\right) \text{ etc.};$$

qui factores cum omnes in $P$ contineantur, dividatur $P$ per $PQ$; erit

$$\tfrac{1}{Q}=(1-x)\left(1-x^3\right)\left(1-x^5\right)\left(1-x^7\right)\left(1-x^9\right) \text{ etc.};$$

ideoque

$$Q=\frac{1}{(1-x)\left(1-x^3\right)\left(1-x^5\right)\left(1-x^7\right)\left(1-x^9\right) \text{ etc.}};$$

quae fractio si evolvatur, prodibit series, in qua quisque coefficiens indicabit, quot variis modis exponens ipsius $x$ per additionem ex numeris imparibus produci possit. Cum igitur haec expressio aequalis sit illi, quam in paragrapho praecedente contemplati sumus, sequitur hinc istud theorema:

*Quot modis datus numerus per additionem formari potest ex omnibus numeris integris inter se inaequalibus, totidmn modis idem numerus formari potent per additionem ex numeris tantum imparibus, sive aequalibus sive inaequalibus.*

327. Cum igitur, ut ante vidimus, sit

$$P=1-x-x^2+x^5+x^7-x^{12}+x^{22}+x^{26}-x^{35}-x^{40}+\text{etc.},$$

erit scribendo $xx$ loco $x$

$$PQ=1-x^2-x^4+x^{10}+x^{14}-x^{24}-x^{30}+x^{44}+x^{52}-\text{etc.}$$

Quocirca erit hanc per illam dividendo

$$Q=\frac{1-x^2-x^4+x^{10}+x^{14}-x^{24}-x^{30}+\text{etc.}}{1-x-x^2+x^5+x^7-x^{12}+x^{22}+x^{26}-\text{etc.}};$$

erit ergo series $Q$ pariter recurrens atque ex serie $\frac{1}{P}$ oritur hanc per

$$1-x^2-x^4+x^{10}+x^{14}-x^{24}-\text{etc.}$$

multiplicando. Nempe, cum sit ex § 324

$$\tfrac{1}{P}=1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+30x^9+\text{etc.},$$

si is multiplicetur per

$$1-x^2-x^4+x^{10}+x^{14}-\text{etc.},$$

fiat

$$\begin{aligned}1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+30x^9+\text{etc.}\\ -x^2-x^3-2x^4-3x^5-5x^6-7x^7-11x^8-15x^9-\text{etc.}\\ -x^4-x^5-2x^6-3x^7-5x^8-7x^9-\text{etc.}\end{aligned}$$

aut

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+\text{etc.}=Q.$$

Hinc ergo, si formatio numerorum per additionem numerorum, sive aequalium sive inaequalium, constet, deducetur formatio numerorum per additionem numerorum inaequalium hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Restant in hoc genere casus quidam memorabiles, quorum evolutio non omni utilitate carebit in numerorum natura cognoscenda. Consideretur nempe haec expressio

$$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\text{ etc.},$$

in qua exponentes ipsius $x$ in ratione dupla progrediuntur. Haec expressio si evolvatur, reperietur quidem haec series

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\text{etc.},$$

quoniam vero dubium esse potest, utrum haec series in infinitum hac lege geometrica progrediatur, hanc ipsam seriem investigemus. Sit igitur

$$P=(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\text{ etc.}$$

ac ponatur series per evolutionem oriunda

$$P=1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\zeta x^6+\eta x^7+\theta x^8+\text{etc.}$$

Patet autem, si loco $x$ scribatur $xx$, tum prodire productum

$$(1+xx)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\text{ etc.}=\tfrac{P}{1+x}.$$

Facta ergo in serie eadem substitutione erit

$$\tfrac{P}{1+x}=1+\alpha x^2+\beta x^4+\gamma x^6+\delta x^8+\varepsilon x^{10}+\zeta x^{12}+\text{etc.}$$

Multiplicetur ergo per $1+x$ eritque

$$P=1+x+\alpha x^2+\alpha x^3+\beta x^4+\beta x^5+\gamma x^6+\gamma x^7+\delta x^8+\delta x^9+\text{etc.};$$

qui valor ipsius $P$ si cum superiori comparetur, habebitur

$$\alpha=1,\ \beta=\alpha,\ \gamma=\alpha,\ \delta=\beta,\ \varepsilon=\beta,\ \zeta=\gamma,\ \eta=\gamma\text{ etc.};$$

erunt ergo omnes coefficientes $=1$ ideoque productum propositum $P$ evolutum dabit seriem geometricam

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\text{ etc}.$$

329. Cum igitur hic omnes ipsius $x$ potestates singulaeque semel occurrant, ex forma producti

$$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\text{ etc.}$$

sequitur omnem numerum integrum ex terminis progressionis geometricae duplae

$$1,\ 2,\ 4,\ 8,\ 16,\ 32\text{ etc}.$$

diversis per additionem formari posse, hocque unico modo.

Nota est haec proprietas in praxi ponderandi. Si enim habeantur pondera $1, 2, 4, 8, 16, 32$ etc. librarum, his solis ponderibus omnia onera ponderari poterunt, nisi partes librae requirant. Sic his decem ponderibus, nempe

$$1^{\#}, 2^{\#}, 4^{\#}, 8^{\#}, 16^{\#}, 32^{\#}, 64^{\#}, 128^{\#}, 256^{\#}, 512^{\#},$$

omnia pondera usque ad $1024^{\#}$ librari possunt, et si unum pondus $1024^{\#}$ addatur, omnibus oneribus usque ad $2048^{\#}$ ponderandis sufficient.

330. Ostendi autem insuper solet in praxi ponderandi paucioribus ponderibus, quae scilicet in ratione geometrica tripla progrediantur, nempe

$$1, 3, 9, 27, 81 \text{ etc.}$$

librarum, pariter omnia onera ponderari posse, nisi opus sit fractionibus. In hac autem praxi pondera non solum uni lanci, sed ambabus, uti necessitas exigit, imponi debent. Nititur ergo ista praxis hoc fundamento, quod ex terminis progressionis geometricae triplae 1, 3, 9, 27, 81 etc. diversis semper sumendis per additionem ac subtractionem omnes omnino numeri produci queant; erit scilicet

$$\begin{array}{l|l|l}
1=1 & 5=9-3-1 & 9=9 \\
2=3-1 & 6=9-3 & 10=9+1 \\
3=3 & 7=9-3+1 & 11=9+3-1 \\
4=3+1 & 8=9-1 & 12=9+3 \\
 & \quad\text{etc.} &
\end{array}$$

331. Ad hanc veritatem ostendendam considero hoc productum infinitum

$$\left(x^{-1}+1+x^{1}\right)\left(x^{-3}+1+x^{3}\right)\left(x^{-9}+1+x^{9}\right)\left(x^{-27}+1+x^{27}\right) \text{ etc.} = P,$$

quod evolutum alias non dabit potestates ipsius $x$, nisi quarum exponentes formari possint ex numeris 1, 3, 9, 27, 81 etc., sive addenda sive subtrahendo. Num vero omnes potestates prodeant singulaeque semel, sic exploro. Sit

$$P = \text{etc.} + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^{1} + \beta x^{2} + \gamma x^{3} + \delta x^{4} + \varepsilon x^{5} + \text{etc}.$$

Manifestum vero est, si $x^{3}$ loco $x$ scribatur, tum prodire

$$\tfrac{P}{x^{-1}+1+x^{1}} = \text{etc.} + bx^{-6} + bx^{-2} + ax^{-1} + 1 + \alpha x^{3} + \beta x^{6} + \gamma x^{9} + \text{etc.}$$

Hinc igitur reperitur

$$P = \text{etc.} + ax^{-4} + ax^{-3} + ax^{-1} + x^{-1} + 1 + x + \alpha x^{2} + \alpha x^{3} + \alpha x^{4} + \beta x^{5} + \beta x^{6} + \beta x^{7} + \text{etc.},$$

quae expressia cum assumpta comparata dabit

$$\alpha = 1,\ \beta = \alpha,\ \gamma = \alpha,\ \delta = \alpha,\ \varepsilon = \beta,\ \zeta = \beta \ \text{ etc.}.$$

et

$$a = 1,\ b = a,\ c = a,\ d = a,\ e = b \ \text{ etc.}.$$

Hinc itaque erit

$$\begin{aligned} P = 1 + x &+ x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \text{ etc.} \\ &+ x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \text{etc.}, \end{aligned}$$

unde patet omnes ipsius $x$ potestates, tam affirmativas quam negativas, hic occurrere atque adeo omnes numeros ex terminis progressionis geometricae triplae vel addendo vel subtrahendo formari posse et unumquemque numerum unico tantum modo.