CHAPTER XIV

THE MULTIPLICATION AND DIVISION OF ANGLES

234. Let some angle or arc be $=z$ in the circle of which the radius $=1$, of which the sine $=x$, the cosine $=y$, and the tangent $=t$; there will be

$$xx+yy=1 \text{ and } t=\tfrac{x}{y}.$$

Therefore since, as we have seen above, both the sines as well as the cosines of the angles $z,\ 2z,\ 3z,\ 4z,\ 5z$ etc. constitute recurring series, the scale of which relation is *2y*, –1, at first the sines of these arcs thus will be had
[*i.e.* $s_n=2ys_{n-1}-s_{n-2}$, with the appropriate boundary conditions for the sequence s_n;]:

$$\begin{aligned}
&\sin.0z=0,\\
&\sin.1z=x,\\
&\sin.2z=2xy,\\
&\sin.3z=4xy^2-x,\\
&\sin.4z=8xy^3-4xy,\\
&\sin.5z=16xy^4-12xy^2+x,\\
&\sin.6z=32xy^5-32xy^3+6xy,\\
&\sin.\,7z=64xy^6-80xy^4+24xy^2-x,\\
&\sin.8z=128xy^7-192xy^5+80xy^3-8xy.
\end{aligned}$$

Hence it is concluded

$$\sin.nz=x\left\{\begin{array}{c}2^{n-1}y^{n-1}-(n-2)2^{n-3}y^{n-3}\\+\frac{(n-3)(n-4)}{1\cdot2}2^{n-5}y^{n-5}-\frac{(n-4)(n-5)(n-6)}{1\cdot2\cdot3}2^{n-7}y^{n-7}\\+\frac{(n-5)(n-6)(n-7)(n-8)}{1\cdot2\cdot3\cdot4}2^{n-9}y^{n-9}-\text{etc.}\end{array}\right\}.$$

235. If we may put the arc $nz=s$, there becomes

$$\sin.nz=\sin.s=\sin.(\pi-s)=\sin.(2\pi+s)=\sin.(3\pi-s)\ \text{etc.};$$

for all these sines are equal to each other. Hence we will obtain more values for [the sine] *x*, which will be

$$\sin.\tfrac{s}{n},\quad \sin.\tfrac{\pi-s}{n},\quad \sin.\tfrac{2\pi+s}{n},\quad \sin.\tfrac{3\pi-s}{n},\quad \sin.\tfrac{4\pi+s}{n},\quad \text{etc.},$$

which all therefore agree equally with the equation found. Moreover just as many different values will be produced for *x*, as there are units in the number *n*, which therefore will be the roots of the equation found. One must be aware therefore, that equal values from the same do not arise, which comes about, while other expressions only are assumed. Therefore with the roots of the equation found from before, a comparison of the same with the terms of the equation provides noteworthy properties. But because it is required for this equation in which only *x* shall be present as the unknown, the value for *y* must be substituted by $\sqrt{(1-xx)}$; from which a two–fold operation will be put in place, provided *n* were an even or odd number.

236. Let *n* be an odd number; because the difference of the arcs $-z, +z, +3z, +5z$ etc. is *2z*, and the cosine of which $=1-2xx$, the scale of the relation of the progression of the sines will be $2-4xx, -1$.
[*i.e.* $s_n = \left(2-4x^2\right)s_{n-1} - s_{n-2}$, with the appropriate boundary conditions for the sequence s_n;]
Hence there becomes

$$\begin{aligned}
&\sin.-z = -x,\\
&\sin.z = x,\\
&\sin.3z = 3x - 4x^3,\\
&\sin.5z = 5x - 20x^3 + 16x^5,\\
&\sin.7z = 7x - 56x^3 + 112x^5 - 64x^7,\\
&\sin.9z = 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9.
\end{aligned}$$

Therefore

$$\sin nz = nx - \tfrac{n(nn-1)}{1\cdot2\cdot3}x^3 + \tfrac{n(nn-1)(nn-9)}{1\cdot2\cdot3\cdot4\cdot5}x^5 - \tfrac{n(nn-1)(nn-9)(nn-25)}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}x^7 + \text{etc.},$$

if indeed *n* were an odd number. And the roots of this equation are

$$\sin.z,\quad \sin.\left(\tfrac{2\pi}{n}+z\right),\quad \sin.\left(\tfrac{4\pi}{n}+z\right),\quad \sin.\left(\tfrac{6\pi}{n}+z\right),\quad \sin.\left(\tfrac{8\pi}{n}+z\right),\quad \text{etc.},$$

the number of which is *n*.

237. Therefore the factors of this equation

$$0=1-\frac{nx}{\sin.nz}+\frac{n(nn-1)x^3}{1\cdot2\cdot3\sin.nz}-\frac{n(nn-1)(nn-9)x^5}{1\cdot2\cdot3\cdot4\cdot5\sin.nz}+\cdots\pm\frac{2^{n-1}x^n}{\sin.nz}$$

(where the upper sign prevails, if n were deficient from being a multiple of four, otherwise the lower) are

$$\left(1-\frac{x}{\sin.z}\right)\left(1-\frac{x}{\sin.\left(\frac{2\pi}{n}+z\right)}\right)\left(1-\frac{x}{\sin.\left(\frac{4\pi}{n}+z\right)}\right)\quad\text{etc.},$$

from which there may be concluded to be

$$\frac{n}{\sin.nz}=\frac{1}{\sin.z}+\frac{1}{\sin.\left(\frac{2\pi}{n}+z\right)}+\frac{1}{\sin.\left(\frac{4\pi}{n}+z\right)}+\frac{1}{\sin.\left(\frac{6\pi}{n}+z\right)}+\text{etc.},$$

while n terms will be found. Truly then the product of all will be

$$\mp\frac{2^{n-1}}{\sin.nz}=\frac{1}{\sin.z\,\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{4\pi}{n}+z\right)\sin.\left(\frac{6\pi}{n}+z\right)\text{etc.}}$$

or

$$\sin.nz=\mp 2^{n-1}\sin.z\,\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{4\pi}{n}+z\right)\sin.\left(\frac{6\pi}{n}+z\right)\text{etc.}$$

And, because the penultimate term shall be missing, there will be

$$0=\sin.z+\sin.\left(\frac{2\pi}{n}+z\right)+\sin.\left(\frac{4\pi}{n}+z\right)+\sin.\left(\frac{6\pi}{n}+z\right)+\sin.\left(\frac{8\pi}{n}+z\right)+\text{etc.}$$

EXAMPLE 1

Therefore if there were $n=3$, these equalities will be produced

$$\begin{aligned}0&=\sin.z+\sin.(120+z)+\sin.(240+z)\\&=\sin.z+\sin.(60-z)-\sin.(60+z),\end{aligned}$$

$$\frac{3}{\sin.3z}=\frac{1}{\sin.z}+\frac{1}{\sin.(120+z)}+\frac{1}{\sin.(240+z)}=\frac{1}{\sin.z}+\frac{1}{\sin.(60-z)}-\frac{1}{\sin.(60+z)},$$

$$\sin.3z = -4\sin.z\,\sin.(120+z)\,\sin.(240+z) = 4\sin.z\,\sin.(60-z)\,\sin.(60+z).$$

Therefore there will be, as we have noted above now,

$$\sin.(60+z) = \sin.z + \sin.(60-z)$$

and

$$3\operatorname{cosec}.3z = \operatorname{cosec}.z + \operatorname{cosec}.(60-z) - \operatorname{cosec}.(60+z).$$

EXAMPLE 2

We may put $n=5$ and these equations will be produced :

$$0 = \sin.z + \sin.\left(\tfrac{2\pi}{5}+z\right) + \sin.\left(\tfrac{4\pi}{5}+z\right) + \sin.\left(\tfrac{6\pi}{5}+z\right) + \sin.\left(\tfrac{8\pi}{5}+z\right)$$

or

$$0 = \sin.z + \sin.\left(\tfrac{2\pi}{5}+z\right) + \sin.\left(\tfrac{\pi}{5}-z\right) - \sin.\left(\tfrac{\pi}{5}+z\right) - \sin.\left(\tfrac{2\pi}{5}+z\right)$$

or

$$0 = \sin.z + \sin.\left(\tfrac{\pi}{5}-z\right) - \sin.\left(\tfrac{\pi}{5}+z\right) - \sin.\left(\tfrac{2\pi}{5}-z\right) + \sin.\left(\tfrac{2\pi}{5}+z\right).$$

While there will be

$$\frac{5}{\sin.5z} = \frac{1}{\sin.z} + \frac{1}{\sin.\left(\frac{\pi}{5}-z\right)} - \frac{1}{\sin.\left(\frac{\pi}{5}+z\right)} - \frac{1}{\sin.\left(\frac{2\pi}{5}-z\right)} + \frac{1}{\sin.\left(\frac{2\pi}{5}+z\right)},$$

$$\sin.5z = 16\,\sin.z\sin.\left(\tfrac{\pi}{5}-z\right)\sin.\left(\tfrac{\pi}{5}+z\right)\sin.\left(\tfrac{2\pi}{5}-z\right)\sin.\left(\tfrac{2\pi}{5}+z\right).$$

EXAMPLE 3

If we put $n = 2m + 1$, in this manner there will be

$$\begin{aligned}0 = \sin.z + \sin.\left(\tfrac{\pi}{n} - z\right) &- \sin.\left(\tfrac{\pi}{n} + z\right)\\ - \sin.\left(\tfrac{2\pi}{n} - z\right) &+ \sin.\left(\tfrac{2\pi}{n} + z\right)\\ + \sin.\left(\tfrac{3\pi}{n} - z\right) &- \sin.\left(\tfrac{3\pi}{n} + z\right),\\ &.\\ &.\\ \pm \sin.\left(\tfrac{m\pi}{n} - z\right) &\mp \sin.\left(\tfrac{m\pi}{n} + z\right)\end{aligned}$$

where the upper sign prevails, if m may be odd, and the lower if it shall be even.
The other equation will be this :

$$\begin{aligned}\frac{n}{\sin.nz} = \frac{1}{\sin.z} + \frac{1}{\sin.\left(\frac{\pi}{n} - z\right)} &- \frac{1}{\sin.\left(\frac{\pi}{n} + z\right)}\\ - \frac{1}{\sin.\left(\frac{2\pi}{n} - z\right)} &+ \frac{1}{\sin.\left(\frac{2\pi}{n} + z\right)}\\ + \frac{1}{\sin.\left(\frac{3\pi}{n} - z\right)} &- \frac{1}{\sin.\left(\frac{3\pi}{n} + z\right)},\\ &.\\ &.\\ \pm \frac{1}{\sin.\left(\frac{m\pi}{n} - z\right)} &\mp \frac{1}{\sin.\left(\frac{m\pi}{n} + z\right)},\end{aligned}$$

which is changed into convenient cosecants. In the third place this product will be had :

$$\begin{aligned}\sin.nz = 2^{2m}\, sin\,.z\, &\sin.\left(\tfrac{\pi}{n} - z\right)\sin.\left(\tfrac{\pi}{n} + z\right)\\ &\sin.\left(\tfrac{2\pi}{n} - z\right)\sin.\left(\tfrac{2\pi}{n} + z\right)\\ &\sin.\left(\tfrac{3\pi}{n} - z\right)\sin.\left(\tfrac{3\pi}{n} + z\right)\\ &.\\ &.\\ &\sin.\left(\tfrac{m\pi}{n} - z\right)\sin.\left(\tfrac{m\pi}{n} + z\right).\end{aligned}$$

238. Now let n be an even number, and since

$$y=\sqrt{(1-xx)} \text{ and } \cos.2z=1-2xx,$$

thus so that the scale of the sines of the series shall be as before $2-4xx,-1$, there will be

$$\sin.0z=0,$$
$$\sin.2z=2x\sqrt{(1-xx)},$$
$$\sin.4z=\left(4x-8x^3\right)\sqrt{(1-xx)},$$
$$\sin.6z=\left(6x-32x^3+32x^5\right)\sqrt{(1-xx)},$$
$$\sin.8z=\left(8x-80x^3+192x^5-128x^7\right)\sqrt{(1-xx)}$$

and generally

$$\sin.nz=\left\{\begin{array}{l} nx-\frac{n(nn-4)}{1\cdot 2\cdot 3}x^3+\frac{n(nn-4)(nn-16)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 \\ -\frac{n(nn-4)(nn-16)(nn-36)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\cdots\pm 2^{n-1}x^{n-1} \end{array}\right\}\sqrt{(1-xx)}$$

with n denoting any even number.

239. In order to make this equation rational, the squares of each side are taken, and an equation of this form will be produced :

$$(\sin.nz)^2=nnxx+Px^4+Qx^6+\cdots-2^{2n-2}x^{2n}$$

or

$$x^{2n}-\cdots-\frac{nn}{2^{2n-2}}xx+\frac{1}{2^{2n-2}}(\sin.nz)^2=0,$$

the roots of which equation will be both positive as well as negative , evidently the expressions

$$\pm\sin.nz,\pm\sin.\left(\frac{\pi}{n}-z\right),\pm\sin.\left(\frac{2\pi}{n}+z\right),\pm\sin.\left(\frac{3\pi}{n}-z\right),\pm\sin.\left(\frac{4\pi}{n}+z\right) \text{ etc.}$$

taken for all n. Therefore since the final term shall be the product of all these roots, by extracting the square root on both sides , there will be

$$\sin.nz=\pm 2^{n-1}\sin.z\,\sin.\left(\frac{\pi}{n}-z\right)\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{3\pi}{n}-z\right)\cdots;$$

where in which cases whichever sign may prevail will be discerned from the particular case.

EXAMPLE

Moreover on substituting for n successively the numbers 2, 4, 6 etc. and by selecting n the different sines will be

$$\begin{aligned}
\sin.2z &= 2\sin.z\sin.\left(\tfrac{\pi}{n}-z\right),\\
\sin.4z &= 8\sin.z\sin.\left(\tfrac{\pi}{4}-z\right)\sin.\left(\tfrac{\pi}{4}+z\right)\sin.\left(\tfrac{2\pi}{4}-z\right),\\
\sin.6z &= 32\sin.z\sin.\left(\tfrac{\pi}{6}-z\right)\sin.\left(\tfrac{\pi}{6}+z\right)\sin.\left(\tfrac{2\pi}{6}-z\right)\\
&\qquad \sin.\left(\tfrac{\pi}{6}+z\right)\sin.\left(\tfrac{3\pi}{6}-z\right),\\
\sin.8z &= 128\sin.z\sin.\left(\tfrac{\pi}{8}-z\right)\sin.\left(\tfrac{\pi}{8}+z\right)\sin.\left(\tfrac{2\pi}{8}-z\right)\sin.\left(\tfrac{2\pi}{8}+z\right)\\
&\qquad \sin.\left(\tfrac{3\pi}{8}-z\right)\sin.\left(\tfrac{3\pi}{8}+z\right)\sin.\left(\tfrac{4\pi}{8}-z\right).
\end{aligned}$$

240. Therefore it is apparent generally to be

$$\begin{aligned}
\sin.nz = 2^{n-1}\sin.z\sin.&\left(\tfrac{\pi}{n}-z\right)\sin.\left(\tfrac{\pi}{n}+z\right)\\
\sin.&\left(\tfrac{2\pi}{n}-z\right)\sin.\left(\tfrac{2\pi}{n}+z\right)\\
\sin.&\left(\tfrac{3\pi}{n}-z\right)\sin.\left(\tfrac{3\pi}{n}+z\right)\\
&.\\
&.\\
\sin.&\left(\tfrac{\pi}{2}-z\right),
\end{aligned}$$

if n were an even number. But if this may be compared with the above, where n was an odd number, so much likeness is taken to be present, so that each may be allowed to be returned into the one. Therefore there will be, whether n should be an even or odd number,

$$\begin{aligned}\sin.nz = 2^{n-1}\sin.z\,\sin.\left(\tfrac{\pi}{n}-z\right)&\sin.\left(\tfrac{\pi}{n}+z\right)\\ \sin.\left(\tfrac{2\pi}{n}-z\right)&\sin.\left(\tfrac{2\pi}{n}+z\right)\\ \sin.\left(\tfrac{3\pi}{n}-z\right)&\sin.\left(\tfrac{3\pi}{n}+z\right)\\ &\text{etc.,}\end{aligned}$$

while just as many factors may be had, as the number n may contain units.

241. These expressions, by which the sines of multiple angles are set out, are able to bring some use to finding the logarithms of sines of multiple angles and likewise to finding more expressions of the sines by factors, such as we have given above. Moreover there will be :

$$\begin{aligned}\sin.z &= 1\sin.z,\\ \sin.2z &= 2\sin.z\,\sin.\left(\tfrac{\pi}{2}-z\right),\\ \sin.3z &= 4\sin.z\,\sin.\left(\tfrac{\pi}{3}-z\right)\sin.\left(\tfrac{\pi}{3}+z\right),\\ \sin.4z &= 8\sin.z\,\sin.\left(\tfrac{\pi}{4}-z\right)\sin.\left(\tfrac{\pi}{4}+z\right)\sin.\left(\tfrac{2\pi}{4}-z\right),\\ \sin.5z &= 16\sin.z\,\sin.\left(\tfrac{\pi}{5}-z\right)\sin.\left(\tfrac{\pi}{5}+z\right)\sin.\left(\tfrac{2\pi}{5}-z\right)\sin.\left(\tfrac{2\pi}{5}+z\right),\\ \sin.6z &= 32\sin.z\,\sin.\left(\tfrac{\pi}{6}-z\right)\sin.\left(\tfrac{\pi}{6}+z\right)\sin.\left(\tfrac{2\pi}{6}-z\right)\sin.\left(\tfrac{2\pi}{6}+z\right)\sin.\left(\tfrac{3\pi}{6}-z\right),\\ &\text{etc.}\end{aligned}$$

242. Since then there shall be sit $\frac{\sin.2nz}{\sin.nz}=2\cos.nz$, the cosines of multiple angles can be expressed in a similar manner by factors :

$$\begin{aligned}\cos.z &= 1\sin.\left(\tfrac{\pi}{2}-z\right),\\ \cos.2z &= 2\sin.\left(\tfrac{\pi}{4}-z\right)\sin.\left(\tfrac{\pi}{4}+z\right),\\ \cos.3z &= 4\sin.\left(\tfrac{\pi}{6}-z\right)\sin.\left(\tfrac{\pi}{6}+z\right)\sin.\left(\tfrac{3\pi}{6}-z\right),\\ \cos.4z &= 8\sin.\left(\tfrac{\pi}{8}-z\right)\sin.\left(\tfrac{\pi}{8}+z\right)\sin.\left(\tfrac{3\pi}{8}-z\right)\sin.\left(\tfrac{3\pi}{8}+z\right),\\ \cos.5z &= 16\sin.\left(\tfrac{\pi}{10}-z\right)\sin.\left(\tfrac{\pi}{10}+z\right)\sin.\left(\tfrac{3\pi}{10}-z\right)\sin.\left(\tfrac{3\pi}{10}+z\right)\sin.\left(\tfrac{5\pi}{10}-z\right),\end{aligned}$$

and generally

$$\begin{aligned}\cos.nz = 2^{n-1}\sin.\left(\tfrac{\pi}{2n}-z\right)\sin.\left(\tfrac{\pi}{2n}+z\right)\\ \sin.\left(\tfrac{3\pi}{2n}-z\right)\sin.\left(\tfrac{3\pi}{2n}+z\right)\\ \sin.\left(\tfrac{5\pi}{2n}-z\right)\sin.\left(\tfrac{5\pi}{2n}+z\right)\\ \text{etc.,}\end{aligned}$$

as long as it may have just as many factors, as the number n may contain units.

243. The same expressions will be produced from a consideration of the cosines of the arcs of multiple angles. For if there were $\cos.z = y$, there will be as follows :

$$\begin{aligned}\cos.0z &= 1,\\ \cos.1z &= y,\\ \cos.2z &= 2y^2-1,\\ \cos.3z &= 4y^3-3y,\\ \cos.4z &= 8y^4-8y^3+1,\\ \cos.5z &= 16y^5-20y^3+5y,\\ \cos.6z &= 32y^6-48y^4+18y^2-1,\\ \cos.7z &= 64y^7-112y^5+56y^3-7y\end{aligned}$$

and generally

$$\begin{aligned}\cos.nz = 2^{n-1}y^n - \tfrac{n}{1}2^{n-3}y^{n-2} + \tfrac{n(n-3)}{1\cdot 2}2^{n-5}y^{n-4} - \tfrac{n(n-4)(n-5)}{1\cdot 2\cdot 3}2^{n-7}y^{n-6}\\ + \tfrac{n(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4}2^{n-9}y^{n-8} - \text{etc.,}\end{aligned}$$

these roots y of which equation, since there shall be

$$\cos.nz = \cos.(2\pi - nz) = \cos.(2\pi + nz) = \cos.(4\pi \pm nz) = \cos.(6\pi \pm nz) \text{ etc.,}$$

will be

$$\cos.z,\quad \cos.\left(\tfrac{2\pi}{n}\pm z\right),\quad \cos.\left(\tfrac{4\pi}{n}\pm z\right),\quad \cos.\left(\tfrac{6\pi}{n}\pm z\right) \text{ etc.,}$$

of which just as many different formulas are required to be selected for y, as the number given; but the whole number given contains just as many ones as n does.

244. Therefore at first it is apparent on account of the second term missing with the exception of the first case $n = 1$, that the sum of all the roots $= 0$. Therefore there will be

$$0 = \cos.z + \cos.\left(\tfrac{2\pi}{n} - z\right) + \cos.\left(\tfrac{2\pi}{n} + z\right) + \cos.\left(\tfrac{4\pi}{n} - z\right) + \cos.\left(\tfrac{4\pi}{n} + z\right) + \text{etc.}$$

with just as many terms taken, as there are units in *n*. But this equality presents itself at once, if *n* were an even number, since any term is cancelled by its negative from the other. Therefore we may consider odd numbers with unity excluded and there will be on account of $\cos.v = -\cos.(\pi - v)$

$$0 = \cos.z - \cos.\left(\tfrac{\pi}{3} - z\right) - \cos.\left(\tfrac{\pi}{3} + z\right),$$

$$0 = \cos.z - \cos.\left(\tfrac{\pi}{5} - z\right) - \cos.\left(\tfrac{\pi}{5} + z\right) + \cos.\left(\tfrac{2\pi}{5} - z\right) + \cos.\left(\tfrac{2\pi}{5} + z\right),$$

$$0 = \cos.z - \cos.\left(\tfrac{\pi}{7} - z\right) - \cos.\left(\tfrac{\pi}{7} + z\right) + \cos.\left(\tfrac{2\pi}{7} - z\right) + \cos.\left(\tfrac{2\pi}{7} + z\right) - \cos.\left(\tfrac{3\pi}{7} - z\right) - \cos.\left(\tfrac{3\pi}{7} + z\right)$$

and generally, if *n* were some odd number, there will be

$$0 = \cos.z - \cos.\left(\tfrac{\pi}{n} - z\right) - \cos.\left(\tfrac{\pi}{n} + z\right) + \cos.\left(\tfrac{2\pi}{n} - z\right) + \cos.\left(\tfrac{2\pi}{n} + z\right) - \cos.\left(\tfrac{3\pi}{n} - z\right) - \cos.\left(\tfrac{3\pi}{n} + z\right)$$
$$+ \cos.\left(\tfrac{4\pi}{7} - z\right) + \cos.\left(\tfrac{4\pi}{7} + z\right) - \text{etc.}$$

with just as many terms taken, as the number *n* contains units. Moreover it is necessary that *n* be an odd number greater than one, as we have now indicated.

245. So that the product from all the roots may be reached, indeed these follow from various expressions, according as *n* were a number either odd or unequally even, or equally even. But all are understood from the general expression found, if the individual sines may be changed into cosines. Clearly there will be

$$\cos.z = 1\cos.z,$$

$$\cos.2z = 2\cos.\left(\tfrac{\pi}{4} + z\right)\cos.\left(\tfrac{\pi}{4} - z\right),$$

$$\cos.3z = 4\cos.\left(\tfrac{2\pi}{6} + z\right)\cos.\left(\tfrac{2\pi}{6} - z\right)\cos.z,$$

$$\cos.4z = 8\cos.\left(\tfrac{3\pi}{8} + z\right)\cos.\left(\tfrac{3\pi}{8} - z\right)\cos.\left(\tfrac{\pi}{8} + z\right)\cos.\left(\tfrac{\pi}{8} - z\right),$$

$$\cos.5z = 16\cos.\left(\tfrac{4\pi}{10} + z\right)\cos.\left(\tfrac{4\pi}{10} - z\right)\cos.\left(\tfrac{2\pi}{10} + z\right)\cos.\left(\tfrac{2\pi}{10} - z\right)\cos.z$$

and generally,

$$\begin{aligned}\cos.nz = \ 2^{n-1} & \cos.\left(\tfrac{n-1}{2n}\pi+z\right)\cos.\left(\tfrac{n-1}{2n}\pi-z\right)\\ & \cos.\left(\tfrac{n-3}{2n}\pi+z\right)\cos.\left(\tfrac{n-3}{2n}\pi-z\right)\\ & \cos.\left(\tfrac{n-5}{2n}\pi+z\right)\cos.\left(\tfrac{n-5}{2n}\pi-z\right)\\ & \cos.\left(\tfrac{n-7}{2n}\pi+z\right)\cos.\left(\tfrac{n-7}{2n}\pi-z\right)\\ & \text{etc.}\end{aligned}$$

with as many factors taken, as there shall be units in the number n.

246. Let n be an odd number and the equation starts from unity ; there will be

$$0 = 1 \mp \frac{ny}{\cos.nz} \pm \text{etc.},$$

where the upper sign prevails, if n were an odd number of the form $4m+1$, the lower, if $n = 4m-1$. Hence there will be

$$+\frac{1}{\cos.z} = \frac{1}{\cos.z}$$

$$-\frac{3}{\cos.3z} = \frac{1}{\cos.z} - \frac{1}{\cos.\left(\frac{\pi}{3}-z\right)} - \frac{1}{\cos.\left(\frac{\pi}{3}+z\right)},$$

$$+\frac{5}{\cos.5z} = \frac{1}{\cos.z} - \frac{1}{\cos.\left(\frac{\pi}{5}-z\right)} - \frac{1}{\cos.\left(\frac{\pi}{5}+z\right)} + \frac{1}{\cos.\left(\frac{2\pi}{5}-z\right)} + \frac{1}{\cos.\left(\frac{2\pi}{5}+z\right)}$$

and generally on putting $n = 2m+1$ there will be

$$\begin{aligned}\frac{n}{\cos.nz} = \frac{2m+1}{\cos.(2m+1)z} = & \frac{1}{\cos.\left(\frac{m}{n}\pi+z\right)} + \frac{1}{\cos.\left(\frac{m}{n}\pi-z\right)}\\ & - \frac{1}{\cos.\left(\frac{m-1}{n}\pi+z\right)} - \frac{1}{\cos.\left(\frac{m-1}{n}\pi-z\right)}\\ & + \frac{1}{\cos.\left(\frac{m-2}{n}\pi+z\right)} + \frac{1}{\cos.\left(\frac{m-2}{n}\pi-z\right)}\\ & - \frac{1}{\cos.\left(\frac{m-3}{n}\pi+z\right)} - \frac{1}{\cos.\left(\frac{m-3}{n}\pi-z\right)}\\ & \text{etc.}\end{aligned}$$

with just as many terms taken, as n contains units.

247. Since therefore there shall be $\frac{1}{\cos.v} = \sec.v$, hence the properties may be deduced for the designated secants ; certainly there will be

$$\sec.z = \sec.z$$
$$3\sec.3z = \sec.\left(\tfrac{\pi}{3} + z\right) + \sec.\left(\tfrac{\pi}{3} - z\right) - \sec.\left(\tfrac{0\pi}{3} + z\right),$$
$$5\sec.5z = \sec.\left(\tfrac{2\pi}{5} + z\right) + \sec.\left(\tfrac{2\pi}{5} - z\right) - \sec.\left(\tfrac{\pi}{5} + z\right) - \sec.\left(\tfrac{\pi}{5} - z\right) + \sec.\left(\tfrac{0\pi}{5} + z\right),$$
$$7\sec.7z = \sec.\left(\tfrac{3\pi}{7} + z\right) + \sec.\left(\tfrac{3\pi}{7} - z\right) - \sec.\left(\tfrac{2\pi}{7} + z\right) - \sec.\left(\tfrac{2\pi}{7} - z\right)$$
$$+ \sec.\left(\tfrac{\pi}{7} + z\right) + \sec.\left(\tfrac{\pi}{7} - z\right) - \sec.\left(\tfrac{0\pi}{7} + z\right)$$

and generally on putting $n = 2m+1$ there will be

$$\begin{aligned} n\sec.nz = {} & \sec.\left(\tfrac{m}{n}\pi + z\right) + \sec.\left(\tfrac{m}{n}\pi - z\right) \\ & - \sec.\left(\tfrac{m-1}{n}\pi + z\right) - \sec.\left(\tfrac{m-1}{n}\pi - z\right) \\ & + \sec.\left(\tfrac{m-2}{n}\pi + z\right) + \sec.\left(\tfrac{m-2}{n}\pi - z\right) \\ & - \sec.\left(\tfrac{m-3}{n}\pi + z\right) - \sec.\left(\tfrac{m-3}{n}\pi - z\right) \\ & + \sec.\left(\tfrac{m-4}{n}\pi + z\right) + \sec.\left(\tfrac{m-4}{n}\pi - z\right) \\ & . \\ & . \\ & \pm \sec.z. \qquad \text{etc.} \end{aligned}$$

248. For the cosecants moreover there will be from § 237

$$\operatorname{cosec}.z = \operatorname{cosec}.z$$
$$3\operatorname{cosec}.3z = \operatorname{cosec}.z + \operatorname{cosec}.\left(\tfrac{\pi}{3} - z\right) - \operatorname{cosec}.\left(\tfrac{\pi}{3} + z\right),$$
$$5\operatorname{cosec}.5z = \operatorname{cosec}.z + \operatorname{cosec}.\left(\tfrac{\pi}{5} - z\right) - \operatorname{cosec}.\left(\tfrac{\pi}{5} + z\right) - \operatorname{cosec}.\left(\tfrac{2\pi}{5} - z\right) + \operatorname{cosec}.\left(\tfrac{2\pi}{5} + z\right),$$
$$7\operatorname{cosec}.7z = \operatorname{cosec}.z + \operatorname{cosec}.\left(\tfrac{\pi}{7} - z\right) - \operatorname{cosec}.\left(\tfrac{\pi}{7} + z\right) - \operatorname{cosec}.\left(\tfrac{2\pi}{7} - z\right) + \operatorname{cosec}.\left(\tfrac{2\pi}{7} + z\right)$$
$$+ \operatorname{cosec}.\left(\tfrac{3\pi}{7} - z\right) - \operatorname{cosec}.\left(\tfrac{3\pi}{7} + z\right)$$

and generally on putting $n = 2m+1$ there will be

$$
\begin{aligned}
n\operatorname{cosec}.nz = \operatorname{cosec}.z &+ \operatorname{cosec}.\left(\tfrac{\pi}{n}-z\right) - \operatorname{cosec}.\left(\tfrac{\pi}{n}+z\right)\\
&- \operatorname{cosec}.\left(\tfrac{2\pi}{n}-z\right) + \operatorname{cosec}.\left(\tfrac{2\pi}{n}+z\right)\\
&+ \operatorname{cosec}.\left(\tfrac{3\pi}{n}-z\right) - \operatorname{cosec}.\left(\tfrac{3\pi}{n}+z\right)\\
&.\\
&.\\
&\pm \operatorname{cosec}.\left(\tfrac{m\pi}{n}-z\right) \pm \operatorname{cosec}.\left(\tfrac{m\pi}{n}+z\right),
\end{aligned}
$$

where the upper sign prevails, if m were an even number, the lower, if m were odd.

249. Since there shall be, as we have seen above,

$$\cos.nz \pm \sqrt{-1}\cdot\sin.nz = \left(\cos.z \pm \sqrt{-1}\cdot\sin.z\right)^n,$$

there will be

$$\cos.nz = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n+\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{2}$$

and

$$\sin.nz = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n-\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{2\sqrt{-1}},$$

therefore

$$\tan.nz = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n-\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n+\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}.$$

We may put

$$\operatorname{tang}.z = \frac{\sin.z}{\cos.z} = t;$$

there will be

$$\tan.nz = \frac{\left(1+t\sqrt{-1}\right)^n-\left(1-t\sqrt{-1}\right)^n}{\left(1+t\sqrt{-1}\right)^n\sqrt{-1}+\left(1-t\sqrt{-1}\right)^n\sqrt{-1}},$$

from which the tangents of multiple angles follow :

$$\begin{aligned} \text{tang.}z &= t, \\ \text{tang.}2z &= \frac{2t}{1-tt}, \\ \text{tang.}3z &= \frac{3t-t^3}{1-3tt}, \\ \text{tang.}4z &= \frac{4t-4t^3}{1-6tt+t^4} \\ \text{tang.}5z &= \frac{5t-10t^3+t^5}{1-10tt+5t^4} \end{aligned}$$

and generally

$$\text{tang.}nz = \frac{nt-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}t^3+\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}t^5-\text{etc.}}{1-\frac{n(n-1)}{1\cdot 2}tt+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}t^4-\text{etc.}}$$

Since now there shall be

$$\text{tang.}nz = \text{tang.}(\pi+nz) = \text{tang.}(2\pi+nz) = \text{tang.}(3\pi+nz) \quad \text{etc.},$$

these will be the values of t or the roots of the equation :

$$\text{tang.}z, \quad \text{tang.}\left(\frac{\pi}{n}+z\right), \quad \text{tang.}\left(\frac{2\pi}{n}+z\right), \quad \text{tang.}\left(\frac{3\pi}{n}+z\right) \quad \text{etc.},$$

the number of which is n.

250. But if the equation should begin from one, there will be

$$0 = 1 - \frac{nt}{\text{tang.}nz} - \frac{n(n-1)tt}{1\cdot 2} + \frac{n(n-1)(n-2)t^3}{1\cdot 2\cdot 3\text{tang.}nz} + \text{etc.}$$

Therefore from a comparison of the coefficients with the roots of the equation there will be

$$\begin{aligned} n\cot.nz = \cot.z + \cot.\left(\frac{\pi}{n}+z\right) + \cot.\left(\frac{2\pi}{n}+z\right) \\ + \cot.\left(\frac{3\pi}{n}+z\right) + \cot.\left(\frac{4\pi}{n}+z\right) + \cdots + \cot.\left(\frac{n-1}{n}\pi+z\right). \end{aligned}$$

From thence the sum of the squares of all these tangents will be

$$= \frac{nn}{(\sin.nz)^2} - n$$

and in a similar manner the higher powers can be defined. But on putting definite numbers in place of n there will be

$$\cot.z = \cot.z,$$
$$2\cot.2z = \cot.z + \cot.\left(\tfrac{\pi}{2}+z\right),$$
$$3\cot.3z = \cot.z + \cot.\left(\tfrac{\pi}{3}+z\right) + \cot.\left(\tfrac{2\pi}{3}+z\right),$$
$$4\cot.4z = \cot.z + \cot.\left(\tfrac{\pi}{4}+z\right) + \cot.\left(\tfrac{2\pi}{4}+z\right) + \cot.\left(\tfrac{3\pi}{4}+z\right),$$
$$5\cot.5z = \cot.z + \cot.\left(\tfrac{\pi}{5}+z\right) + \cot.\left(\tfrac{2\pi}{5}+z\right) + \cot.\left(\tfrac{3\pi}{5}+z\right) + \cot.\left(\tfrac{4\pi}{5}+z\right).$$

251. Truly because there is $\cot.v = -\cot.(\pi - v)$, there will be

$$\cot.z = \cot.z,$$
$$2\cot.2z = \cot.z - \cot.\left(\tfrac{\pi}{2}-z\right),$$
$$3\cot.3z = \cot.z - \cot.\left(\tfrac{\pi}{3}-z\right) + \cot.\left(\tfrac{\pi}{3}+z\right),$$
$$4\cot.4z = \cot.z - \cot.\left(\tfrac{\pi}{4}-z\right) + \cot.\left(\tfrac{\pi}{4}+z\right) - \cot.\left(\tfrac{2\pi}{4}-z\right),$$
$$5\cot.5z = \cot.z - \cot.\left(\tfrac{\pi}{5}-z\right) + \cot.\left(\tfrac{\pi}{5}+z\right) - \cot.\left(\tfrac{2\pi}{5}-z\right) + \cot.\left(\tfrac{2\pi}{5}+z\right).$$

and generally

$$\begin{aligned} n\cot.nz = \cot.z &- \cot.\left(\tfrac{\pi}{n}-z\right) + \cot.\left(\tfrac{\pi}{n}+z\right)\\ &- \cot.\left(\tfrac{2\pi}{n}-z\right) + \cot.\left(\tfrac{2\pi}{n}+z\right)\\ &- \cot.\left(\tfrac{3\pi}{n}-z\right) + \cot.\left(\tfrac{3\pi}{n}+z\right)\\ &- \text{etc.},\end{aligned}$$

while just as many terms may be taken, as the number n contains units.

252. We may begin the equation found from the greatest power, where in the first place the cases are to be distinguished, in which n is either an even or an odd number. Let the number n be odd, $n = 2m+1$; there will be

$$t - \text{tang.}z = 0,$$
$$t^3 - 3tt\,\text{tang.}3z - 3t + \text{tang.}3z = 0,$$
$$t^5 - 5t^4\ \text{tang.}5z - 10t^3 + 10tt\,\text{tang.}5z + 5t - \text{tang.}5z = 0$$

and generally

$$t^n - nt^{n\text{-}1}\text{tang.}nz - \cdots \mp \text{tang.}nz = 0,$$

where the upper sign – prevails, if m shall be an even number, the lower +, if m shall be odd. Therefore there will be, from the coefficient of the second term :

$$\text{tang.}z = \text{tang.}z,$$
$$3\text{tang.}3z = \text{tang.}z + \text{tang.}\left(\tfrac{\pi}{3} + z\right) + \text{tang.}\left(\tfrac{2\pi}{3} + z\right),$$
$$5\text{tang.}5z = \text{tang.}z + \text{tang.}\left(\tfrac{\pi}{5} + z\right) + \text{tang.}\left(\tfrac{2\pi}{5} + z\right) + \text{tang.}\left(\tfrac{3\pi}{5} + z\right) + \text{tang.}\left(\tfrac{4\pi}{5} + z\right)$$
etc.

253. Therefore since there shall be $\text{tang.}v = -\ \text{tang.}(n - v)$, angles greater than a right angle may be reduced to angles less than a right angle, and there will be

$$\text{tang.}z = \text{tang.}z,$$
$$3\text{tang.}3z = \text{tang.}z - \text{tang.}\left(\tfrac{\pi}{3} - z\right) + \text{tang.}\left(\tfrac{\pi}{3} + z\right),$$
$$5\text{tang.}5z = \text{tang.}z - \text{tang.}\left(\tfrac{\pi}{5} - z\right) + \text{tang.}\left(\tfrac{\pi}{5} + z\right) - \text{tang.}\left(\tfrac{2\pi}{5} - z\right) + \text{tang.}\left(\tfrac{2\pi}{5} + z\right),$$
$$7\text{tang.}7z = \text{tang.}z - \text{tang.}\left(\tfrac{\pi}{7} - z\right) + \text{tang.}\left(\tfrac{\pi}{7} + z\right) - \text{tang.}\left(\tfrac{2\pi}{7} - z\right) + \text{tang.}\left(\tfrac{2\pi}{7} + z\right)$$
$$- \text{tang.}\left(\tfrac{3\pi}{7} - z\right) + \text{tang.}\left(\tfrac{3\pi}{7} + z\right)$$

and generally, if $n = 2m + 1$, there will be

$$n\text{tang.}nz = \text{tang.}z - \text{tang.}\left(\tfrac{\pi}{n} - z\right) + \text{tang.}\left(\tfrac{\pi}{n} + z\right) - \text{tang.}\left(\tfrac{2\pi}{n} - z\right) + \text{tang.}\left(\tfrac{2\pi}{n} + z\right)$$
$$- \text{tang.}\left(\tfrac{3\pi}{n} - z\right) + \text{tang.}\left(\tfrac{3\pi}{n} + z\right)$$
$$\vdots$$
$$- \text{tang.}\left(\tfrac{m\pi}{n} - z\right) + \text{tang.}\left(\tfrac{m\pi}{n} + z\right)$$

254.Then truly the product from all these tangents will be $= \text{tang.}nz$,
therefore so that by the number of negative signs alternately even and odd the ambiguities of the above signs are removed. Thus there will be

$$\begin{aligned}\text{tang}.z &= \text{tang}.z,\\ \text{tang}.3z &= \text{tang}.z\ \text{tang}.\left(\tfrac{\pi}{3}-z\right)\text{tang}.\left(\tfrac{\pi}{3}+z\right),\\ \text{tang}.5z &= \text{tang}.z\ \text{tang}.\left(\tfrac{\pi}{5}-z\right)\text{tang}.\left(\tfrac{\pi}{5}+z\right)\text{tang}.\left(\tfrac{2\pi}{5}-z\right)\text{tang}.\left(\tfrac{2\pi}{5}+z\right)\end{aligned}$$

and generally, if $n=2m+1$, there will be

$$\begin{aligned}\text{tang}.nz = \text{tang}.z\ &\text{tang}.\left(\tfrac{\pi}{n}-z\right)\text{tang}.\left(\tfrac{\pi}{n}+z\right)\\ &\text{tang}.\left(\tfrac{2\pi}{n}-z\right)\text{tang}.\left(\tfrac{2\pi}{n}+z\right)\\ &\text{tang}.\left(\tfrac{3\pi}{n}-z\right)\text{tang}.\left(\tfrac{3\pi}{n}+z\right)\\ &.\\ &.\\ &\text{tang}.\left(\tfrac{m\pi}{n}-z\right)\text{tang}.\left(\tfrac{m\pi}{n}+z\right)\end{aligned}$$

255. Now let the number n be even and by beginning from the greatest power there will be

$$\begin{aligned}&tt+2t\cot.2z-1=0,\\ &t^4+4t^3\cot.4z-6tt-4t\cot.4z+1=0\end{aligned}$$

and generally, if $n=2m$, there will be

$$t^n+nt^{n-1}\cot.nz-\cdots\mp 1=0\ ,$$

where the upper sign – prevails, if m shall be an odd number, the lower +, if m shall be even. Therefore by comparing the roots with the coefficient of the second term, there will be

$$\begin{aligned}-2\cot.2z &= \text{tang}.z+\text{tang}.\left(\tfrac{\pi}{2}+z\right),\\ -4\cot.4z &= \text{tang}.z+\text{tang}.\left(\tfrac{\pi}{4}+z\right)+\text{tang}.\left(\tfrac{2\pi}{4}+z\right)+\text{tang}.\left(\tfrac{2\pi}{4}+z\right)+\text{tang}.\left(\tfrac{3\pi}{4}+z\right),\\ -6\cot.6z &= \text{tang}.z+\text{tang}.\left(\tfrac{\pi}{6}+z\right)+\text{tang}.\left(\tfrac{\pi}{6}+z\right)+\text{tang}.\left(\tfrac{2\pi}{6}+z\right)+\text{tang}.\left(\tfrac{3\pi}{6}+z\right)\\ &\quad+\text{tang}.\left(\tfrac{4\pi}{6}+z\right)+\text{tang}.\left(\tfrac{5\pi}{6}+z\right)\\ &\quad\text{etc.}\end{aligned}$$

256. Since there shall be $\text{tang.}v = -\ \text{tang.}(\pi - v)$, the equations will be of the following forms :

$$2\cot.2z = -\text{tang.}z + \text{tang.}\left(\tfrac{\pi}{2} - z\right),$$
$$4\cot.4z = -\text{tang.}z + \text{tang.}\left(\tfrac{\pi}{4} - z\right) - \text{tang.}\left(\tfrac{\pi}{4} + z\right) + \text{tang.}\left(\tfrac{2\pi}{4} - z\right),$$
$$6\cot.6z = -\text{tang.}z + \text{tang.}\left(\tfrac{\pi}{6} - z\right) - \text{tang.}\left(\tfrac{\pi}{6} + z\right) + \text{tang.}\left(\tfrac{2\pi}{6} - z\right) - \text{tang.}\left(\tfrac{2\pi}{6} + z\right) + \text{tang.}\left(\tfrac{3\pi}{6} - z\right)$$

and generally, if $n = 2m$, there will be

$$\begin{aligned} n\cot.nz = -\text{tang.}z &+ \text{tang.}\left(\tfrac{\pi}{n} - z\right) - \text{tang.}\left(\tfrac{\pi}{n} + z\right) \\ &+ \text{tang.}\left(\tfrac{2\pi}{n} - z\right) - \text{tang.}\left(\tfrac{2\pi}{n} + z\right) \\ &+ \text{tang.}\left(\tfrac{3\pi}{n} - z\right) - \text{tang.}\left(\tfrac{3\pi}{n} + z\right) \\ &\ \ . \\ &\ \ . \\ &+ \text{tang.}\left(\tfrac{m\pi}{n} - z\right). \end{aligned}$$

257. Again by these forms the ambiguity produced from all the roots are removed and therefore there will be

$$1 = \text{tang.}z\ \text{tang.}\left(\tfrac{\pi}{2} - z\right),$$
$$1 = \text{tang.}z\ \text{tang.}\left(\tfrac{\pi}{4} - z\right) \text{tang.}\left(\tfrac{\pi}{4} + z\right) \text{tang.}\left(\tfrac{2\pi}{4} - z\right),$$
$$1 = \text{tang.}z\ \text{tang.}\left(\tfrac{\pi}{6} - z\right) \text{tang.}\left(\tfrac{\pi}{6} + z\right) \text{tang.}\left(\tfrac{2\pi}{6} - z\right) \text{tang.}\left(\tfrac{2\pi}{6} + z\right) \text{tang.}\left(\tfrac{3\pi}{6} - z\right)$$
etc.

Truly an account of the equations at once meets the eye without difficulty, since two angles are to be found always, of which the one is the complement of the other to a right angle. Therefore the tangents of the two angles of this kind give a product $= 1$ and thus the products of all must be equal to one.

258. Because the sines and cosines of angles constituting an arithmetic progression provide a recurring sequence, by the preceding chapter the sum of some number of sines and cosines of this kind will be shown. Let the angles in arithmetic progression be

$$a,\ a+b,\ a+2b,\ a+3b,\ a+4b,\ a+5b \quad \text{etc}.$$

and first the sum of the sines of these angles is sought of the progression to infinity ; therefore there is put

$$s = \sin.a + \sin.(a+b) + \sin.(a+2b) + \sin.(a+3b) + \text{etc.},$$

and because this series is recurring, the scale of its relation is $2\cos.b, -1$,
[*i.e.* $s_n = 2\cos.b\, s_{n-1} - s_{n-2}$, with appropriate boundary conditions for the sequence s_n;]
this series will arise from the expansion of the fractions, the denominator of which is

$$1 - 2z\cos.b + zz$$

on putting $z = 1$. Truly this fraction will be

$$= \frac{\sin.a + z\left(\sin.(a+b) - 2\sin.a\cos.b\right)}{1 - 2z\cos.b + zz};$$

whereby on making $z = 1$ there will be

$$s = \frac{\sin.a + \sin.(a+b) - 2\sin.a\cos.b}{2 - 2\cos.b} = \frac{\sin.a - \sin.(a-b)}{2(1-\cos.b)}$$

on account of

$$2\sin.a\cos.b = \sin.(a+b) + \sin.(a-b).$$

But since there shall be

$$\sin.f - \sin.g = 2\cos.\tfrac{f+g}{2}\sin.\tfrac{f-g}{2},$$

there will be

$$\sin.a - \sin.(a-b) = 2\cos.\left(a - \tfrac{1}{2}b\right)\sin.\tfrac{1}{2}b;$$

but

$$1 - \cos.b = 2\left(\sin.\tfrac{1}{2}b\right)^2,$$

from which there becomes

$$s = \frac{\cos.\left(a - \frac{1}{2}b\right)}{2\sin.\frac{1}{2}b}.$$

259. And thus hence the sum of the sines of whatever amount can be assigned, the arcs of which proceed in an arithmetical progression. Truly the sum of this progression is sought

$$\sin.a + \sin.(a+b) + \sin.(a+2b) + \sin.(a+3b) + \cdots + \sin.(a+nb).$$

Because the sum of this progression continued to infinity is

$$s = \frac{\cos.\left(a - \frac{1}{2}b\right)}{2\sin.\frac{1}{2}b}.$$

these final terms following to infinity will be considered :

$$\sin.\left(a + (n+1)b\right) + \sin.\left(a + (n+2)b\right) + \sin.\left(a + (n+3)b\right) + \text{etc.};$$

because the sum of these sines is

$$= \frac{\cos.\left(a + \left(n + \frac{1}{2}\right)b\right)}{2\sin.\frac{1}{2}b}.$$

if this be taken from the former, the sum sought will remain. Evidently, if there were,

$$\sin.a + \sin.(a+b) + \sin.(a+2b) + \sin.(a+3b) + \cdots + \sin.(a+nb),$$

the sum will be

$$s = \frac{\cos.\left(a - \frac{1}{2}b\right) - \cos.\left(a + \left(n + \frac{1}{2}\right)b\right)}{2\sin.\frac{1}{2}b} = \frac{\sin.\left(a + \frac{1}{2}nb\right)\sin.\frac{1}{2}(n+1)b}{\sin.\frac{1}{2}b}.$$

260. In a like manner if the sum of the cosines may be considered and there is put

$$s = \cos.a + \cos.(a+b) + \cos.(a+2b) + \cos.(a+3b) + \text{etc.}$$

to infinity will be

$$s = \frac{\cos.a + z\left(\cos.(a+b) - 2\cos.a\cos.b\right)}{1 - 2z\cos.b + zz},$$

and on putting $z = 1$, from

$$2\cos.a\cos.b = \cos.(a-b) + \cos.(a+b)$$

it becomes

$$s = \frac{\cos.a - \cos.(a-b)}{2(1 - \cos.b)}$$

But there is

$$\cos.f - \cos.g = 2\sin.\frac{f+g}{2}\sin.\frac{g-f}{2};$$

from which there becomes

$$\cos.a - \cos.(a-b) = -2\sin.\left(a - \tfrac{1}{2}b\right)\sin.\tfrac{1}{2}b,$$

and on account of

$$1-\cos.b=2\left(\sin.\tfrac{1}{2}b\right)^2$$

there will be

$$s=-\frac{\sin.\left(a-\frac{1}{2}b\right)}{2\sin.\frac{1}{2}b}.$$

Whereby, since in a similar manner, the sum of this series

$$\cos.\left(a+(n+1)b\right)+\cos.\left(a+(n+2)b\right)+\cos.\left(a+(n+3)b\right)+\text{etc.}$$

shall be

$$s=-\frac{\sin.\left(a+\left(n+\frac{1}{2}\right)b\right)}{2\sin.\frac{1}{2}b},$$

if this latter series is may be taken from that former one, the sum of this series shall be left

$$s=\cos.a+\cos.(a+b)+\cos.(a+2b)+\cos.(a+3b)+\text{etc.}$$

and it will be

$$s=\frac{-\sin.\left(a-\frac{1}{2}b\right)+\sin.\left(a+\left(n+\frac{1}{2}\right)b\right)}{2\sin.\frac{1}{2}b}=\frac{\cos.\left(a+\frac{1}{2}nb\right)\sin.\frac{1}{2}(n+1)b}{\sin.\frac{1}{2}b}.$$

261. Most other questions about the sines, cosines, and tangents may be resolved from the principles presented ; they are of which kind, if the squares or higher powers of the sines, cosines or tangents must be summed; truly because these may be derived similarly from the coefficients remaining of the above equations, here with these I shall not delay a long time. But so that which principles may reach to these latter sums, whichever power of the sines and cosines to be observed which may be set out in terms of single sines and cosines, we set out briefly so that it may be seen more clearly.

262. Towards bringing this about, it will help to produce these lemmas from the preceding :

$$2\sin.a\sin.z=\cos.(a-z)-\cos.(a+z),$$
$$2\cos.a\sin.z=\sin.(a+z)-\sin.(a-z),$$
$$2\sin.a\cos.z=\sin.(a+z)+\sin.(a-z),$$
$$2\cos.a\cos.z=\cos.(a-z)+\cos.(a+z).$$

Hence therefore the first powers of the sines are found :

$$\sin.z = \sin.z,$$
$$2\left(\sin.z\right)^2 = 1 - \cos.2z,$$
$$4\left(\sin.z\right)^3 = 3\sin.z - \sin.3z,$$
$$8\left(\sin.z\right)^4 = 3 - 4\cos.2z + \cos.4z,$$
$$16\left(\sin.z\right)^5 = 10\sin.z - 5\sin.3z + \sin.5z,$$
$$32\left(\sin.z\right)^6 = 10 - 15\cos.2z + 6\cos.4z - \cos.6z,$$
$$64\left(\sin.z\right)^7 = 35\sin.z - 21\sin.3z + 7\sin.5z - \sin.7z,$$
$$128\left(\sin.z\right)^8 = 35 - 56\cos.2z + 28\cos.4z - 8\cos.6z + \cos.8z,$$
$$256\left(\sin.z\right)^9 = 126\sin.z - 84\sin.3z + 36\sin.5z - 9\sin.7z + \sin.9z$$
etc.

The law, by which these coefficients are progressing, is seen to be related to the last parts of the binomial expansion, except for the absolute number in the even powers which shall be only half of that, which the single powers present in the binomial
[Thus, e.g. in the second from last line, the binomial coefficient would be 70 rather than 35, etc.].

263. The powers of cosines may be defined in a like manner :

$$\cos.z = \cos.z,$$
$$2\left(\cos.z\right)^2 = 1 - \cos.2z,$$
$$4\left(\cos.z\right)^3 = 3\cos.z + \cos.3z,$$
$$8\left(\cos.z\right)^4 = 3 + 4\cos.2z + \cos.4z,$$
$$16\left(\cos.z\right)^5 = 10\cos.z + 5\cos.3z + \cos.5z,$$
$$32\left(\cos.z\right)^6 = 10 + 15\cos.2z + 6\cos.4z + \cos.6z,$$
$$64\left(\cos.z\right)^7 = 35\cos.z + 21\cos.3z + 7\cos.5z + \cos.7z,$$
etc.

Here the same warnings are to be made regarding the law of the progression, that we have observed for the sines.

CAPUT XIV

DE MULTIPLICATIONE AC DIVISIONE ANGULORUM

234. Sit angulus vel arcus in circulo, cuius radius $=1$, quicunque $=z$, eius sinus $=x$, cosinus $=y$ et tangens $=t$; erit

$$xx+yy=1 \text{ et } t=\tfrac{x}{y}.$$

Cum igitur, uti supra vidimus, tam sinus quam cosinus angulorum *z, 2z, 3z, 4z, 5z* etc. constituant seriem recurrentem, cuius scala relationis est *2y*, –1, primum sinus horum arcuum ita se habebunt:

$$\begin{aligned}
&\sin.0z=0,\\
&\sin.1z=\quad x,\\
&\sin.2z=\quad 2xy,\\
&\sin.3z=\quad 4xy^2-x,\\
&\sin.4z=\quad 8xy^3-4xy,\\
&\sin.5z=\ 16xy^4-12xy^2\ +x,\\
&\sin.6z=\ 32xy^5-32xy^3\ +6xy,\\
&\sin.\,7z=\ 64xy^6-80xy^4\ +24xy^2-x,\\
&\sin.8z=128xy^7-192xy^5+80xy^3-8xy.
\end{aligned}$$

Hinc concluditur fore

$$\sin.nz=x\left\{\begin{array}{c}2^{n-1}\,y^{n-1}-(n-2)2^{n-3}y^{n-3}\\ +\frac{(n-3)(n-4)}{1\cdot 2}2^{n-5}\,y^{n-5}-\frac{(n-4)(n-5)(n-6)}{1\cdot 2\cdot 3}2^{n-7}\,y^{n-7}\\ +\frac{(n-5)(n-6)(n-7)(n-8)}{1\cdot 2\cdot 3\cdot 4}2^{n-9}\,y^{n-9}-\text{etc.}\end{array}\right\}.$$

235. Si ponamus arcum $nz=s$, erit

$$\sin.nz=\sin.s=\sin.(\pi-s)=\sin.(2\pi+s)=\sin.(3\pi-s)\ \text{etc.};$$

hi enim sinus omnes sunt inter se aequales. Hinc obtinemus plures valores pro *x*, qui erunt

$$\sin.\tfrac{s}{n},\quad \sin.\tfrac{\pi-s}{n},\quad \sin.\tfrac{2\pi+s}{n},\quad \sin.\tfrac{3\pi-s}{n},\quad \sin.\tfrac{4\pi+s}{n},\quad \text{etc.},$$

qui ergo omnes aequationi inventae aeque conveniunt. Tot autem prodibunt diversi pro x valores, quot numerus n continet unitates, qui propterea erunt radices aequationis inventae. Cavendum ergo est, ne valores aequales pro iisdem habeantur, quod fiet, dum alternae tantum expressiones assumantur. Cognitis igitur radicibus aequationis a posteriori, earum comparatio cum terminis aequationis notatu dignas praebebit proprietates. Quoniam autem ad hoc aequatio, in qua tantum x tamquam incognita insit, requiritur, pro y suus valor $\sqrt{(1-xx)}$ substitui debet; unde duplex operatio instituenda erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar; quia arcuum $-z, +z, +3z, +5z$ etc. differentia est $2z$ huiusque cosinus $=1-2xx$, erit progressionis sinuum scala relationis haec $2-4xx, -1$. Hinc erit

$$\begin{aligned}
&\sin.-z = -x,\\
&\sin.z = x,\\
&\sin.3z = 3x - 4x^3,\\
&\sin.5z = 5x - 20x^3 + 16x^5,\\
&\sin.7z = 7x - 56x^3 + 112x^5 - 64x^7,\\
&\sin.9z = 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9.
\end{aligned}$$

Ergo

$$\sin nz = nx - \tfrac{n(nn-1)}{1\cdot 2\cdot 3}x^3 + \tfrac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 - \tfrac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \text{etc.},$$

siquidem n fuerit numerus impar. Huiusque aequationis radices sunt

$$\sin.z,\quad \sin.\left(\tfrac{2\pi}{n}+z\right),\quad \sin.\left(\tfrac{4\pi}{n}+z\right),\quad \sin.\left(\tfrac{6\pi}{n}+z\right),\quad \sin.\left(\tfrac{8\pi}{n}+z\right),\quad \text{etc.},$$

quarum numerus est n.

237. Huius ergo aequationis

$$0 = 1 - \tfrac{nx}{\sin.nz} + \tfrac{n(nn-1)x^3}{1\cdot 2\cdot 3\sin.nz} - \tfrac{n(nn-1)(nn-9)x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\sin.nz} + \cdots \pm \tfrac{2^{n-1}x^{n}}{\sin.nz}$$

(ubi signum superius valet, si n unitate deficiat a multiplo quaternari, contra inferius) factores sunt

$$\left(1-\tfrac{x}{\sin.z}\right)\left(1-\tfrac{x}{\sin.\left(\frac{2\pi}{n}+z\right)}\right)\left(1-\tfrac{x}{\sin.\left(\frac{4\pi}{n}+z\right)}\right)\quad \text{etc.},$$

ex quibus concluditur fore

$$\frac{n}{\sin.nz}=\frac{1}{\sin.z}+\frac{1}{\sin.\left(\frac{2\pi}{n}+z\right)}+\frac{1}{\sin.\left(\frac{4\pi}{n}+z\right)}+\frac{1}{\sin.\left(\frac{6\pi}{n}+z\right)}+\text{etc.},$$

donec habeantur n termini. Tum vero productum omnium erit

$$\mp\frac{2^{n-1}}{\sin.nz}=\frac{1}{\sin.z\,\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{4\pi}{n}+z\right)\sin.\left(\frac{6\pi}{n}+z\right)\text{etc.}}$$

seu

$$\sin.nz=\mp 2^{n-1}\sin.z\,\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{4\pi}{n}+z\right)\sin.\left(\frac{6\pi}{n}+z\right)\text{etc.}$$

Et, quia terminus penultimus deest, erit

$$0=\sin.z+\sin.\left(\frac{2\pi}{n}+z\right)+\sin.\left(\frac{4\pi}{n}+z\right)+\sin.\left(\frac{6\pi}{n}+z\right)+\sin.\left(\frac{8\pi}{n}+z\right)+\text{etc.}$$

EXEMPLUM 1

Si ergo fuerit $n=3$, prodibunt hae aequalitates

$$\begin{aligned}0&=\sin.z+\sin.(120+z)+\sin.(240+z)\\&=\sin.z+\sin.(60-z)-\sin.(60+z),\end{aligned}$$

$$\frac{3}{\sin.3z}=\frac{1}{\sin.z}+\frac{1}{\sin.(120+z)}+\frac{1}{\sin.(240+z)}=\frac{1}{\sin.z}+\frac{1}{\sin.(60-z)}-\frac{1}{\sin.(60+z)},$$

$$\sin.3z=-4\sin.z\,\sin.(120+z)\sin.(240+z)=4\sin.z\,\sin.(60-z)\sin.(60+z).$$

Erit ergo, uti iam supra notavimus,

$$\sin.(60+z)=\sin.z+\sin.(60-z)$$

et

$$3\,\text{cosec}.3z=\text{cosec}.z+\text{cosec}.(60-z)-\text{cosec}.(60+z).$$

EXEMPLUM 2

Ponamus esse $n=5$ atque prodibunt hae aequationes

$$0=\sin.z+\sin.\left(\tfrac{2\pi}{5}+z\right)+\sin.\left(\tfrac{4\pi}{5}+z\right)+\sin.\left(\tfrac{6\pi}{5}+z\right)+\sin.\left(\tfrac{8\pi}{5}+z\right)$$

seu

$$0=\sin.z+\sin.\left(\tfrac{2\pi}{5}+z\right)+\sin.\left(\tfrac{\pi}{5}-z\right)-\sin.\left(\tfrac{\pi}{5}+z\right)-\sin.\left(\tfrac{2\pi}{5}+z\right)$$

seu

$$0=\sin.z+\sin.\left(\tfrac{\pi}{5}-z\right)-\sin.\left(\tfrac{\pi}{5}+z\right)-\sin.\left(\tfrac{2\pi}{5}-z\right)+\sin.\left(\tfrac{2\pi}{5}+z\right).$$

Deinde erit

$$\frac{5}{\sin.5z}=\frac{1}{\sin.z}+\frac{1}{\sin.\left(\frac{\pi}{5}-z\right)}-\frac{1}{\sin.\left(\frac{\pi}{5}+z\right)}-\frac{1}{\sin.\left(\frac{2\pi}{5}-z\right)}+\frac{1}{\sin.\left(\frac{2\pi}{5}+z\right)},$$

$$\sin.5z=16\sin.z\sin.\left(\tfrac{\pi}{5}-z\right)\sin.\left(\tfrac{\pi}{5}+z\right)\sin.\left(\tfrac{2\pi}{5}-z\right)\sin.\left(\tfrac{2\pi}{5}+z\right).$$

EXEMPLUM 3

Hoc modo, si ponamus $n=2m+1$, erit

$$\begin{aligned}0=\sin.z+\sin.\left(\tfrac{\pi}{n}-z\right)\ &-\ \sin.\left(\tfrac{\pi}{n}+z\right)\\ -\sin.\left(\tfrac{2\pi}{n}-z\right)&+\sin.\left(\tfrac{2\pi}{n}+z\right)\\ +\sin.\left(\tfrac{3\pi}{n}-z\right)&-\sin.\left(\tfrac{3\pi}{n}+z\right),\\ &.\\ &.\\ \pm\sin.\left(\tfrac{m\pi}{n}-z\right)&\mp\sin.\left(\tfrac{m\pi}{n}+z\right)\end{aligned}$$

ubi signa superiora valent, si m sit numerus impar, inferiora, si sit par.
Altera aequatio erit haec

$$\begin{aligned}\frac{n}{\sin.nz} = \frac{1}{\sin.z} + \frac{1}{\sin.\left(\frac{\pi}{n}-z\right)} &- \frac{1}{\sin.\left(\frac{\pi}{n}+z\right)} \\ - \frac{1}{\sin.\left(\frac{2\pi}{n}-z\right)} &+ \frac{1}{\sin.\left(\frac{2\pi}{n}+z\right)} \\ + \frac{1}{\sin.\left(\frac{3\pi}{n}-z\right)} &- \frac{1}{\sin.\left(\frac{3\pi}{n}+z\right)}, \\ &. \\ &. \\ \pm \frac{1}{\sin.\left(\frac{m\pi}{n}-z\right)} &\mp \frac{1}{\sin.\left(\frac{m\pi}{n}+z\right)},\end{aligned}$$

quae ad cosecantes commode transfertur. Tertio habetur hoc productum

$$\begin{aligned}\sin.nz = 2^{2m}\, sin.z\, &\sin.\left(\frac{\pi}{n}-z\right)\sin.\left(\frac{\pi}{n}+z\right) \\ &\sin.\left(\frac{2\pi}{n}-z\right)\sin.\left(\frac{2\pi}{n}+z\right) \\ &\sin.\left(\frac{3\pi}{n}-z\right)\sin.\left(\frac{3\pi}{n}+z\right) \\ &. \\ &. \\ &\sin.\left(\frac{m\pi}{n}-z\right)\sin.\left(\frac{m\pi}{n}+z\right).\end{aligned}$$

238. Sit n nunc numerus par, et quoniam est

$$y = \sqrt{(1-xx)} \text{ et } \cos.2z = 1-2xx,$$

ita ut seriei sinuum sit scala relationis ut ante $2-4xx, -1$, erit

$$\begin{aligned}&\sin.0z = 0, \\ &\sin.2z = 2x\sqrt{(1-xx)}, \\ &\sin.4z = \left(4x-8x^3\right)\sqrt{(1-xx)}, \\ &\sin.6z = \left(6x-32x^3+32x^5\right)\sqrt{(1-xx)}, \\ &\sin.8z = \left(8x-80x^3+192x^5-128x^7\right)\sqrt{(1-xx)}\end{aligned}$$

et generaliter

$$\sin.nz = \left\{ \begin{matrix} nx - \frac{n(nn-4)}{1\cdot2\cdot3}x^3 + \frac{n(nn-4)(nn-16)}{1\cdot2\cdot3\cdot4\cdot5}x^5 \\ -\frac{n(nn-4)(nn-16)(nn-36)}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}x^7 + \cdots \pm 2^{n-1}x^{n-1} \end{matrix} \right\} \sqrt{(1-xx)}$$

denotante n numerum quemcunque parem.

239. Ad aequationem hanc rationalem efficiendam sumantur utrinque quadrata ac prodibit huiusmodi aequatio

$$(\sin.nz)^2 = nnxx + Px^4 + Qx^6 + \cdots - 2^{2n-2}x^{2n}$$

seu

$$x^{2n} - \cdots - \frac{nn}{2^{2n-2}}xx + \frac{1}{2^{2n-2}}(\sin.nz)^2 = 0,$$

cuius aequationis radices erunt tam affirmativae quam negativae, scilicet

$$\pm\sin.nz, \pm\sin.\left(\frac{\pi}{n}-z\right), \pm\sin.\left(\frac{2\pi}{n}+z\right), \pm\sin.\left(\frac{3\pi}{n}-z\right), \pm\sin.\left(\frac{4\pi}{n}+z\right) \text{ etc.}$$

sumendo omnino n huiusmodi expressiones. Cum igitur ultimus terminus sit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

$$\sin.nz = \pm 2^{n-1}\sin.z \sin.\left(\frac{\pi}{n}-z\right)\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{3\pi}{n}-z\right)\cdots;$$

ubi quibus casibus utrumvis signum valeat, ex casibus particularibus erit dispiciendum.

EXEMPLUM

Substituendo autem pro n successive numeros 2, 4, 6 etc. et eligendo n sinus diversos erit

$$\sin.2z = 2\sin.z\sin.\left(\frac{\pi}{n}-z\right),$$

$$\sin.4z = 8\sin.z\sin.\left(\frac{\pi}{4}-z\right)\sin.\left(\frac{\pi}{4}+z\right)\sin.\left(\frac{2\pi}{4}-z\right),$$

$$\sin.6z = 32\sin.z\sin.\left(\frac{\pi}{6}-z\right)\sin.\left(\frac{\pi}{6}+z\right)\sin.\left(\frac{2\pi}{6}-z\right)$$
$$\sin.\left(\frac{\pi}{6}+z\right)\sin.\left(\frac{3\pi}{6}-z\right),$$

$$\sin.8z = 128\sin.z\sin.\left(\frac{\pi}{8}-z\right)\sin.\left(\frac{\pi}{8}+z\right)\sin.\left(\frac{2\pi}{8}-z\right)\sin.\left(\frac{2\pi}{8}+z\right)$$
$$\sin.\left(\frac{3\pi}{8}-z\right)\sin.\left(\frac{3\pi}{8}+z\right)\sin.\left(\frac{4\pi}{8}-z\right).$$

240. Patet ergo fore generatim

$$\begin{aligned}\sin.nz = 2^{n-1}\sin.z\,&\sin.\left(\tfrac{\pi}{n}-z\right)\sin.\left(\tfrac{\pi}{n}+z\right)\\&\sin.\left(\tfrac{2\pi}{n}-z\right)\sin.\left(\tfrac{2\pi}{n}+z\right)\\&\sin.\left(\tfrac{3\pi}{n}-z\right)\sin.\left(\tfrac{3\pi}{n}+z\right)\\&\quad .\\&\quad .\\&\sin.\left(\tfrac{\pi}{2}-z\right),\end{aligned}$$

si n fuerit numerus par. Quodsi autem haec cum superiori, ubi n erat numerus impar, comparetur, tanta similitudo adesse deprehenditur, ut utramque in unam redigere liceat. Erit ergo, sive n fuerit numerus par sive impar,

$$\begin{aligned}\sin.nz = 2^{n-1}\sin.z\,&\sin.\left(\tfrac{\pi}{n}-z\right)\sin.\left(\tfrac{\pi}{n}+z\right)\\&\sin.\left(\tfrac{2\pi}{n}-z\right)\sin.\left(\tfrac{2\pi}{n}+z\right)\\&\sin.\left(\tfrac{3\pi}{n}-z\right)\sin.\left(\tfrac{3\pi}{n}+z\right)\\&\qquad \text{etc.},\end{aligned}$$

donec tot habeantur factores, quot numerus n continet unitates.

241. Expressiones istae, quibus sinus angulorum multiplorum per factores exponuntur, non parum utilitatis afferre possunt ad logarithmos sinuum angulorum multiplorum inveniendos itemque ad plures expressiones sinuum per factores, quales supra dedimus, reperiendas. Erit autem

$$\begin{aligned}\sin.z &= 1\sin.z,\\ \sin.2z &= 2\sin.z\sin.\left(\tfrac{\pi}{2}-z\right),\\ \sin.3z &= 4\sin.z\sin.\left(\tfrac{\pi}{3}-z\right)\sin.\left(\tfrac{\pi}{3}+z\right),\\ \sin.4z &= 8\sin.z\sin.\left(\tfrac{\pi}{4}-z\right)\sin.\left(\tfrac{\pi}{4}+z\right)\sin.\left(\tfrac{2\pi}{4}-z\right),\\ \sin.5z &= 16\sin.z\sin.\left(\tfrac{\pi}{5}-z\right)\sin.\left(\tfrac{\pi}{5}+z\right)\sin.\left(\tfrac{2\pi}{5}-z\right)\sin.\left(\tfrac{2\pi}{5}+z\right),\\ \sin.6z &= 32\sin.z\sin.\left(\tfrac{\pi}{6}-z\right)\sin.\left(\tfrac{\pi}{6}+z\right)\sin.\left(\tfrac{2\pi}{6}-z\right)\sin.\left(\tfrac{2\pi}{6}+z\right)\sin.\left(\tfrac{3\pi}{6}-z\right),\\ &\text{etc.}\end{aligned}$$

242. Cum deinde sit $\frac{\sin.2nz}{\sin.nz} = 2\cos.nz$, cosinus angulorum multiplorum simili modo per factores exprimentur:

$$\begin{aligned}
\cos.z &= 1\sin.\left(\tfrac{\pi}{2} - z\right),\\
\cos.2z &= 2\sin.\left(\tfrac{\pi}{4} - z\right)\sin.\left(\tfrac{\pi}{4} + z\right),\\
\cos.3z &= 4\sin.\left(\tfrac{\pi}{6} - z\right)\sin.\left(\tfrac{\pi}{6} + z\right)\sin.\left(\tfrac{3\pi}{6} - z\right),\\
\cos.4z &= 8\sin.\left(\tfrac{\pi}{8} - z\right)\sin.\left(\tfrac{\pi}{8} + z\right)\sin.\left(\tfrac{3\pi}{8} - z\right)\sin.\left(\tfrac{3\pi}{8} + z\right),\\
\cos.5z &= 16\sin.\left(\tfrac{\pi}{10} - z\right)\sin.\left(\tfrac{\pi}{10} + z\right)\sin.\left(\tfrac{3\pi}{10} - z\right)\sin.\left(\tfrac{3\pi}{10} + z\right)\sin.\left(\tfrac{5\pi}{10} - z\right),
\end{aligned}$$

et generaliter

$$\begin{aligned}
\cos.nz = 2^{n-1}&\sin.\left(\tfrac{\pi}{2n} - z\right)\sin.\left(\tfrac{\pi}{2n} + z\right)\\
&\sin.\left(\tfrac{3\pi}{2n} - z\right)\sin.\left(\tfrac{3\pi}{2n} + z\right)\\
&\sin.\left(\tfrac{5\pi}{2n} - z\right)\sin.\left(\tfrac{5\pi}{2n} + z\right)\\
&\text{etc.},
\end{aligned}$$

quoad tot habeantur factores, quot numerus n continet unitates.

243. Eaedem expressiones prodibunt ex consideratione cosinuum arcuum multiplorum. Si enim fuerit $\cos.z = y$, erit, ut sequitur:

$$\begin{aligned}
\cos.0z &= 1,\\
\cos.1z &= y,\\
\cos.2z &= 2y^2 - 1,\\
\cos.3z &= 4y^3 - 3y,\\
\cos.4z &= 8y^4 - 8y^3 + 1,\\
\cos.5z &= 16y^5 - 20y^3 + 5y,\\
\cos.6z &= 32y^6 - 48y^4 + 18y^2 - 1,\\
\cos.7z &= 64y^7 - 112y^5 + 56y^3 - 7y
\end{aligned}$$

et generaliter

$$\cos.nz = 2^{n-1}y^n - \tfrac{n}{1}2^{n-3}y^{n-2} + \tfrac{n(n-3)}{1\cdot 2}2^{n-5}y^{n-4} - \tfrac{n(n-4)(n-5)}{1\cdot 2\cdot 3}2^{n-7}y^{n-6} + \tfrac{n(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4}2^{n-9}y^{n-8} - \text{etc.},$$

cuius aequationis, cum sit

$$\cos.nz = \cos.(2\pi - nz) = \cos.(2\pi + nz) = \cos.(4\pi \pm nz) = \cos.(6\pi \pm nz)\ \text{etc.},$$

erunt radices ipsius y hae

$$\cos.z,\quad \cos.\left(\tfrac{2\pi}{n} \pm z\right),\quad \cos.\left(\tfrac{4\pi}{n} \pm z\right),\quad \cos.\left(\tfrac{6\pi}{n} \pm z\right)\ \text{etc.},$$

quarum formularum tot diversae sunt pro y eligendae, quot dantur; dantur autem tot, quot n continet unitates.

244. Primum igitur patet ob terminum secundum deficientem excepto casu $n = 1$ fore summam harum radicum omnium $= 0$. Erit ergo

$$0 = \cos.z + \cos.\left(\tfrac{2\pi}{n} - z\right) + \cos.\left(\tfrac{2\pi}{n} + z\right) + \cos.\left(\tfrac{4\pi}{n} - z\right) + \cos.\left(\tfrac{4\pi}{n} + z\right) + \text{etc.}$$

sumendo tot terminos, quot n continet unitates. Haec autem aequalitas sponte se offert, si n sit numerus par, cum quivis terminus ab alio sui negativo destruatur. Contemplemur ergo numeros impares unitate exclusa eritque ob $\cos.v = -\cos.(\pi - v)$

$$0 = \cos.z - \cos.\left(\tfrac{\pi}{3} - z\right) - \cos.\left(\tfrac{\pi}{3} + z\right),$$

$$0 = \cos.z - \cos.\left(\tfrac{\pi}{5} - z\right) - \cos.\left(\tfrac{\pi}{5} + z\right) + \cos.\left(\tfrac{2\pi}{5} - z\right) + \cos.\left(\tfrac{2\pi}{5} + z\right),$$

$$0 = \cos.z - \cos.\left(\tfrac{\pi}{7} - z\right) - \cos.\left(\tfrac{\pi}{7} + z\right) + \cos.\left(\tfrac{2\pi}{7} - z\right) + \cos.\left(\tfrac{2\pi}{7} + z\right) - \cos.\left(\tfrac{3\pi}{7} - z\right) - \cos.\left(\tfrac{3\pi}{7} + z\right)$$

et generaliter, si fuerit n numerus impar quicunque, erit

$$0 = \cos.z - \cos.\left(\tfrac{\pi}{n} - z\right) - \cos.\left(\tfrac{\pi}{n} + z\right) + \cos.\left(\tfrac{2\pi}{n} - z\right) + \cos.\left(\tfrac{2\pi}{n} + z\right) - \cos.\left(\tfrac{3\pi}{n} - z\right) - \cos.\left(\tfrac{3\pi}{n} + z\right) + \cos.\left(\tfrac{4\pi}{7} - z\right) + \cos.\left(\tfrac{4\pi}{7} + z\right) - \text{etc.}$$

sumendo tot terminos, quot numerus n continet unitates. Oportet autem n esse numerum imparem unitate maiorem, uti iam monuimus.

245. Quod ad productum ex omnibus attinet, variae quidem prodeunt expressiones, prout n fuerit numerus vel impar vel impariter par vel pariter par. Omnes autem comprehenduntur in expressione generali inventa, si singuli sinus in cosinus transmutentur. Erit scilicet

$$\begin{aligned}
\cos.z &= 1\cos.z,\\
\cos.2z &= 2\cos.\left(\tfrac{\pi}{4}+z\right)\cos.\left(\tfrac{\pi}{4}-z\right),\\
\cos.3z &= 4\cos.\left(\tfrac{2\pi}{6}+z\right)\cos.\left(\tfrac{2\pi}{6}-z\right)\cos.z,\\
\cos.4z &= 8\cos.\left(\tfrac{3\pi}{8}+z\right)\cos.\left(\tfrac{3\pi}{8}-z\right)\cos.\left(\tfrac{\pi}{8}+z\right)\cos.\left(\tfrac{\pi}{8}-z\right),\\
\cos.5z &= 16\cos.\left(\tfrac{4\pi}{10}+z\right)\cos.\left(\tfrac{4\pi}{10}-z\right)\cos.\left(\tfrac{2\pi}{10}+z\right)\cos.\left(\tfrac{2\pi}{10}-z\right)\cos.z
\end{aligned}$$

et generaliter

$$\begin{aligned}
\cos.nz = 2^{n-1}&\cos.\left(\tfrac{n-1}{2n}\pi+z\right)\cos.\left(\tfrac{n-1}{2n}\pi-z\right)\\
&\cos.\left(\tfrac{n-3}{2n}\pi+z\right)\cos.\left(\tfrac{n-3}{2n}\pi-z\right)\\
&\cos.\left(\tfrac{n-5}{2n}\pi+z\right)\cos.\left(\tfrac{n-5}{2n}\pi-z\right)\\
&\cos.\left(\tfrac{n-7}{2n}\pi+z\right)\cos.\left(\tfrac{n-7}{2n}\pi-z\right)\\
&\text{etc.}
\end{aligned}$$

sumptis tot factoribus, quot numerus n continet unitates.

246. Sit n numerus impar atque aequatio incipiatur ab unitate; erit

$$0 = 1 \mp \frac{ny}{\cos.nz} \pm \text{etc.},$$

ubi signum superius valet, si n fuerit numerus impar formae $4m+1$, inferius, si $n = 4m-1$. Hinc erit

$$\begin{aligned}
&+\frac{1}{\cos.z} = \frac{1}{\cos.z}\\
&-\frac{3}{\cos.3z} = \frac{1}{\cos.z} - \frac{1}{\cos.\left(\frac{\pi}{3}-z\right)} - \frac{1}{\cos.\left(\frac{\pi}{3}+z\right)},\\
&+\frac{5}{\cos.5z} = \frac{1}{\cos.z} - \frac{1}{\cos.\left(\frac{\pi}{5}-z\right)} - \frac{1}{\cos.\left(\frac{\pi}{5}+z\right)} + \frac{1}{\cos.\left(\frac{2\pi}{5}-z\right)} + \frac{1}{\cos.\left(\frac{2\pi}{5}+z\right)}
\end{aligned}$$

et generaliter posito $n = 2m+1$ erit

$$\frac{n}{\cos.nz}=\frac{2m+1}{\cos.(2m+1)z}=\frac{1}{\cos.\left(\frac{m}{n}\pi+z\right)}+\frac{1}{\cos.\left(\frac{m}{n}\pi-z\right)}$$
$$-\frac{1}{\cos.\left(\frac{m-1}{n}\pi+z\right)}-\frac{1}{\cos.\left(\frac{m-1}{n}\pi-z\right)}$$
$$+\frac{1}{\cos.\left(\frac{m-2}{n}\pi+z\right)}+\frac{1}{\cos.\left(\frac{m-2}{n}\pi-z\right)}$$
$$-\frac{1}{\cos.\left(\frac{m-3}{n}\pi+z\right)}-\frac{1}{\cos.\left(\frac{m-3}{n}\pi-z\right)}$$
$$\text{etc.}$$

sumendis tot terminis, quot n continet unitates.

247. Cum ergo sit $\frac{1}{\cos.v}=\sec.v$, hinc pro secantibus insignes proprietates deducuntur; erit nempe

$$\sec.z=\sec.z$$
$$3\sec.3z=\sec.\left(\frac{\pi}{3}+z\right)+\sec.\left(\frac{\pi}{3}-z\right)-\sec.\left(\frac{0\pi}{3}+z\right),$$
$$5\sec.5z=\sec.\left(\frac{2\pi}{5}+z\right)+\sec.\left(\frac{2\pi}{5}-z\right)-\sec.\left(\frac{\pi}{5}+z\right)-\sec.\left(\frac{\pi}{5}-z\right)+\sec.\left(\frac{0\pi}{5}+z\right),$$
$$7\sec.7z=\sec.\left(\frac{3\pi}{7}+z\right)+\sec.\left(\frac{3\pi}{7}-z\right)-\sec.\left(\frac{2\pi}{7}+z\right)-\sec.\left(\frac{2\pi}{7}-z\right)$$
$$+\sec.\left(\frac{\pi}{7}+z\right)+\sec.\left(\frac{\pi}{7}-z\right)-\sec.\left(\frac{0\pi}{7}+z\right)$$

et generaliter posito $n=2m+1$ erit

$$n\sec.nz=\sec.\left(\frac{m}{n}\pi+z\right)+\sec.\left(\frac{m}{n}\pi-z\right)$$
$$-\sec.\left(\frac{m-1}{n}\pi+z\right)-\sec.\left(\frac{m-1}{n}\pi-z\right)$$
$$+\sec.\left(\frac{m-2}{n}\pi+z\right)+\sec.\left(\frac{m-2}{n}\pi-z\right)$$
$$-\sec.\left(\frac{m-3}{n}\pi+z\right)-\sec.\left(\frac{m-3}{n}\pi-z\right)$$
$$+\sec.\left(\frac{m-4}{n}\pi+z\right)+\sec.\left(\frac{m-4}{n}\pi-z\right)$$
$$.$$
$$.$$
$$\pm\sec.z.$$
$$\text{etc.}$$

248. Pro cosecantibus autem erit ex § 237

$$\mathrm{cosec.}z = \mathrm{cosec.}z$$
$$3\mathrm{cosec.}3z = \mathrm{cosec.}z + \mathrm{cosec.}\left(\tfrac{\pi}{3} - z\right) - \mathrm{cosec.}\left(\tfrac{\pi}{3} + z\right),$$
$$5\mathrm{cosec.}5z = \mathrm{cosec.}z + \mathrm{cosec.}\left(\tfrac{\pi}{5} - z\right) - \mathrm{cosec.}\left(\tfrac{\pi}{5} + z\right) - \mathrm{cosec.}\left(\tfrac{2\pi}{5} - z\right) + \mathrm{cosec.}\left(\tfrac{2\pi}{5} + z\right),$$
$$7\mathrm{cosec.}7z = \mathrm{cosec.}z + \mathrm{cosec.}\left(\tfrac{\pi}{7} - z\right) - \mathrm{cosec.}\left(\tfrac{\pi}{7} + z\right) - \mathrm{cosec.}\left(\tfrac{2\pi}{7} - z\right) + \mathrm{cosec.}\left(\tfrac{2\pi}{7} + z\right)$$
$$+\mathrm{cosec.}\left(\tfrac{3\pi}{7} - z\right) - \mathrm{cosec.}\left(\tfrac{3\pi}{7} + z\right)$$

et generaliter ponendo $n = 2m + 1$ erit

$$\begin{aligned} n\mathrm{cosec.}nz = \mathrm{cosec.}z &+ \mathrm{cosec.}\left(\tfrac{\pi}{n} - z\right) - \mathrm{cosec.}\left(\tfrac{\pi}{n} + z\right) \\ &- \mathrm{cosec.}\left(\tfrac{2\pi}{n} - z\right) + \mathrm{cosec.}\left(\tfrac{2\pi}{n} + z\right) \\ &+ \mathrm{cosec.}\left(\tfrac{3\pi}{n} - z\right) - \mathrm{cosec.}\left(\tfrac{3\pi}{n} + z\right) \\ &\cdot \\ &\cdot \\ &\pm \mathrm{cosec.}\left(\tfrac{m\pi}{n} - z\right) \pm \mathrm{cosec.}\left(\tfrac{m\pi}{n} + z\right), \end{aligned}$$

ubi signa superiora valent, si *m* fuerit numerus par, inferiora, si *m* sit impar.

249. Cum sit, uti supra vidimus,

$$\cos.nz \pm \sqrt{-1}\cdot\sin.nz = \left(\cos.z \pm \sqrt{-1}\cdot\sin.z\right)^n,$$

erit

$$\cos.nz = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n + \left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{2}$$

et

$$\sin.nz = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n - \left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{2\sqrt{-1}},$$

ergo

$$\tan.nz = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n - \left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n + \left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}.$$

Ponamus

$$\mathrm{tang.}z = \frac{\sin.z}{\cos.z} = t;$$

erit

$$\tan.nz = \frac{\left(1+t\sqrt{-1}\right)^n - \left(1-t\sqrt{-1}\right)^n}{\left(1+t\sqrt{-1}\right)^n\sqrt{-1} + \left(1-t\sqrt{-1}\right)^n\sqrt{-1}},$$

unde oriuntur tangentes angulorum multiplorum sequentes

$$\text{tang}.z = t,$$
$$\text{tang}.2z = \frac{2t}{1-tt},$$
$$\text{tang}.3z = \frac{3t-t^3}{1-3tt},$$
$$\text{tang}.4z = \frac{4t-4t^3}{1-6tt+t^4}$$
$$\text{tang}.5z = \frac{5t-10t^3+t^5}{1-10tt+5t^4}$$

et generaliter

$$\text{tang}.nz = \frac{nt - \frac{n(n-1)(n-2)}{1\cdot2\cdot3}t^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot2\cdot3\cdot4\cdot5}t^5 - \text{etc.}}{1 - \frac{n(n-1)}{1\cdot2}tt + \frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}t^4 - \text{etc.}}$$

Cum iam sit

$$\text{tang}.nz = \text{tang}.(\pi + nz) = \text{tang}.(2\pi + nz) = \text{tang}.(3\pi + nz) \quad \text{etc.},$$

erunt valores ipsius t seu radices aequationis hae

$$\text{tang}.z, \quad \text{tang}.\left(\tfrac{\pi}{n} + z\right), \quad \text{tang}.\left(\tfrac{2\pi}{n} + z\right), \quad \text{tang}.\left(\tfrac{3\pi}{n} + z\right) \quad \text{etc.},$$

quarum numerus est n.

250. Quodsi aequatio ab unitate incipiat, erit

$$0 = 1 - \frac{nt}{\text{tang}.nz} - \frac{n(n-1)tt}{1\cdot2} + \frac{n(n-1)(n-2)t^3}{1\cdot2\cdot3\,\text{tang}.nz} + \text{etc.}$$

Ex comparatione ergo coefficientium cum radicibus erit

$$n\cot.nz = \cot.z + \cot.\left(\tfrac{\pi}{n} + z\right) + \cot.\left(\tfrac{2\pi}{n} + z\right) + \cot.\left(\tfrac{3\pi}{n} + z\right) + \cot.\left(\tfrac{4\pi}{n} + z\right) + \cdots + \cot.\left(\tfrac{n-1}{n}\pi + z\right).$$

Deinde erit summa quadratorum harum cotangentium omnium

$$=\frac{nn}{(\sin.nz)^2}-n$$

similique modo ulteriores potestates possunt definiri. Ponendo autem loco n numeros definitos erit

$$\begin{aligned}
\cot.z &= \cot.z,\\
2\cot.2z &= \cot.z+\cot.\left(\tfrac{\pi}{2}+z\right),\\
3\cot.3z &= \cot.z+\cot.\left(\tfrac{\pi}{3}+z\right)+\cot.\left(\tfrac{2\pi}{3}+z\right),\\
4\cot.4z &= \cot.z+\cot.\left(\tfrac{\pi}{4}+z\right)+\cot.\left(\tfrac{2\pi}{4}+z\right)+\cot.\left(\tfrac{3\pi}{4}+z\right),\\
5\cot.5z &= \cot.z+\cot.\left(\tfrac{\pi}{5}+z\right)+\cot.\left(\tfrac{2\pi}{5}+z\right)+\cot.\left(\tfrac{3\pi}{5}+z\right)+\cot.\left(\tfrac{4\pi}{5}+z\right).
\end{aligned}$$

251. Quia vero est $\cot.v=-\cot.(\pi-v)$, erit

$$\begin{aligned}
\cot.z &= \cot.z,\\
2\cot.2z &= \cot.z-\cot.\left(\tfrac{\pi}{2}-z\right),\\
3\cot.3z &= \cot.z-\cot.\left(\tfrac{\pi}{3}-z\right)+\cot.\left(\tfrac{\pi}{3}+z\right),\\
4\cot.4z &= \cot.z-\cot.\left(\tfrac{\pi}{4}-z\right)+\cot.\left(\tfrac{\pi}{4}+z\right)-\cot.\left(\tfrac{2\pi}{4}-z\right),\\
5\cot.5z &= \cot.z-\cot.\left(\tfrac{\pi}{5}-z\right)+\cot.\left(\tfrac{\pi}{5}+z\right)-\cot.\left(\tfrac{2\pi}{5}-z\right)+\cot.\left(\tfrac{2\pi}{5}+z\right).
\end{aligned}$$

et generaliter

$$\begin{aligned}
n\cot.nz = \cot.z &-\cot.\left(\tfrac{\pi}{n}-z\right)+\cot.\left(\tfrac{\pi}{n}+z\right)\\
&-\cot.\left(\tfrac{2\pi}{n}-z\right)+\cot.\left(\tfrac{2\pi}{n}+z\right)\\
&-\cot.\left(\tfrac{3\pi}{n}-z\right)+\cot.\left(\tfrac{3\pi}{n}+z\right)\\
&-\text{etc.},
\end{aligned}$$

donec tot habeantur termini, quot numerus n continet unitates.

252. Incipiamus aequationem inventam a potestate summa, ubi primum distinguendi sunt casus, quibus n est vel numerus par vel impar. Sit n numerus impar, $n=2m+1$; erit

$$t - \text{tang.}z = 0,$$
$$t^3 - 3tt\text{tang.}3z - 3t + \text{tang.}3z = 0,$$
$$t^5 - 5t^4\ \text{tang.}5z - 10t^3 + 10tt\text{tang.}5z + 5t - \text{tang.}5z = 0$$

et generaliter

$$t^n - nt^{n\text{-}1}\text{tang.}nz - \cdots \mp \text{tang.}nz = 0,$$

ubi signum superius – valet, si m sit numerus par, inferius +, si m sit numerus impar. Erit ergo ex coefficiente secundi termini

$$\text{tang.}z = \text{tang.}z,$$
$$3\text{tang.}3z = \text{tang.}z + \text{tang.}\left(\tfrac{\pi}{3} + z\right) + \text{tang.}\left(\tfrac{2\pi}{3} + z\right),$$
$$5\text{tang.}5z = \text{tang.}z + \text{tang.}\left(\tfrac{\pi}{5} + z\right) + \text{tang.}\left(\tfrac{2\pi}{5} + z\right) + \text{tang.}\left(\tfrac{3\pi}{5} + z\right) + \text{tang.}\left(\tfrac{4\pi}{5} + z\right)$$
$$\text{etc.}$$

253. Cum igitur sit $\text{tang.}v = -\ \text{tang.}(n - v)$, anguli recto maiores ad angulos recto minores reducuntur eritque

$$\text{tang.}z = \text{tang.}z,$$
$$3\text{tang.}3z = \text{tang.}z - \text{tang.}\left(\tfrac{\pi}{3} - z\right) + \text{tang.}\left(\tfrac{\pi}{3} + z\right),$$
$$5\text{tang.}5z = \text{tang.}z - \text{tang.}\left(\tfrac{\pi}{5} - z\right) + \text{tang.}\left(\tfrac{\pi}{5} + z\right) - \text{tang.}\left(\tfrac{2\pi}{5} - z\right) + \text{tang.}\left(\tfrac{2\pi}{5} + z\right),$$
$$7\text{tang.}7z = \text{tang.}z - \text{tang.}\left(\tfrac{\pi}{7} - z\right) + \text{tang.}\left(\tfrac{\pi}{7} + z\right) - \text{tang.}\left(\tfrac{2\pi}{7} - z\right) + \text{tang.}\left(\tfrac{2\pi}{7} + z\right)$$
$$- \text{tang.}\left(\tfrac{3\pi}{7} - z\right) + \text{tang.}\left(\tfrac{3\pi}{7} + z\right)$$

et generaliter, si $n = 2m + 1$, erit

$$n\text{tang.}nz = \text{tang.}z - \text{tang.}\left(\tfrac{\pi}{n} - z\right) + \text{tang.}\left(\tfrac{\pi}{n} + z\right) - \text{tang.}\left(\tfrac{2\pi}{n} - z\right) + \text{tang.}\left(\tfrac{2\pi}{n} + z\right)$$
$$- \text{tang.}\left(\tfrac{3\pi}{n} - z\right) + \text{tang.}\left(\tfrac{3\pi}{n} + z\right)$$
$$\vdots$$
$$- \text{tang.}\left(\tfrac{m\pi}{n} - z\right) + \text{tang.}\left(\tfrac{m\pi}{n} + z\right)$$

254.Tum vero productum ex his tangentibus omnibus erit $= \text{tang}.nz$,
propterea quod per signorum negativorum numerum alternatim parem et imparem superior signorum ambiguitas tollitur. Sic erit

$$\text{tang}.z = \text{tang}.z,$$
$$\text{tang}.3z = \text{tang}.z\ \text{tang}.\left(\tfrac{\pi}{3} - z\right)\text{tang}.\left(\tfrac{\pi}{3} + z\right),$$
$$\text{tang}.5z = \text{tang}.z\ \text{tang}.\left(\tfrac{\pi}{5} - z\right)\text{tang}.\left(\tfrac{\pi}{5} + z\right)\text{tang}.\left(\tfrac{2\pi}{5} - z\right)\text{tang}.\left(\tfrac{2\pi}{5} + z\right)$$

et generaliter, si $n = 2m+1$, erit

$$\begin{aligned}\text{tang}.nz = \text{tang}.z\ &\text{tang}.\left(\tfrac{\pi}{n} - z\right)\text{tang}.\left(\tfrac{\pi}{n} + z\right)\\ &\text{tang}.\left(\tfrac{2\pi}{n} - z\right)\text{tang}.\left(\tfrac{2\pi}{n} + z\right)\\ &\text{tang}.\left(\tfrac{3\pi}{n} - z\right)\text{tang}.\left(\tfrac{3\pi}{n} + z\right)\\ &.\\ &.\\ &\text{tang}.\left(\tfrac{m\pi}{n} - z\right)\text{tang}.\left(\tfrac{m\pi}{n} + z\right)\end{aligned}$$

255. Sit iam n numerus par atque incipiendo a potestate summa erit

$$tt + 2t\cot.2z - 1 = 0,$$
$$t^4 + 4t^3\cot.4z - 6tt - 4t\cot.4z + 1 = 0$$

et generaliter, si $n = 2m$, erit

$$t^n + nt^{n-1}\cot.nz - \cdots \mp 1 = 0,$$

ubi signum superius – valet, si m sit numerus impar, inferius +, si m sit par. Comparando ergo radices cum coefficiente secundi termini erit

$$-2\cot.2z = \text{tang}.z + \text{tang}.\left(\tfrac{\pi}{2} + z\right),$$
$$-4\cot.4z = \text{tang}.z + \text{tang}.\left(\tfrac{\pi}{4} + z\right) + \text{tang}.\left(\tfrac{2\pi}{4} + z\right) + \text{tang}.\left(\tfrac{2\pi}{4} + z\right) + \text{tang}.\left(\tfrac{3\pi}{4} + z\right),$$
$$\begin{aligned}-6\cot.6z = \text{tang}.z &+ \text{tang}.\left(\tfrac{\pi}{6} + z\right) + \text{tang}.\left(\tfrac{\pi}{6} + z\right) + \text{tang}.\left(\tfrac{2\pi}{6} + z\right) + \text{tang}.\left(\tfrac{3\pi}{6} + z\right)\\ &+ \text{tang}.\left(\tfrac{4\pi}{6} + z\right) + \text{tang}.\left(\tfrac{5\pi}{6} + z\right)\end{aligned}$$

etc.

256. Cum sit $\tan g.v = -\tan g.(\pi - v)$, sequentes formabuntur aequationes

$$2\cot.2z = -\tan g.z + \tan g.\left(\tfrac{\pi}{2} - z\right),$$
$$4\cot.4z = -\tan g.z + \tan g.\left(\tfrac{\pi}{4} - z\right) - \tan g.\left(\tfrac{\pi}{4} + z\right) + \tan g.\left(\tfrac{2\pi}{4} - z\right),$$
$$6\cot.6z = -\tan g.z + \tan g.\left(\tfrac{\pi}{6} - z\right) - \tan g.\left(\tfrac{\pi}{6} + z\right) + \tan g.\left(\tfrac{2\pi}{6} - z\right) - \tan g.\left(\tfrac{2\pi}{6} + z\right) + \tan g.\left(\tfrac{3\pi}{6} - z\right)$$

et generaliter, si $n = 2m$, erit

$$\begin{aligned} n\cot.nz = -\tan g.z &+ \tan g.\left(\tfrac{\pi}{n} - z\right) - \tan g.\left(\tfrac{\pi}{n} + z\right) \\ &+ \tan g.\left(\tfrac{2\pi}{n} - z\right) - \tan g.\left(\tfrac{2\pi}{n} + z\right) \\ &+ \tan g.\left(\tfrac{3\pi}{n} - z\right) - \tan g.\left(\tfrac{3\pi}{n} + z\right) \\ &\;. \\ &\;. \\ &+ \tan g.\left(\tfrac{m\pi}{n} - z\right). \end{aligned}$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus destruitur eritque idcirco

$$1 = \tan g.z\,\tan g.\left(\tfrac{\pi}{2} - z\right),$$
$$1 = \tan g.z\,\tan g.\left(\tfrac{\pi}{4} - z\right)\tan g.\left(\tfrac{\pi}{4} + z\right)\tan g.\left(\tfrac{2\pi}{4} - z\right),$$
$$1 = \tan g.z\,\tan g.\left(\tfrac{\pi}{6} - z\right)\tan g.\left(\tfrac{\pi}{6} + z\right)\tan g.\left(\tfrac{2\pi}{6} - z\right)\tan g.\left(\tfrac{2\pi}{6} + z\right)\tan g.\left(\tfrac{3\pi}{6} - z\right)$$

etc.

Harum vero aequationum ratio statim sponte in oculos incurrit, cum perpetuo bini anguli reperiantur, quorum alter est alterius complementum ad rectum. Huiusmodi ergo binorum angulorum tangentes productum dant $=1$ ideoque omnium productum unitati debet esse aequale.

258. Quoniam sinus et cosinus angulorum progressionem arithmeticam constituentium seriem recurrentem praebent, per caput praecedens summa huiusmodi sinuum et cosinuum quotcunque exhiberi poterit. Sint anguli in arithmetica progressione

$$a,\ a+b,\ a+2b,\ a+3b,\ a+4b,\ a+5b \ \text{ etc.}$$

et quaeratur primo summa sinuum horum angulorum in infinitum progredientium;

ponatur ergo

$$s = \sin.a + \sin.(a+b) + \sin.(a+2b) + \sin.(a+3b) + \text{etc.},$$

et quia haec series est recurrens, cuius scala relationis est $2\cos.b, -1$, orietur haec series ex evolutione fractionis, cuius denominator est

$$1 - 2z\cos.b + zz$$

posito $z = 1$. Ipsa vero fractio erit

$$= \frac{\sin.a + z(\sin.(a+b) - 2\sin.a\cos.b)}{1 - 2z\cos.b + zz};$$

quare facto $z = 1$ erit

$$s = \frac{\sin.a + \sin.(a+b) - 2\sin.a\cos.b}{2 - 2\cos.b} = \frac{\sin.a - \sin.(a-b)}{2(1-\cos.b)}$$

ob

$$2\sin.a\cos.b = \sin.(a+b) + \sin.(a-b).$$

Cum autem sit

$$\sin.f - \sin.g = 2\cos.\tfrac{f+g}{2}\sin.\tfrac{f-g}{2},$$

erit

$$\sin.a - \sin.(a-b) = 2\cos.\left(a - \tfrac{1}{2}b\right)\sin.\tfrac{1}{2}b;$$

at

$$1 - \cos.b = 2\left(\sin.\tfrac{1}{2}b\right)^2,$$

unde erit

$$s = \frac{\cos.\left(a - \frac{1}{2}b\right)}{2\sin.\frac{1}{2}b}.$$

259. Hinc itaque summa quotcunque sinuum, quorum arcus in arithmetica progressione incedunt, assignari poterit. Quaeratur nempe summa huius progressionis

$$\sin.a + \sin.(a+b) + \sin.(a+2b) + \sin.(a+3b) + \cdots + \sin.(a+nb).$$

Quia summa huius progressionis in infinitum continuatae est

$$s = \frac{\cos.\left(a - \frac{1}{2}b\right)}{2\sin.\frac{1}{2}b}.$$

considerentur termini ultimum sequentes in infinitum hi

$$\sin.\big(a+(n+1)b\big)+\sin.\big(a+(n+2)b\big)+\sin.\big(a+(n+3)b\big)+\text{etc.};$$

quia horum sinuum summa est

$$=\frac{\cos.\big(a+(n+\frac{1}{2})b\big)}{2\sin.\frac{1}{2}b}.$$

si haec a priori subtrahatur, remanebit summa quaesita. Scilicet, si fuerit

$$\sin.a+\sin.(a+b)+\sin.(a+2b)+\sin.(a+3b)+\cdots+\sin.(a+nb),$$

erit

$$s=\frac{\cos.(a-\frac{1}{2}b)-\cos.\big(a+(n+\frac{1}{2})b\big)}{2\sin.\frac{1}{2}b}=\frac{\sin.(a+\frac{1}{2}nb)\sin.\frac{1}{2}(n+1)b}{\sin.\frac{1}{2}b}.$$

260. Pari modo si consideretur summa cosinuum atque ponatur
$s=\cos.a+\cos.(a+b)+\cos.(a+2b)+\cos.(a+3b)+\text{etc.}.$ in infinitum,
erit

$$s=\frac{\cos.a+z\big(\cos.(a+b)-2\cos.a\cos.b\big)}{1-2z\cos.b+zz}$$

posito $z=1$. Quare ob

$$2\cos.a\cos.b=\cos.(a-b)+\cos.(a+b)$$

fiet

$$s=\frac{\cos.a-\cos.(a-b)}{2(1-\cos.b)}$$

At est

$$\cos.f-\cos.g=2\sin.\frac{f+g}{2}\sin.\frac{g-f}{2};$$

unde erit

$$\cos.a-\cos.(a-b)=-2\sin.\big(a-\tfrac{1}{2}b\big)\sin.\tfrac{1}{2}b,$$

et ob

$$1-\cos.b=2\big(\sin.\tfrac{1}{2}b\big)^2$$

erit

$$s=-\frac{\sin.(a-\frac{1}{2}b)}{2\sin.\frac{1}{2}b}.$$

Quare, cum simili modo sit huius seriei

$$\cos.\big(a+(n+1)b\big)+\cos.\big(a+(n+2)b\big)+\cos.\big(a+(n+3)b\big)+\text{etc.}$$

summa

$$s = -\frac{\sin.\left(a+\left(n+\frac{1}{2}\right)b\right)}{2\sin.\frac{1}{2}b},$$

si haec ab illa subtrahatur, relinquetur summa huius seriei

$$s = \cos.a + \cos.(a+b) + \cos.(a+2b) + \cos.(a+3b) + \text{etc.}$$

eritque

$$s = \frac{-\sin.\left(a-\frac{1}{2}b\right)+\sin.\left(a+\left(n+\frac{1}{2}\right)b\right)}{2\sin.\frac{1}{2}b} = \frac{\cos.\left(a+\frac{1}{2}nb\right)\sin.\frac{1}{2}(n+1)b}{\sin.\frac{1}{2}b}.$$

261. Plurimae aliae quaestiones circa sinus et tangentes ex principlis allatis resolvi possent; cuiusmodi sunt, si quadrata altioresve potestates sinuum tangentiumve summari deberent; verum quia haec ex reliquis aequationum superiorum coefficientibus similiter derivantur, iis hic diutius non immoror. Quod autem ad has postremas summationes attinet, notandum est quamcunque sinuum cosinuumque potestatem per singulos sinus cosinusve explicari posse, quod, ut clarius perspiciatur, breviter exponamus.

262. Ad hoc expediendum iuvabit ex praecedentibus haec lemmata deprompsisse

$$2\sin.a\sin.z = \cos.(a-z) - \cos.(a+z),$$
$$2\cos.a\sin.z = \sin.(a+z) - \sin.(a-z),$$
$$2\sin.a\cos.z = \sin.(a+z) + \sin.(a-z),$$
$$2\cos.a\cos.z = \cos.(a-z) + \cos.(a+z).$$

Hinc igitur primum potestates sinuum reperiuntur:

$$\sin.z = \sin.z,$$
$$2(\sin.z)^2 = 1 - \cos.2z,$$
$$4(\sin.z)^3 = 3\sin.z - \sin.3z,$$
$$8(\sin.z)^4 = 3 - 4\cos.2z + \cos.4z,$$
$$16(\sin.z)^5 = 10\sin.z - 5\sin.3z + \sin.5z,$$
$$32(\sin.z)^6 = 10 - 15\cos.2z + 6\cos.4z - \cos.6z,$$
$$64(\sin.z)^7 = 35\sin.z - 21\sin.3z + 7\sin.5z - \sin.7z,$$
$$128(\sin.z)^8 = 35 - 56\cos.2z + 28\cos.4z - 8\cos.6z + \cos.8z,$$
$$256(\sin.z)^9 = 126\sin.z - 84\sin.3z + 36\sin.5z - 9\sin.7z + \sin.9z$$
etc.

Lex, qua hi coefficientes progrediuntur, ex unciis binomii elevati intelligitur, nisi quod numerus absolutus in potestatibus paribus semissis tantum sit eius, quem unciae praebent.

263. Pari modo potestates cosinuum definientur:

$$\cos.z = \cos.z,$$
$$2(\cos.z)^2 = 1 - \cos.2z,$$
$$4(\cos.z)^3 = 3\cos.z + \cos.3z,$$
$$8(\cos.z)^4 = 3 + 4\cos.2z + \cos.4z,$$
$$16(\cos.z)^5 = 10\cos.z + 5\cos.3z + \cos.5z,$$
$$32(\cos.z)^6 = 10 + 15\cos.2z + 6\cos.4z + \cos.6z,$$
$$64(\cos.z)^7 = 35\cos.z + 21\cos.3z + 7\cos.5z + \cos.7z,$$
etc.

Hic ratione legis progressionis eadem sunt monenda, quae circa sinus notavimus.