# CHAPTER XIII

# ABOUT RECURRING SERIES

211. For this kind of series, which De Moivre was accustomed to call *recurring*, I refer here to all series, which arise from the expansion of each fractional function by putting in place actual division. For above now we have shown how these series can be prepared, so that any term may be determined from some number of terms according to a certain constant law, which law depends on the denominator of the fractional function. But since now I will show how some fractional function may be resolved into other simpler forms, hence also a recurring series may be resolved into other simpler series. Therefore in this chapter the resolution of any recurring series of any order is proposed to be set out in terms of simpler series.

212. Let this be the proposed general fractional function

$$\frac{a+bz+czz+dz^3+\text{etc.}}{1-\alpha bz-\beta zz-\gamma z^3-\delta z^4-\text{etc.}},$$

which may be expanded out by division into this recurring series :

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.};$$

the coefficients of which may be progressing in some manner, as we have shown above. But if now that fractional function may be resolved into its simple fractions and each may be set out in a recurring series, it is evident that the sum of all these series arising from the partial fractions must be equal to the recurring series

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.}$$

Therefore the partial fractions, which we have shown how to discover above, will give the partial series, of which the natures may be seen easily on account of the simplicity ; but all partial series taken together produce the proposed recurring series, from which its nature will be understood completely.

213. These shall be the recurring series arising from the individual partial fractions:

$$\begin{array}{l} a + bz + czz + dz^3 + ez^4 + \text{etc.}, \\ a' + b'z + c'zz + d'z^3 + e'z^4 + \text{etc.}, \\ a'' + b''z + c''zz + d''z^3 + e''z^4 + \text{etc.}, \\ a''' + b'''z + c'''zz + d'''z^3 + e'''z^4 + \text{etc.}, \\ \text{etc.} \end{array}$$

Because these series taken together must be equal to this series :

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

it is necessary that there shall be

$$\begin{array}{l} A = a + a' + a'' + a''' + \text{etc.}, \\ B = b + b' + b'' + b''' + \text{etc.}, \\ C = c + c' + c'' + c''' + \text{etc.}, \\ D = d + d' + d'' + d''' + \text{etc.} \\ \text{etc.} \end{array}$$

Hence, if the coefficients of the power $z^n$ of the individual series arising from the partial fractions are able to be defined, the sum of these will give the coefficient of the power $z^n$ in the recurring series $A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$

214. Some doubt may arise here, whether, if two series of this kind were equal to each other, by necessity thence it may follow that the coefficients of the same powers of *z* may be equal to each other,

$$A + Bz + Cz^2 + Dz^3 + \text{etc.} = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \mathfrak{D}z^3 + \text{etc.}$$

or whether there shall be $A = \mathfrak{A}$, $B = \mathfrak{B}$, $C = \mathfrak{C}$, $D = \mathfrak{D}$, etc. But this doubt may be removed easily, if we demand carefully that this equality must be sustained, whatever the value of the variable *z* may hold. Therefore let $z = 0$ and evidently $A = \mathfrak{A}$. Therefore from these with the equal terms on both sides subtracted, and the remaining equations divided by *z* there will be found :

$$B + Cz + Dz^2 + \text{etc.} = \mathfrak{B} + \mathfrak{C}z + \mathfrak{D}z^2 + \text{etc.},$$

from which $B = \mathfrak{B}$ follows; moreover, in a similar manner it will be shown to be the case that $C = \mathfrak{C}$, $D = \mathfrak{D}$ and thus again indefinitely.

215. Therefore, we may consider the series, which arise from partial fractions in which some partial fraction proposed is resolved. And indeed in the first place it is apparent the fraction

$$\frac{\mathfrak{A}}{1-pz}$$

gives the series

$$\mathfrak{A} + \mathfrak{A}pz + \mathfrak{A}p^2z^2 + \mathfrak{A}p^3z^3 + \text{etc.},$$

the general term of which is

$$\mathfrak{A}p^nz^n;$$

for this is usually called the *general term*, because from that all the numbers on being substituted in place of *n* successively give rise to all the terms of the series.
Then from the fraction

$$\frac{\mathfrak{A}}{(1-pz)^2}$$

the series arises

$$\mathfrak{A} + 2\mathfrak{A}pz + 3\mathfrak{A}p^2z^2 + 4\mathfrak{A}p^3z^3 + \text{etc.},$$

the general term of which is

$$(n+1)\mathfrak{A}p^nz^n.$$

Then from the fraction

$$\frac{\mathfrak{A}}{(1-pz)^3}$$

the series arises

$$\mathfrak{A} + 3\mathfrak{A}pz + 6\mathfrak{A}p^2z^2 + 10\mathfrak{A}p^3z^3 + \text{etc.}$$

the general term of which is

$$\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^nz^n.$$

Moreover generally the fraction

$$\frac{\mathfrak{A}}{(1-pz)^k}$$

gives rise to this series :

$$\mathfrak{A}+k\mathfrak{A}pz+\frac{k(k+1)}{1\cdot 2}\mathfrak{A}p^2z^2+\frac{k(k+1)(k+2)}{1\cdot 2\cdot 3}\mathfrak{A}p^3z^3+\text{etc.},$$

of which the general term is

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1\cdot 2\cdot 3\cdots(k-1)}\mathfrak{A}p^nz^n.$$

But from the progression of the series this term is deduced also

$$\frac{k(k+1)(k+2)\cdots(n+k-1)}{1\cdot 2\cdot 3\cdots n}\mathfrak{A}p^nz^n,$$

Truly this expression is equal to that, that which will become apparent by cross–multiplication put in place; for there becomes

$$1\cdot 2\cdot 3\cdots n(n+1)\cdots(n+k-1)=1\cdot 2\cdot 3\cdots(k-1)k\cdots(k+n-1),$$

which is an identical equation.

216. Therefore just as many partial fractions of this kind $\frac{\mathfrak{A}}{(1-pz)^k}$ are come upon in the resolution of fractional functions into to partial fractions, as one can assign to the total general term of the recurrent series arising, from that fractional function

$$A+Bz+Cz^2+Dz^3+\text{etc.},$$

evidently which will be the sum of the series of the terms of the general series, which arises from the partial fractions.

## EXAMPLE 1

To find the general term of the recurring series, which arises from this fraction

$$\frac{1-z}{1-z-2zz}.$$

Hence the series arising [from division] is

$$1+0z+2zz+2z^3+6z^4+10z^4+22z^6+42z^7+86z^8+\text{etc}.$$

Towards finding the coefficient of the general power $z^n$, the fraction $\frac{1-z}{1-z-2zz}$ may be resolved into

$$\frac{\frac{2}{3}}{1+z}+\frac{\frac{1}{3}}{1-2z},$$

from which the general term sought arises

$$\left(\tfrac{2}{3}(-1)^n+\tfrac{1}{3}\cdot 2^n\right)z^n=\tfrac{2^n\pm 2}{3}z^n$$

where the + sign prevails, if *n* shall be an even number, and the – sign, if *n* shall be odd.

EXAMPLE 2

To find the general term of the recurring series, which arises from the fraction

$$\frac{1-z}{1-5z+6zz},$$

or of this series

$$1+4z+14zz+46z^3+146z^4+454z^4+\text{etc}.$$

On account of the denominator $=(1-2z)(1-3z)$, the fraction is resolved into these terms

$$=\frac{-1}{1-2z}+\frac{2}{1-3z},$$

from which the general term becomes

$$2\cdot 3^n z^n-2^n z^n=\left(2\cdot 3^n-2^n\right)z^n.$$

EXAMPLE 3

To find the general term of this series

$$1+3z+4zz+7z^3+11z^4+18z^4+29z^5+47z^6+\text{etc.},$$

which arises from the expansion of the fraction

$$\frac{1+2z}{1-z-zz}.$$

On account of the factors of the denominator

$$1-\frac{1+\sqrt{5}}{2}z \text{ and } 1-\frac{1-\sqrt{5}}{2}z$$

by resolution

$$\frac{\frac{1+\sqrt{5}}{2}}{1-\frac{1+\sqrt{5}}{2}z}+\frac{\frac{1-\sqrt{5}}{2}}{1-\frac{1-\sqrt{5}}{2}z}$$

will appear, from which the general term will be

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}z^n+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}z^n.$$

EXAMPLE 4

To find the general term of this series

$$a+(\alpha a+b)z+\left(\alpha^2a+\alpha b+\beta a\right)z^2+\left(\alpha^3a+\alpha^2b+2\alpha\beta a+\beta b\right)z^3+\text{etc.},$$

which arises from the expansion of the fraction

$$\frac{a+bz}{1-\alpha z-\beta zz}.$$

These two fractions arise by resolution :

$$\frac{\left(a\left(\sqrt{(\alpha\alpha+4\beta)}+\alpha\right)+2b\right):2\sqrt{(\alpha\alpha+4\beta)}}{1-\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2}z}+\frac{\left(a\left(\sqrt{(\alpha\alpha+4\beta)}-\alpha\right)-2b\right):2\sqrt{(\alpha\alpha+4\beta)}}{1-\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2}z};$$

hence the general term will be [from the geometric progressions involved:]

$$\frac{a\left(\sqrt{(\alpha\alpha+4\beta)}+\alpha\right)+2b:}{2\sqrt{(\alpha\alpha+4\beta)}}\left(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2}\right)^n z^n+\frac{a\left(\sqrt{(\alpha\alpha+4\beta)}-\alpha\right)-2b:}{2\sqrt{(\alpha\alpha+4\beta)}}\left(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2}\right)^n$$

From which all of the recurring series, of which any term may be determined by the two preceding, will be able to define the general terms readily.

EXAMPLE 5

To find the general term of this series

$$1+z+2z^2+2z^3+3z^4+3z^5+4z^6+4z^7+\text{etc.},$$

which arises from the fraction

$$\frac{1}{1-z-zz+z^3}=\frac{1}{(1-z)^2(1+z)}.$$

Though the law of the progression thus may be clear from examination, so that it may not need an explanation, yet the fractions

$$\frac{\frac{1}{2}}{(1-z)^2}+\frac{\frac{1}{4}}{1-z}+\frac{\frac{1}{4}}{1+z}$$

arising by resolution give this general term

$$\tfrac{1}{2}(n+1)z^n+\tfrac{1}{4}z^n+\tfrac{1}{4}(-1)^n z^n=\tfrac{2n+3\pm 1}{4}z^n$$

where the upper sign prevails if $n$ should be an even number, and the lower, if $n$ were odd.

217. With this agreed on, the general terms of all recurring series are able to be shown, because it is allowed to resolve the partial fractions into partial fractions of this kind. But if moreover we wish to avoid imaginary expressions, on many occasions partial fractions of this kind will be come upon

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz},\quad \frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2},\cdots\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^k};$$

from the expansion series of this kind arise, it is required to be seen. And in the first place indeed, on account of

$$\cos.n\varphi=2\cos.\varphi\cos.(n-1)\varphi-\cos.(n-2)\varphi,$$

the fraction

$$\frac{\mathfrak{A}}{1-2pz\cos.\varphi+ppzz}$$

by expansion will give

$$\begin{aligned}&\mathfrak{A}+2\mathfrak{A}pz\cos.\varphi+2\mathfrak{A}ppzz\cos.2\varphi+2\mathfrak{A}p^3z^3\cos.3\varphi+2\mathfrak{A}p^4z^4\cos.4\varphi+\text{etc.}\\&\qquad+\mathfrak{A}ppzz\qquad\qquad+2\mathfrak{A}p^3z^3\cos.\varphi\ \ +2\mathfrak{A}p^4z^4\cos.2\varphi+\text{etc.}\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\mathfrak{A}p^4z^4\qquad\quad+\text{etc.}\\&\qquad\qquad\qquad\qquad\text{etc.,}\end{aligned}$$

of which the general term is not yet readily apparent.

218. Therefore so that we may reach the goal, we will consider these two series

$$Ppz\sin.\varphi+Pp^2z^2\sin.2\varphi+Pp^3z^3\sin.3\varphi+Pp^4z^4\sin.4\varphi+\text{etc.},$$
$$Q+Qpz\cos.\varphi+Qp^2z^2\cos.2\varphi+Qp^3z^3\cos.3\varphi+Qp^4z^4\cos.4\varphi+\text{etc.},$$

which two series certainly are produced from the expansion of the fraction, of which the denominator is

$$1-2pz\cos.\varphi+ppzz.$$

And the former expression certainly arises from this fraction :

$$\frac{Ppz\sin.\varphi}{1-2pz\cos.\varphi+ppzz},$$

and truly the latter from this :

$$\frac{Q-Qpz\cos.\varphi}{1-2pz\cos.\varphi+ppzz}$$

These two fractions may be added and the sum

$$\frac{Q+Ppz\sin.\varphi-Qpz\cos.\varphi}{1-2pz\cos.\varphi+ppzz}$$

will give the series, of which the general term will be

$$\left(P\sin.n\varphi+Q\cos.n\varphi\right)p^nz^n.$$

But this fraction is made equal to the proposed fraction

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz};$$

there will be

$$Q=\mathfrak{A}\quad\text{and}\quad P=\mathfrak{A}\cot.\varphi+\mathfrak{B}\operatorname{cosec}.\varphi.$$

Therefore from this fraction

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

the general term of the proposed series will be

$$\frac{\mathfrak{A}\cos.\varphi\sin.n\varphi+\mathfrak{B}\sin.n\varphi+\mathfrak{A}\sin.\varphi\cos.n\varphi}{\sin.\varphi}p^nz^n=\frac{\mathfrak{A}\sin.(n+1)\varphi+\mathfrak{B}\sin.n\varphi}{\sin.\varphi}p^nz^n.$$

219. Towards finding the general term, if the denominator of the fraction were a power so that

$$(1-2pz\cos.\varphi+ppzz)^k,$$

it will be appropriate that the fraction be resolved into two fractions even if imaginary

$$\frac{a}{\left(1-\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)pz\right)^k}+\frac{b}{\left(1-\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)pz\right)^k},$$

the term of the general series likewise of which taken, arising from these will be

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1.2.3\cdots(k-1)}\left(cos.n\varphi+\sqrt{-1}\cdot\sin.n\varphi\right)ap^nz^n$$
$$+\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1.2.3\cdots(k-1)}\left(cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi\right)bp^nz^n.$$

Let

$$a+b=f,\quad a-b=\frac{g}{\sqrt{-1}},$$

so that there shall be

$$\frac{f\sqrt{-1}+g}{2\sqrt{-1}}\quad\text{and}\quad b=\frac{f\sqrt{-1}-g}{2\sqrt{-1}},$$

and this expression

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1.2.3\cdots(k-1)}\left(f\,cos.n\varphi+g\sin.n\varphi\right)p^nz^n$$

will be the general term of the series, which arises from these fractions

$$\frac{\frac{1}{2}f+\frac{1}{2\sqrt{-1}}g}{\left(1-\left(cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)pz\right)^k}+\frac{\frac{1}{2}f-\frac{1}{2\sqrt{-1}}g}{\left(1-\left(cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)pz\right)^k},$$

or which arises from this single fraction

$$\frac{\left\{\begin{array}{l} f-kfpz\cos.\varphi+\frac{k(k-1)}{1\cdot2}fp^2z^2\cos.2\varphi-\frac{k(k-1)(k-2)}{1\cdot2\cdot3}fp^3z^3\cos.3\varphi+\text{etc.}\\ \quad kgpz\sin.\varphi-\frac{k(k-1)}{1\cdot2}gp^2z^2\sin.2\varphi+\frac{k(k-1)(k-2)}{1\cdot2\cdot3}gp^3z^3\sin.3\varphi-\text{etc.}\end{array}\right\}}{(1-2pz\cos.\varphi+ppzz)^k}$$

220. Therefore on putting $k=2$, the general term of this series

$$\frac{f-2pz(f\cos.\varphi-g\sin.\varphi)+ppzz(f\cos.2\varphi-g\sin.2\varphi)}{(1-2pz\cos.\varphi+ppzz)^2}$$

arising from this fraction will be

$$(n+1)(f\ \cos.n\varphi+g\ \sin.n\varphi)\,p^nz^n$$

But the general term of the series arising from this fraction, see § 218,

$$\frac{a}{1-2pz\ \cos.\varphi+ppzz}$$

or this

$$\frac{a-2apz\cos.\varphi+appzz}{(1-2pz\cos.\varphi+ppzz)^2}$$

is

$$\frac{a\sin.(n+1)\varphi}{\sin.\varphi}p^nz^n.$$

These fractions may be added in turn and putting

$$a+f=\mathfrak{A},$$
$$2a\cos.\varphi+2f cos.\varphi-2g\sin.\varphi=-\mathfrak{B}$$

and

$$a+f\ \cos.2\varphi-g\ \sin.2\varphi=0;$$

hence there will be

$$g=\frac{\mathfrak{B}+2\mathfrak{A}\cos.\varphi}{2\sin.\varphi},$$

$$a=\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{1-\cos.2\varphi}=\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^2}$$

and

$$f=\frac{-\mathfrak{A}\cos.2\varphi-\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^2}$$

and

$$g=\frac{\mathfrak{B}\sin.\varphi+\mathfrak{A}\sin.2\varphi}{2(\sin.\varphi)^2}.$$

On this account, the general term of the series arising from this fraction

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2}$$

is

$$\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^3}\sin.(n+1)\varphi\cdot p^nz^n+(n+1)\frac{\mathfrak{B}\sin.\varphi\sin.n\varphi+\mathfrak{A}\sin.2\varphi\sin.n\varphi-\mathfrak{B}\cos.\varphi\cos.n\varphi-\mathfrak{A}\cos.2\varphi\cos.n\varphi}{2(\sin.\varphi)^2}\cdot p^nz^n$$
$$=-\frac{(n+1)(\mathfrak{A}\cos.(n+2)\varphi+\mathfrak{B}\cos.(n+1)\varphi)}{2(\sin.\varphi)^2}\cdot p^nz^n+\frac{(\mathfrak{A}+\mathfrak{B}\cos.\varphi)\sin.(n+1)\varphi}{2(\sin.\varphi)^3}\cdot p^nz^n$$
$$=\frac{\frac{1}{2}(n+3)\sin.(n+1)\varphi-\frac{1}{2}(n+1)\sin.(n+3)\varphi}{2(\sin.\varphi)^3}\cdot\mathfrak{A}p^nz^n+\frac{\frac{1}{2}(n+2)\sin.n\varphi-\frac{1}{2}n\sin.(n+2)\varphi}{2(\sin.\varphi)^3}\cdot\mathfrak{B}p^nz^n.$$

Therefore the general term itself sought of the series

$$=\frac{(n+3)\sin.(n+1)\varphi-(n+1)\sin.(n+3)\varphi}{4(\sin.\varphi)^3}\cdot\mathfrak{A}p^nz^n+\frac{(n+2)\sin.n\varphi-n\sin.(n+2)\varphi}{4(\sin.\varphi)^3}\cdot\mathfrak{B}p^nz^n$$

which arises from the fraction

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2}.$$

221. Let $k=3$, and the general term of the series arising from this fraction

$$\frac{f-3pz(f\cos.\varphi-g\sin.\varphi)+3ppzz(f\cos.2\varphi-g\sin.2\varphi)-p^3z^3(f\cos.3\varphi-g\sin.3\varphi)}{(1-2pz\cos.\varphi+ppzz)^3}$$

will be

$$\frac{(n+1)(n+2)}{1\cdot 2}(f\cos.n\varphi+g\sin.n\varphi)p^nz^n.$$

Then the general term of the series arising from the fraction

$$\frac{a+bpz}{(1-2pz\cos.\varphi+ppzz)^2}$$

or from this

$$\frac{a-(2a\cos.\varphi-b)pz+(a-2b\cos.\varphi)ppzz+bp^3z^3}{(1-2pz\cos.\varphi+ppzz)^3}$$

is

$$=\frac{(n+3)\sin.(n+1)\varphi-(n+1)\sin.(n+3)\varphi}{4(\sin.\varphi)^3}\cdot ap^nz^n+\frac{(n+2)\sin.n\varphi-n\sin.(n+2)\varphi}{4(\sin.\varphi)^3}\cdot bp^nz^n$$

These fractions may be added and the numerator is put $=\mathfrak{A}$; there will be

$$a+f=\mathfrak{A},$$
$$3f\cos.\varphi-3g\sin.\varphi+2a\cos.\varphi-b=0,$$
$$3f\cos.2\varphi-3g\sin.2\varphi+a-2b\cos.\varphi=0$$

and

$$b=f\cos.3\varphi-g\sin.3\varphi;$$

hence there will be

$$a=\frac{f\cos.3\varphi-g\sin.3\varphi-3f\cos.\varphi+3g\sin.\varphi}{2\cos.\varphi}$$
$$=2g(\sin.\varphi)^2\operatorname{tang}.\varphi-f-2f(\sin.\varphi)^2.$$

.
Then

$$\frac{f}{g}=\frac{\sin.5\varphi-2\sin.3\varphi+\sin.\varphi}{\cos.5\varphi-2\cos.3\varphi+\cos.\varphi}$$

is found, and

$$a+f=\mathfrak{A}=2g(\sin.\varphi)^2\operatorname{tang}.\varphi-2f(\sin.\varphi)^2,$$

therefore

$$\frac{\mathfrak{A}}{2(\sin.\varphi)^2}=\frac{g\sin.\varphi-f\cos.\varphi}{\cos.\varphi};$$

from which finally there emerges

$$f=\frac{\mathfrak{A}(\sin.\varphi-2\sin.3\varphi+\sin.5\varphi)}{16(\sin.\varphi)^5},$$
$$g=\frac{\mathfrak{A}(\cos.\varphi-2\cos.3\varphi+\cos.5\varphi)}{16(\sin.\varphi)^5}.$$

On account of

$$16(\sin.\varphi)^5 = \sin.5\varphi - 5\ \sin.3\varphi + 10\ \sin.\varphi$$

there will be

$$a = \frac{\mathfrak{A}(9\sin.\varphi - 3\sin.3\varphi)}{16(\sin.\varphi)^5}$$

and

$$b = \frac{\mathfrak{A}(-\sin.2\varphi + \sin.2\varphi)}{16(\sin.\varphi)^5} = 0.$$

But there is

$$3\ \sin.\varphi - \sin.3\varphi = 4(\sin.\varphi)^3,$$

therefore

$$a = \frac{3\mathfrak{A}}{4(\sin.\varphi)^2}.$$

On account of which the general term will be

$$\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^n z^n \frac{\sin.(n+1)\varphi - 2\sin.(n+3)\varphi + \sin.(n+5)\varphi}{16(\sin.\varphi)^5}$$
$$+ 3\mathfrak{A}p^n z^n \frac{(n+3)\sin.(n+1)\varphi - (n+1)\sin.(n+3)\varphi}{16(\sin.\varphi)^5}$$

$$= \frac{\mathfrak{A}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{matrix} \frac{(n+4)(n+5)}{1\cdot 2}\sin.(n+1)\varphi - \frac{2(n+1)(n+5)}{1\cdot 2}\sin.(n+3)\varphi \\ + \frac{(n+1)(n+2)}{1\cdot 2}\sin.(n+5)\varphi \end{matrix}\right\}.$$

222. Therefore the general term, which arises from this fraction

$$\frac{\mathfrak{A} + \mathfrak{B}pz}{(1 - 2pz\cos.\varphi + ppzz)^3},$$

here will be

$$\frac{\mathfrak{A}p^nz^n}{16(\sin.\varphi)^5}\left\{\begin{array}{c}\frac{(n+5)(n+4)}{1\cdot 2}\sin.(n+1)\varphi-\frac{2(n+1)(n+5)}{1\cdot 2}\sin.(n+3)\varphi\\ +\frac{(n+1)(n+2)}{1\cdot 2}\sin.(n+5)\varphi\end{array}\right\}$$

$$+\frac{\mathfrak{B}p^nz^n}{16(\sin.\varphi)^5}\left\{\begin{array}{c}\frac{(n+4)(n+3)}{1\cdot 2}\sin.n\varphi-\frac{2n(n+4)}{1\cdot 2}\sin.(n+2)\varphi\\ +\frac{n(n+1)}{1\cdot 2}\sin.(n+4)\varphi\end{array}\right\}.$$

And by progressing further, the general term of the series which arises from this fraction

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^4},$$

will be this

$$\frac{\mathfrak{A}p^nz^n}{64(\sin.\varphi)^5}\left\{\begin{array}{l}\frac{(n+7)(n+6)(n+5)}{1\cdot 2\cdot 3}\sin.(n+1)\varphi-\frac{3(n+1)(n+7)(n+6)}{1\cdot 2\cdot 3}\sin.(n+3)\varphi\\ +\frac{3(n+1)(n+2)(n+7)}{1\cdot 2\cdot 3}\sin.(n+5)\varphi-\frac{(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3}\sin.(n+7)\varphi\end{array}\right\}$$

$$+\frac{\mathfrak{B}p^nz^n}{64(\sin.\varphi)^7}\left\{\begin{array}{l}\frac{(n+6)(n+5)(n+4)}{1\cdot 2\cdot 3}\sin.n\varphi-\frac{3n(n+6)(n+5)}{1\cdot 2\cdot 3}\sin.(n+2)\varphi\\ +\frac{3n(n+1)(n+6)}{1\cdot 2\cdot 3}\sin.(n+4)\varphi-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin.(n+6)\varphi\end{array}\right\}.$$

Moreover from these expressions it is understood readily, in what manner the form of the general terms may progress for higher powers. Towards a thorough understanding of these expressions, truly it is appropriate to note that

$$\sin.\varphi = \sin.\varphi,$$
$$4(\sin.\varphi)^3 = 3\sin.\varphi - \sin.3\varphi,$$
$$16(\sin.\varphi)^5 = 10\sin.\varphi - 5\sin.3\varphi + \sin.5\varphi,$$
$$64(\sin.\varphi)^7 = 35\sin.\varphi - 21\sin.3\varphi + 7\sin.5\varphi - \sin.7\varphi,$$
$$256(\sin.\varphi)^9 = 126\sin.\varphi - 84\sin.3\varphi + 36\sin.5\varphi - 9\sin 7\varphi + \sin.9\varphi$$

etc.

223. Therefore with this agreed to, all the fractional functions are able to be resolved into real partial fractions, likewise the general terms of all the recurring series can be shown by real expressions. So that which may appear clearer, the following examples have been adjoined.

EXAMPLE 1

From the fraction

$$\frac{1}{(1-z)(1-zz)(1-z^3)} = \frac{1}{1-z-zz+z^4+z^5-z^6}$$

this recurring series arises

$$1 + z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + 7z^6 + 8z^7 + 10z^8 + 12z^9 + \text{etc.},$$

the general term of which is desired.

The following proposed ordered fraction arises

$$\frac{1}{(1-z)^3(1+z)(1+z+zz)},$$

which is resolved into these fractions

$$\frac{1}{6(1-z)^3} + \frac{1}{4(1-z)^2} + \frac{17}{72(1-z)} + \frac{1}{8(1+z)} + \frac{2+z}{9(1+z+zz)}.$$

The first of these $\frac{1}{6(1-z)^3}$ gives the general term

$$\frac{(n+1)(n+2)}{1\cdot 2}\cdot\frac{1}{6}z^n = \frac{nn+3n+2}{12}z^n,$$

the second gives $\frac{1}{4(1-z)^2}$ gives

$$\frac{n+1}{4}z^n,$$

the third $\frac{17}{72(1-z)}$ gives

$$\tfrac{17}{72} z^n ,$$

the fourth $\frac{1}{8(1+z)}$ gives

$$\tfrac{1}{8}(-1)^n z^n .$$

Truly the fifth $\frac{2+z}{9(1+z+zz)}$ compared with the form

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz},$$

gives

$$p=-1,\ \ \varphi=\tfrac{\pi}{3}=60^0,\ \ \mathfrak{A}=+\tfrac{2}{9}\ \text{ et }\ \mathfrak{B}=-\tfrac{1}{9},$$

from which the general term arises

$$\frac{2\sin.(n+1)\varphi-sin.n\varphi}{9sin.\varphi}(-1)^n z^n=\frac{4\sin.(n+1)\varphi-2sin.n\varphi}{9\sqrt{3}}(-1)^n z^n$$
$$=\frac{4\sin.\frac{(n+1)\pi}{3}-2sin.\frac{n\pi}{3}}{9\sqrt{3}}(-1)^n z^n .$$

All these expressions may be gathered together into one sum and the proposed general term sought of the proposed series will be produced

$$\left(\tfrac{nn}{12}+\tfrac{n}{2}+\tfrac{47}{72}\right)z^n \pm\tfrac{1}{8}z^n \pm\frac{4\sin.\frac{(n+1)\pi}{3}-2sin.\frac{n\pi}{3}}{9\sqrt{3}}z^n ,$$

where the upper sign prevails, if *n* is an even number, and the lower if it is odd. Where it is to be observed, if *n* should be a number of the form *3m,* to become :

$$\frac{4\sin.\frac{(n+1)\pi}{3}-2sin.\frac{n\pi}{3}}{9\sqrt{3}}=\pm\tfrac{2}{9};$$

if there should b $n=3m+1$, this expression will become $=\mp\frac{1}{9}$ ; but if $n=3m+2$, it will become this expression $\mp\frac{1}{9}$ according as *n* should be either an even or odd number. From these the nature of the series thus can be set out, so that,

| if | then the general term will become |
| --- | --- |
| $n = 6m + 0$ | $\left(\frac{nn}{12} + \frac{n}{2} + 1\right) z^n$ |
| $n = 6m + 1$ | $\left(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}\right) z^n$ |
| $n = 6m + 2$ | $\left(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}\right) z^n$ |
| $n = 6m + 3$ | $\left(\frac{nn}{12} + \frac{n}{2} + \frac{3}{4}\right) z^n$ |
| $n = 6m + 4$ | $\left(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}\right) z^n$ |
| $n = 6m + 5$ | $\left(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}\right) z^n$ |

Thus if $n = 50$, the form $n = 6m + 2$ prevails and the term of the series $= 234z^{50}$.

EXAMPLE 2

From the fraction

$$\frac{1+z+zz}{1\text{-}z\text{-}z^4+z^5}$$

this recurring series arises :

$$1 + 2z + 3zz + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + \text{etc.},$$

the general term of which is required to be found.

The proposed fraction is reduced to this form

$$\frac{1+z+zz}{(1-z)^2(1+z)(1+zz)},$$

which therefore is resolved into these partial fractions

$$\frac{3}{4(1-z)^2} + \frac{3}{8(1-z)} + \frac{1}{8(1+z)} + \frac{-1+z}{4(1+zz)}.$$

The first of these $\frac{3}{4(1-z)^2}$ gives the general term

$$\frac{3(n+1)}{4} z^n$$

the second $\frac{3}{8(1-z)}$ gives

$$\frac{3}{8} z^n$$

the third $\frac{1}{8(1+z)}$ gives

$$\tfrac{1}{8}(-1)^n z^n$$

and the fourth $\frac{-1+z}{4(1+zz)}$ compared with the form

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz},$$

gives

$$p=1,\ \ \varphi=0 \quad\text{and}\quad \varphi=\tfrac{\pi}{2},\ \ \mathfrak{A}=-\tfrac{1}{4},\ \ \mathfrak{B}=+\tfrac{1}{4},$$

from which the general term shall be

$$=\left(-\tfrac{1}{4}\sin.\tfrac{(n+1)\pi}{2}+\tfrac{1}{4}\sin.\tfrac{n\pi}{2}\right)z^n.$$

Whereby on collecting the terms the general term sought will be :

$$=\left(\tfrac{3n}{4}+\tfrac{9}{8}\right)z^n\pm\tfrac{1}{8}z^n-\tfrac{1}{4}\left(\sin.\tfrac{(n+1)\pi}{2}-\tfrac{1}{4}\sin.\tfrac{n\pi}{2}\right)z^n.$$

Hence

| if | the general term will be |
|---|---|
| $n=4m+0$ | $\left(\frac{3n}{4}+1\right)z^n$ |
| $n=4m+1$ | $\left(\frac{3n}{4}+\frac{5}{4}\right)z^n$ |
| $n=4m+2$ | $\left(\frac{3n}{4}+\frac{3}{2}\right)z^n$ |
| $n=4m+3$ | $\left(\frac{3n}{4}+\frac{3}{4}\right)z^n$ |

Thus if $n=50$, $n=4m+2$ will prevail and the term will be $=39z^{50}$.

224. Therefore for the recurring series proposed, because that fraction is recognised easily from which it originates, the general term may be found following the given precepts. Moreover from the law of the recurring series, from which any term is defined from the preceding, the denominator is known at once and the factors of which fractions provide the form of the general term; indeed only the coefficients are determined by the numerator. Truly let this recurring series be proposed

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.},$$

the law of which progression, from which any term may be determined from some number of the preceding, may provide this denominator of the fraction

$$1-\alpha z-\beta z^2-\gamma z^3,$$

thus so that there shall be

$$D=\alpha C+\beta B+\gamma A\ ,\ E=\alpha D+\beta C+\gamma B,\ F=\alpha E+\beta D+\gamma C\ \text{ etc.},$$

which multipliers $+\alpha,\ +\beta,\ +\gamma$ are said to constitute a scale of the relation by De Moivre. Therefore the law of the progression has been put in place by the scale of the relation and the scale of the relation at once provides the denominator of the fraction, from the resolution of which the proposed recurring series originates.

225. Therefore to find the general term or the coefficient of the indefinite power $z^n$ the factors of the denominator $1-\alpha z-\beta z^2-\gamma z^3$ must be found, either simple or two-fold, if we wish to avoid imaginary quantities. In the first place all these simple factors will be unequal amongst themselves and real

$$(1-pz)(1-qz)(1-rz)$$

and the fraction generating the proposed series will be resolved into

$$\frac{\mathfrak{A}}{1-pz}+\frac{\mathfrak{B}}{1-qz}+\frac{\mathfrak{C}}{1-rz};$$

from which the general term of the series will be

$$\left(\mathfrak{A}p^n+\mathfrak{B}q^n+\mathfrak{C}r^n\right)z^n.$$

If two factors were equal, truly $q=p$, then the general term will be of this kind

$$\left(\left(\mathfrak{A}(n+1)+\mathfrak{B}\right)p^n+\mathfrak{C}r^n\right)z^n,$$

and if in addition there were $r=q=p$, the general term will be

$$\left(\mathfrak{A}\frac{(n+1)(n+1)}{12}+\mathfrak{B}(n+1)+\mathfrak{C}\right)p^n z^n$$

But if truly the denominator $1-\alpha z-\beta z^2-\gamma z^3$ should have a two-fold factor, so that there shall be

$$=(1-pz)(1-2qs\cos.\varphi+qqzz),$$

then the general term will be

$$\left(\mathfrak{A}p^n+\frac{\mathfrak{B}\sin.(n+1)\varphi+\mathfrak{C}\sin.n\varphi}{\sin.\varphi}q^n\right)z^n.$$

Therefore since with the numbers 0, 1, 2 put in place successively for $n$ the terms $A,\ Bz,\ Cz^2$, must be produced and the values of the letters $\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C}$ will be determined.

226. Let the scale be from two members or by which a term may be determined from two preceding terms, thus so that there shall be

$$C=\alpha B-\beta A,\ D=\alpha C-\beta B,\ E=\alpha D-\beta C \text{ etc.},$$

and it is evident this recurring series, which shall be

$$A+Bz+Cz^2+Dz^3+Ez^4+\cdots+Pz^n+Qz^{n+1}+\text{etc.},$$

arises from the fraction, the denominator of which shall be

$$1-\alpha z+\beta zz.$$

The factors of this denominator shall be

$$(1-pz)(1-qz);$$

and there will be

$$p+q=\alpha \text{ and } pq=\beta$$

and the general term of the series will be

$$\left(\mathfrak{A}p^n+\mathfrak{B}q^n\right)z^n$$

Hence on making $n=0$ there will be

$$A=\mathfrak{A}+\mathfrak{B}$$

and on making $n = 1$ there will be

$$B=\mathfrak{A}p+\mathfrak{B}q,$$

from which there becomes

$$Aq - B = \mathfrak{A}(q - p)$$

and

$$\mathfrak{A} = \frac{Aq-B}{q-p} \text{ and } \mathfrak{B} = \frac{Ap-B}{p-q}$$

But with the values $\mathfrak{A}$ and $\mathfrak{B}$ found there will be

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n \text{ et } Q = \mathfrak{A}p^{n+1} + \mathfrak{B}q^{n+1}.$$

Then truly there will be

$$\mathfrak{A}\mathfrak{B} = \frac{BB-\alpha AB+\beta AA}{4\beta-\alpha\alpha}.$$

227. Hence the manner can be deduced how any term can be formed from a single preceding term, since according to this rule two terms are required for the progression. For since there shall be

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n \text{ and } Q = \mathfrak{A}p \cdot p^n + \mathfrak{B}q \cdot q^n,$$

there will be

$$Pq - Q = \mathfrak{A}(q - p)p^n \text{ and } Pp - Q = \mathfrak{B}(p - q)q^n,$$

These expressions may be multiplied by each other and there will becomes

$$P^2 pq - (p+q)PQ + QQ + \mathfrak{A}\mathfrak{B}(p-q)^2 p^n q^n = 0.$$

But there is

$$p + q = \alpha, \quad pq = \beta, \quad (p-q)^2 = (p+q)^2 - 4pq = \alpha\alpha - 4\beta \text{ and } p^n q^n = \beta^n.$$

With which substituted the expression becomes

$$\beta P^2 - \alpha PQ + QQ = (\beta AA - \alpha AB + BB)\beta^n$$

or

$$\frac{QQ-\alpha PQ+\beta PP}{BB-\alpha AB+\beta AA} = \beta^n,$$

which is a significant property of recurring series, any term of which is determined by the two preceding terms. But with any term known $P$ the following will be

$$Q = \tfrac{1}{2}\alpha P + \sqrt{\left(\left(\tfrac{1}{4}\alpha^2 - \beta\right)P^2 + (BB - \alpha AB + \beta AA)\beta^n\right)},$$

which expression, even if it appears to become irrational, yet always is rational, because irrational terms do not occur in the series.

228. Again from any two given contiguous terms $Pz^n$ and $Qz^{n+1}$ a much more remote term $Xz^{2n}$ can be assigned conveniently. For let there be put

$$X = fP^2 + gPQ - h\mathfrak{A}\mathfrak{B}\beta^n.$$

Because

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n \text{ and } Q = \mathfrak{A}p \cdot p^n + \mathfrak{B}q \cdot q^n, \text{ and also } X = \mathfrak{A}p^{2n} + \mathfrak{B}q^{2n},$$

there becomes as follows :

$$\begin{array}{rlll} fP^2 = & f\mathfrak{A}^2 p^{2n} & + f\mathfrak{B}^2 q^{2n} & + 2f\mathfrak{A}\mathfrak{B}\beta^n \\ qPQ = & g\mathfrak{A}^2 p \cdot p^{2n} & + g\mathfrak{B}^2 q \cdot q^{2n} & + g\mathfrak{A}\mathfrak{B}\alpha\beta^n \\ -h\mathfrak{A}\mathfrak{B}\beta^n = & & & - h\mathfrak{A}\mathfrak{B}\beta^n \\ \hline X = & \mathfrak{A}p^{2n} + \mathfrak{B}q^{2n} & & \end{array}$$

Therefore there becomes

$$f + gp = \tfrac{1}{\mathfrak{A}}, \quad f + gq = \tfrac{1}{\mathfrak{B}}, \quad \text{and } h = 2f + g\alpha,$$

from which

$$g = \tfrac{\mathfrak{B}-\mathfrak{A}}{\mathfrak{A}\mathfrak{B}(p-q)} \quad \text{and} \quad f = \tfrac{\mathfrak{A}p-\mathfrak{B}q}{\mathfrak{A}\mathfrak{B}(p-q)}.$$

But there is

$$\mathfrak{B} - \mathfrak{A} = \tfrac{\alpha A-2B}{p-q} \quad \text{and} \quad \mathfrak{A}p - \mathfrak{B}q = \tfrac{\alpha B-2A\beta}{p-q}.$$

Therefore

$$f = \tfrac{\alpha B-2A\beta}{\mathfrak{A}\mathfrak{B}(\alpha\alpha-4\beta)} \quad \text{et} \quad g = \tfrac{\alpha A-2B}{\mathfrak{A}\mathfrak{B}(\alpha\alpha-4\beta)}$$

or

$$f = \tfrac{2A\beta-\alpha B}{BB-\alpha AB+\beta AA} \quad \text{and} \quad g = \tfrac{2B-\alpha A}{BB-\alpha AB+\beta AA};$$

and thus

$$h = \tfrac{(4\beta-\alpha\alpha)A}{BB-\alpha AB+\beta AA}.$$

Therefore there will be

$$X = \tfrac{(2A\beta-\alpha B)P^2+(2B-\alpha A)PQ}{BB-\alpha AB+\beta AA} - A\beta^n.$$

Indeed in a similar manner there is found :

$$X = \frac{(\alpha\beta A - (\alpha\alpha - 2\beta)B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \beta AA)} - \frac{2B\beta^n}{\alpha}.$$

With these joined together through the elimination of the term $\beta^n$ there is found:

$$X = \frac{(\beta A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \beta AA}.$$

229. In a like manner if the following terms may be put in place :

$$A + Bz + Cz^2 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \cdots + Xz^{2n} + Yz^{2n+1} + Zz^{2n+2} + \text{etc.},$$

there will be

$$Z = \frac{(\beta A - \alpha B)Q^2 + 2BQR - ARR}{BB - \alpha AB + \beta AA}$$

and on account of $R = \alpha Q - \beta P$ there will be

$$Z = \frac{-\beta\beta AP^2 + 2\beta(\alpha A - B)PQ + (\alpha B - (\alpha\alpha - \beta)A)Q^2}{BB - \alpha AB + \beta AA}$$

But $Z = \alpha Y - \beta X$, therefore $Y = \frac{Z + \beta X}{\alpha}$; from which there becomes

$$Y = \frac{-\beta BP^2 + 2\beta APQ + (B - \alpha A)QQ}{BB - \alpha AB + \beta AA}.$$

Thus therefore again from $X$ and $Y$ in a similar manner it will be possible to define the coefficients of the powers $z^{4n}$ and $z^{4n+1}$ and hence of these $z^{8n}$ and $z^{8n+1}$, and thus henceforth.

EXAMPLE

This recurring series shall be proposed :

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \cdots + Pz^n + Qz^{n+1} + \text{etc.};$$

since any coefficient of which shall be the sum of the two preceding, the denominator of the fraction producing this series shall be

$$1 - z - zz;$$

and thus

$$a=1,\quad \beta=-1 \text{ and } A=1,\quad B=3,$$

from which there becomes

$$BB-\alpha AB+\beta AA=5.$$

From which at first there arises

$$Q=\frac{P+\sqrt{\left(5PP+20(-1)^n\right)}}{2}=\frac{P+\sqrt{(5PP\pm 20)}}{2},$$

where the upper sign prevails, if $n$ shall be an even number, and the lower if odd. Thus if $n=4$, because $P=11$, there will be

$$Q=\frac{11+\sqrt{(5\cdot 121+20)}}{2}=\frac{11+25}{2}=18.$$

Again if the coefficient of the term $z^{2n}$ shall be $X$, there will be

$$X=\frac{-4PP+6PQ-QQ}{5};$$

therefore the coefficient of the power $z^8$ will be

$$\frac{-4\cdot 121+6\cdot 198-324}{5}=76.$$

But since there shall be

$$Q=\frac{P+\sqrt{(5PP\pm 20)}}{2},$$

there will be

$$QQ=\frac{3PP\pm 10+P\sqrt{(5PP\pm 20)}}{2}$$

and thus

$$X=\frac{-PP\mp 2+P\sqrt{(5PP\pm 20)}}{2}.$$

Therefore from any term of the series $Pz^n$ these will be obtained :

$$\frac{P+\sqrt{(5PP\pm 20)}}{2}z^{n+1} \text{ and } \frac{-PP\mp 2+P\sqrt{(5PP\pm 20)}}{2}z^{2n}.$$

230. In a similar manner for recurring series, any term of which may be determined from the three preceding terms, it is possible to define any term from the two preceding terms. Indeed a recurring series of this kind shall be

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{etc.}$$

of which the scale of the relation shall be $\alpha, -\beta, +\gamma$ or which arises from the fraction, the denominator of which shall be

$$= 1 - \alpha z + \beta z^2 - \gamma z^3.$$

But if now the terms $P$, $Q$, $R$ may be expressed in the same manner by the factors of this denominator, which shall be

$$(1 - pz)(1 - qz)(1 - rz),$$

so that the coefficients shall be

$$P = \mathfrak{A}p^n \quad + \mathfrak{B}q^n \quad + \mathfrak{C}r^n,$$
$$Q = \mathfrak{A}p \cdot p^n \ + \mathfrak{B}q \cdot q^n \ + \mathfrak{C}r \cdot r^n$$

and

$$R = \mathfrak{A}p^2 \cdot p^n + \mathfrak{B}q^2 \cdot q^n + \mathfrak{C}r^2 \cdot r^n,$$

on account of

$$p + q + r = \alpha, \quad pq + pr + qr = \beta, \quad pqr = \gamma$$

this proportion will be found :

$$\begin{array}{llll} R^3 - 2\alpha QR^2 & +(\alpha\alpha + \beta)Q^2R & - (\alpha\beta - \gamma)Q^3 & : \gamma^n \\ \quad + \beta P & - (\alpha\beta + 3\gamma)PQ & + (\alpha\gamma + \beta\beta)PQ^2 & \\ & + \alpha\gamma P^2 & - 2\beta\gamma P^2 Q & \\ & & + \gamma\gamma P^3 & \\ = C^3 - 2\alpha BC^2 & + (\alpha\alpha + \beta)B^2C & - (\alpha\beta - \gamma)B^3 & : 1. \\ \quad + \beta A & - (\alpha\beta + 3\gamma)AB & + (\alpha\gamma + \beta\beta)AB^2 & \\ & + \alpha\gamma A^2 & - 2\beta\gamma A^2B & \\ & & + \gamma\gamma A^3 & \end{array}$$

Therefore the term *R* depends on the two previous terms *P* and *Q*, and to find it requires the resolution of a cubic equation.

231. From these observations concerned with the general terms of recurring series it remains, that we may investigate the sums of the same series. And indeed in the first place it is evident that the sum of a recurring series extending to infinity is equal to the fraction which arises from that series ; since the denominator of which fraction may be apparent from the law of the progression itself, it remains for us to determine the numerator. Thus this series shall be proposed :

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \text{etc.}$$

the law of the progression of which progression provides this denominator

$$= 1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4.$$

We may assume the fraction of the sum of the series to infinity to be equal to

$$= \frac{a + bz + cz^2 + dz^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4};$$

from witch since the proposed series must arise, there will be by comparison

$$\begin{aligned} a &= A, \\ b &= B - \alpha A, \\ c &= C - \alpha B + \beta A, \\ d &= D - \alpha C + \beta B - \gamma A. \end{aligned}$$

Hence the sum sought will be

$$\frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 + (D - \alpha C + \beta B - \gamma A)z^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

232. Hence it is understood readily, how the sum of a recurring series as far as to a given term must be found. Clearly supposing the sum of the series only to the term $Pz^n$, and putting

$$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots + Pz^n.$$

Because the sum to infinity of this series is agreed upon, the sum of the terms following beyond $Pz^n$ shall be sought, which shall be :

$$t = Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + Tz^{n+4} + \text{etc.};$$

this series divided by $z^{n+1}$ gives a recurring series equal to the proposed, the sum of which therefore will be :

$$t = \frac{Qz^{n+1}+(R-\alpha Q)z^{n+2}+(S-\alpha R+\beta Q)z^{n+3}+(T-\alpha S+\beta R-\gamma Q)z^{n+4}}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}.$$

From which the sum sought emerges :

$$s = \frac{A+(B-\alpha A)z+(C-\alpha B+\beta A)z^2+(D-\alpha C+\beta B-\gamma A)z^3}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}$$

$$-\frac{Qz^{n+1}+(R-\alpha Q)z^{n+2}+(S-\alpha R+\beta Q)z^{n+3}+(T-\alpha S+\beta R-\gamma Q)z^{n+4}}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}$$

233. But if therefore the scale of the relation were from the two members $\alpha, -\beta$, the sum of the series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots + Pz^n,$$

which arises from the fraction

$$\frac{A+(B-\alpha A)z}{1-\alpha z+\beta zz}$$

will be

$$\frac{A+(B-\alpha A)z-Qz^{n+1}-(R-\alpha Q)z^{n+2}}{1-\alpha z+\beta zz}.$$

Or, from the nature of the series

$$R = \alpha Q - \beta P,$$

from which the sum will be produced

$$\frac{A+(B-\alpha A)z-Qz^{n+1}+\beta z^{n+2}}{1-\alpha z+\beta zz}$$

EXAMPLE

Let the series be proposed :

$$1+3z+4z^2+7z^3+\cdots+Pz^n,$$

where there is

$$\alpha=1,\quad \beta=-1,\quad A=1,\quad B=3;$$

the sum of this will be

$$\frac{1+2z-Qz^{n+1}-Pz^{n+2}}{1-z-zz}.$$

Truly on putting $z=1$ the sum of the series will be

$$1+3+4+7+11+\cdots+P=P+Q-3.$$

Therefore the sum of the final and of the following term exceeds the sum of the series by three. Because truly there is :

$$Q=\frac{P+\sqrt{(5PP\pm 20)}}{2}$$

the sum of the series will be

$$1+3+4+7+11+\cdots+P=\frac{3P-6+\sqrt{(5PP\pm 20)}}{2}.$$

Therefore the sum of the series can be shown by the final term only.

# CAPUT XIII

# DE SERIEBUS RECURRENTIBUS

211. Ad hoc serierum genus, quas Moivreus *recurrentes* vocare solet, hic refero omnes series, quae ex evolutione functionis cuiusque fractae per divisionem actualem instituta nascuntur. Supra enim iam ostendimus has series ita esse comparatas, ut quivis terminus ex aliquot praecedentibus secundum legem quandam constantem determinetur, quae lex a denominatore functionis fractae pendet. Cum autem nunc functionem quamcunque fractam in alias simpliciores resolvere docuerim, hinc series quoque recurrens in alias simpliciores resolvetur. In hoc igitur capite propositum est serierum recurrentium cuiusvis gradus resolutionem in simpliciores exponere.

212. Sit proposita ista functio fracta genuina

$$\frac{a+bz+czz+dz^3+\text{etc.}}{1-\alpha bz-\beta zz-\gamma z^3-\delta z^4-\text{etc.}},$$

quae per divisionem evolvatur in hanc seriem recurrentem

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.};$$

cuius coefficientes quemadmodum progrediantur, supra est ostensum. Quodsi iam functio illa fracta resolvatur in fractiones suas simplices et unaquaeque in seriem recurrentem evolvatur, manifestum est summam omnium harum serierum ex fractionibus partialibus ortarum aequalem esse debere seriei recurrenti

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.}$$

Fractiones ergo partiales, quas supra invenire docuimus, dabunt series partiales, quarum indoles ob simplicitatem facile perspicitur; omnes autem series partiales iunctim sumptae producent seriem recurrentem propositam, unde et huius natura penitius cognoscetur.

213. Sint series recurrentes ex singulis fractionibus partialibus ortae hae

$$a + bz + czz + dz^3 + ez^4 + \text{etc.},$$
$$a' + b'z + c'zz + d'z^3 + e'z^4 + \text{etc.},$$
$$a'' + b''z + c''zz + d''z^3 + e''z^4 + \text{etc.},$$
$$a''' + b'''z + c'''zz + d'''z^3 + e'''z^4 + \text{etc.},$$
$$\text{etc.}$$

Quoniam hae series iunctim sumptae aequales esse debent huic

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

necesse est, ut sit

$$A = a + a' + a'' + a''' + \text{etc.},$$
$$B = b + b' + b'' + b''' + \text{etc.},$$
$$C = c + c' + c'' + c''' + \text{etc.},$$
$$D = d + d' + d'' + d''' + \text{etc.}$$
$$\text{etc.}$$

Hinc, si singularum serierum ex fractionibus partialibus ortarum definiri queant coefficientes potestatis $z^n$, horum summa dabit coefficientem potestatis $z^n$ in serie recurrente $A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$

214. Dubium hic suboriri posset, an, si duae huiusmodi series fuerint inter se aequales necessario inde sequatur coefficientes similium potestatum ipsius *z* inter se esse aequales,

$$A + Bz + Cz^2 + Dz^3 + \text{etc.} = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \mathfrak{D}z^3 + \text{etc.}$$

seu an sit $A = \mathfrak{A}$, $B = \mathfrak{B}$, $C = \mathfrak{C}$, $D = \mathfrak{D}$, etc. Hoc autem dubium facile tolletur, si perpendamus hanc aequalitatem subsistere debere, quemcunque valorem obtineat variabilis *z*. Sit igitur $z = 0$ atque manifestum est fore $A = \mathfrak{A}$. His ergo terminis aequalibus utrinque sublatis ac reliqua aequatione per *z* divisa habebitur

$$B + Cz + Dz^2 + \text{etc.} = \mathfrak{B} + \mathfrak{C}z + \mathfrak{D}z^2 + \text{etc.},$$

unde sequitur fore $B = \mathfrak{B}$; simili autem modo ostendetur esse $C = \mathfrak{C}$, $D = \mathfrak{D}$ et ita porro in infinitum.

215. Contemplemur ergo series, quae ex fractionibus partialibus, in quas fractio quaepiam. proposita resolvitur, oriuntur. Ac primo quidem patet fractionem

$$\frac{\mathfrak{A}}{1-pz}$$

dare seriem

$$\mathfrak{A}+\mathfrak{A}pz+\mathfrak{A}p^2z^2+\mathfrak{A}p^3z^3+\text{etc.},$$

cuius terminus generalis est

$$\mathfrak{A}p^nz^n;$$

haec enim expressio vocari solet *terminus generalis,* quoniam ex ea loco *n* numeros omnes successive substituendo omnes seriei termini nascuntur.
Deinde ex fractione

$$\frac{\mathfrak{A}}{(1-pz)^2}$$

oritur series

$$\mathfrak{A}+2\mathfrak{A}pz+3\mathfrak{A}p^2z^2+4\mathfrak{A}p^3z^3+\text{etc.},$$

cuius terminus generalis est

$$(n+1)\mathfrak{A}p^nz^n.$$

Tum ex fractione

$$\frac{\mathfrak{A}}{(1-pz)^3}$$

oritur series

$$\mathfrak{A}+3\mathfrak{A}pz+6\mathfrak{A}p^2z^2+10\mathfrak{A}p^3z^3+\text{etc.}$$

cuius terminus generalis est

$$\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^nz^n.$$

Generatim autem fractio

$$\frac{\mathfrak{A}}{(1-pz)^k}$$

praebet seriem hanc

$$\mathfrak{A}+k\mathfrak{A}pz+\frac{k(k+1)}{1\cdot 2}\mathfrak{A}p^2z^2+\frac{k(k+1)(k+2)}{1\cdot 2\cdot 3}\mathfrak{A}p^3z^3+\text{etc.},$$

cuius terminus generalis est

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1\cdot 2\cdot 3\cdots(k-1)}\mathfrak{A}p^nz^n.$$

Ex ipsa autem seriei progressione colligitur hic idem terminus

$$\frac{k(k+1)(k+2)\cdots(n+k-1)}{1\cdot2\cdot3\cdots n}\mathfrak{A}p^n z^n$$

Haec vero expressio illi est aequalis, id quod multiplicatione per crucem instituta patebit; fiet enim

$$1\cdot 2\cdot 3\cdots n(n+1)\cdots(n+k-1)=1\cdot 2\cdot 3\cdots(k-1)k\cdots(k+n-1),$$

quae est aequatio identica.

216. Quoties ergo in resolutione functionum fractarum ad huiusmodi fractiones partiales $\frac{\mathfrak{A}}{(1-pz)^k}$ pervenitur, toties seriei recurrentis ex illa functione fracta ortae

$$A+Bz+Cz^2+Dz^3+\text{etc.}$$

terminus generalis assignari poterit, quippe qui erit summa terminorum generalium serierum, quae ex fractionibus partialibus nascuntur.

EXEMPLUM 1

Invenire terminum generalem seriei recurrentis, quae ex hac fractione

$$\frac{1-z}{1-z-2zz}$$

nascitur.

Series hinc nata est

$$1+0z+2zz+2z^3+6z^4+10z^4+22z^6+42z^7+86z^8+\text{etc}$$

.

Ad coefficientem potestatis generalis $z^n$ inveniendum fractio $\frac{1-z}{1-z-2zz}$ resolvatur in

$$\frac{\frac{2}{3}}{1+z}+\frac{\frac{1}{3}}{1-2z},$$

unde oritur terminus generalis quaesitus

$$\left(\tfrac{2}{3}(-1)^n+\tfrac{1}{3}\cdot 2^n\right)z^n=\frac{2^n\pm 2}{3}z^n$$

ubi signum + valet, si *n* sit numerus par, signum –, si *n* sit impar.

EXEMPLUM 2

Invenire terminum generalem seriei recurrentis, quae oritur ex fractione

$$\frac{1-z}{1-5z+6zz},$$

seu seriei huius

$$1+4z+14zz+46z^3+146z^4+454z^4+\text{etc}.$$

Ob denominatorem $=(1-2z)(1-3z)$ resolvitur fractio in has

$$=\frac{-1}{1-2z}+\frac{2}{1-3z},$$

ex quibus fit terminus generalis

$$2\cdot 3^n z^n-2^n z^n=\left(2\cdot 3^n-2^n\right)z^n.$$

EXEMPLUM 3

Invenire terminum generalem seriei huius

$$1+3z+4zz+7z^3+11z^4+18z^4+29z^5+47z^6+\text{etc.},$$

quae oritur ex evolutione fractionis

$$\frac{1+2z}{1-z-zz}.$$

Ob denominatoris factores

$$1-\frac{1+\sqrt{5}}{2}z \text{ et } 1-\frac{1-\sqrt{5}}{2}z$$

per resolutionem prodeunt

$$\frac{\frac{1+\sqrt{5}}{2}}{1-\frac{1+\sqrt{5}}{2}z}+\frac{\frac{1-\sqrt{5}}{2}}{1-\frac{1-\sqrt{5}}{2}z}$$

unde erit terminus generalis

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}z^n+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}z^n.$$

EXEMPLUM 4

Invenire terminum generalem seriei huius

$$a+(\alpha a+b)z+\left(\alpha^2 a+\alpha b+\beta a\right)z^2+\left(\alpha^3 a+\alpha^2 b+2\alpha\beta a+\beta b\right)z^3+\text{etc.},$$

quae oritur ex evolutione fractionis

$$\frac{a+bz}{1-\alpha z-\beta zz}.$$

Per resolutionem oriuntur hae duae fractiones :

$$\frac{\left(a\left(\sqrt{(\alpha\alpha+4\beta)}+\alpha\right)+2b\right):2\sqrt{(\alpha\alpha+4\beta)}}{1-\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2}z}+\frac{\left(a\left(\sqrt{(\alpha\alpha+4\beta)}-\alpha\right)-2b\right):2\sqrt{(\alpha\alpha+4\beta)}}{1-\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2}z};$$

hinc terminus generalis erit

$$\frac{a\left(\sqrt{(\alpha\alpha+4\beta)}+\alpha\right)+2b:}{2\sqrt{(\alpha\alpha+4\beta)}}\left(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2}\right)^n z^n+\frac{a\left(\sqrt{(\alpha\alpha+4\beta)}-\alpha\right)-2b:}{2\sqrt{(\alpha\alpha+4\beta)}}\left(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2}\right)^n$$

Ex quo omnium serierum recurrentium, quarum quisque terminus per duos praecedentes determinatur, termini generales expedite definiri poterunt.

EXEMPLUM 5

Invenire terminum generalem huius seriei

$$1+z+2z^2+2z^3+3z^4+3z^5+4z^6+4z^7+\text{etc.},$$

quae oritur ex fractione

$$\frac{1}{1-z-zz+z^3}=\frac{1}{(1-z)^2(1+z)}.$$

Quanquam lex progressionis primo intuitu ita est manifesta, ut explicatione non indigeat, tamen fractiones per resolutionem ortae

$$\frac{\frac{1}{2}}{(1-z)^2}+\frac{\frac{1}{4}}{1-z}+\frac{\frac{1}{4}}{1+z}$$

dant hunc terminum generalem

$$\tfrac{1}{2}(n+1)z^n+\tfrac{1}{4}z^n+\tfrac{1}{4}(-1)^n z^n=\tfrac{2n+3\pm 1}{4}z^n$$

ubi signum superius valet, si *n* fuerit numerus par, inferius, si *n* fuerit impar.

217. Hoc pacto omnium serierum recurrentium termini generales exhiberi possunt, quoniam omnes fractiones in huiusmodi fractiones partiales simplices resolvere licet. Quodsi autem expressiones imaginarias vitare velimus, saepenumero ad huiusmodi fractiones partiales pervenietur

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz},\quad \frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2},\cdots\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^k};$$

ex quarum evolutione cuiusmodi series nascantur, videndum est. Ac primo quidem ob

$$\cos.n\varphi=2\cos.\varphi\cos.(n-1)\varphi-\cos.(n-2)\varphi$$

fractio

$$\frac{\mathfrak{A}}{1-2pz\cos.\varphi+ppzz}$$

evoluta dabit

$$\begin{aligned}\mathfrak{A}+2\mathfrak{A}pz\cos.\varphi+2\mathfrak{A}ppzz\cos.2\varphi+2\mathfrak{A}p^3z^3\cos.3\varphi+2\mathfrak{A}p^4z^4\cos.4\varphi+\text{etc.}\\ +\mathfrak{A}ppzz\qquad +2\mathfrak{A}p^3z^3\cos.\varphi\quad +2\mathfrak{A}p^4z^4\cos.2\varphi+\text{etc.}\\ +\mathfrak{A}p^4z^4\qquad +\text{etc.}\\ \text{etc.,}\end{aligned}$$

cuius seriei terminus generalis non tam facile apparet.

218. Quo igitur ad scopum perveniamus, consideremus has duas series

$$Ppz\sin.\varphi+Pp^2z^2\sin.2\varphi+Pp^3z^3\sin.3\varphi+Pp^4z^4\sin.4\varphi+\text{etc.},$$
$$Q+Qpz\cos.\varphi+Qp^2z^2\cos.2\varphi+Qp^3z^3\cos.3\varphi+Qp^4z^4\cos.4\varphi+\text{etc.},$$

quae duae series utique nascuntur ex evolutione fractionis, cuius denominator est

$$1-2pz\cos.\varphi+ppzz.$$

Ac prior quidem oritur ex hac fractione

$$\frac{Ppz\sin.\varphi}{1-2pz\cos.\varphi+ppzz},$$

posterior vero ex hac

$$\frac{Q-Qpz\cos.\varphi}{1-2pz\cos.\varphi+ppzz}$$

Addantur hae duae fractiones atque summa

$$\frac{Q+Ppz\sin.\varphi-Qpz\cos.\varphi}{1-2pz\cos.\varphi+ppzz}$$

dabit seriem, cuius terminus generalis erit

$$\left(P\sin.n\varphi+Q\cos.n\varphi\right)p^n z^n.$$

Fiat autem haec fractio propositae

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

aequalis; erit

$$Q=\mathfrak{A}\quad \text{et}\quad P=\mathfrak{A}\cot.\varphi+\mathfrak{B}\operatorname{cosec}.\varphi..$$

Seriei ergo ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

ortae terminus generalis erit

$$\frac{\mathfrak{A}\cos.\varphi\sin.n\varphi+\mathfrak{B}\sin.n\varphi+\mathfrak{A}\sin.\varphi\cos.n\varphi}{\sin.\varphi}p^n z^n=\frac{\mathfrak{A}\sin.(n+1)\varphi+\mathfrak{B}\sin.n\varphi}{\sin.\varphi}p^n z^n.$$

219. Ad terminum generalem inveniendum, si denominator fractionis fuerit potestas ut

$$\left(1-2pz\cos.\varphi+ppzz\right)^k,$$

conveniet hanc fractionem resolvi in duas etsi imaginarias

$$\frac{a}{\left(1-\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)pz\right)^k}+\frac{b}{\left(1-\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)pz\right)^k},$$

quarum simul sumptarum terminus generalis seriei ex ipsis ortae erit

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1.2.3\cdots(k-1)}\left(cos.n\varphi+\sqrt{-1}\cdot\sin.n\varphi\right)ap^nz^n$$
$$+\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1.2.3\cdots(k-1)}\left(cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi\right)bp^nz^n.$$

Sit

$$a+b=f,\quad a-b=\frac{g}{\sqrt{-1}},$$

ut sit

$$\frac{f\sqrt{-1}+g}{2\sqrt{-1}}\quad \text{and}\quad b=\frac{f\sqrt{-1}-g}{2\sqrt{-1}},$$

eritque haec expressio

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1.2.3\cdots(k-1)}\left(f\,cos.n\varphi+g\sin.n\varphi\right)p^nz^n$$

terminus generalis seriei, quae oritur ex his fractionibus

$$\frac{\frac{1}{2}f+\frac{1}{2\sqrt{-1}}g}{\left(1-\left(cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)pz\right)^k}+\frac{\frac{1}{2}f-\frac{1}{2\sqrt{-1}}g}{\left(1-\left(cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)pz\right)^k},$$

seu quae oritur ex hac fractione una

$$\frac{\left\{\begin{array}{l} f-kfpz\cos.\varphi+\frac{k(k-1)}{1\cdot2}fp^2z^2\cos.2\varphi-\frac{k(k-1)(k-2)}{1\cdot2\cdot3}fp^3z^3\cos.3\varphi+\text{etc.}\\ \quad kgpz\sin.\varphi-\frac{k(k-1)}{1\cdot2}gp^2z^2\sin.2\varphi+\frac{k(k-1)(k-2)}{1\cdot2\cdot3}gp^3z^3\sin.3\varphi-\text{etc.}\end{array}\right\}}{(1-2pz\cos.\varphi+ppzz)^k}$$

220. Posito ergo $k=2$ erit seriei ex hac fractione

$$\frac{f-2pz(f\cos.\varphi-g\sin.\varphi)+ppzz(f\cos.2\varphi-g\sin.2\varphi)}{(1-2pz\cos.\varphi+ppzz)^2}$$

ortae terminus generalis

$$(n+1)(f\cos.n\varphi+g\sin.n\varphi)p^nz^n$$

At seriei ex hac fractione

$$\frac{a}{1-2pz\cos.\varphi+ppzz}$$

seu hac

$$\frac{a-2apz\cos.\varphi+appzz}{(1-2pz\cos.\varphi+ppzz)^2}$$

ortae terminus generalis est

$$\frac{a\sin.(n+1)\varphi}{\sin.\varphi}p^n z^n.$$

Addantur hae fractiones invicem ac ponatur

$$a+f=\mathfrak{A},$$
$$2a\cos.\varphi+2f\cos.\varphi-2g\sin.\varphi=-\mathfrak{B}$$

et

$$a+f\cos.2\varphi-g\sin.2\varphi=0;$$

hinc erit

$$g=\frac{\mathfrak{B}+2\mathfrak{A}\cos.\varphi}{2\sin.\varphi},$$

$$a=\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{1-\cos.2\varphi}=\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^2}$$

et

$$f=\frac{-\mathfrak{A}\cos.2\varphi-\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^2}$$

et

$$g=\frac{\mathfrak{B}\sin.\varphi+\mathfrak{A}\sin.2\varphi}{2(\sin.\varphi)^2}.$$

Hanc ob rem seriei ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2}$$

ortae terminus generalis est

$$\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^3}\sin.(n+1)\varphi\cdot p^n z^n+(n+1)\frac{\mathfrak{B}\sin.\varphi\sin.n\varphi+\mathfrak{A}\sin.2\varphi\sin.n\varphi-\mathfrak{B}\cos.\varphi\cos.n\varphi-\mathfrak{A}\cos.2\varphi\cos.n\varphi}{2(\sin.\varphi)^2}\cdot p^n z^n$$
$$=-\frac{(n+1)(\mathfrak{A}\cos.(n+2)\varphi+\mathfrak{B}\cos.(n+1)\varphi)}{2(\sin.\varphi)^2}\cdot p^n z^n+\frac{(\mathfrak{A}+\mathfrak{B}\cos.\varphi)\sin.(n+1)\varphi}{2(\sin.\varphi)^3}\cdot p^n z^n$$
$$=\frac{\frac{1}{2}(n+3)\sin.(n+1)\varphi-\frac{1}{2}(n+1)\sin.(n+3)\varphi}{2(\sin.\varphi)^3}\cdot\mathfrak{A}p^n z^n+\frac{\frac{1}{2}(n+2)\sin.n\varphi-\frac{1}{2}n\sin.(n+2)\varphi}{2(\sin.\varphi)^3}\cdot\mathfrak{B}p^n z^n.$$

Est ergo iste terminus generalis quaesitus

$$=\frac{(n+3)\sin.(n+1)\varphi-(n+1)\sin.(n+3)\varphi}{4(\sin.\varphi)^3}\cdot\mathfrak{A}p^n z^n+\frac{(n+2)\sin.n\varphi-n\sin.(n+2)\varphi}{4(\sin.\varphi)^3}\cdot\mathfrak{B}p^n z^n$$

seriei, quae oritur ex fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2}.$$

221. Sit $k=3$ eritque seriei ex hac fractione ortae

$$\frac{f-3pz(f\cos.\varphi-g\sin.\varphi)+3ppzz(f\cos.2\varphi-g\sin.2\varphi)-p^3z^3(f\cos.3\varphi-g\sin.3\varphi)}{(1-2pz\cos.\varphi+ppzz)^3}$$

terminus generalis

$$\frac{(n+1)(n+2)}{1\cdot 2}(f\cos.n\varphi+g\sin.n\varphi)p^n z^n.$$

Deinde seriei ex fractione

$$\frac{a+bpz}{(1-2pz\cos.\varphi+ppzz)^2}$$

seu ex hac

$$\frac{a-(2a\cos.\varphi-b)pz+(a-2b\cos.\varphi)ppzz+bp^3z^3}{(1-2pz\cos.\varphi+ppzz)^3}$$

ortae terminus generalis est

$$=\frac{(n+3)\sin.(n+1)\varphi-(n+1)\sin.(n+3)\varphi}{4(\sin.\varphi)^3}\cdot ap^n z^n+\frac{(n+2)\sin.n\varphi-n\sin.(n+2)\varphi}{4(\sin.\varphi)^3}\cdot bp^n z^n$$

Addantur hae fractiones ac ponatur numerator $=\mathfrak{A}$; erit

$$a+f=\mathfrak{A},$$
$$3f\cos.\varphi-3g\sin.\varphi+2a\cos.\varphi-b=0,$$
$$3f\cos.2\varphi-3g\sin.2\varphi+a-2b\cos.\varphi=0$$

et

$$b=f\ \cos.3\varphi-g\ \sin.3\varphi;$$

hinc erit

$$a=\frac{f\cos.3\varphi-g\sin.3\varphi-3f\cos.\varphi+3g\sin.\varphi}{2\cos.\varphi}$$
$$=2g(\sin.\varphi)^2\tan g.\varphi-f-2f(\sin.\varphi)^2.$$

.
Deinde reperitur

$$\frac{f}{g}=\frac{\sin.5\varphi-2\sin.3\varphi+\sin.\varphi}{\cos.5\varphi-2\cos.3\varphi+\cos.\varphi}$$

et

$$a+f=\mathfrak{A}=2g(\sin.\varphi)^2\tan g.\varphi-2f(\sin.\varphi)^2,$$

ergo

$$\frac{\mathfrak{A}}{2(\sin.\varphi)^2}=\frac{g\sin.\varphi-f\cos.\varphi}{\cos.\varphi};$$

ex quibus tandem oritur

$$f=\frac{\mathfrak{A}(\sin.\varphi-2\sin.3\varphi+\sin.5\varphi)}{16(\sin.\varphi)^5},$$
$$g=\frac{\mathfrak{A}(\cos.\varphi-2\cos.3\varphi+\cos.5\varphi)}{16(\sin.\varphi)^5}.$$

Ob

$$16(\sin.\varphi)^5=\sin.5\varphi-5\sin.3\varphi+10\sin.\varphi$$

erit

$$a=\frac{\mathfrak{A}(9\sin.\varphi-3\sin.3\varphi)}{16(\sin.\varphi)^5}$$

et

$$b=\frac{\mathfrak{A}(-\sin.2\varphi+\sin.2\varphi)}{16(\sin.\varphi)^5}=0.$$

Est autem

$$3\sin.\varphi-\sin.3\varphi=4(\sin.\varphi)^3,$$

ergo

$$a = \frac{3\mathfrak{A}}{4(\sin.\varphi)^2}.$$

Quocirca erit terminus generalis

$$\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^n z^n \frac{\sin.(n+1)\varphi - 2\sin.(n+3)\varphi + \sin.(n+5)\varphi}{16(\sin.\varphi)^5}$$
$$+ 3\mathfrak{A}p^n z^n \frac{(n+3)\sin.(n+1)\varphi - (n+1)\sin.(n+3)\varphi}{16(\sin.\varphi)^5}$$

$$= \frac{\mathfrak{A}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{array}{l} \frac{(n+4)(n+5)}{1\cdot 2}\sin.(n+1)\varphi - \frac{2(n+1)(n+5)}{1\cdot 2}\sin.(n+3)\varphi \\ + \frac{(n+1)(n+2)}{1\cdot 2}\sin.(n+5)\varphi \end{array}\right\}.$$

222. Seriei ergo, quae oritur ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi + ppzz)^3},$$

terminus generalis erit hic

$$\frac{\mathfrak{A}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{array}{l} \frac{(n+5)(n+4)}{1\cdot 2}\sin.(n+1)\varphi - \frac{2(n+1)(n+5)}{1\cdot 2}\sin.(n+3)\varphi \\ + \frac{(n+1)(n+2)}{1\cdot 2}\sin.(n+5)\varphi \end{array}\right\}$$

$$+ \frac{\mathfrak{B}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{array}{l} \frac{(n+4)(n+3)}{1\cdot 2}\sin.n\varphi - \frac{2n(n+4)}{1\cdot 2}\sin.(n+2)\varphi \\ + \frac{n(n+1)}{1\cdot 2}\sin.(n+4)\varphi \end{array}\right\}.$$

Atque ulterius progrediendo seriei, quae oritur ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi + ppzz)^4},$$

terminus generalis erit hic

$$+\frac{\mathfrak{A}p^n z^n}{64(\sin.\varphi)^5}\left\{\begin{aligned}&\frac{(n+7)(n+6)(n+5)}{1\cdot 2\cdot 3}\sin.(n+1)\varphi-\frac{3(n+1)(n+7)(n+6)}{1\cdot 2\cdot 3}\sin.(n+3)\varphi\\&+\frac{3(n+1)(n+2)(n+7)}{1\cdot 2\cdot 3}\sin.(n+5)\varphi-\frac{(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3}\sin.(n+7)\varphi\end{aligned}\right\}$$

$$+\frac{\mathfrak{B}p^n z^n}{64(\sin.\varphi)^7}\left\{\begin{aligned}&\frac{(n+6)(n+5)(n+4)}{1\cdot 2\cdot 3}\sin.n\varphi-\frac{3n(n+6)(n+5)}{1\cdot 2\cdot 3}\sin.(n+2)\varphi\\&+\frac{3n(n+1)(n+6)}{1\cdot 2\cdot 3}\sin.(n+4)\varphi-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin.(n+6)\varphi\end{aligned}\right\}.$$

Ex his autem expressionibus facile intelligitur, quemadmodum formae terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expressionum penitius inspiciendam notari convenit esse

$$\begin{aligned}\sin.\varphi &= \sin.\varphi,\\ 4(\sin.\varphi)^3 &= 3\sin.\varphi-\sin.3\varphi,\\ 16(\sin.\varphi)^5 &= 10\sin.\varphi-5\sin.3\varphi+\sin.5\varphi,\\ 64(\sin.\varphi)^7 &= 35\sin.\varphi-21\sin.3\varphi+7\sin.5\varphi-\sin.7\varphi,\\ 256(\sin.\varphi)^9 &= 126\sin.\varphi-84\sin.3\varphi+36\sin.5\varphi-9\sin 7\varphi+\sin.9\varphi\end{aligned}$$

etc.

223. Cum igitur hoc pacto omnes functiones fractae in fractiones partiales reales resolvi queant, simul omnium serierum recurrentium termini generales per expressiones reales exhiberi poterunt. Quod quo clarius appareat, exempla sequentia adiuncta sunt.

EXEMPLUM 1

Ex fractione

$$\frac{1}{(1-z)(1-zz)(1-z^3)}=\frac{1}{1-z-zz+z^4+z^5-z^6}$$

oritur ista series recurrens

$$1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+\text{etc.},$$

cuius terminus generalis desideratur.

Fractio proposita secundum factores ordinata fit

$$\frac{1}{(1-z)^3(1+z)(1+z+zz)},$$

quae resolvitur in has fractiones

$$\frac{1}{6(1-z)^3}+\frac{1}{4(1-z)^2}+\frac{17}{72(1-z)}+\frac{1}{8(1+z)}+\frac{2+z}{9(1+z+zz)}.$$

Harum prima $\frac{1}{6(1-z)^3}$ dat terminum generalem

$$\frac{(n+1)(n+2)}{1\cdot 2}\cdot\frac{1}{6}z^n=\frac{nn+3n+2}{12}z^n,$$

secunda $\frac{1}{4(1-z)^2}$ dat

$$\frac{n+1}{4}z^n,$$

tertia $\frac{17}{72(1-z)}$ dat

$$\frac{17}{72}z^n,$$

quarta $\frac{1}{8(1+z)}$ dat

$$\frac{1}{8}(-1)^n z^n.$$

Quinta vero $\frac{2+z}{9(1+z+zz)}$ comparata cum forma

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz},$$

dat

$$p=-1,\quad \varphi=\frac{\pi}{3}=60^0,\quad \mathfrak{A}=+\frac{2}{9}\quad \text{et}\quad \mathfrak{B}=-\frac{1}{9},$$

unde oritur terminus generalis

$$\frac{2\sin.(n+1)\varphi-\sin.n\varphi}{9\sin.\varphi}(-1)^n z^n=\frac{4\sin.(n+1)\varphi-2\sin.n\varphi}{9\sqrt{3}}(-1)^n z^n$$
$$=\frac{4\sin.\frac{(n+1)\pi}{3}-2\sin.\frac{n\pi}{3}}{9\sqrt{3}}(-1)^n z^n.$$

Colligantur hae expressiones omnes in unam summam ac prodibit seriei propositae terminus generalis quaesitus

$$\left(\frac{nn}{12}+\frac{n}{2}+\frac{47}{72}\right)z^n\pm\frac{1}{8}z^n\pm\frac{4\sin.\frac{(n+1)\pi}{3}-2\sin.\frac{n\pi}{3}}{9\sqrt{3}}z^n,$$

ubi signa superiora valent, si *n* numerus par, inferiora, sin impar. Ubi notandum est, si fuerit *n* numerus formae *3m*, fore

$$\frac{4\sin.\frac{(n+1)\pi}{3} - 2\sin.\frac{n\pi}{3}}{9\sqrt{3}} = \pm\frac{2}{9};$$

si fuerit $n = 3m+1$, erit haec expressio $= \mp\frac{1}{9}$ ; at si $n = 3m+2$, erit ista expressio $\mp\frac{1}{9}$ prout *n* fuerit numerus vel par vel impar. Ex his natura seriei ita explicari potest, ut,

| si fuerit | terminus generalis futurus sit |
|---|---|
| $n = 6m+0$ | $\left(\frac{nn}{12}+\frac{n}{2}+1\right)z^n$ |
| $n = 6m+1$ | $\left(\frac{nn}{12}+\frac{n}{2}+\frac{5}{12}\right)z^n$ |
| $n = 6m+2$ | $\left(\frac{nn}{12}+\frac{n}{2}+\frac{2}{3}\right)z^n$ |
| $n = 6m+3$ | $\left(\frac{nn}{12}+\frac{n}{2}+\frac{3}{4}\right)z^n$ |
| $n = 6m+4$ | $\left(\frac{nn}{12}+\frac{n}{2}+\frac{2}{3}\right)z^n$ |
| $n = 6m+5$ | $\left(\frac{nn}{12}+\frac{n}{2}+\frac{5}{12}\right)z^n$ |

Sic si fuerit $n = 50$, valet forma $n = 6m+2$ eritque terminus seriei $= 234z^{50}$.

## EXEMPLUM 2

Ex fractione

$$\frac{1+z+zz}{1-z-z^4+z^5}$$

oritur ista series recurrens

$$1+2z+3zz+3z^3+4z^4+5z^5+6z^6+6z^7+7z^8+\text{etc.},$$

cuius terminum generalem invenire oportet.

Fractio proposita ad hanc formam reducitur

$$\frac{1+z+zz}{(1-z)^2(1+z)(1+zz)},$$

quae propterea resolvitur in has fractiones partiales

$$\frac{3}{4(1-z)^2}+\frac{3}{8(1-z)}+\frac{1}{8(1+z)}+\frac{-1+z}{4(1+zz)}.$$

Harum prima $\frac{3}{4(1-z)^2}$ dat terminum generalem

$$\frac{3(n+1)}{4}z^n$$

secunda $\frac{3}{8(1-z)}$ dat

$$\frac{3}{8}z^n$$

tertia $\frac{1}{8(1+z)}$ dat

$$\frac{1}{8}(-1)^n z^n$$

et quarta $\frac{-1+z}{4(1+zz)}$ comparata cum forma

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz},$$

dat

$$p=1,\ \ \varphi=0\ \text{ et }\ \varphi=\frac{\pi}{2},\ \ \mathfrak{A}=-\frac{1}{4},\ \ \mathfrak{B}=+\frac{1}{4},$$

unde fit terminus generalis

$$=\left(-\frac{1}{4}\sin.\frac{(n+1)\pi}{2}+\frac{1}{4}\sin.\frac{n\pi}{2}\right)z^n.$$

Quare colligendo erit terminus generalis quaesitus

$$=\left(\frac{3n}{4}+\frac{9}{8}\right)z^n\pm\frac{1}{8}z^n-\frac{1}{4}\left(\sin.\frac{(n+1)\pi}{2}-\frac{1}{4}\sin.\frac{n\pi}{2}\right)z^n.$$

Hinc

| si fuerit | terminus generalis futurus sit |
|---|---|

| | |
|---|---|
| $n = 4m + 0$ | $\left(\frac{3n}{4} + 1\right) z^n$ |
| $n = 4m + 1$ | $\left(\frac{3n}{4} + \frac{5}{4}\right) z^n$ |
| $n = 4m + 2$ | $\left(\frac{3n}{4} + \frac{3}{2}\right) z^n$ |
| $n = 4m + 3$ | $\left(\frac{3n}{4} + \frac{3}{4}\right) z^n$ |

Ita si $n = 50$, valebit $n = 4m + 2$ eritque terminus $= 39z^{50}$.

224. Proposita ergo serie recurrente, quoniam illa fractio, unde oritur, facile cognoscitur, eius terminus generalis secundum praecepta data reperietur. Ex lege autem seriei recurrentis, qua quisque terminus ex praecedentibus definitur, statim innotescit denominator fractionis huiusque factores praebebunt formam termini generalis; per numeratorem enim tantum coefficientes determinantur. Sit nempe proposita haec series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

cuius lex progressionis, qua unusquisque terminus ex aliquot praecedentibus determinatur, praebeat hunc fractionis denominatorem

$$1 - \alpha z - \beta z^2 - \gamma z^3,$$

ita ut sit

$$D = \alpha C + \beta B + \gamma A\ ,\ E = \alpha D + \beta C + \gamma B,\ F = \alpha E + \beta D + \gamma C \ \text{ etc.},$$

qui multiplicatores $+\alpha$, $+\beta$, $+\gamma$ a MOIVREO *scalam relationis* constituere dicuntur. Lex ergo progressionis posita est in scala relationis atque scala relationis statim praebet denominatorem fractionis, ex cuius resolutione proposita series recurrens oritur.

225. Ad terminum ergo generalem seu coefficientem potestatis indefinitae $z^n$ inveniendum quaeri debent denominatoris $1 - \alpha z - \beta z^2 - \gamma z^3$ factores vel simplices vel duplices, si imaginarios vitare velimus. Sint primo factores simplices omnes inter se inaequales et reales hi $(1 - pz)(1 - qz)(1 - rz)$ atque fractio generans seriem propositam resolvetur in

$$\frac{\mathfrak{A}}{1 - pz} + \frac{\mathfrak{B}}{1 - qz} + \frac{\mathfrak{C}}{1 - rz};$$

unde seriei terminus generalis erit

$$\left(\mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n\right) z^n.$$

Si duo factores fuerint aequales, nempe $q = p$, tum terminus generalis huiusmodi erit

$$\left(\left(\mathfrak{A}(n+1)+\mathfrak{B}\right)p^n + \mathfrak{C}r^n\right)z^n,$$

et si insuper fuerit $r = q = p$, erit terminus generalis

$$\left(\mathfrak{A}\tfrac{(n+1)(n+1)}{12}+\mathfrak{B}(n+1)+\mathfrak{C}\right)p^n z^n$$

Quodsi vero denominator $1-\alpha z-\beta z^2-\gamma z^3$ duplicem habeat factorem, ut sit

$$=(1-pz)(1-2qs\cos.\varphi+qqzz),$$

tum terminus generalis erit

$$\left(\mathfrak{A}p^n + \tfrac{\mathfrak{B}\sin.(n+1)\varphi+\mathfrak{C}\sin.n\varphi}{\sin.\varphi}q^n\right)z^n.$$

Cum igitur positis pro $n$ successive numeris 0, 1, 2 prodire debeant termini $A,\ Bz,\ Cz^2$, hinc valores litterarum $\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C}$ determinabuntur.

226. Sit scala relationis bimembris seu determinetur quisque terminus per duos praecedentes, ita ut sit

$$C=\alpha B-\beta A,\ D=\alpha C-\beta B,\ E=\alpha D-\beta C \text{ etc.},$$

atque manifestum est seriem hanc recurrentem, quae sit

$$A+Bz+Cz^2+Dz^3+Ez^4+\cdots+Pz^n+Qz^{n+1}+\text{etc.},$$

oriri ex fractione, cuius denominator sit

$$1-\alpha z+\beta zz.$$

Sint huius denominatoris factores

$$(1-pz)(1-qz)$$

erit

$$p+q=\alpha \quad \text{et} \quad pq=\beta$$

atque seriei terminus generalis erit

$$\left(\mathfrak{A}p^{n}+\mathfrak{B}q^{n}\right)z^{n}$$

Hinc facto $n=0$ erit

$$A=\mathfrak{A}+\mathfrak{B}$$

et facto $n = 1$ erit

$$B=\mathfrak{A}p+\mathfrak{B}q,$$

unde fit

$$Aq-B=\mathfrak{A}\left(q-p\right)$$

et

$$\mathfrak{A}=\frac{Aq-B}{q-p} \text{ and } \mathfrak{B}=\frac{Ap-B}{p-q}$$

Inventis autem valoribus $\mathfrak{A}$ et $\mathfrak{B}$ erit

$$P=\mathfrak{A}p^{n}+\mathfrak{B}q^{n} \text{ et } Q=\mathfrak{A}p^{n+1}+\mathfrak{B}q^{n+1}.$$

Tum vero erit

$$\mathfrak{A}\mathfrak{B}=\frac{BB-\alpha AB+\beta AA}{4\beta-\alpha\alpha}.$$

227. Hinc deduci potest modus quemvis terminum ex unico praecedente formandi, cum ad hoc per legem progressionis duo requirantur. Cum enim sit

$$P=\mathfrak{A}p^{n}+\mathfrak{B}q^{n} \text{ et } Q=\mathfrak{A}p\cdot p^{n}+\mathfrak{B}q\cdot q^{n},$$

erit

$$Pq-Q=\mathfrak{A}\left(q-p\right)p^{n} \text{ et } Pp-Q=\mathfrak{B}\left(p-q\right)q^{n},$$

Multiplicentur hae expressiones in se invicem eritque

$$P^{2}pq-\left(p+q\right)PQ+QQ+\mathfrak{A}\mathfrak{B}\left(p-q\right)^{2}p^{n}q^{n}=0.$$

At est

$$p+q=\alpha,\quad pq=\beta,\quad \left(p-q\right)^{2}=\left(p+q\right)^{2}-4pq=\alpha\alpha-4\beta \text{ et } p^{n}q^{n}=\beta^{n}.$$

Quibus substitutis habebitur

$$\beta P^{2}-\alpha PQ+QQ=\left(\beta AA-\alpha AB+BB\right)\beta^{n}$$

seu

$$\frac{QQ-\alpha PQ+\beta PP}{BB-\alpha AB+\beta AA}=\beta^{n},$$

quae est insignis proprietas serierum recurrentium, quarum quisque terminus per duos praecedentes determinatur. At cognito quovis termino $P$ erit sequens

$$Q=\tfrac{1}{2}\alpha P+\sqrt{\left(\left(\tfrac{1}{4}\alpha^{2}-\beta\right)P^{2}+\left(BB-\alpha AB+\beta AA\right)\beta^{n}\right)},$$

quae expressio, etsi speciem irrationalitatis prae se fert, tamen semper est rationalis, propterea quod termini irrationales in serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibusvis $pz^{n}$ et $Qz^{n+1}$ commode assignari potest terminus multo magis remotus $Xz^{2n}$. Ponatur enim

$$X= fP^{2}+gPQ-h\mathfrak{A}\mathfrak{B}\beta^{n}.$$

Quoniam est

$$P=\mathfrak{A}p^{n}+\mathfrak{B}q^{n} \text{ et } Q=\mathfrak{A}p\cdot p^{n}+\mathfrak{B}q\cdot q^{n} \text{ atque } X= \mathfrak{A}p^{2n}+\mathfrak{B}q^{2n},$$

erit ut sequitur:

$$\begin{array}{rlll} fP^{2}= & f\mathfrak{A}^{2}p^{2n} & +f\mathfrak{B}^{2}q^{2n} & +2f\mathfrak{A}\mathfrak{B}\beta^{n}\\ qPQ= & g\mathfrak{A}^{2}p\cdot p^{2n} & +g\mathfrak{B}^{2}q\cdot q^{2n} & +g\mathfrak{A}\mathfrak{B}\alpha\beta^{n}\\ -h\mathfrak{A}\mathfrak{B}\beta^{n}= & & & -h\mathfrak{A}\mathfrak{B}\beta^{n}\\ \hline X= & \mathfrak{A}p^{2n}+\mathfrak{B}q^{2n} & & \end{array}$$

Fiet ergo

$$f+gp=\tfrac{1}{\mathfrak{A}},\quad f+gq=\tfrac{1}{\mathfrak{B}},\quad \text{et}\quad h=2f+g\alpha,$$

unde

$$g=\tfrac{\mathfrak{B}-\mathfrak{A}}{\mathfrak{A}\mathfrak{B}(p-q)} \quad\text{et}\quad f=\tfrac{\mathfrak{A}p-\mathfrak{B}q}{\mathfrak{A}\mathfrak{B}(p-q)}.$$

At est

$$\mathfrak{B}-\mathfrak{A}=\tfrac{\alpha A-2B}{p-q} \quad\text{et}\quad \mathfrak{A}p-\mathfrak{B}q=\tfrac{\alpha B-2A\beta}{p-q}.$$

Ergo

$$f=\tfrac{\alpha B-2A\beta}{\mathfrak{A}\mathfrak{B}(\alpha\alpha-4\beta)} \quad\text{et}\quad g=\tfrac{\alpha A-2B}{\mathfrak{A}\mathfrak{B}(\alpha\alpha-4\beta)}$$

seu

$$f=\tfrac{2A\beta-\alpha B}{BB-\alpha AB+\beta AA} \quad\text{et}\quad g=\tfrac{2B-\alpha A}{BB-\alpha AB+\beta AA};$$

ideoque

$$h=\tfrac{(4\beta-\alpha\alpha)A}{BB-\alpha AB+\beta AA}.$$

Eritque ergo

$$X = \frac{(2A\beta-\alpha B)P^2+(2B-\alpha A)PQ}{BB-\alpha AB+\beta AA} - A\beta^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\beta A-(\alpha\alpha-2\beta)B)P^2+(2B-\alpha A)Q^2}{\alpha(BB-\alpha AB+\beta AA)} - \frac{2B\beta^n}{\alpha}.$$

His coniungendis per eliminationem termini $\beta^n$ reperitur

$$X = \frac{(\beta A-\alpha B)P^2+2BPQ-AQQ}{BB-\alpha AB+\beta AA}.$$

229. Simili modo si statuantur termini sequentes

$$A+Bz+Cz^2+\cdots+Pz^n+Qz^{n+1}+Rz^{n+2}+\cdots+Xz^{2n}+Yz^{2n+1}+Zz^{2n+2}+\text{etc.},$$

erit

$$Z = \frac{(\beta A-\alpha B)Q^2+2BQR-ARR}{BB-\alpha AB+\beta AA}$$

et ob $R=\alpha Q-\beta P$ erit

$$Z = \frac{-\beta\beta AP^2+2\beta(\alpha A-B)PQ+(\alpha B-(\alpha\alpha-\beta)A)Q^2}{BB-\alpha AB+\beta AA}$$

At est $Z=\alpha Y-\beta X$, ergo $Y=\frac{Z+\beta X}{\alpha}$; unde fit

$$Y = \frac{-\beta BP^2+2\beta APQ+(B-\alpha A)QQ}{BB-\alpha AB+\beta AA}.$$

Sic igitur porro ex $X$ et $Y$ definiri poterunt simili modo coefficientes potestatum $z^{4n}$ et $z^{4n+1}$ hincque ipsarum $z^{8n}$ et $z^{8n+1}$, et ita porro.

EXEMPLUM

Sit proposita ista series recurrens

$$1+3z+4z^2+7z^3+11z^4+18z^5+\cdots+Pz^n+Qz^{n+1}+\text{etc.};$$

cuius cum quilibet coefficiens sit summa duorum praecedentium, erit denominator fractionis hanc seriem producentis

$$1-z-zz;$$

ideoque

$$a = 1, \quad \beta = -1 \text{ et } A = 1, \quad B = 3,$$

unde fit

$$BB - \alpha AB + \beta AA = 5.$$

Ex quo orietur primum

$$Q = \frac{P + \sqrt{\left(5PP + 20(-1)^n\right)}}{2} = \frac{P + \sqrt{(5PP \pm 20)}}{2},$$

ubi signum superius valet, si $n$ sit numerus par, inferius, si impar. Sic si $n = 4$, ob $P = 11$ erit

$$Q = \frac{11 + \sqrt{(5 \cdot 121 + 20)}}{2} = \frac{11 + 25}{2} = 18.$$

Si porro coefficiens termini $z^{2n}$ sit $X$, erit

$$X = \frac{-4PP + 6PQ - QQ}{5};$$

ergo potestatis $z^8$ coefficiens erit

$$\frac{-4 \cdot 121 + 6 \cdot 198 - 324}{5} = 76.$$

Cum autem sit

$$Q = \frac{P + \sqrt{(5PP \pm 20)}}{2},$$

erit

$$QQ = \frac{3PP \pm 10 + P\sqrt{(5PP \pm 20)}}{2}$$

ideoque

$$X = \frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2}.$$

Ex termino ergo seriei quocunque $Pz^n$ obtinentur hi

$$\frac{P + \sqrt{(5PP \pm 20)}}{2} z^{n+1} \text{ et } \frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2} z^{2n}.$$

230. Simili modo in seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri potest. Sit enim series huiusmodi recurrens

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{etc.}$$

cuius scala relationis sit $\alpha, -\beta, +\gamma$ seu quae oriatur ex fractione, cuius denominator

$$= 1 - \alpha z + \beta z^2 - \gamma z^3.$$

Quodsi iam termini *P, Q, R* eodem modo per factores huius denominatoris, qui sint

$$(1 - pz)(1 - qz)(1 - rz),$$

exprimantur, ut sit

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n,$$
$$Q = \mathfrak{A}p \cdot p^n + \mathfrak{B}q \cdot q^n + \mathfrak{C}r \cdot r^n$$

et

$$R = \mathfrak{A}p^2 \cdot p^n + \mathfrak{B}q^2 \cdot q^n + \mathfrak{C}r^2 \cdot r^n,$$

ob

$$p + q + r = \alpha, \quad pq + pr + qr = \beta, \quad pqr = \gamma$$

reperietur haec proportio

$$\begin{aligned}
&R^3 - 2\alpha QR^2 + (\alpha\alpha + \beta)Q^2R - (\alpha\beta - \gamma)Q^3 : \gamma^n \\
&\quad + \beta P - (\alpha\beta + 3\gamma)PQ + (\alpha\gamma + \beta\beta)PQ^2 \\
&\quad + a\gamma P^2 - 2\beta\gamma P^2Q \\
&\quad + \gamma\gamma P^3 \\
&= C^3 - 2\alpha BC^2 + (\alpha\alpha + \beta)B^2C - (\alpha\beta - \gamma)B^3 : 1. \\
&\quad + \beta A - (\alpha\beta + 3\gamma)AB + (\alpha\gamma + \beta\beta)AB^2 \\
&\quad + \alpha\gamma A^2 - 2\beta\gamma A^2B \\
&\quad + \gamma\gamma A^3
\end{aligned}$$

Pendet ergo inventio termini *R* ex duobus praecedentibus *P* et *Q* a resolutione aequationis cubicae.

231. His de terminis generalibus serierum recurrentium notatis superest, ut earundem serierum summas investigemus. Ac prima quidem manifestum est summam seriei recurrentis in infinitum extensae aequalem esse fractioni, ex qua oritur; cuius fractionis cum denominator ex ipsa progressionis lege pateat, reliquum est, ut numeratorem definiamus. Sit itaque proposita haec series

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+Gz^6+\text{etc.}$$

cuius lex progressionis praebeat hunc denominatorem

$$=1-\alpha z+\beta z^2-\gamma z^3+\delta z^4.$$

Sumamus fractionem summae seriei in infinitum aequalem esse

$$=\frac{a+bz+cz^2+dz^3}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4};$$

ex qua cum series proposita oriri debeat, erit comparando

$$\begin{aligned}&a=A,\\&b=B-\alpha A,\\&c=C-\alpha B+\beta A,\\&d=D-\alpha C+\beta B-\gamma A.\end{aligned}$$

Hinc erit summa quaesita

$$\frac{A+(B-\alpha A)z+(C-\alpha B+\beta A)z^2+(D-\alpha C+\beta B-\gamma A)z^3}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}.$$

232. Hinc facile intelligitur, quemadmodum seriei recurrentis summa ad datum terminum usque inveniri debeat. Quaeratur scilicet seriei modo assumptae summa ad terminum $Pz^n$ atque ponatur

$$s=A+Bz+Cz^2+Dz^3+Ez^4+\cdots+Pz^n.$$

Quoniam huius seriei summa in infinitum constat, quaeratur summa terminorum ultimum $Pz^n$ in infinitum sequentium, qui sint

$$t=Qz^{n+1}+Rz^{n+2}+Sz^{n+3}+Tz^{n+4}+\text{etc.};$$

haec series per $z^{n+1}$ divisa dat seriem recurrentem propositae aequalem, cuius propterea summa erit

$$t=\frac{Qz^{n+1}+(R-\alpha Q)z^{n+2}+(S-\alpha R+\beta Q)z^{n+3}+(T-\alpha S+\beta R-\gamma Q)z^{n+4}}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}.$$

Unde orietur summa quaesita

$$s = \frac{A+(B-\alpha A)z+(C-\alpha B+\beta A)z^2+(D-\alpha C+\beta B-\gamma A)z^3}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}$$

$$-\frac{Qz^{n+1}+(R-\alpha Q)z^{n+2}+(S-\alpha R+\beta Q)z^{n+3}+(T-\alpha S+\beta R-\gamma Q)z^{n+4}}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}$$

233. Quodsi ergo scala relationis fuerit bimembris $\alpha,-\beta$, seriei

$$A+Bz+Cz^2+Dz^3+Ez^4+\cdots+Pz^n,$$

quae oritur ex fractione

$$\frac{A+(B-\alpha A)z}{1-\alpha z+\beta zz}$$

summa erit

$$\frac{A+(B-\alpha A)z-Qz^{n+1}-(R-\alpha Q)z^{n+2}}{1-\alpha z+\beta zz}.$$

At est ex natura seriei

$$R=\alpha Q-\beta P,$$

unde prodibit summa

$$\frac{A+(B-\alpha A)z-Qz^{n+1}+\beta z^{n+2}}{1-\alpha z+\beta zz}$$

EXEMPLUM

Sit proposita series

$$1+3z+4z^2+7z^3+\cdots+Pz^n,$$

ubi est

$$\alpha=1,\quad \beta=-1,\quad A=1,\quad B=3;$$

erit huius summa

$$\frac{1+2z-Qz^{n+1}-Pz^{n+2}}{1-z-zz}.$$

Posito vero $z = 1$ erit summa seriei

$$1 + 3 + 4 + 7 + 11 + \cdots + P = P + Q - 3.$$

Summa ergo termini ultimi et sequentis ternario excedit summam seriei. Quia vero est

$$Q = \frac{P + \sqrt{(5PP \pm 20)}}{2}$$

erit summa seriei

$$1 + 3 + 4 + 7 + 11 + \cdots + P = \frac{3P - 6 + \sqrt{(5PP \pm 20)}}{2}.$$

Ex solo ergo termino ultimo summa potest exhiberi.