CHAPTER XII

CONCERNING THE REAL EXPANSION OF FRACTIONAL FUNCTIONS.

199. Now above, in the second chapter, a method has examined any fractional function being resolved into just as many parts as its denominator may have simple factors ; indeed these present the denominators of all the partial fractions. From which it is evident, if a denominator may have simple imaginary parts, the fractions thence also arising shall be imaginary ; therefore in these cases it will be of little help to have a real fraction resolved into imaginary parts. Therefore [Ch. IX] I have shown that for all integral functions, the denominator of any fraction of such, however great the number of simple imaginary factors may be present, yet may be resolved into real two–fold factors, or of the second dimension ; in this manner the resolution of fractional imaginary quantities can be avoided, if we may assume for the denominators, two–fold factors rather than simple principal denominators.

200. Therefore this fractional function shall be proposed $\frac{M}{N}$, from which just as many simple fractions may be elicited following the method put in place above, as the denominator N may have simple real factors. But in place of the imaginary factors there shall be this expression

$$pp-2pqz\cos.\varphi+qqzz,$$

for a factor of N ; and because in this calculation it is required to consider the numerator and denominator to be set out in this form, this shall be the proposed fraction

$$\frac{A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}}{(pp-2pqz\cos.\varphi+qqzz)(\alpha+\beta z+\gamma zz+\delta z^3+\text{etc.})}$$

and this partial fraction arises, that may be put in place from the factor of the denominator $pp-2pqz\cos.\varphi+qqzz$,

$$\frac{\mathfrak{A}+\mathfrak{a}z}{pp-2pqz\cos.\varphi+qqzz}$$

because indeed the variable z in the denominator has two dimensions, in the numerator it will only have one dimension, not truly several; for the integral function may contain several, that it will be necessary to elicit in turn.

201. For the sake of brevity the numerator shall be

$$A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}=M$$

and the other factor of the denominator

$$a+\beta z+\gamma z^2+\text{etc.}=Z\,;$$

the other part Z arising from the factor of the denominator may be put $=\frac{Y}{Z}$ and there will be

$$Y=\frac{M-\mathfrak{A}Z-\mathfrak{a}zZ}{pp-2pqz\cos.\varphi+qqzz}$$

[*i.e.* the whole fraction

$$\frac{A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}}{(pp-2pqz\cos.\varphi+qqzz)(\alpha+\beta z+\gamma zz+\delta z^3+\text{etc.})}=\frac{M}{(pp-2pqz\cos.\varphi+qqzz)Z}$$
$$=\frac{\mathfrak{A}+\mathfrak{a}z}{pp-2pqz\cos.\varphi+qqzz}+\frac{Y}{Z}.]$$

which expression Y must be an integral function of z, and thus it is necessary that $M-\mathfrak{A}Z-\mathfrak{a}zZ$ shall be divisible by $pp-2pqz\cos.\varphi+qqzz$. Therefore $M-\mathfrak{A}Z-\mathfrak{a}zZ$ shall vanish, if there is put

$$pp-2pqz\cos.\varphi+qqzz=0\,,$$

that is, if there may be put both [§ 146]

$$z=\frac{p}{q}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)$$

as well as

$$z=\frac{p}{q}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right).$$

Let $\frac{p}{q}=f$ and there becomes [§ 133]

$$z^n=f^n\left(\cos.n\varphi\pm\sqrt{-1}\cdot\sin.n\varphi\right).$$

Therefore the double value substituted for z will give a two–fold equation, from which both the unknown constants $\mathfrak{A}$ and $\mathfrak{a}$ may be defined.

202. Therefore with this substitution made, the equation

$$M=\mathfrak{A}Z+\mathfrak{a}zZ$$

expanded out will give this equation

$$\begin{aligned}&A+Bf\cos.\varphi+Cff\cos.2\varphi+Df^3\cos.3\varphi+\text{etc.}\\&\pm\left(Bf\sin.\varphi+Cff\sin.2\varphi+Df^3\sin.3\varphi+\text{etc.}\right)\sqrt{-1}\\&=\mathfrak{A}\left(\alpha+\beta f\cos.\varphi+\gamma ff\cos.2\varphi+\delta f^3\cos.3\varphi+\text{etc.}\right)\\&\quad\pm\mathfrak{A}\left(\beta f\sin.\varphi+\gamma ff\sin.2\varphi+\delta f^3\sin.3\varphi+\text{etc.}\right)\sqrt{-1}\\&\quad+\mathfrak{a}\left(\alpha f\cos.\varphi+\beta ff\cos.2\varphi+\gamma f^3\cos.3\varphi+\text{etc.}\right)\\&\quad\pm\mathfrak{a}\left(\alpha f\sin.\varphi+\beta ff\sin.2\varphi+\gamma f^3\sin.3\varphi+\text{etc.}\right)\sqrt{-1}.\end{aligned}$$

To abbreviate the calculation, let

$$\begin{aligned}A+Bf\cos.\varphi+Cff\cos.2\varphi+Df^3\cos.3\varphi+\text{etc.}&=\mathfrak{P},\\Bf\sin.\varphi+Cff\sin.2\varphi+Df^3\sin.3\varphi+\text{etc.}&=\mathfrak{p},\\\alpha+\beta f\cos.\varphi+\gamma ff\cos.2\varphi+\delta f^3\cos.3\varphi+\text{etc.}&=\mathfrak{Q},\\\beta f\sin.\varphi+\gamma ff\sin.2\varphi+\delta f^3\sin.3\varphi+\text{etc.}&=\mathfrak{q},\\\alpha f\cos.\varphi+\beta ff\cos.2\varphi+\gamma f^3\cos.3\varphi+\text{etc.}&=\mathfrak{R},\\\alpha f\sin.\varphi+\beta ff\sin.2\varphi+\gamma f^3\sin.3\varphi+\text{etc.}&=\mathfrak{r}\end{aligned}$$

and with these in place, the equation [$M=\mathfrak{A}Z+\mathfrak{a}zZ$] becomes

$$\mathfrak{P}\pm\mathfrak{p}\sqrt{-1}=\mathfrak{A}\mathfrak{Q}\pm\mathfrak{A}\mathfrak{q}\sqrt{-1}+\mathfrak{a}\mathfrak{R}\pm\mathfrak{a}\mathfrak{r}\sqrt{-1}\,.$$

203. On account of the ambiguity of the signs, these two equations arise :

$$\mathfrak{P}=\mathfrak{A}\mathfrak{Q}+\mathfrak{a}\mathfrak{R}\,,$$
$$\mathfrak{p}=\mathfrak{A}\mathfrak{q}+\mathfrak{a}\mathfrak{r}\,,$$

from which the unknowns $\mathfrak{A}$ and $\mathfrak{a}$ thus are defined, so that there shall be

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{r} - \mathfrak{p}\mathfrak{R}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}} \text{ and } \mathfrak{a} = \frac{\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}}{\mathfrak{q}\mathfrak{R} - \mathfrak{Q}\mathfrak{r}}$$

Therefore for the proposed fraction

$$\frac{M}{(pp - 2pqz\cos.\varphi + qqzz)Z}$$

the partial fraction arising from that may be defined by the following rule:

$$\frac{\mathfrak{A} + \mathfrak{a}z}{pp - 2pqz\cos.\varphi + qqzz}.$$

By putting $f = \frac{p}{q}$ and with the individual terms expanded out, it becomes as follows :
On putting

$$z^n = f^n\cos.n\varphi, \quad M \text{ becomes } = \mathfrak{P},$$
$$z^n = f^n\sin.n\varphi, \quad M \text{ becomes } = \mathfrak{p},$$
$$z^n = f^n\cos.n\varphi, \quad Z \text{ becomes } = \mathfrak{Q},$$
$$z^n = f^n\sin.n\varphi, \quad Zz \text{ becomes } = \mathfrak{q},$$
$$z^n = f^n\cos.n\varphi, \quad zZ \text{ becomes } = \mathfrak{R},$$
$$z^n = f^n\sin.n\varphi, \quad zZ \text{ becomes } = \mathfrak{r}.$$

[*i.e*. the real and imaginary parts of the terms are found, for some *n*.]

With the values $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ found in this way, there becomes

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{r} - \mathfrak{p}\mathfrak{R}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}} \text{ and } \mathfrak{a} = \frac{\mathfrak{p}\mathfrak{Q} - \mathfrak{P}\mathfrak{q}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}}.$$

EXAMPLE 1

Let this fraction be proposed to be resolved :

$$\frac{zz}{(1 - z + zz)(1 + z^4)},$$

from the denominator factor $1 - z + zz$ of which, it shall be required to define the [fractional] part

$$\frac{\mathfrak{A} + \mathfrak{a}z}{1 - z + zz}$$

And indeed in the first place this factor compared with the general form $pp - 2pqz\cos.\varphi + qqzz$ gives:

$$p = 1, \quad q = 1 \quad \text{and} \quad \cos.\varphi = \tfrac{1}{2},$$

from which there becomes

$$\varphi = 60^0 = \tfrac{\pi}{3}.$$

And thus because there is

$$M = zz,\ Z = 1 + z^4 \ \text{ and } \ f = 1,$$

there will be

$$\mathfrak{P} = \cos.\tfrac{2\pi}{3} = -\tfrac{1}{2},\ \ \mathfrak{p} = \tfrac{\sqrt{3}}{2},$$

$$\mathfrak{Q} = 1 + \cos.\tfrac{4\pi}{3} = \tfrac{1}{2},\ \ \mathfrak{q} = -\tfrac{\sqrt{3}}{2},$$

$$\mathfrak{R} = \cos.\tfrac{2\pi}{3} + \cos.\tfrac{5\pi}{3} = 1,\ \ \mathfrak{r} = 0.$$

From these is found

$$\mathfrak{A} = -1 \ \text{ and } \ \mathfrak{a} = 0$$

and thus the fraction sought is

$$\frac{-1}{1 - z + zz}$$

and the complement of this shall be

$$\frac{1 - z + zz}{1 + z^4};$$

of which the denominator $1 + z^4$ since it may have the factors $1 + z\sqrt{2} + zz$ and $1 - z\sqrt{2} + zz$, can be undertaken to be resolved anew ; moreover there becomes $\varphi = \frac{\pi}{4}$ and in the first case $f = -1$, in the second $f = +1$.

EXAMPLE 2

Let this fraction be proposed to be resolved
:

$$\frac{1 + z + zz}{\left(1 + z\sqrt{2} + zz\right)\left(1 - z\sqrt{2} + zz\right)}$$

and there will be

$$M = 1 + z + zz;$$

and for the first factor there will be found:

$$f=-1,\quad \varphi=\tfrac{\pi}{4}\quad \text{and}\quad Z=1-z\sqrt{2}+zz$$

from which there becomes, [on substituting the real and imaginary parts of z into the above table :]

$$\begin{aligned}
\mathfrak{P}&=1-\cos.\tfrac{\pi}{4}+\cos.\tfrac{2\pi}{4}=\tfrac{\sqrt{2}-1}{\sqrt{2}},\\
\mathfrak{p}&=-\sin.\tfrac{\pi}{4}+\sin.\tfrac{2\pi}{4}=\tfrac{\sqrt{2}-1}{\sqrt{2}},\\
\mathfrak{Q}&=1+\sqrt{2}\cdot\cos.\tfrac{\pi}{4}+\cos.\tfrac{\pi}{4}=2,\\
\mathfrak{q}&=+\sqrt{2}\cdot\sin.\tfrac{\pi}{4}+\sin.\tfrac{2\pi}{4}=2,\\
\mathfrak{R}&=-\cos.\tfrac{\pi}{4}-\sqrt{2}\cdot\cos.\tfrac{2\pi}{4}-\cos.\tfrac{3\pi}{4}=0,\\
\mathfrak{r}&=-\sin.\tfrac{\pi}{4}-\sqrt{2}\cdot\sin.\tfrac{2\pi}{4}-\sin.\tfrac{3\pi}{4}=-2\sqrt{2}.
\end{aligned}$$

From these there is found:

$$\mathfrak{Q}\mathfrak{r}-\mathfrak{q}\mathfrak{R}=-2\sqrt{2}$$

and

$$\mathfrak{A}=\tfrac{-1}{2\sqrt{2}}\ \text{et}\ \ \mathfrak{a}=0,$$

from which, with the factor of the denominator $1+z\sqrt{2}+zz$, this partial fraction will arise

$$\frac{\left(\sqrt{2}-1\right):2\sqrt{2}}{1+z\sqrt{2}+zz}$$

Moreover the other factor will give this in a similar manner

$$\frac{\left(\sqrt{2}+1\right):2\sqrt{2}}{1-z\sqrt{2}+zz}.$$

Hence the first proposed function

$$\frac{1+z+zz}{\left(1+z\sqrt{2}+zz\right)\left(1-z\sqrt{2}+zz\right)}$$

is resolved into these partial fractions :

$$\frac{-1}{1-z+zz}+\frac{\left(\sqrt{2}-1\right):2\sqrt{2}}{1+z\sqrt{2}+zz}+\frac{\left(\sqrt{2}+1\right):2\sqrt{2}}{1-z\sqrt{2}+zz}.$$

EXAMPLE 3

Let this fraction be proposed to be resolved

$$\frac{1+2z+zz}{\left(1-\frac{8}{5}z+zz\right)\left(1+2z+3zz\right)}.$$

For the factor of the denominator $1-\frac{8}{5}z+zz$ this fraction may arise :

$$\frac{\mathfrak{A}+\mathfrak{a}z}{1-\frac{8}{5}z+zz}$$

and there will be

$$p=1,\ q=1,\ \cos.\varphi=\tfrac{4}{5},$$

from which

$$f=1,\quad M=1+2z+zz,\quad Z=1+2z+3zz\,.$$

Truly because here the ratio of the angle φ to a right angle is not determined, the sines and cosines of the multiples must be found in turn. Since there shall be

$$\cos.\varphi=\tfrac{4}{5},\qquad \text{there will be}\quad \sin.\varphi=\tfrac{3}{5},$$
$$\cos.2\varphi=\tfrac{7}{25},\qquad \sin.2\varphi=\tfrac{24}{25},$$
$$\cos.3\varphi=-\tfrac{44}{125}\qquad \sin.3\varphi=\tfrac{117}{125};$$

hence there shall be

$$\mathfrak{P}=1+2\cdot\tfrac{4}{5}+\tfrac{7}{25}=\tfrac{72}{25},$$
$$\mathfrak{p}=2\cdot\tfrac{3}{5}+\tfrac{24}{25}=\tfrac{54}{25},$$
$$\mathfrak{Q}=1+2\cdot\tfrac{4}{5}+3\cdot\tfrac{7}{25}=\tfrac{86}{25},$$
$$\mathfrak{q}=2\cdot\tfrac{3}{5}+3\cdot\tfrac{24}{25}=\tfrac{102}{25},$$
$$\mathfrak{R}=\tfrac{4}{5}+2\cdot\tfrac{7}{25}-3\cdot\tfrac{44}{125}=\tfrac{38}{25},$$
$$\mathfrak{r}=\tfrac{4}{5}+2\cdot\tfrac{24}{25}+3\cdot\tfrac{117}{25}=\tfrac{666}{25}$$

and thus

$$\mathfrak{Q}\mathfrak{r}-\mathfrak{q}\mathfrak{R}=\tfrac{53400}{25\cdot125}=\tfrac{2136}{125}.$$

Therefore

$$\mathfrak{Q}=\frac{1836}{2136}=\frac{153}{178},\quad \mathfrak{a}=-\frac{540}{2136}=-\frac{45}{178}.$$

Whereby the fraction arising from the factor $1-\frac{8}{5}z+zz$ will be

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}.$$

In the same way we may find the fraction corresponding to the other factor ; there will be

$$p=1,\quad q=-\sqrt{3}\quad \text{and}\quad \cos.\varphi=\frac{1}{\sqrt{3}},$$

therefore

$$f=-\frac{1}{\sqrt{3}},\quad M=1+2z+zz\quad \text{and}\quad Z=1-\frac{8}{5}z+zz.$$

On account of which there becomes :

$$\cos.\varphi=\frac{1}{\sqrt{3}},\qquad \sin.\varphi=\frac{\sqrt{2}}{\sqrt{3}},$$
$$\cos.2\varphi=-\frac{1}{3},\qquad \sin.2\varphi=\frac{2\sqrt{2}}{\sqrt{3}},$$
$$\cos.3\varphi=-\frac{5}{3\sqrt{3}}\qquad \sin.3\varphi=\frac{\sqrt{2}}{3\sqrt{3}},$$

consequently

$$\mathfrak{P}=1-\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}+\frac{1}{3}\cdot-\frac{1}{3}=\frac{2}{9},$$
$$\mathfrak{p}=-\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}+\frac{1}{3}\cdot\frac{2\sqrt{2}}{\sqrt{3}}=-\frac{4\sqrt{2}}{9},$$
$$\mathfrak{Q}=1+\frac{8}{5\sqrt{3}}\cdot\frac{1}{\sqrt{3}}+\frac{1}{3}\cdot-\frac{1}{3}=\frac{64}{45},$$
$$\mathfrak{q}=+\frac{8}{5\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}+\frac{1}{3}\cdot\frac{2\sqrt{2}}{\sqrt{3}}=\frac{34\sqrt{2}}{45},$$
$$\mathfrak{R}=-\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}-\frac{8}{5\cdot3}\cdot-\frac{1}{3}-\frac{1}{3\sqrt{3}}\cdot-\frac{5}{3\sqrt{3}}=\frac{4}{135},$$
$$\mathfrak{r}=-\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}-\frac{8}{5\cdot3}\cdot\frac{2\sqrt{2}}{3}-\frac{1}{3\sqrt{3}}\cdot\frac{\sqrt{2}}{3\sqrt{3}}=-\frac{98\sqrt{2}}{135}$$

and thus

$$\mathfrak{Q}\mathfrak{r}-\mathfrak{q}\mathfrak{R}=-\frac{712\sqrt{2}}{675};$$

therefore there becomes

$$\mathfrak{A}=\frac{100}{712}=\frac{25}{178},\quad \mathfrak{a}=\frac{540}{712}=\frac{135}{178}.$$

Therefore the proposed fraction

$$\frac{1+2z+zz}{\left(1-\frac{8}{5}z+zz\right)\left(1+2z+3zz\right)}$$

is resolved into

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}+\frac{5(5+27z):178}{1+2z+3zz}.$$

204. But the values of the letters $\mathfrak{R}$ and $\mathfrak{r}$ can be defined from the letters $\mathfrak{Q}$ and $\mathfrak{q}$. [*i.e.* the real and imaginary parts of the term Z are $\mathfrak{Q}$ and $\mathfrak{q}$; and of zZ, $\mathfrak{R}$ and $\mathfrak{r}$.]
For since there shall be

$$\mathfrak{Q}=\alpha+\beta\, f cos.\varphi+\gamma\, ff cos.2\varphi+\delta f^{3} cos.3\varphi+\text{etc.}$$
$$\mathfrak{q}=\beta f\ sin.\varphi+\gamma\, ff\ sin.2\varphi+\delta f^{3} sin.3\varphi+\text{etc.}$$

there will be

$$\mathfrak{Q}cos.\varphi-\mathfrak{q}\,sin.\varphi=\alpha cos.\varphi+\beta\, f cos.2\varphi+\gamma\, ff cos.3\varphi+\text{etc.}$$

and thus

$$\mathfrak{R}=f\left(\mathfrak{Q}cos.\varphi-\mathfrak{q}\,sin.\varphi\right).$$

Then there will be

$$\mathfrak{Q}\sin.\varphi+\mathfrak{q}\cos.\varphi=\alpha\sin.\varphi+\beta\, f sin.2\varphi+\gamma f^{2}\sin.3\varphi+\text{etc.},$$

therefore

$$\mathfrak{r}=f\left(\mathfrak{Q}\sin.\varphi+\mathfrak{q}\cos.\varphi\right).$$

From these again there shall be

$$\mathfrak{Q}\mathfrak{r}-\mathfrak{q}\mathfrak{R}=\left(\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}\right) f sin.\varphi\,,$$

$$\mathfrak{P}\mathfrak{r}-\mathfrak{q}\mathfrak{R}=\left(\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}\right) f\sin\varphi+\left(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}\right) f\cos.\varphi\,;$$

and consequently there will be

$$\mathfrak{A}=\frac{\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}}{\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}}+\frac{\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}}{\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}}\cdot\frac{\cos.\varphi}{\sin\varphi},$$
$$\mathfrak{a}=\frac{-\mathfrak{P}\mathfrak{q}+\mathfrak{p}\mathfrak{Q}}{\left(\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}\right) f sin.\varphi}.$$

Whereby from the factor of the denominator $pp-2pqz\cos.\varphi+qqzz$ this partial fraction arises

$$\frac{\left(\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}\right) f sin.\varphi+\left(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}\right)\left(f\ cos.\varphi-z\right)}{\left(pp-2pqz\cos.\varphi+qqzz\right)\left(\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}\right) f sin.\varphi}$$

or this, on account of $f=\frac{p}{q}$:

$$\frac{(\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q})p\sin.\varphi+(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q})(p\cos.\varphi-qz)}{(pp-2pqz\cos.\varphi+qqzz)(\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q})p\sin.\varphi}.$$

205. This [above] partial fraction therefore arises from the proposed function

$$\frac{M}{(pp-2pqz\cos.\varphi+qqzz)Z}$$

with the factor of the denominator $pp-2pqz\cos.\varphi+qqzz$ and the following letters $\mathfrak{P},\mathfrak{p},\mathfrak{Q}$ and $\mathfrak{q}$ are found from the functions M and Z only : On putting

$$z^n=\frac{p^n}{q^n}.\cos.n\varphi \text{ there shall be } M=\mathfrak{P} \text{ and } Z=\mathfrak{Q}$$

and on putting

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \text{ there shall be } M=\mathfrak{p} \text{ and } Z=\mathfrak{q},$$

where it is to be observes the functions M and Z, before this substitution is made, must all be expanded out, so that they have forms of this kind

$$M=A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}$$

and

$$Z=\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\text{etc.};$$

and thus there will be

$$\mathfrak{P}=A+B\frac{p}{q}\cos.\varphi+C\frac{p^2}{q^2}\cos.2\varphi+D\frac{p^3}{q^3}\cos.3\varphi+\text{ etc.},$$

$$\mathfrak{p}=\qquad B\frac{p}{q}\sin.\varphi+C\frac{p^2}{q^2}\sin.2\varphi+D\frac{p^3}{q^3}\sin.3\varphi+\text{ etc.},$$

$$\mathfrak{Q}=\alpha+\beta\frac{p}{q}\cos.\varphi+\gamma\frac{p^2}{q^2}\cos.2\varphi+\delta\frac{p^3}{q^3}\cos.3\varphi+\text{ etc.},$$

$$\mathfrak{q}=\qquad \beta\frac{p}{q}\sin.\varphi+\gamma\frac{p^2}{q^2}\sin.2\varphi+\delta\frac{p^3}{q^3}\sin.3\varphi+\text{ etc.}$$

206. But from the preceding it is understood this resolution does not have a place, if the function Z at this stage has the same factor $pp-2pqz\cos.\varphi+qqzz$ included within it ; for in this case in the equation

$$M=\mathfrak{A}Z+\alpha zZ$$

with the substitution made

$$z^n = f^n\left(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi\right)$$

the quantity Z itself will vanish and therefore nothing may be deduced. On account of which if the denominator of the fractional function $\frac{M}{N}$ may have the factor $(pp - 2pqz\cos.\varphi + qqzz)^2$ or a higher power, there will be a need for the resolution of the individual powers. Therefore let

$$N = (pp - 2pqz\cos.\varphi + qqzz)^2 Z,$$

and from the factor $(pp - 2pqz\cos.\varphi + qqzz)^2$ of the denominator, two partial fractions of this kind will emerge

$$\frac{\mathfrak{A}+\mathfrak{a}z}{(pp-2pqz\cos.\varphi+qqzz)^2} + \frac{\mathfrak{B}+\mathfrak{b}z}{pp-2pqz\cos.\varphi+qqzz},$$

where it is necessary to determine the constant letters $\mathfrak{A}, \mathfrak{a}, \mathfrak{B}$ and $\mathfrak{b}$.

207. With these in place this expression

$$\frac{M-(\mathfrak{A}+\mathfrak{a}z)Z-(\mathfrak{B}+\mathfrak{b}z)Z(pp-2pqz\cos.\varphi+qqzz)}{(pp-2pqz\cos.\varphi+qqzz)^2}$$

must be a whole function and on this account the numerator will be divisible by the denominator [§ 43]. Therefore in the first place this expression

$$M - \mathfrak{A}Z - \mathfrak{a}zZ$$

must be divisible by $pp - 2pqz\cos.\varphi + qqzz$; which since it shall be the preceding case, the letters $\mathfrak{A}$ and $\mathfrak{a}$ will be found in the same way too.

Whereby on putting

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \text{ there shall be } M = \mathfrak{P} \text{ and } Z = \mathfrak{N}$$

and on putting

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \text{ there shall be } M = \mathfrak{p} \text{ and } Z = \mathfrak{n}.$$

And with these made, the second rule given above will become :

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{N}+\mathfrak{p}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{P}\mathfrak{n}-\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{a} = \frac{-\mathfrak{P}\mathfrak{n}+\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

208. Therefore with $\mathfrak{A}$ and $\mathfrak{a}$ found in this way, the whole function

$$\frac{M-(\mathfrak{A}+\mathfrak{a}z)Z}{pp-2pqz\cos.\varphi+qqzz}$$

is made, which shall be $=P$, and remains, so that

$$P-\mathfrak{B}Z-\mathfrak{b}zZ$$

emerges divisible by $pp-2pqz\cos.\varphi+qqzz$; which expression since it shall be similar to the preceding, if on putting

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \text{ becomes } P=\mathfrak{R}$$

and on putting

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \text{ becomes } P=\mathfrak{r},$$

there will become

$$\mathfrak{B} = \frac{\mathfrak{R}\mathfrak{N}+\mathfrak{r}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{R}\mathfrak{n}-\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{b} = \frac{-\mathfrak{R}\mathfrak{n}+\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

209. Hence now generally it is possible to conclude, in what way the resolution must be put in place, if the denominator of the proposed function $\frac{M}{N}$ should have the factor

$$\left(pp-2pqz\cos.\varphi+qqzz\right)^k.$$

For there shall be

$$N=\left(pp-2pqz\cos.\varphi+qqzz\right)^k Z,$$

thus so that this shall be the fractional function required to be resolved :

$$\frac{M}{\left(pp-2pqz\cos.\varphi+qqzz\right)^k Z}.$$

Therefore the factor of the denominator $\left(pp-2pqz\cos.\varphi+qqzz\right)^k$ presents these parts :

$$\frac{\mathfrak{A}+\mathfrak{a}z}{(pp-2pqz\cos.\varphi+qqzz)^k}+\frac{\mathfrak{B}+\mathfrak{b}z}{(pp-2pqz\cos.\varphi+qqzz)^{k-1}}$$
$$+\frac{\mathfrak{C}+\mathfrak{c}z}{(pp-2pqz\cos.\varphi+qqzz)^{k-2}}+\frac{\mathfrak{D}+\mathfrak{d}z}{(pp-2pqz\cos.\varphi+qqzz)^{k-2}}+\text{etc.}$$

Now on putting

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \text{ there shall be } M=\mathfrak{M} \text{ and } Z=\mathfrak{N}$$

and on putting

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \text{ there shall be } M=\mathfrak{m} \text{ and } Z=\mathfrak{n};$$

there becomes

$$\mathfrak{A}=\frac{\mathfrak{M}\mathfrak{N}+\mathfrak{m}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2}+\frac{\mathfrak{M}\mathfrak{n}-\mathfrak{m}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{a}=\frac{-\mathfrak{M}\mathfrak{n}+\mathfrak{m}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

Then there will be named :

$$\frac{M-(\mathfrak{A}+\mathfrak{a}z)Z}{pp-2pqz\cos.\varphi+qqzz}=P$$

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \text{ shall be } M=\mathfrak{P}$$

and on putting

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \text{ there shall be } P=\mathfrak{p};$$

there will be

$$\mathfrak{B}=\frac{\mathfrak{P}\mathfrak{N}+\mathfrak{m}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2}+\frac{\mathfrak{P}\mathfrak{n}-\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{b}=\frac{-\mathfrak{P}\mathfrak{n}+\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

Then there may be named :

$$\frac{P-(\mathfrak{B}+\mathfrak{b}z)Z}{pp-2pqz\cos.\varphi+qqzz}=Q$$

and on putting

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \text{ there shall be } Q=\mathfrak{Q};$$

and on putting

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \text{ there shall be } Q=\mathfrak{q};$$

there becomes

$$\mathfrak{C} = \frac{\mathfrak{Q}\mathfrak{N}+\mathfrak{q}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{Q}\mathfrak{n}-\mathfrak{q}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{c} = \frac{-\mathfrak{Q}\mathfrak{n}+\mathfrak{q}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

Again there may be named

$$\frac{Q-(\mathfrak{C}+\mathfrak{c}z)Z}{pp-2pqz\cos.\varphi+qqzz} = R$$

and on putting

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \text{ there shall be } R = \mathfrak{R}$$

and on putting

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \text{ there shall be } R = \mathfrak{r};$$

there becomes

$$\mathfrak{D} = \frac{\mathfrak{R}\mathfrak{N}+\mathfrak{r}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{R}\mathfrak{n}-\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{d} = \frac{-\mathfrak{R}\mathfrak{n}+\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

And by progressing in this way, until the numerator of the last of the fractions shall be determined, the denominator of which shall be $pp-2pqz\cos.\varphi+qqzz$.

EXAMPLE

Let this fractional function be proposed :

$$\frac{z-z^3}{(1+zz)^4(1+z^4)}$$

from the factor of the denominator $(1+zz)^4$ these fractional parts may arise :

$$\frac{\mathfrak{A}+\mathfrak{a}z}{(1+zz)^4} + \frac{\mathfrak{B}+\mathfrak{b}z}{(1+zz)^3} + \frac{\mathfrak{C}+\mathfrak{c}z}{(1+zz)^2} + \frac{\mathfrak{D}+\mathfrak{d}z}{1+zz}.$$

By comparison therefore there will be put in place :

$$p=1,\ q=1,\ \cos.\varphi=0 \text{ and thus } \varphi=\frac{\pi}{2}$$

and again

$$M = z-z^3 \text{ and } Z = 1+z^4$$

Hence there will be

$$\mathfrak{M}=0,\ \mathfrak{m}=2,\ \mathfrak{N}=2,\ \mathfrak{n}=0 \text{ and } \sin.\varphi=1.$$

And thus hence there is found:

$$\mathfrak{A} = -\tfrac{4}{4}\cdot 0 = 0, \text{ and } \mathfrak{a} = 1,$$

therefore

$$\mathfrak{A} + \mathfrak{a}z = z$$

and hence

$$P = \tfrac{z - z^3 - z - z^5}{1 + zz} = -z^3$$

and

$$\mathfrak{P} = 0, \ \mathfrak{p} = 1,$$

from which there is found

$$\mathfrak{B} = 0 \text{ and } \mathfrak{b} = \tfrac{1}{2}.$$

Therefore

$$\mathfrak{B} + \mathfrak{b}z = \tfrac{1}{2}z$$

and

$$Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^5}{1+zz} = -\tfrac{1}{2}z - \tfrac{1}{2}z^3,$$

from which

$$\mathfrak{Q} = 0 \text{ and } \mathfrak{q} = 0,$$

therefore

$$\mathfrak{C} = 0 \text{ and } \mathfrak{c} = 0.$$

And hence

$$R = \frac{-\frac{1}{2}z - \frac{1}{2}zz^3}{1+zz} = -\tfrac{1}{2}z,$$

therefore

$$\mathfrak{R} = 0 \text{ and } \mathfrak{r} = -\tfrac{1}{2},$$

from which there becomes

$$\mathfrak{D} = 0 \text{ and } \mathfrak{d} = -\tfrac{1}{4}.$$

On account of which these are the fractions sought :

$$\frac{z}{(1+zz)^4} + \frac{z}{2(1+zz)^3} - \frac{z}{4(1+zz)}.$$

Truly the numerator of the remaining fraction is :

$$S = \frac{R - (\mathfrak{D} + \mathfrak{d}z)Z}{1+zz} = -\tfrac{1}{4}z + \tfrac{1}{4}z^3,$$

which therefore will be

$$\frac{-z+z^3}{4(1+z^4)}.$$

210. Therefore by this method likewise the fraction of the complement becomes known, which, taken together with the fraction found, produces the proposed fraction itself. Clearly if all the partial fractions of the fraction

$$\frac{M}{(pp-2pqz\cos.\varphi+qqzz)^k Z}$$

were found arising from the factor $(pp-2pqz\cos.\varphi+qqzz)^k$, for which the forms are the values of the functions *P, Q, R, S, T,* if the series of these letters may be continued further, for which there is a need for finding the numerators it follows that the final will be the numerator of the remaining fraction having the denominator Z; surely, if $k=1$, the remaining fraction will be $\frac{P}{Z}$; if $k=2$, the remaining fraction will be $\frac{Q}{Z}$; if $k=3$, that will be $\frac{R}{Z}$, and thus henceforth. But for this remaining fraction having the denominator Z that will be able to be found by the final rules.

CAPUT XII

DE REALI FUNCTIONUM FRACTARUM EVOLUTIONE

199. Iam supra, in capite secundo, methodus est tradita functionem quamcunque fractam in tot partes resolvendi, quot eius denominator habeat factores simplices; hi enim praebent denominatores fractionum illarum partialium. Ex quo manifestum est, si denominator quos habeat factores simplices imaginarios, fractiones quoque inde ortas fore imaginarias; his ergo casibus parum iuvabit fractionem realem in imaginarias resolvisse. Cum igitur ostendissem [cap. IX] omnem functionem integram, qualis est denominator cuiusvis fractionis, quantumvis factoribus simplicibus imaginariis scateat, tamen in factores duplices, seu secundae dimensionis, reales semper resolvi posse, hoc modo in resolutione fractionum quantitates imaginariae evitari poterunt, si pro denominatoribus fractionum partialium non factores denominatoris principalis simplices, sed duplices reales assumamus.

200. Sit igitur proposita haec functio fracta $\frac{M}{N}$ ex qua tot fractiones simplices secundum methodum supra [cap.11] expositam eliciantur, quot denominator N habuerit factores simplices reales. Sit autem loco imaginariorum haec expressio

$$pp-2pqz\cos.\varphi+qqzz$$

factor ipsius N, et quoniam in hoc negotio numeratorem et denominatorem in forma, evoluta contemplari oportet, sit haec fractio proposita

$$\frac{A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}}{(pp-2pqz\cos.\varphi+qqzz)(\alpha+\beta z+\gamma zz+\delta z^3+\text{etc.})}$$

ac ponatur fractio partialis ex denominatoris factore $pp-2pqz\cos.\varphi+qqzz$; $+qqzz$ oriunda haec

$$\frac{\mathfrak{A}+\mathfrak{a}z}{(pp-2pqz\cos.\varphi+qqzz)}$$

quoniam enim variabilis z in denominatore duas habet dimensiones, in numeratore unam habere poterit, non vero plures; alias enim integra functio contineretur, quam seorsim elici oportet.

201. Sit brevitatis gratia numerator

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.} = M$$

et alter denominatoris factor

$$a + \beta z + \gamma z^2 + \text{etc.} = Z\,;$$

ponatur altera pars ex denominatoris factore Z oriunda $= \frac{Y}{Z}$ eritque

$$Y = \frac{M - \mathfrak{A}Z - \mathfrak{a}zZ}{pp - 2pqz\cos.\varphi + qqzz}$$

quae expressio functio integra ipsius z esse debet, ideoque necesse est, ut $M - \mathfrak{A}Z - \mathfrak{a}zZ$ divisibile sit per $pp - 2pqz\cos.\varphi + qqzz$. Evanescet ergo $M - \mathfrak{A}Z - \mathfrak{a}zZ$, si ponatur

$$pp - 2pqz\cos.\varphi + qqzz = 0\,,$$

hoc est, si ponatur [§ 146] tam

$$z = \frac{p}{q}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)$$

quam

$$z = \frac{p}{q}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right)$$

sit $\frac{p}{q} = f$ eritque [§ 133]

$$z^n = f^n\left(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi\right).$$

Duplex ergo hic valor pro z substitutus duplicem dabit aequationem, unde ambas incognitas constantes $\mathfrak{A}$ et $\mathfrak{a}$ definire licet.

202. Facta ergo hac substitutione aequatio

$$M = \mathfrak{A}Z + \mathfrak{a}zZ$$

evoluta hanc duplicem dabit aequationem

$$\begin{aligned}&A+Bf\cos.\varphi+Cff\cos.2\varphi+Df^3\cos.3\varphi+\text{etc.}\\&\pm\left(Bf\sin.\varphi+Cff\sin.2\varphi+Df^3\sin.3\varphi+\text{etc.}\right)\sqrt{-1}\\&=\mathfrak{A}\left(\alpha+\beta f\cos.\varphi+\gamma ff\cos.2\varphi+\delta f^3\cos.3\varphi+\text{etc.}\right)\\&\quad\pm\mathfrak{A}\left(\beta f\sin.\varphi+\gamma ff\sin.2\varphi+\delta f^3\sin.3\varphi+\text{etc.}\right)\sqrt{-1}\\&\quad+\mathfrak{a}\left(\alpha f\cos.\varphi+\beta ff\cos.2\varphi+\gamma f^3\cos.3\varphi+\text{etc.}\right)\\&\quad\pm\mathfrak{a}\left(\alpha f\sin.\varphi+\beta ff\sin.2\varphi+\gamma f^3\sin.3\varphi+\text{etc.}\right)\sqrt{-1}.\end{aligned}$$

Sit ad calculum abbreviandum

$$\begin{aligned}A+Bf\cos.\varphi+Cff\cos.2\varphi+Df^3\cos.3\varphi+\text{etc.}&=\mathfrak{P},\\Bf\sin.\varphi+Cff\sin.2\varphi+Df^3\sin.3\varphi+\text{etc.}&=\mathfrak{p},\\\alpha+\beta f\cos.\varphi+\gamma ff\cos.2\varphi+\delta f^3\cos.3\varphi+\text{etc.}&=\mathfrak{Q},\\\beta f\sin.\varphi+\gamma ff\sin.2\varphi+\delta f^3\sin.3\varphi+\text{etc.}&=\mathfrak{q},\\\alpha f\cos.\varphi+\beta ff\cos.2\varphi+\gamma f^3\cos.3\varphi+\text{etc.}&=\mathfrak{R},\\\alpha f\sin.\varphi+\beta ff\sin.2\varphi+\gamma f^3\sin.3\varphi+\text{etc.}&=\mathfrak{r}\end{aligned}$$

eritque his positis

$$\mathfrak{P}\pm\mathfrak{p}\sqrt{-1}=\mathfrak{A}\mathfrak{Q}\pm\mathfrak{A}\mathfrak{q}\sqrt{-1}+\mathfrak{a}\mathfrak{R}\pm\mathfrak{a}\mathfrak{r}\sqrt{-1}\,.$$

203. Ob signorum ambiguitatem hae duae oriuntur aequationes

$$\mathfrak{P}=\mathfrak{A}\mathfrak{Q}+\mathfrak{a}\mathfrak{R}\,,$$
$$\mathfrak{p}=\mathfrak{A}\mathfrak{q}+\mathfrak{a}\mathfrak{r}\,,$$

ex quibus incognitae $\mathfrak{A}$ et $\mathfrak{a}$ ita definiuntur, ut sit

$$\mathfrak{A}=\tfrac{\mathfrak{P}\mathfrak{r}-\mathfrak{p}\mathfrak{R}}{\mathfrak{Q}\mathfrak{r}-\mathfrak{q}\mathfrak{R}}\text{ et }\mathfrak{a}=\tfrac{\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}}{\mathfrak{q}\mathfrak{R}-\mathfrak{Q}\mathfrak{r}}$$

Proposita ergo fractione

$$\frac{M}{\left(pp-2pqz\cos.\varphi+qqzz\right)Z}$$

per sequentem regulam fractio partialis ex ea, oriunda

$$\frac{\mathfrak{A}+\mathfrak{a}z}{pp-2pqz\cos.\varphi+qqzz}$$

definietur. Posito $f=\frac{p}{q}$ et evolutis singulis terminis fiat, ut sequitur:
Posito

$$z^n = f^n\cos.n\varphi \quad \text{sit} \quad M=\mathfrak{P},$$
$$z^n = f^n\sin.n\varphi \quad \text{sit} \quad M=\mathfrak{p},$$
$$z^n = f^n\cos.n\varphi \quad \text{sit} \quad Z=\mathfrak{Q},$$
$$z^n = f^n\sin.n\varphi \quad \text{sit} \quad Z=\mathfrak{q},$$
$$z^n = f^n\cos.n\varphi \quad \text{sit} \quad zZ=\mathfrak{R},$$
$$z^n = f^n\sin.n\varphi \quad \text{sit} \quad zZ=\mathfrak{r}.$$

Inventis hoc modo valoribus $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ erit

$$\mathfrak{A}=\frac{\mathfrak{P}\mathfrak{r}-\mathfrak{p}\mathfrak{R}}{\mathfrak{Q}\mathfrak{r}-\mathfrak{q}\mathfrak{R}} \quad \text{et} \quad \mathfrak{a}=\frac{\mathfrak{p}\mathfrak{Q}-\mathfrak{P}\mathfrak{q}}{\mathfrak{Q}\mathfrak{r}-\mathfrak{q}\mathfrak{R}}.$$

EXEMPLUM 1

Sit proposita haec functio fracta

$$\frac{zz}{(1-z+zz)(1+z^4)},$$

ex qua partem a denominatoris factore $1-z+zz$ oriundam definire oporteat, quae sit

$$\frac{\mathfrak{A}+\mathfrak{a}z}{1-z+zz}$$

Ac primo quidem hic factor cum forma generali $pp-2pqz\cos.\varphi+qqzz$ comparatus dat

$$p=1, \quad q=1 \quad \text{et} \quad \cos.\varphi=\frac{1}{2},$$

unde fit

$$\varphi=60^0=\frac{\pi}{3}.$$

Quia itaque est

$$M=zz,\ Z=1+z^4 \ \text{et} \ f=1,$$

erit

$$\mathfrak{P} = \cos.\tfrac{2\pi}{3} = -\tfrac{1}{2}, \quad \mathfrak{p} = \tfrac{\sqrt{3}}{2},$$
$$\mathfrak{Q} = 1 + \cos.\tfrac{4\pi}{3} = \tfrac{1}{2}, \quad \mathfrak{q} = -\tfrac{\sqrt{3}}{2},$$
$$\mathfrak{R} = \cos.\tfrac{2\pi}{3} + \cos.\tfrac{5\pi}{3} = 1, \quad \mathfrak{r} = 0.$$

Ex his invenitur

$$\mathfrak{A} = -1 \text{ et } \mathfrak{a} = 0$$

ideoque fractio quaesita est

$$\frac{-1}{1-z+zz}$$

huiusque complementum erit

$$\frac{1-z+zz}{1+z^4};$$

cuius denominator $1+z^4$ cum habeat factores $1+z\sqrt{2}+zz$ et $1-z\sqrt{2}+zz$, resolutio denuo suscipi potest; fit autem $\varphi = \frac{\pi}{4}$ et priori casu $f = -1$, posteriori $f = +1$.

EXEMPLUM 2

Sit igitur proposita haec fractio resolvenda

$$\frac{1+z+zz}{\left(1+z\sqrt{2}+zz\right)\left(1-z\sqrt{2}+zz\right)}$$

et erit

$$M = 1 + z + zz;$$

et pro priore factore habebitur

$$f = -1, \quad \varphi = \tfrac{\pi}{4} \text{ et } Z = 1 - z\sqrt{2} + zz$$

unde erit

$$\mathfrak{P} = 1 - \cos.\tfrac{\pi}{4} + \cos.\tfrac{2\pi}{4} = \tfrac{\sqrt{2}-1}{\sqrt{2}},$$
$$\mathfrak{p} = -\sin.\tfrac{\pi}{4} + \sin.\tfrac{2\pi}{4} = \tfrac{\sqrt{2}-1}{\sqrt{2}},$$
$$\mathfrak{Q} = 1 + \sqrt{2}\cdot\cos.\tfrac{\pi}{4} + \cos.\tfrac{\pi}{4} = 2,$$
$$\mathfrak{q} = +\sqrt{2}\cdot\sin.\tfrac{\pi}{4} + \sin.\tfrac{2\pi}{4} = 2,$$
$$\mathfrak{R} = -\cos.\tfrac{\pi}{4} - \sqrt{2}\cdot\cos.\tfrac{2\pi}{4} - \cos.\tfrac{3\pi}{4} = 0,$$
$$\mathfrak{r} = -\sin.\tfrac{\pi}{4} - \sqrt{2}\cdot\sin.\tfrac{2\pi}{4} - \sin.\tfrac{3\pi}{4} = -2\sqrt{2}.$$

Ex his reperitur

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = -2\sqrt{2}$$

et

$$\mathfrak{A} = \frac{-1}{2\sqrt{2}} \text{ et } \mathfrak{a} = 0,$$

unde ex denominatoris factore $1 + z\sqrt{2} + zz$ haec orietur fractio partialis

$$\frac{(\sqrt{2}-1) : 2\sqrt{2}}{1+z\sqrt{2}+zz}$$

Alter autem factor dabit simili modo hanc

$$\frac{(\sqrt{2}+1) : 2\sqrt{2}}{1-z\sqrt{2}+zz}.$$

Hinc functio primum proposita

$$\frac{1+z+zz}{(1+z\sqrt{2}+zz)(1-z\sqrt{2}+zz)}$$

resolvitur in has

$$\frac{-1}{1-z+zz} + \frac{(\sqrt{2}-1) : 2\sqrt{2}}{1+z\sqrt{2}+zz} + \frac{(\sqrt{2}+1) : 2\sqrt{2}}{1-z\sqrt{2}+zz}.$$

EXEMPLUM 3

Sit proposita haec fractio resolvenda

$$\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}.$$

Pro factore denominatoris $1 - \frac{8}{5}z + zz$ oriatur ista fractio

$$\frac{\mathfrak{A}+\mathfrak{a}z}{1-\frac{8}{5}z+zz}$$

eritque

$$p = 1,\ q = 1,\ \cos.\varphi = \frac{4}{5},$$

unde

$$f = 1, \quad M = 1 + 2z + zz, \quad Z = 1 + 2z + 3zz.$$

Quia vero hic ratio anguli φ ad rectum non constat, sinus et cosinus eius multiplorum seorsim debent investigari. Cum sit

$$\cos.\varphi = \tfrac{4}{5}, \qquad \text{erit} \quad \sin.\varphi = \tfrac{3}{5},$$
$$\cos.2\varphi = \tfrac{7}{25}, \qquad \sin.2\varphi = \tfrac{24}{25},$$
$$\cos.3\varphi = -\tfrac{44}{125} \qquad \sin.3\varphi = \tfrac{117}{125};$$

hinc fit

$$\mathfrak{P} = 1 + 2\cdot\tfrac{4}{5} + \tfrac{7}{25} = \tfrac{72}{25},$$
$$\mathfrak{p} = 2\cdot\tfrac{3}{5} + \tfrac{24}{25} = \tfrac{54}{25},$$
$$\mathfrak{Q} = 1 + 2\cdot\tfrac{4}{5} + 3\cdot\tfrac{7}{25} = \tfrac{86}{25},$$
$$\mathfrak{q} = 2\cdot\tfrac{3}{5} + 3\cdot\tfrac{24}{25} = \tfrac{102}{25},$$
$$\mathfrak{R} = \tfrac{4}{5} + 2\cdot\tfrac{7}{25} - 3\cdot\tfrac{44}{125} = \tfrac{38}{25},$$
$$\mathfrak{r} = \tfrac{4}{5} + 2\cdot\tfrac{24}{25} + 3\cdot\tfrac{117}{25} = \tfrac{666}{25}$$

ideoque

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = \tfrac{53400}{25\cdot125} = \tfrac{2136}{125}..$$

Ergo

$$\mathfrak{Q} = \tfrac{1836}{2136} = \tfrac{153}{178}, \quad \mathfrak{a} = -\tfrac{540}{2136} = -\tfrac{45}{178}.$$

Quare fractio ex factore $1 - \frac{8}{5}z + zz$ oriunda, erit

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}.$$

Quaeramus simili modo fractionem alteri factori respondentem; erit

$$p = 1, \quad q = -\sqrt{3} \quad \text{et} \quad \cos.\varphi = \tfrac{1}{\sqrt{3}},$$

ergo

$$f = -\tfrac{1}{\sqrt{3}}, \quad M = 1 + 2z + zz \quad \text{et} \quad Z = 1 - \tfrac{8}{5}z + zz.$$

Fiet autem ob

$$\cos.\varphi = \tfrac{1}{\sqrt{3}}, \qquad \sin.\varphi = \tfrac{\sqrt{2}}{\sqrt{3}},$$
$$\cos.2\varphi = -\tfrac{1}{3}, \qquad \sin.2\varphi = \tfrac{2\sqrt{2}}{\sqrt{3}},$$
$$\cos.3\varphi = -\tfrac{5}{3\sqrt{3}} \qquad \sin.3\varphi = \tfrac{\sqrt{2}}{3\sqrt{3}},$$

conseqneuter

$$\mathfrak{P} = 1 - \tfrac{2}{\sqrt{3}} \cdot \tfrac{1}{\sqrt{3}} + \tfrac{1}{3} \cdot -\tfrac{1}{3} = \tfrac{2}{9},$$
$$\mathfrak{p} = -\tfrac{2}{\sqrt{3}} \cdot \tfrac{\sqrt{2}}{\sqrt{3}} + \tfrac{1}{3} \cdot \tfrac{2\sqrt{2}}{\sqrt{3}} = -\tfrac{4\sqrt{2}}{9},$$
$$\mathfrak{Q} = 1 + \tfrac{8}{5\sqrt{3}} \cdot \tfrac{1}{\sqrt{3}} + \tfrac{1}{3} \cdot -\tfrac{1}{3} = \tfrac{64}{45},$$
$$\mathfrak{q} = +\tfrac{8}{5\sqrt{3}} \cdot \tfrac{\sqrt{2}}{\sqrt{3}} + \tfrac{1}{3} \cdot \tfrac{2\sqrt{2}}{\sqrt{3}} = \tfrac{34\sqrt{2}}{45},$$
$$\mathfrak{R} = -\tfrac{1}{\sqrt{3}} \cdot \tfrac{1}{\sqrt{3}} - \tfrac{8}{5\cdot 3} \cdot -\tfrac{1}{3} - \tfrac{1}{3\sqrt{3}} \cdot -\tfrac{5}{3\sqrt{3}} = \tfrac{4}{135},$$
$$\mathfrak{r} = -\tfrac{1}{\sqrt{3}} \cdot \tfrac{\sqrt{2}}{\sqrt{3}} - \tfrac{8}{5\cdot 3} \cdot \tfrac{2\sqrt{2}}{3} - \tfrac{1}{3\sqrt{3}} \cdot \tfrac{\sqrt{2}}{3\sqrt{3}} = -\tfrac{98\sqrt{2}}{135}$$

ideoque

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = -\tfrac{712\sqrt{2}}{675};$$

fiet ergo

$$\mathfrak{A} = \tfrac{100}{712} = \tfrac{25}{178}, \quad \mathfrak{a} = \tfrac{540}{712} = \tfrac{135}{178}.$$

Fractio ergo proposita

$$\frac{1+2z+zz}{\left(1-\frac{8}{5}z+zz\right)\left(1+2z+3zz\right)}$$

resolvitur in

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz} + \frac{5(5+27z):178}{1+2z+3zz}.$$

204. Possunt autem valores litterarum $\mathfrak{R}$ et $\mathfrak{r}$ ex litteris $\mathfrak{Q}$ et $\mathfrak{q}$ definiri. Cum enim sit

$$\mathfrak{Q} = \alpha + \beta\, fcos.\varphi + \gamma\, ffcos.2\varphi + \delta\, f^3 cos.3\varphi + \text{etc.}$$
$$\mathfrak{q} = \beta\, f\, sin.\varphi + \gamma\, ff\, sin.2\varphi + \delta\, f^3 sin.3\varphi + \text{etc.}$$

erit

$$\mathfrak{Q}cos.\varphi - \mathfrak{q}\,sin.\varphi = \alpha cos.\varphi + \beta\, fcos.2\varphi + \gamma\, ffcos.3\varphi + \text{etc.}$$

ideoque

$$\mathfrak{R} = f\left(\mathfrak{Q}cos.\varphi - \mathfrak{q}\,sin.\varphi\right).$$

Deinde erit

$$\mathfrak{Q}\sin.\varphi + \mathfrak{q}\cos.\varphi = \alpha\sin.\varphi + \beta\, fsin.2\varphi + \gamma\, f^2\sin.3\varphi + \text{etc.},$$

ergo

$$\mathfrak{r} = f\left(\mathfrak{Q}\sin.\varphi + \mathfrak{q}\cos.\varphi\right).$$

Ex his porro fit

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = \left(\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q}\right) fsin.\varphi\ ,$$

$$\mathfrak{P}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = \left(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q}\right) f\sin\varphi + \left(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}\right) f\cos.\varphi\ ;$$

eritque consequenter

$$\mathfrak{A} = \tfrac{\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}}{\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}} + \tfrac{\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}}{\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}} \cdot \tfrac{\cos.\varphi}{\sin\varphi},$$
$$\mathfrak{a} = \tfrac{-\mathfrak{P}\mathfrak{q}+\mathfrak{p}\mathfrak{Q}}{\left(\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}\right) fsin.\varphi}.$$

Quare ex denominatoris factore $pp - 2pqz\cos.\varphi + qqzz$ nascitur ista fractio partialis

$$\frac{\left(\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}\right) fsin.\varphi + \left(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}\right)\left(f\ cos.\varphi - z\right)}{\left(pp-2pqz\cos.\varphi+qqzz\right)\left(\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}\right) fsin.\varphi}.$$

seu ob $f = \frac{p}{q}$ haec

$$\frac{\left(\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}\right) psin.\varphi + \left(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}\right)\left(p\,cos.\varphi - qz\right)}{\left(pp-2pqz\cos.\varphi+qqzz\right)\left(\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}\right) psin.\varphi}.$$

205. Oritur ergo haec fractio partialis ex functionis propositae

$$\frac{M}{\left(pp-2pqz\cos.\varphi+qqzz\right)Z}$$

factore denominatoris $pp - 2pqz\cos.\varphi + qqzz$ atque litterae $\mathfrak{P}, \mathfrak{p}, \mathfrak{Q}$ et $\mathfrak{q}$ sequenti modo ex functionibus *M* et *Z* inveniuntur: Posito

$$z^n = \tfrac{p^n}{q^n}.\cos.n\varphi \ \text{ sit } \ M = \mathfrak{P} \text{ et } Z = \mathfrak{Q}$$

et posito

$$z^n = \tfrac{p^n}{q^n}\sin.n\varphi \ \text{ sit } \ M = \mathfrak{p} \text{ et } Z = \mathfrak{q}\,,$$

ubi notandum est functiones *M* et *Z*, antequam haec substitutio fiat, omnino evolvi debere, ut huiusmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

et

$$Z = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \text{etc.};$$

eritque ideo

$$\mathfrak{P} = A + B\tfrac{p}{q}\cos.\varphi + C\tfrac{p^2}{q^2}\cos.2\varphi + D\tfrac{p^3}{q^3}\cos.3\varphi + \text{ etc.},$$
$$\mathfrak{p} = \quad B\tfrac{p}{q}\sin.\varphi + C\tfrac{p^2}{q^2}\sin.2\varphi + D\tfrac{p^3}{q^3}\sin.3\varphi + \text{ etc.},$$
$$\mathfrak{Q} = \alpha + \beta\tfrac{p}{q}\cos.\varphi + \gamma\tfrac{p^2}{q^2}\cos.2\varphi + \delta\tfrac{p^3}{q^3}\cos.3\varphi + \text{ etc.},$$
$$\mathfrak{p} = \quad \beta\tfrac{p}{q}\sin.\varphi + \gamma\tfrac{p^2}{q^2}\sin.2\varphi + \delta\tfrac{p^3}{q^3}\sin.3\varphi + \text{ etc.}$$

206. Ex praecedentibus autem intelligitur hanc resolutionem locum habere non posse, si functio Z eundem factorem $pp - 2pqz\cos.\varphi + qqzz$ adhuc in se complectatur; hoc enim casu in aequatione

$$M = \mathfrak{A}Z + \alpha zZ$$

facta substitutione

$$z^n = f^n\left(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi\right)$$

ipsa quantitas Z evanesceret nihilque propterea colligi posset. Quamobrem si functionis fractae $\frac{M}{N}$ denominator habeat factorem $\left(pp - 2pqz\cos.\varphi + qqzz\right)^2$ vel altiorem potestatem, peculiari opus erit resolutione. Sit igitur

$$N = \left(pp - 2pqz\cos.\varphi + qqzz\right)^2 Z$$

atque ex denominatoris factore $\left(pp - 2pqz\cos.\varphi + qqzz\right)^2$ orientur huiusmodi duae fractiones partiales

$$\frac{\mathfrak{A}+\mathfrak{a}z}{\left(pp-2pqz\cos.\varphi+qqzz\right)^2} + \frac{\mathfrak{B}+\mathfrak{b}z}{pp-2pqz\cos.\varphi+qqzz},$$

ubi litteras constantes $\mathfrak{A}, \mathfrak{a}, \mathfrak{B}$ et $\mathfrak{b}$ determinari oportet.

207. His positis debebit ista expressio

$$\frac{M-(\mathfrak{A}+\mathfrak{a}z)Z-(\mathfrak{B}+\mathfrak{b}z)Z\left(pp-2pqz\cos.\varphi+qqzz\right)}{\left(pp-2pqz\cos.\varphi+qqzz\right)^2}$$

esse functio integra et hanc ob rem numerator divisibilis erit per denominatorem [§ 43]. Primum ergo haec expressio

$$M - \mathfrak{A}Z - \mathfrak{a}zZ$$

divisibilis esse debet per $pp - 2pqz\cos.\varphi + qqzz$; qui cum sit casus praecedens, eodem quoque modo litterae $\mathfrak{A}$ et $\mathfrak{a}$ determinabuntur.

Quare posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \text{ sit } M = \mathfrak{P} \text{ et } Z = \mathfrak{N}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \text{ sit } M = \mathfrak{p} \text{ et } Z = \mathfrak{n}.$$

Hisque factis secundum regulam supra datam erit

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{N}+\mathfrak{p}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{P}\mathfrak{n}-\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{a} = \frac{-\mathfrak{P}\mathfrak{n}+\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

208. Inventis ergo hoc modo $\mathfrak{A}$ et $\mathfrak{a}$ fiet

$$\frac{M-(\mathfrak{A}+\mathfrak{a}z)Z}{pp-2pqz\cos.\varphi+qqzz}$$

functio integra, quae sit $=P$, atque superest, ut

$$P - \mathfrak{B}Z - \mathfrak{b}zZ$$

divisibile evadat per $pp - 2pqz\cos.\varphi + qqzz$; quae expressio cum similis sit praecedenti, si posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \text{ vocetur } P = \mathfrak{R}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \text{ vocetur } P = \mathfrak{r},$$

erit

$$\mathfrak{B} = \frac{\mathfrak{R}\mathfrak{N}+\mathfrak{r}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{R}\mathfrak{n}-\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{b} = \frac{-\mathfrak{R}\mathfrak{n}+\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

209. Hinc iam generaliter concludere licet, quomodo resolutio institui debeat, si denominator functionis propositae $\frac{M}{N}$ factorem habeat

$$\left(pp-2pqz\cos.\varphi+qqzz\right)^k.$$

Sit enim

$$N=\left(pp-2pqz\cos.\varphi+qqzz\right)^k Z,$$

ita ut haec resolvenda sit functio fracta

$$\frac{M}{\left(pp-2pqz\cos.\varphi+qqzz\right)^k Z}.$$

Praebeat ergo factor denominatoris $\left(pp-2pqz\cos.\varphi+qqzz\right)^k$ has partes

$$\frac{\mathfrak{A}+\mathfrak{a}z}{\left(pp-2pqz\cos.\varphi+qqzz\right)^k}+\frac{\mathfrak{B}+\mathfrak{b}z}{\left(pp-2pqz\cos.\varphi+qqzz\right)^{k-1}}$$
$$+\frac{\mathfrak{C}+\mathfrak{c}z}{\left(pp-2pqz\cos.\varphi+qqzz\right)^{k-2}}+\frac{\mathfrak{D}+\mathfrak{d}z}{\left(pp-2pqz\cos.\varphi+qqzz\right)^{k-2}}+\text{etc.}$$

Iam posito

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \text{ sit } M=\mathfrak{M} \text{ et } Z=\mathfrak{N}$$

et posito

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \text{ sit } M=\mathfrak{m} \text{ et } Z=\mathfrak{n};$$

erit

$$\mathfrak{A}=\frac{\mathfrak{M}\mathfrak{N}+\mathfrak{m}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2}+\frac{\mathfrak{M}\mathfrak{n}-\mathfrak{m}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{a}=\frac{-\mathfrak{M}\mathfrak{n}+\mathfrak{m}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

Deinde vocetur

$$\frac{M-(\mathfrak{A}+\mathfrak{a}z)Z}{pp-2pqz\cos.\varphi+qqzz}=P$$

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \text{ sit } M=\mathfrak{P}$$

atque posito

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \text{ sit } P=\mathfrak{p};$$

erit

$$\mathfrak{B} = \frac{\mathfrak{P}\mathfrak{N}+\mathfrak{m}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{P}\mathfrak{n}-\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{b} = \frac{-\mathfrak{P}\mathfrak{n}+\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

Tum vocetur

$$\frac{P-(\mathfrak{B}+\mathfrak{b}z)Z}{pp-2pqz\cos.\varphi+qqzz} = Q$$

atque posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad Q = \mathfrak{Q}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad Q = \mathfrak{q};$$

erit

$$\mathfrak{C} = \frac{\mathfrak{Q}\mathfrak{N}+\mathfrak{q}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{Q}\mathfrak{n}-\mathfrak{q}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{c} = \frac{-\mathfrak{Q}\mathfrak{n}+\mathfrak{q}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

Porro vocetur

$$\frac{Q-(\mathfrak{C}+\mathfrak{c}z)Z}{pp-2pqz\cos.\varphi+qqzz} = R$$

atque posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad R = \mathfrak{R}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad R = \mathfrak{r};$$

erit

$$\mathfrak{D} = \frac{\mathfrak{R}\mathfrak{N}+\mathfrak{r}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2} + \frac{\mathfrak{R}\mathfrak{n}-\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$
$$\mathfrak{d} = \frac{-\mathfrak{R}\mathfrak{n}+\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

Hocque modo progrediendum est, donec ultimae fractionis, cuius denominator est $pp-2pqz\cos.\varphi+qqzz$, numerator fuerit determinatus.

EXEMPLUM

Sit ista proposita functio fracta

$$\frac{z-z^3}{(1+zz)^4(1+z^4)}$$

ex cuius denominatoris factore $(1+zz)^4$ oriantur hae fractiones partiales

$$\frac{\mathfrak{A}+\mathfrak{a}z}{(1+zz)^4}+\frac{\mathfrak{B}+\mathfrak{b}z}{(1+zz)^3}+\frac{\mathfrak{C}+\mathfrak{c}z}{(1+zz)^2}+\frac{\mathfrak{D}+\mathfrak{d}z}{1+zz}.$$

Comparatione ergo instituta erit

$$p=1,\ q=1,\ \cos.\varphi=0 \text{ ideoque } \varphi=\tfrac{\pi}{2}$$

porroque

$$M=z-z^3 \text{ et } Z=1+z^4$$

Hinc erit

$$\mathfrak{M}=0,\ \mathfrak{m}=2,\ \mathfrak{N}=2,\ \mathfrak{n}=0 \text{ et } \sin.\varphi=1.$$

Hinc itaque invenitur

$$\mathfrak{A}=-\tfrac{4}{4}\cdot 0=0, \text{ et } \mathfrak{a}=1,$$

ergo

$$\mathfrak{A}+\mathfrak{a}z=z$$

hincque

$$P=\frac{z-z^3-z-z^5}{1+zz}=-z^3$$

et

$$\mathfrak{P}=0,\ \mathfrak{p}=1,$$

unde reperitur

$$\mathfrak{B}=0 \text{ et } \mathfrak{b}=\tfrac{1}{2}.$$

Ergo

$$\mathfrak{B}+\mathfrak{b}z=\tfrac{1}{2}z$$

et

$$Q=\frac{-z^3-\frac{1}{2}z-\frac{1}{2}z^5}{1+zz}=-\tfrac{1}{2}z-\tfrac{1}{2}z^3,$$

unde

$$\mathfrak{Q}=0 \text{ et } \mathfrak{q}=0,$$

ergo

$$\mathfrak{C}=0 \text{ et } \mathfrak{c}=0.$$

Hincque

$$R = \frac{-\frac{1}{2}z - \frac{1}{2}zz^3}{1+zz} = -\frac{1}{2}z,$$

ergo

$$\mathfrak{R} = 0 \text{ et } \mathfrak{r} = -\frac{1}{2},$$

unde fit

$$\mathfrak{D} = 0 \text{ et } \mathfrak{d} = -\frac{1}{4}.$$

Quamobrem fractiones quaesitae sunt hae

$$\frac{z}{(1+zz)^4} + \frac{z}{2(1+zz)^3} - \frac{z}{4(1+zz)}.$$

Reliquae vero fractionis numerator est

$$S = \frac{R - (\mathfrak{D} + \mathfrak{d}z)Z}{1+zz} = -\frac{1}{4}z + \frac{1}{4}z^3$$

quae ergo erit

$$\frac{-z+z^3}{4(1+z^4)}.$$

210. Hac ergo methodo simul innotescit fractio complementi, quae cum inventis coniuncta producat fractionem propositam ipsam. Scilicet si fractionis

$$\frac{M}{(pp - 2pqz\cos.\varphi + qqzz)^k Z}$$

inventae fuerint omnes fractiones partiales ex factore $(pp - 2pqz\cos.\varphi + qqzz)^k$ oriundae, pro quibus formati sunt valores functionum *P, Q, R, S, T,* si harum litterarum series ulterius continuetur, erit ea, quae ultimam, qua opus est ad numeratores inveniendos, sequitur, numerator reliquae fractionis denominatorem *Z* habentis; nempe, si $k = 1$, erit reliqua fractio $\frac{P}{Z}$; si $k = 2$, erit reliqua fractio $\frac{Q}{Z}$; si $k = 3$, erit ea $\frac{R}{Z}$, et ita porro. Inventa autem hac reliqua fractione denominatorem *Z* habente ea per has regulas ulterius resolvi poterit.