CHAPTER XI

CONCERNING OTHER INFINITE EXPRESSIONS OF ARCS AND SINES

184. Because above, (§ 158) with z denoting the arc of some circle, we have observed that

$$\sin.z = z\left(1-\frac{zz}{\pi\pi}\right)\left(1-\frac{zz}{4\pi\pi}\right)\left(1-\frac{zz}{9\pi\pi}\right)\left(1-\frac{zz}{16\pi\pi}\right) \text{ etc.}$$

and

$$\cos.z = \left(1-\frac{4zz}{\pi\pi}\right)\left(1-\frac{4zz}{9\pi\pi}\right)\left(1-\frac{4zz}{25\pi\pi}\right)\left(1-\frac{4zz}{49\pi\pi}\right) \text{ etc.},$$

we may consider the arc to be $z=\frac{m\pi}{n}$; there will become

$$\sin.\frac{m\pi}{n} = \frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right) \text{ etc.}$$

and

$$\cos.\frac{m\pi}{n} = \left(1-\frac{4mm}{nn}\right)\left(1-\frac{4mm}{9nn}\right)\left(1-\frac{4mm}{25nn}\right)\left(1-\frac{4mm}{49nn}\right) \text{ etc.}$$

Or *2n* may be put in place of *n*, so that these expressions may appear

$$\sin.\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{36nn-mm}{36nn}\cdot\text{etc.},$$

$$\cos.\frac{m\pi}{2n} = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn}\cdot\text{etc.},$$

which resolved into simple factors will give :

$$\sin.\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{2n-m}{2n}\cdot\frac{2n+m}{2n}\cdot\frac{4n-m}{4n}\cdot\frac{4n+m}{4n}\cdot\frac{6n-m}{6n}\cdot\text{etc.},$$

$$\cos.\frac{m\pi}{2n} = \frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}\cdot\frac{5n-m}{5n}\cdot\frac{5n+m}{5n}\cdot\text{etc.}$$

Now $n-m$ may be put in place of *m;* because there is

$$\sin.\frac{(n-m)\pi}{2n} = \cos.\frac{m\pi}{2n} \text{ and } \cos.\frac{(n-m)\pi}{2n} = \sin.\frac{m\pi}{2n},$$

these expressions become :

$$\cos.\tfrac{m\pi}{2n} = \tfrac{(n-m)\pi}{2n}\cdot\tfrac{n+m}{2n}\cdot\tfrac{3n-m}{2n}\cdot\tfrac{3n+m}{4n}\cdot\tfrac{5n-m}{4n}\cdot\tfrac{5n+m}{6n}\cdot\text{etc.},$$

$$\sin.\tfrac{m\pi}{2n} = \tfrac{m}{n}\cdot\tfrac{2n-m}{n}\cdot\tfrac{2n+m}{3n}\cdot\tfrac{4n-m}{3n}\cdot\tfrac{4n+m}{5n}\cdot\tfrac{6n-m}{5n}\cdot\text{etc.}$$

185. Therefore since both expressions may be obtained for the sine and cosine of the angle $\frac{m\pi}{2n}$, if these may be compared by dividing one by the other, there will be

$$1 = \tfrac{\pi}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{6}\cdot\tfrac{7}{8}\cdot\tfrac{9}{8}\cdot\text{etc.}.$$

and thus

$$\tfrac{\pi}{2} = \tfrac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot\text{etc.}}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot\text{etc.}},$$

which is the expression for the perimeter of a circle, which Wallis found in his *Arithmetica infinitorum.* Moreover innumerable expressions similar to this can be shown with the help of the expression for the sine ; for that may be deduced to become :

$$\tfrac{\pi}{2} = \tfrac{n}{m}\sin.\tfrac{m\pi}{2n}\cdot\tfrac{2n}{2n-m}\cdot\tfrac{2n}{2n+m}\cdot\tfrac{4n}{4n-m}\cdot\tfrac{4n}{4n+m}\cdot\tfrac{6n}{6n-m}\cdot\text{etc.},$$

which on putting $\frac{m}{n}=1$ produces that formula of Wallis itself. Therefore let $\frac{m}{n}=\frac{1}{2}$; on account of $\sin.\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$, there becomes

$$\tfrac{\pi}{2} = \tfrac{\sqrt{2}}{1}\cdot\tfrac{4}{3}\cdot\tfrac{4}{5}\cdot\tfrac{8}{7}\cdot\tfrac{8}{9}\cdot\tfrac{12}{11}\cdot\tfrac{12}{13}\cdot\tfrac{16}{15}\cdot\tfrac{16}{17}\cdot\text{etc.}$$

Let $\frac{m}{n}=\frac{1}{3}$; because $\sin.\frac{1}{6}\pi = \frac{1}{2}$, there becomes

$$\tfrac{\pi}{2} = \tfrac{3}{2}\cdot\tfrac{6}{5}\cdot\tfrac{6}{7}\cdot\tfrac{12}{11}\cdot\tfrac{12}{13}\cdot\tfrac{18}{17}\cdot\tfrac{18}{19}\cdot\tfrac{24}{23}\cdot\text{etc.}$$

But if the Wallisian expression may be divided by that, when $\frac{m}{n}=\frac{1}{2}$, there arises

$$\sqrt{2} = \tfrac{2\cdot 2\cdot 6\cdot 6\cdot 10\cdot 10\cdot 14\cdot 14\cdot 18\cdot 18\cdot\text{etc.}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17\cdot 19\cdot\text{etc.}}.$$

186. Because the tangent of each angle may be equal to the sine divided by the cosine, the tangent also can be expressed by infinite factors in this manner. But if moreover the first expression of the sine may be divided by the other expression for the cosine, there will be

$$\text{tang.}\frac{m\pi}{2n}=\frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\frac{4n+m}{5n-m}\cdot\text{etc.},$$

$$\text{cot.}\frac{m\pi}{2n}=\frac{n-m}{m}\cdot\frac{n+m}{2n-m}\cdot\frac{3n-m}{2n+m}\cdot\frac{3n+m}{4n-m}\cdot\frac{5n-m}{4n+m}\cdot\text{etc.}$$

The secants and cosecants also will be expressed in a similar manner :

$$\text{sec.}\frac{m\pi}{2n}=\frac{n}{n-m}\cdot\frac{n}{n+m}\cdot\frac{3n}{3n-m}\cdot\frac{3n}{3n+m}\cdot\frac{5n}{5n-m}\cdot\frac{5n}{5n+m}\cdot\text{etc.}$$

$$\text{cosec.}\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{n}{2n-m}\cdot\frac{3n}{2n+m}\cdot\frac{3n}{4n-m}\cdot\frac{5n}{4n+m}\cdot\frac{5n}{6n-m}\cdot\text{etc.}$$

But if the other formulas of the sines and cosines may be combined, there will be

$$\text{tang.}\frac{m\pi}{2n}=\frac{\pi}{2}\cdot\frac{m}{n-m}\cdot\frac{1(2n-m)}{2(n+m)}\cdot\frac{3(2n+m)}{2(3n-m)}\cdot\frac{3(4n-m)}{4(3n+m)}\cdot\text{etc.},$$

$$\text{cot.}\frac{m\pi}{2n}=\frac{\pi}{2}\cdot\frac{n-m}{m}\cdot\frac{1(n+m)}{2(2n-m)}\cdot\frac{3(3n-m)}{2(2n+m)}\cdot\frac{3(3n+m)}{4(4n-m)}\cdot\text{etc.},$$

$$\text{sec.}\frac{m\pi}{2n}=\frac{2}{\pi}\cdot\frac{n}{n-m}\cdot\frac{2n}{n+m}\cdot\frac{2n}{3n-m}\cdot\frac{4n}{3n+m}\cdot\frac{4n}{5n-m}\cdot\text{etc.},$$

$$\text{cosec.}\frac{m\pi}{2n}=\frac{2}{\pi}\cdot\frac{n}{m}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}\cdot\text{etc.}$$

187. If k may be written in place of m and in a like manner the sine and cosine of the angle $\frac{k\pi}{2n}$ may be defined, and the former may be divided by these expressions, these formulas will appear :

$$\frac{\sin.\frac{m\pi}{2n}}{\sin.\frac{k\pi}{2n}}=\frac{m}{k}\cdot\frac{2n-m}{2n-k}\cdot\frac{2n+m}{2n+k}\cdot\frac{4n-m}{4n-k}\cdot\frac{4n+m}{4n+k}\cdot\text{etc.}$$

$$\frac{\sin.\frac{m\pi}{2n}}{\cos.\frac{k\pi}{2n}}=\frac{m}{n-k}\cdot\frac{2n-m}{n+k}\cdot\frac{2n+m}{3n-k}\cdot\frac{4n-m}{3n+k}\cdot\frac{4n+m}{5n-k}\cdot\text{etc.},$$

$$\frac{\cos.\frac{m\pi}{2n}}{\cos.\frac{k\pi}{2n}}=\frac{n-m}{n-k}\cdot\frac{n+m}{n+k}\cdot\frac{3n-m}{3n-k}\cdot\frac{3n+m}{3n+k}\cdot\frac{5n-m}{5n-k}\cdot\text{etc.},$$

$$\frac{\cos.\frac{m\pi}{2n}}{\sin.\frac{k\pi}{2n}}=\frac{n-m}{k}\cdot\frac{n+m}{2n-k}\cdot\frac{3n-m}{2n+k}\cdot\frac{3n+m}{4n-k}\cdot\frac{5n-m}{4n+k}\cdot\text{etc.}$$

Therefore with an angle $\frac{k\pi}{2n}$ of this kind assumed, the sine and cosine of which may be given, one will be able to determine the sine and cosine of any other angle $\frac{m\pi}{2n}$.

188. Therefore reciprocally the true values of expressions of this kind can be assigned, which are made from infinite factors, either by the periphery of the circle or by the sine or cosine of the angles given, which itself is of some consequence, since even now other methods may not be present, with the aid of which the values of infinite products of this kind can be shown. Truly the remaining expressions of this kind bring little of use to the values both of π itself as well as to the sines or cosines of the angles $\frac{m\pi}{2n}$ being elicited by approximation. For although these factors

$$\frac{\pi}{2} = 2\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right) \text{ etc.}$$

may be multiplied by themselves in decimal fractions without difficulty, yet exceedingly many terms must be taken into the computation, if we wish to find the value of π just to ten figures only.

189. But the particular use of expressions of this kind even if of infinitudes depends on the finding of logarithms, in which calculation the use of factors is so great, so that without these the calculation of logarithms would be most difficult. And indeed in the first place, since there shall be

$$\pi = 4\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right) \text{ etc.},$$

with logarithms taken it will become

$$l\pi = l4 + l\left(1-\frac{1}{9}\right) + l\left(1-\frac{1}{25}\right) + l\left(1-\frac{1}{49}\right) + \text{ etc.},$$

or

$$l\pi = l2 - l\left(1-\frac{1}{4}\right) - l\left(1-\frac{1}{16}\right) - l\left(1-\frac{1}{36}\right) - \text{etc.},$$

[Derived from the infinite products $\cos.\frac{m\pi}{2n} = \frac{(n-m)\pi}{2n}\cdot\frac{n+m}{2n}\cdot\frac{3n-m}{2n}\cdot\frac{3n+m}{4n}\cdot\frac{5n-m}{4n}\cdot\frac{5n+m}{6n}\cdot$ etc. with $m = 2n$, and from $\sin.\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{36nn-mm}{36nn}\cdot$ etc. with $m = n$ respectively.]

whether common logarithms or hyperbolic logarithms may be taken. Because indeed common logarithms are found readily from hyperbolic logarithms, the agreed abbreviation to be used will be for the hyperbolic logarithm of π to be found.

190. Therefore since with hyperbolic logarithms taken there shall be

$$l(1-x) = -x - \tfrac{xx}{2} - \tfrac{x^3}{3} - \tfrac{x^4}{4} - \text{etc.},$$

if the individual terms may be set out in this manner, there will be

$$\begin{aligned} l\pi = l4 &- \tfrac{1}{9} - \tfrac{1}{2\cdot 9^2} - \tfrac{1}{3\cdot 9^3} - \tfrac{1}{4\cdot 9^4} - \text{etc.} \\ &- \tfrac{1}{25} - \tfrac{1}{2\cdot 25^2} - \tfrac{1}{3\cdot 25^3} - \tfrac{1}{4\cdot 25^4} - \text{etc.} \\ &- \tfrac{1}{49} - \tfrac{1}{2\cdot 49^2} - \tfrac{1}{3\cdot 49^3} - \tfrac{1}{4\cdot 49^4} - \text{etc.} \end{aligned}$$

In these series, with a number of the same kind descending vertically indefinitely, give rise to series the sums of which we have now found above [§ 169, 170] ; whereby, if for the sake of brevity we may put

$$\begin{aligned} A &= 1 + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \text{ etc.}, \\ B &= 1 + \tfrac{1}{3^4} + \tfrac{1}{5^4} + \tfrac{1}{7^4} + \tfrac{1}{9^4} + \text{ etc.}, \\ C &= 1 + \tfrac{1}{3^6} + \tfrac{1}{5^6} + \tfrac{1}{7^6} + \tfrac{1}{9^6} + \text{ etc.}, \\ D &= 1 + \tfrac{1}{3^8} + \tfrac{1}{5^8} + \tfrac{1}{7^8} + \tfrac{1}{9^8} + \text{etc.} \end{aligned}$$

the expansion becomes

$$l\pi = l4 - (A-1) - \tfrac{1}{2}(B-1) - \tfrac{1}{3}(C-1) - \tfrac{1}{4}(D-1) - \text{etc.}$$

Truly with the sums found above expressed approximately

$$\begin{aligned} A &= 1{,}23370\ 05501\ 36169\ 82735\ 431, \\ B &= 1{,}01467\ 80316\ 04192\ 05454\ 625, \\ C &= 1{,}00144\ 70766\ 40942\ 12190\ 647, \\ D &= 1{,}00015\ 51790\ 25296\ 11930\ 298, \\ E &= 1{,}00001\ 70413\ 63044\ 82548\ 818, \\ F &= 1{,}00000\ 18858\ 48583\ 11957\ 590, \\ G &= 1{,}00000\ 02092\ 40519\ 21150\ 010, \\ H &= 1{,}00000\ 00232\ 37157\ 37915\ 670, \\ I &= 1{,}00000\ 00025\ 81437\ 55665\ 977, \\ K &= 1{,}00000\ 00002\ 86807\ 69745\ 558, \\ L &= 1{,}00000\ 00000\ 31866\ 77514\ 044, \\ M &= 1{,}00000\ 00000\ 03540\ 72294\ 392, \\ N &= 1{,}00000\ 00000\ 00393\ 41246\ 691, \\ O &= 1{,}00000\ 00000\ 00043\ 71244\ 859, \end{aligned}$$

$$P = 1{,}00000\ 00000\ 00004\ 85693\ 682,$$
$$Q = 1{,}00000\ 00000\ 00000\ 53965\ 957,$$
$$R = 1{,}00000\ 00000\ 00000\ 05996\ 217,$$
$$S = 1{,}00000\ 00000\ 00000\ 00666\ 246,$$
$$T = 1{,}00000\ 00000\ 00000\ 00074\ 027,$$
$$V = 1{,}00000\ 00000\ 00000\ 00008\ 225,$$
$$W = 1{,}00000\ 00000\ 00000\ 00000\ 914,$$
$$X = 1{,}00000\ 00000\ 00000\ 00000\ 102.$$

Hence without to much tedious calculation, the hyperbolic logarithm of π becomes :

$$= 1{,}14472\ 98858\ 49400\ 17414\ 345\,;$$

which if it may be multiplied by 0,43429 etc., will give rise to the common logarithm of π ,

$$= 0{,}49714\ 98726\ 94133\ 85435\ 128\,.$$

191. Because again we have expressed both the sine as well as the cosine of the angle $\frac{m\pi}{2n}$ by an infinite number of factors, we will be able to express the logarithm of each conveniently. Moreover from the formulas found [§ 184], in the first place there will be

$$l\sin.\tfrac{m\pi}{2n} = l\pi + l\tfrac{m}{2n} \ + \ l\left(1-\tfrac{mm}{4nn}\right) + l\left(1-\tfrac{mm}{16nn}\right) + l\left(1-\tfrac{mm}{36nn}\right) + \text{etc.},$$
$$l\cos.\tfrac{m\pi}{2n} = l\left(1-\tfrac{mm}{nn}\right) + l\left(1-\tfrac{mm}{9nn}\right) + l\left(1-\tfrac{mm}{25nn}\right) + l\left(1-\tfrac{mm}{49nn}\right) + \text{etc.}$$

Hence initially as before the hyperbolic logarithms are expressed readily by maximally converging series. But besides not having the necessity to multiply infinite series, by the act of taking logarithms we may relinquish the initial terms set out, and there will be

$$\begin{aligned} l\sin.\tfrac{m\pi}{2n} &= l\pi + lm \ + \ l(2n-m) + l(2n+m) - l8 - 3ln \\ &- \tfrac{mm}{16nn} - \tfrac{m^4}{2\cdot 16^2 n^4} - \tfrac{m^6}{3\cdot 16^3 n^6} - \tfrac{m^8}{4\cdot 16^4 n^8} - \text{etc.} \\ &- \tfrac{mm}{36nn} - \tfrac{m^4}{2\cdot 36^2 n^4} - \tfrac{m^6}{3\cdot 36^3 n^6} - \tfrac{m^8}{4\cdot 36^4 n^8} - \text{etc.} \\ &- \tfrac{mm}{64nn} - \tfrac{m^4}{2\cdot 64^2 n^4} - \tfrac{m^6}{3\cdot 64^3 n^6} - \tfrac{m^8}{4\cdot 64^4 n^8} - \text{etc.}, \\ &\qquad \text{etc.}, \end{aligned}$$

$$
\begin{aligned}
l\cos.\tfrac{m\pi}{2n} = {} & l(n-m)+l(n+m)-3ln \\
& -\tfrac{mm}{9nn}-\tfrac{m^4}{2\cdot 9^2 n^4}-\tfrac{m^6}{3\cdot 9^3 n^6}-\tfrac{m^8}{4\cdot 9^4 n^8}-\text{etc.} \\
& -\tfrac{mm}{25nn}-\tfrac{m^4}{2\cdot 25^2 n^4}-\tfrac{m^6}{3\cdot 25^3 n^6}-\tfrac{m^8}{4\cdot 25^4 n^8}-\text{etc.} \\
& -\tfrac{mm}{49nn}-\tfrac{m^4}{2\cdot 49^2 n^4}-\tfrac{m^6}{3\cdot 49^3 n^6}-\tfrac{m^8}{4\cdot 49^4 n^8}-\text{etc.} \\
& \text{etc.}
\end{aligned}
$$

192. Therefore the even individual powers of $\frac{m}{n}$ occur in these series, which are multiplied by the series, the sums of which we have assigned above.
Clearly there will be

$$
\begin{aligned}
l\sin.\tfrac{m\pi}{2n} = {} & lm+l(2n-m)+l(2n+m)-3ln-l8 \\
& -\tfrac{mm}{nn}\left(\tfrac{1}{4^2}+\tfrac{1}{6^2}+\tfrac{1}{8^2}+\tfrac{1}{10^2}+\tfrac{1}{12^2}+\text{etc.}\right) \\
& -\tfrac{m^4}{2n^4}\left(\tfrac{1}{4^4}+\tfrac{1}{6^4}+\tfrac{1}{8^4}+\tfrac{1}{10^4}+\tfrac{1}{12^4}+\text{etc.}\right) \\
& -\tfrac{m^6}{3n^6}\left(\tfrac{1}{4^6}+\tfrac{1}{6^6}+\tfrac{1}{8^6}+\tfrac{1}{10^6}+\tfrac{1}{12^6}+\text{etc.}\right) \\
& -\tfrac{m^8}{4n^8}\left(\tfrac{1}{4^8}+\tfrac{1}{6^8}+\tfrac{1}{8^8}+\tfrac{1}{10^8}+\tfrac{1}{12^8}+\text{etc.}\right) \\
& \text{etc.,}
\end{aligned}
$$

$$
\begin{aligned}
l\cos.\tfrac{m\pi}{2n} = {} & l(n-m)+l(n+m)-2\,ln \\
& -\tfrac{mm}{nn}\left(\tfrac{1}{3^2}+\tfrac{1}{5^2}+\tfrac{1}{7^2}+\tfrac{1}{9^2}+\text{etc.}\right) \\
& -\tfrac{m^4}{2n^4}\left(\tfrac{1}{3^4}+\tfrac{1}{5^4}+\tfrac{1}{7^4}+\tfrac{1}{9^4}+\text{etc.}\right) \\
& -\tfrac{m^6}{3n^6}\left(\tfrac{1}{3^6}+\tfrac{1}{5^6}+\tfrac{1}{7^6}+\tfrac{1}{9^6}+\text{etc.}\right) \\
& -\tfrac{m^8}{4n^8}\left(\tfrac{1}{3^8}+\tfrac{1}{5^8}+\tfrac{1}{7^8}+\tfrac{1}{9^8}+\text{etc.}\right) \\
& \text{etc.}
\end{aligned}
$$

The sums of the latter series have been shown in the above manner (§ 190) ; indeed the first series may be derived from these, but, so that they may be able to be transferred for use more easily, I have added the sums of these here equally.

193. But if therefore for the sake of brevity we may put

$$\alpha = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \text{etc.},$$
$$\beta = \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \text{etc.},$$
$$\gamma = \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \text{etc.},$$
$$\delta = \frac{1}{2^8} + \frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \text{etc.},$$
$$\text{etc.},$$

these sums will be expressed by number approximately :

$$\alpha = 0{,}41123\ 35167\ 12056\ 60911\ 810,$$
$$\beta = 0{,}06764\ 52021\ 06946\ 13696\ 975,$$
$$\gamma = 0{,}01589\ 59853\ 43507\ 01780\ 804,$$
$$\delta = 0{,}00392\ 21771\ 72648\ 22007\ 570,$$
$$\varepsilon = 0{,}00097\ 75337\ 64773\ 25984\ 896,$$
$$\zeta = 0{,}00024\ 42007\ 04724\ 92872\ 273,$$
$$\eta = 0{,}00006\ 10388\ 94539\ 49332\ 915,$$
$$\theta = 0{,}00001\ 52590\ 22251\ 27271\ 502,$$
$$\iota = 0{,}00000\ 38147\ 11827\ 44318\ 008,$$
$$\kappa = 0{,}00000\ 09536\ 75226\ 17534\ 053,$$
$$\lambda = 0{,}00000\ 02384\ 18635\ 95259\ 255,$$
$$\mu = 0{,}00000\ 00596\ 04648\ 32831\ 556,$$
$$\nu = 0{,}00000\ 00149\ 01161\ 41589\ 813,$$
$$\xi = 0{,}00000\ 00037\ 25290\ 31233\ 986,$$
$$o = 0{,}00000\ 00009\ 31322\ 57548\ 284,$$
$$\pi = 0{,}00000\ 00002\ 32830\ 64370\ 808,$$
$$\rho = 0{,}00000\ 00000\ 58207\ 66091\ 686,$$
$$\sigma = 0{,}00000\ 00000\ 14551\ 91522\ 858,$$
$$\tau = 0{,}00000\ 00000\ 03637\ 97880\ 710,$$
$$\upsilon = 0{,}00000\ 00000\ 00909\ 49470\ 177,$$
$$\varphi = 0{,}00000\ 00000\ 00227\ 37367\ 544,$$
$$\chi = 0{,}00000\ 00000\ 00056\ 84341\ 886,$$
$$\psi = 0{,}00000\ 00000\ 00014\ 21085\ 472,$$
$$\omega = 0{,}00000\ 00000\ 00003\ 55271\ 368.$$

The remaining sums decrease in the quadruple ratio.

194. Therefore with these called upon in aid, there will be

$$l\sin.\tfrac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3ln + 3l\pi - l8$$
$$-\tfrac{mm}{nn}\left(\alpha - \tfrac{1}{2^2}\right) - \tfrac{m^4}{2n^4}\left(\beta - \tfrac{1}{2^4}\right) - \tfrac{m^6}{3n^6}\left(\gamma - \tfrac{1}{2^6}\right) - \text{etc.},$$
$$l\cos.\tfrac{m\pi}{2n} = l(n-m) + l(n+m) - 2ln$$
$$-\tfrac{mm}{nn}(A-1) - \tfrac{m^4}{2n^4}(B-1) - \tfrac{m^6}{3n^6}(C-1) - \text{etc.};$$

therefore because the logarithms $l\pi$ and $l8$ are given, there will be

The hyperbolic Logarithm of the sine of the angle $\frac{m}{n}90^0$

$$
\begin{aligned}
&= lm + l(2n-m) + l(2n+m) - 3ln \\
&\quad - \quad 0{,}93471\ 16558\ 30435\ 75411 \\
&\quad - \tfrac{m^2}{n^2}\cdot 0{,}16123\ 35167\ 12056\ 60912 \\
&\quad - \tfrac{m^4}{n^4}\cdot 0{,}00257\ 26010\ 53473\ 06848 \\
&\quad - \tfrac{m^6}{n^6}\cdot 0{,}00009\ 03284\ 47835\ 67260 \\
&\quad - \tfrac{m^8}{n^8}\cdot 0{,}00000\ 39817\ 93162\ 05502 \\
&\quad - \tfrac{m^{10}}{n^{10}}\cdot 0{,}00000\ 01942\ 52954\ 65197 \\
&\quad - \tfrac{m^{12}}{n^{12}}\cdot 0{,}00000\ 00100\ 13287\ 48812 \\
&\quad - \tfrac{m^{14}}{n^{14}}\cdot 0{,}00000\ 00005\ 34041\ 35619 \\
&\quad - \tfrac{m^{16}}{n^{16}}\cdot 0{,}00000\ 00000\ 29148\ 59659 \\
&\quad - \tfrac{m^{18}}{n^{18}}\cdot 0{,}00000\ 00000\ 01617\ 97980 \\
&\quad - \tfrac{m^{20}}{n^{20}}\cdot 0{,}00000\ 00000\ 00090\ 97691 \\
&\quad - \tfrac{m^{22}}{n^{22}}\cdot 0{,}00000\ 00000\ 00005\ 16828 \\
&\quad - \tfrac{m^{24}}{n^{24}}\cdot 0{,}00000\ 00000\ 00000\ 29608 \\
&\quad - \tfrac{m^{26}}{n^{26}}\cdot 0{,}00000\ 00000\ 00000\ 01708 \\
&\quad - \tfrac{m^{28}}{n^{28}}\cdot 0{,}00000\ 00000\ 00000\ 00099 \\
&\quad - \tfrac{m^{30}}{n^{30}}\cdot 0{,}00000\ 00000\ 00000\ 00006.
\end{aligned}
$$

But *The hyperbolic logarithm of the cosine of the angle* $\frac{m}{n}90^0$

$$
\begin{aligned}
&= l(n-m) + l(n+m) - 2ln \\
&-\tfrac{m^2}{n^2}\cdot 0{,}23370\ 05501\ 36169\ 82735 \\
&-\tfrac{m^4}{n^4}\cdot 0{,}00733\ 90158\ 02096\ 02727 \\
&-\tfrac{m^6}{n^6}\cdot 0{,}00048\ 23588\ 80314\ 04064 \\
&-\tfrac{m^8}{n^8}\cdot 0{,}00003\ 87947\ 56324\ 02983 \\
&-\tfrac{m^{10}}{n^{10}}\cdot 0{,}00000\ 34082\ 72608\ \ 96510 \\
&-\tfrac{m^{12}}{n^{12}}\cdot 0{,}00000\ 03143\ 08097\ 18660 \\
&-\tfrac{m^{14}}{n^{14}}\cdot 0{,}00000\ 00298\ 91502\ 74450 \\
&-\tfrac{m^{16}}{n^{16}}\cdot 0{,}00000\ 00029\ 04644\ 67239 \\
&-\tfrac{m^{18}}{n^{18}}\cdot 0{,}00000\ 00002\ 86826\ 39518 \\
&-\tfrac{m^{20}}{n^{20}}\cdot 0{,}00000\ 00000\ 28680\ 76975 \\
&-\tfrac{m^{22}}{n^{22}}\cdot 0{,}00000\ 00000\ 02896\ 97956 \\
&-\tfrac{m^{24}}{n^{24}}\cdot 0{,}00000\ 00000\ 00295\ 06025 \\
&-\tfrac{m^{26}}{n^{26}}\cdot 0{,}00000\ 00000\ 00030\ 26250 \\
&-\tfrac{m^{28}}{n^{28}}\cdot 0{,}00000\ 00000\ 00003\ 12232 \\
&-\tfrac{m^{30}}{n^{30}}\cdot 0{,}00000\ 00000\ 00000\ 32380 \\
&-\tfrac{m^{32}}{n^{32}}\cdot 0{,}00000\ 00000\ 00000\ 03373 \\
&-\tfrac{m^{34}}{n^{34}}\cdot 0{,}00000\ 00000\ 00000\ 00353 \\
&-\tfrac{m^{36}}{n^{36}}\cdot 0{,}00000\ 00000\ 00000\ 00037 \\
&-\tfrac{m^{38}}{n^{38}}\cdot 0{,}00000\ 00000\ 00000\ 00004.
\end{aligned}
$$

195. If these hyperbolic logarithms of the sines and cosines may be multiplied by 0,43429 44819 etc., the common logarithms of the same will be produced referred to the radius $=1$. Because truly in tables the logarithm of the whole sine is accustomed to be put in place $=10$, so that logarithm tables of sines and cosines may be obtained, after multiplication 10 must be added. Hence there will be

Common logarithm of the sine of the angle $\frac{m}{n}90^0$

$$
\begin{aligned}
&= lm + l(2n-m) + l(2n+m) - 3ln \\
&\quad + \quad 9{,}59405\ 98857\ 02190 \\
&\quad - \tfrac{m^2}{n^2}\cdot 0{,}07002\ 28266\ 05902 \\
&\quad - \tfrac{m^4}{n^4}\cdot 0{,}00111\ 72664\ 41662 \\
&\quad - \tfrac{m^6}{n^6}\cdot 0{,}00003\ 92291\ 46454 \\
&\quad - \tfrac{m^8}{n^8}\cdot 0{,}00000\ 17292\ 70798 \\
&\quad - \tfrac{m^{10}}{n^{10}}\cdot 0{,}00000\ 00843\ 62986 \\
&\quad - \tfrac{m^{12}}{n^{12}}\cdot 0{,}00000\ 00043\ 48715 \\
&\quad - \tfrac{m^{14}}{n^{14}}\cdot 0{,}00000\ 00002\ 31931 \\
&\quad - \tfrac{m^{16}}{n^{16}}\cdot 0{,}00000\ 00000\ 12659 \\
&\quad - \tfrac{m^{18}}{n^{18}}\cdot 0{,}00000\ 00000\ 00703 \\
&\quad - \tfrac{m^{20}}{n^{20}}\cdot 0{,}00000\ 00000\ 00040.
\end{aligned}
$$

The common Logarithm of the cosine of the angle $\frac{m}{n}90^0$

$$
\begin{aligned}
&= l(n-m) + l(n+m) - 2ln \\
&+ \quad 10{,}00000\ 00000\ 00000 \\
&- \tfrac{m^2}{n^2}\cdot 0{,}10149\ 48593\ 41893 \\
&- \tfrac{m^4}{n^4}\cdot 0{,}00318\ 72940\ 65451 \\
&- \tfrac{m^6}{n^6}\cdot 0{,}00020\ 94858\ 00017 \\
&- \tfrac{m^8}{n^8}\cdot 0{,}00001\ 68483\ 48598 \\
&- \tfrac{m^{10}}{n^{10}}\cdot 0{,}00000\ 14801\ 93987 \\
&- \tfrac{m^{12}}{n^{12}}\cdot 0.00000\ 01365\ 02272 \\
&- \tfrac{m^{14}}{n^{14}}\cdot 0{,}00000\ 00129\ 81715 \\
&- \tfrac{m^{16}}{n^{16}}\cdot 0{,}00000\ 00012\ 61471 \\
&- \tfrac{m^{18}}{n^{18}}\cdot 0{,}00000\ 00001\ 24567 \\
&- \tfrac{m^{20}}{n^{20}}\cdot 0{,}00000\ 00000\ 12456 \\
&- \tfrac{m^{22}}{n^{22}}\cdot 0{,}00000\ 00000\ 01258 \\
&- \tfrac{m^{24}}{n^{24}}\cdot 0{,}00000\ 00000\ 00128 \\
&- \tfrac{m^{26}}{n^{26}}\cdot 0{,}00000\ 00000\ 00013.
\end{aligned}
$$

196. Therefore with the aid of these formulas the logarithms of the sines and cosines of any angle can be found, both hyperbolic as well as common, even with the sines and cosines of these themselves unknown. Moreover the logarithms of the tangents, cotangents, secants and cosecants are found from the logarithms of the sines and cosines by subtraction alone, on account of which there will be no need for special formulas for these. Moreover it is to be noted that the hyperbolic logarithms of the numbers $m,\ n,\ n-m,\ n+m$ etc. are required to be taken when the hyperbolic logarithms of the sines and cosines are sought, and moreover the common logarithms, when such have been investigated with the aid of the latter formulas. Besides $m:n$ may denote the ratio, that a proposed angle has to a right angle ; and thus since the sines of angles greater than half a right angle may be made equal to the cosine of the angle less than the half right

angle in turn, the fraction $\frac{m}{n}$ need never be taken greater than $\frac{1}{2}$ and on this account these terms converge much more, as a half suffices to be put in place.

197. Before we shall leave this argument, we may uncover a more suitable way the tangents and secants of any angles may be found, than was shown in the preceding chapter. For whatever the tangents and secants may be determined by the sines and cosines, this still is made by a division, which operation is exceedingly tiresome with so great numbers. And indeed both the tangents and cotangents we have now shown above (§ 135), truly cannot be reduced to the account of the formulas in that place, as we have reserved for this chapter.

198. Therefore from § 181, in the first expression we have elicited for the tangent of the angle $\frac{m\pi}{2n}$. Since indeed there shall be

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}=\frac{\pi}{4mn}\text{tang.}\frac{m\pi}{2n},$$

there will be

$$\text{tang.}\frac{m\pi}{2n}=\frac{4mn}{\pi}\left(\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}\right).$$

Since then there shall be

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn}\cot.\frac{m\pi}{n},$$

if we may write *2n* for *n*, there becomes

$$\cot.\frac{m\pi}{2n}=\frac{2n}{m\pi}-\frac{4mn}{\pi}\left(\frac{1}{4nn-mm}+\frac{1}{16nn-mm}+\frac{1}{36nn-mm}+\text{etc.}\right).$$

These fractions besides the first may be changed into infinite series, clearly which are easily handled in calculations ; there becomes

$$\begin{aligned}\text{tang.}\frac{m\pi}{2n}=\frac{mn}{nn-mm}\cdot\frac{4}{\pi}&+\frac{4}{\pi}\left(\frac{m}{3^2n}+\frac{m^3}{3^4n^3}+\frac{m^5}{3^6n^5}+\text{etc.}\right)\\&+\frac{4}{\pi}\left(\frac{m}{5^2n}+\frac{m^3}{5^4n^3}+\frac{m^5}{5^6n^5}+\text{etc.}\right)\\&+\frac{4}{\pi}\left(\frac{m}{7^2n}+\frac{m^3}{7^4n^3}+\frac{m^5}{7^6n^5}+\text{etc.}\right)\\&\qquad\text{etc.},\end{aligned}$$

$$\cot.\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{2}{\pi}-\frac{mn}{4nn-mm}\cdot\frac{4}{\pi}-\frac{4}{\pi}\left(\frac{m}{4^2n}+\frac{m^3}{4^4n^3}+\frac{m^5}{4^6n^5}+\text{etc.}\right)$$
$$-\frac{4}{\pi}\left(\frac{m}{8^2n}+\frac{m^3}{8^4n^3}+\frac{m^5}{8^6n^5}+\text{etc.}\right)$$
$$\text{etc.}$$

198a But from the known value of π there is found

$$\frac{1}{\pi}=0{,}31830\ 98861\ 83790\ 67153\ 77675\ 26745\ 028724,$$

then here the same series arise, which we have indicated above [§ 190 and 193] by the letters A, B, C, D etc. and $\alpha, \beta, \gamma, \delta$ etc. Therefore with these observed there will be

$$\text{tang.}\frac{m\pi}{2n}=\frac{mn}{nn-mm}\cdot\frac{4}{\pi}$$
$$+\frac{m}{n}\cdot\frac{4}{\pi}(A-1)+\frac{m^3}{n^3}\cdot\frac{4}{\pi}(B-1)+\frac{m^5}{n^5}\cdot\frac{4}{\pi}(C-1)+\frac{m^7}{n^7}\cdot\frac{4}{\pi}(D-1)+\text{etc.}$$

Then for the cotangent

$$\cot.\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{2}{\pi}-\frac{4mn}{4nn-mm}\cdot\frac{1}{\pi}$$
$$-\frac{m}{n}\cdot\frac{4}{\pi}\left(\alpha-\frac{1}{2^2}\right)-\frac{m^3}{n^3}\cdot\frac{4}{\pi}\left(\beta-\frac{1}{2^4}\right)-\frac{m^5}{n^5}\cdot\frac{4}{\pi}\left(\gamma-\frac{1}{2^6}\right)-\text{etc.},$$

and from these formulas the expressions have arisen, which we have given above for the tangent and cotangent (§ 135) ; likewise truly (§ 137) we have shown, how from the tangents and cotangents found by addition and subtraction alone, how the secants and cosecants may be found. Therefore with the aid of these general rules a canon of sines, tangents, and secants, and of the logarithms of these may be possible to be calculated much easier, than indeed that has been done by the first originators.

CAPUT XI

DE ALIIS ARCUUM ATQUE SINUUM EXPRESSIONIBUS INFINITIS

184. Quoniam supra (§ 158) denotante *z* arcum circuli quemcunque vidimus esse

$$\sin.z = z\left(1-\tfrac{zz}{\pi\pi}\right)\left(1-\tfrac{zz}{4\pi\pi}\right)\left(1-\tfrac{zz}{9\pi\pi}\right)\left(1-\tfrac{zz}{16\pi\pi}\right) \text{ etc.}$$

et

$$\cos.z = \left(1-\tfrac{4zz}{\pi\pi}\right)\left(1-\tfrac{4zz}{9\pi\pi}\right)\left(1-\tfrac{4zz}{25\pi\pi}\right)\left(1-\tfrac{4zz}{49\pi\pi}\right) \text{ etc.}$$

ponamus esse arcum $z = \frac{m\pi}{n}$; erit

$$\sin.\tfrac{m\pi}{n} = \tfrac{m\pi}{n}\left(1-\tfrac{mm}{nn}\right)\left(1-\tfrac{mm}{4nn}\right)\left(1-\tfrac{mm}{9nn}\right)\left(1-\tfrac{mm}{16nn}\right) \text{ etc.}$$

et

$$\cos.\tfrac{m\pi}{n} = \left(1-\tfrac{4mm}{nn}\right)\left(1-\tfrac{4mm}{9nn}\right)\left(1-\tfrac{4mm}{25nn}\right)\left(1-\tfrac{4mm}{49nn}\right) \text{ etc.}$$

Vel ponatur *2n* loco *n,* ut prodeant hae expressiones

$$\sin.\tfrac{m\pi}{2n} = \tfrac{m\pi}{2n}\cdot\tfrac{4nn-mm}{4nn}\cdot\tfrac{16nn-mm}{16nn}\cdot\tfrac{36nn-mm}{36nn}\cdot\text{etc.},$$

$$\cos.\tfrac{m\pi}{2n} = \tfrac{nn-mm}{nn}\cdot\tfrac{9nn-mm}{9nn}\cdot\tfrac{25nn-mm}{25nn}\cdot\tfrac{49nn-mm}{49nn}\cdot\text{etc.},$$

quae in factores simplices resolutae dant

$$\sin.\tfrac{m\pi}{2n} = \tfrac{m\pi}{2n}\cdot\tfrac{2n-m}{2n}\cdot\tfrac{2n+m}{2n}\cdot\tfrac{4n-m}{4n}\cdot\tfrac{4n+m}{4n}\cdot\tfrac{6n-m}{6n}\cdot\text{etc.},$$

$$\cos.\tfrac{m\pi}{2n} = \tfrac{n-m}{n}\cdot\tfrac{n+m}{n}\cdot\tfrac{3n-m}{3n}\cdot\tfrac{3n+m}{3n}\cdot\tfrac{5n-m}{5n}\cdot\tfrac{5n+m}{5n}\cdot\text{etc.}$$

Ponatur $n-m$ loco *m;* quia est

$$\sin.\frac{(n-m)\pi}{2n}=\cos.\frac{m\pi}{2n} \text{ et } \cos.\frac{(n-m)\pi}{2n}=\sin.\frac{m\pi}{2n},$$

provenient hae expressiones

$$\cos.\frac{m\pi}{2n}=\frac{(n-m)\pi}{2n}\cdot\frac{n+m}{2n}\cdot\frac{3n-m}{2n}\cdot\frac{3n+m}{4n}\cdot\frac{5n-m}{4n}\cdot\frac{5n+m}{6n}\cdot\text{etc.},$$

$$\sin.\frac{m\pi}{2n}=\frac{m}{n}\cdot\frac{2n-m}{n}\cdot\frac{2n+m}{3n}\cdot\frac{4n-m}{3n}\cdot\frac{4n+m}{5n}\cdot\frac{6n-m}{5n}\cdot\text{etc.}$$

185. Cum igitur pro sinu et cosinu anguli $\frac{m\pi}{2n}$ binae habeantur expressiones, si eae inter se comparentur dividendo, erit

$$1=\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdot\frac{9}{8}\cdot\text{etc.}.$$

ideoque

$$\frac{\pi}{2}=\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12\cdot12\cdot\text{etc.}}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot11\cdot13\cdot\text{etc.}},$$

quae est expressio pro peripheria circuli, quam Wallisius invenit in *Arithmetica infinitorum.* Similes autem huic innumeras expressiones exhibere licet ope primae expressionis pro sinu; ex ea enim deducitur fore

$$\frac{\pi}{2}=\frac{n}{m}\sin.\frac{m\pi}{2n}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}\cdot\frac{6n}{6n-m}\cdot\text{etc.},$$

quae posito $\frac{m}{n}=1$ praebet illam ipsam Wallisii formulam. Sit ergo $\frac{m}{n}=\frac{1}{2}$; ob $\sin.\frac{1}{4}\pi=\frac{1}{\sqrt{2}}$ erit

$$\frac{\pi}{2}=\frac{\sqrt{2}}{1}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\frac{16}{15}\cdot\frac{16}{17}\cdot\text{etc.}$$

Sit $\frac{m}{n}=\frac{1}{3}$. ob $\sin.\frac{1}{6}\pi=\frac{1}{2}$ erit

$$\frac{\pi}{2}=\frac{3}{2}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\frac{18}{17}\cdot\frac{18}{19}\cdot\frac{24}{23}\cdot\text{etc.}$$

Quodsi expressio Wallisiana dividatur per illam, ubi $\frac{m}{n}=\frac{1}{2}$, erit

$$\sqrt{2}=\frac{2\cdot2\cdot6\cdot6\cdot10\cdot10\cdot14\cdot14\cdot18\cdot18\cdot\text{etc.}}{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15\cdot17\cdot19\cdot\text{etc.}}.$$

186. Quoniam tangens cuiusque anguli aequatur sinui per cosinum diviso, tangens quoque per huiusmodi factores infinitos exprimi poterit. Quodsi autem prima sinus expressio dividatur per alteram cosinus expressionem, erit

$$\text{tang.}\frac{m\pi}{2n}=\frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\frac{4n+m}{5n-m}\cdot\text{etc.},$$

$$\text{cot.}\frac{m\pi}{2n}=\frac{n-m}{m}\cdot\frac{n+m}{2n-m}\cdot\frac{3n-m}{2n+m}\cdot\frac{3n+m}{4n-m}\cdot\frac{5n-m}{4n+m}\cdot\text{etc.}$$

Simili modo autem secantes et cosecantes exprimentur

$$\text{sec.}\frac{m\pi}{2n}=\frac{n}{n-m}\cdot\frac{n}{n+m}\cdot\frac{3n}{3n-m}\cdot\frac{3n}{3n+m}\cdot\frac{5n}{5n-m}\cdot\frac{5n}{5n+m}\cdot\text{etc.}$$

$$\text{cosec.}\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{n}{2n-m}\cdot\frac{3n}{2n+m}\cdot\frac{3n}{4n-m}\cdot\frac{5n}{4n+m}\cdot\frac{5n}{6n-m}\cdot\text{etc.}$$

Sin autem alterae sinuum et cosinuum formulae combinentur, erit

$$\text{tang.}\frac{m\pi}{2n}=\frac{\pi}{2}\cdot\frac{m}{n-m}\cdot\frac{1(2n-m)}{2(n+m)}\cdot\frac{3(2n+m)}{2(3n-m)}\cdot\frac{3(4n-m)}{4(3n+m)}\cdot\text{etc.},$$

$$\text{cot.}\frac{m\pi}{2n}=\frac{\pi}{2}\cdot\frac{n-m}{m}\cdot\frac{1(n+m)}{2(2n-m)}\cdot\frac{3(3n-m)}{2(2n+m)}\cdot\frac{3(3n+m)}{4(4n-m)}\cdot\text{etc.},$$

$$\text{sec.}\frac{m\pi}{2n}=\frac{2}{\pi}\cdot\frac{n}{n-m}\cdot\frac{2n}{n+m}\cdot\frac{2n}{3n-m}\cdot\frac{4n}{3n+m}\cdot\frac{4n}{5n-m}\cdot\text{etc.},$$

$$\text{cosec.}\frac{m\pi}{2n}=\frac{2}{\pi}\cdot\frac{n}{m}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}\cdot\text{etc.}$$

187. Si loco *m* scribatur *k* similique modo anguli $\frac{k\pi}{2n}$ sinus et cosinus definiantur ac per has expressiones illae priores dividantur, prodibunt istae formulae

$$\frac{\sin.\frac{m\pi}{2n}}{\sin.\frac{k\pi}{2n}}=\frac{m}{k}\cdot\frac{2n-m}{2n-k}\cdot\frac{2n+m}{2n+k}\cdot\frac{4n-m}{4n-k}\cdot\frac{4n+m}{4n+k}\cdot\text{etc.}$$

$$\frac{\sin.\frac{m\pi}{2n}}{\cos.\frac{k\pi}{2n}}=\frac{m}{n-k}\cdot\frac{2n-m}{n+k}\cdot\frac{2n+m}{3n-k}\cdot\frac{4n-m}{3n+k}\cdot\frac{4n+m}{5n-k}\cdot\text{etc.},$$

$$\frac{\cos.\frac{m\pi}{2n}}{\cos.\frac{k\pi}{2n}}=\frac{n-m}{n-k}\cdot\frac{n+m}{n+k}\cdot\frac{3n-m}{3n-k}\cdot\frac{3n+m}{3n+k}\cdot\frac{5n-m}{5n-k}\cdot\text{etc.},$$

$$\frac{\cos.\frac{m\pi}{2n}}{\sin.\frac{k\pi}{2n}}=\frac{n-m}{k}\cdot\frac{n+m}{2n-k}\cdot\frac{3n-m}{2n+k}\cdot\frac{3n+m}{4n-k}\cdot\frac{5n-m}{4n+k}\cdot\text{etc.}$$

Sumpto ergo $\frac{k\pi}{2n}$ eiusmodi angulo, cuius sinus et cosinus dentur, per hos licebit alius cuiuscunque anguli $\frac{m\pi}{2n}$ sinum et cosinum determinare.

188. Vicissim igitur huiusmodi expressionum, quae ex factoribus infinitis constant, valores veri vel per circuli peripheriam vel per sinus et cosinus angulorum datorum assignari possunt, quod ipsum non parvi est momenti, cum etiamnunc aliae methodi non constent, quarum ope huiusmodi productorum infinitorum valores exhiberi queant. Ceterum vera huiusmodi expressiones parum utilitatis afferunt ad valores cum ipsius π tum sinuum cosinuumve angulorum $\frac{m\pi}{2n}$ per approximationem eruendos. Quanquam enim isti factores

$$\frac{\pi}{2}=2\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right)\text{ etc.}$$

in fractionibus decimalibus non difficulter in se multiplicantur, tamen nimis multi termini in computum duci deberent, si valorem ipsius π ad decem tantum figuras iustum invenire vellemus.

189. Praecipuus autem usus huiusmodi expressionum etsi infinitarum in inventione logarithmorum versatur, in quo negotio factorum utilitas tanta est, ut sine illis logarithmorum supputatio esset difficillima. Ac prima quidem, cum sit

$$\pi=4\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right)\text{ etc.},$$

erit sumendis logarithmis

$$l\pi=l4+l\left(1-\frac{1}{9}\right)+l\left(1-\frac{1}{25}\right)+l\left(1-\frac{1}{49}\right)+\text{ etc.},$$

vel

$$l\pi = l2 - l\left(1-\tfrac{1}{4}\right) - l\left(1-\tfrac{1}{16}\right) - l\left(1-\tfrac{1}{36}\right) - \text{etc.},$$

sive logarithmi communes sive hyperbolici sumantur. Quoniam vero ex logarithmis hyperbolicis vulgares facile reperiuntur, insigne compendium adhiberi poterit ad logarithmum hyperbolicum ipsius π inveniendum.

190. Cum igitur logarithmis hyperbolicis sumendis sit

$$l(1-x) = -x - \tfrac{xx}{2} - \tfrac{x^3}{3} - \tfrac{x^4}{4} - \text{etc.},$$

si hoc modo singuli termini evolvantur, erit

$$\begin{aligned} l\pi = l4 &- \tfrac{1}{9} - \tfrac{1}{2\cdot 9^2} - \tfrac{1}{3\cdot 9^3} - \tfrac{1}{4\cdot 9^4} - \text{etc.} \\ &- \tfrac{1}{25} - \tfrac{1}{2\cdot 25^2} - \tfrac{1}{3\cdot 25^3} - \tfrac{1}{4\cdot 25^4} - \text{etc.} \\ &- \tfrac{1}{49} - \tfrac{1}{2\cdot 49^2} - \tfrac{1}{3\cdot 49^3} - \tfrac{1}{4\cdot 49^4} - \text{etc.} \end{aligned}$$

In his seriebus numero infinitis verticaliter descendendo eiusmodi prodeunt series, quarum summas supra [§ 169, 170] iam invenimus; quare, si brevitatis gratia ponamus

$$\begin{aligned} A &= 1 + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \text{ etc.}, \\ B &= 1 + \tfrac{1}{3^4} + \tfrac{1}{5^4} + \tfrac{1}{7^4} + \tfrac{1}{9^4} + \text{ etc.}, \\ C &= 1 + \tfrac{1}{3^6} + \tfrac{1}{5^6} + \tfrac{1}{7^6} + \tfrac{1}{9^6} + \text{ etc.}, \\ D &= 1 + \tfrac{1}{3^8} + \tfrac{1}{5^8} + \tfrac{1}{7^8} + \tfrac{1}{9^8} + \text{etc.} \end{aligned}$$

erit

$$l\pi = l4 - (A-1) - \tfrac{1}{2}(B-1) - \tfrac{1}{3}(C-1) - \tfrac{1}{4}(D-1) - \text{etc.}$$

Est vero summis supra inventis proxime exprimendis

$$\begin{aligned} A &= 1{,}23370\ 05501\ 36169\ 82735\ 431, \\ B &= 1{,}01467\ 80316\ 04192\ 05454\ 625, \\ C &= 1{,}00144\ 70766\ 40942\ 12190\ 647, \\ D &= 1{,}00015\ 51790\ 25296\ 11930\ 298, \\ E &= 1{,}00001\ 70413\ 63044\ 82548\ 818, \\ F &= 1{,}00000\ 18858\ 48583\ 11957\ 590, \end{aligned}$$

G =1,00000 02092 40519 21150 010,
H =1,00000 00232 37157 37915 670,
I =1,00000 00025 81437 55665 977,
K =1,00000 00002 86807 69745 558,
L =1,00000 00000 31866 77514 044,
M =1,00000 00000 03540 72294 392,
N =1,00000 00000 00393 41246 691,
O =1,00000 00000 00043 71244 859,
P =1,00000 00000 00004 85693 682,
Q =1,00000 00000 00000 53965 957,
R =1,00000 00000 00000 05996 217,
S = 1,00000 00000 00000 00666 246,
T = 1,00000 00000 00000 00074 027,
V = 1,00000 00000 00000 00008 225,
W =1,00000 00000 00000 00000 914,
X = 1,00000 00000 00000 00000 102.

Hine sine taedioso calculo reperitur logarithmus hyperbolicus ipsius

$$=1{,}14472\ 98858\ 49400\ 17414\ 345\,;$$

qui si multiplicetur per 0,43429 etc., prodit logarithmus vulgaris ipsius

$$=0{,}49714\ 98726\ 94133\ 85435\ 128\,.$$

191. Quia porro tam sinum quam cosinum anguli $\frac{m\pi}{2n}$ expressum habemus per factores numero infinitos, utriusque logarithmum commode exprimere poterimus. Erit autem ex formulis primo [§ 184] inventis

$$l\sin.\tfrac{m\pi}{2n} = l\pi + l\tfrac{m}{2n} + l\left(1-\tfrac{mm}{4nn}\right) + l\left(1-\tfrac{mm}{16nn}\right) + l\left(1-\tfrac{mm}{36nn}\right) + \text{etc.},$$
$$l\cos.\tfrac{m\pi}{2n} = l\left(1-\tfrac{mm}{nn}\right) + l\left(1-\tfrac{mm}{9nn}\right) + l\left(1-\tfrac{mm}{25nn}\right) + l\left(1-\tfrac{mm}{49nn}\right) + \text{etc.}$$

Hinc primum logarithmi hyperbolici ut ante per series maxime convergentes facile exprimuntur. Ne autem praeter necessitatem series infinitas multiplicemus, terminos priores actu in logarithmis involutos relinquamus eritque

$$
\begin{aligned}
l\sin.\tfrac{m\pi}{2n} = {} & l\pi + lm + l(2n-m) + l(2n+m) - l8 - 3ln \\
& - \tfrac{mm}{16nn} - \tfrac{m^4}{2\cdot 16^2 n^4} - \tfrac{m^6}{3\cdot 16^3 n^6} - \tfrac{m^8}{4\cdot 16^4 n^8} - \text{etc.} \\
& - \tfrac{mm}{36nn} - \tfrac{m^4}{2\cdot 36^2 n^4} - \tfrac{m^6}{3\cdot 36^3 n^6} - \tfrac{m^8}{4\cdot 36^4 n^8} - \text{etc.} \\
& - \tfrac{mm}{64nn} - \tfrac{m^4}{2\cdot 64^2 n^4} - \tfrac{m^6}{3\cdot 64^3 n^6} - \tfrac{m^8}{4\cdot 64^4 n^8} - \text{etc.}, \\
& \text{etc.},
\end{aligned}
$$

$$
\begin{aligned}
l\cos.\tfrac{m\pi}{2n} = {} & l(n-m) + l(n+m) - 3ln \\
& - \tfrac{mm}{9nn} - \tfrac{m^4}{2\cdot 9^2 n^4} - \tfrac{m^6}{3\cdot 9^3 n^6} - \tfrac{m^8}{4\cdot 9^4 n^8} - \text{etc.} \\
& - \tfrac{mm}{25nn} - \tfrac{m^4}{2\cdot 25^2 n^4} - \tfrac{m^6}{3\cdot 25^3 n^6} - \tfrac{m^8}{4\cdot 25^4 n^8} - \text{etc.} \\
& - \tfrac{mm}{49nn} - \tfrac{m^4}{2\cdot 49^2 n^4} - \tfrac{m^6}{3\cdot 49^3 n^6} - \tfrac{m^8}{4\cdot 49^4 n^8} - \text{etc.} \\
& \text{etc.}
\end{aligned}
$$

192. Occurrunt ergo in his seriebus singulae potestates pares ipsius $\frac{m}{n}$, quae sunt multiplicatae per series, quarum summas iam supra assignavimus.
Erit nempe

$$
\begin{aligned}
l\sin.\tfrac{m\pi}{2n} = {} & lm + l(2n-m) + l(2n+m) - 3ln - l8 \\
& - \tfrac{mm}{nn}\left(\tfrac{1}{4^2} + \tfrac{1}{6^2} + \tfrac{1}{8^2} + \tfrac{1}{10^2} + \tfrac{1}{12^2} + \text{etc.}\right) \\
& - \tfrac{m^4}{2n^4}\left(\tfrac{1}{4^4} + \tfrac{1}{6^4} + \tfrac{1}{8^4} + \tfrac{1}{10^4} + \tfrac{1}{12^4} + \text{etc.}\right) \\
& - \tfrac{m^6}{3n^6}\left(\tfrac{1}{4^6} + \tfrac{1}{6^6} + \tfrac{1}{8^6} + \tfrac{1}{10^6} + \tfrac{1}{12^6} + \text{etc.}\right) \\
& - \tfrac{m^8}{4n^8}\left(\tfrac{1}{4^8} + \tfrac{1}{6^8} + \tfrac{1}{8^8} + \tfrac{1}{10^8} + \tfrac{1}{12^8} + \text{etc.}\right) \\
& \text{etc.},
\end{aligned}
$$

$$
\begin{aligned}
&l\cos.\tfrac{m\pi}{2n} = l(n-m) + l(n+m) - 2\,ln \\
&-\tfrac{mm}{nn}\left(\tfrac{1}{3^2}+\tfrac{1}{5^2}+\tfrac{1}{7^2}+\tfrac{1}{9^2}+\text{etc.}\right) \\
&-\tfrac{m^4}{2n^4}\left(\tfrac{1}{3^4}+\tfrac{1}{5^4}+\tfrac{1}{7^4}+\tfrac{1}{9^4}+\text{etc.}\right) \\
&-\tfrac{m^6}{3n^6}\left(\tfrac{1}{3^6}+\tfrac{1}{5^6}+\tfrac{1}{7^6}+\tfrac{1}{9^6}+\text{etc.}\right) \\
&-\tfrac{m^8}{4n^8}\left(\tfrac{1}{3^8}+\tfrac{1}{5^8}+\tfrac{1}{7^8}+\tfrac{1}{9^8}+\text{etc.}\right) \\
&\qquad\qquad\text{etc.}
\end{aligned}
$$

Serierum posteriorum modo ante (§ 190) summae sunt exhibitae; priores series quidem ex his derivari possent, at, quo facilius ad usum transferri queant, earum summas pariter hic adiiciam.

193. Quodsi ergo brevitatis gratia ponamus

$$
\begin{aligned}
\alpha &= \tfrac{1}{2^2}+\tfrac{1}{4^2}+\tfrac{1}{6^2}+\tfrac{1}{8^2}+\text{etc.}, \\
\beta &= \tfrac{1}{2^4}+\tfrac{1}{4^4}+\tfrac{1}{6^4}+\tfrac{1}{8^4}+\text{etc.}, \\
\gamma &= \tfrac{1}{2^6}+\tfrac{1}{4^6}+\tfrac{1}{6^6}+\tfrac{1}{8^6}+\text{etc.}, \\
\delta &= \tfrac{1}{2^8}+\tfrac{1}{4^8}+\tfrac{1}{6^8}+\tfrac{1}{8^8}+\text{etc.}, \\
&\quad\text{etc.,}
\end{aligned}
$$

erunt summae in numeris proxime expressae hae:

$$\alpha = 0{,}41123\ 35167\ 12056\ 60911\ 810,$$
$$\beta = 0{,}06764\ 52021\ 06946\ 13696\ 975,$$
$$\gamma = 0{,}01589\ 59853\ 43507\ 01780\ 804,$$
$$\delta = 0{,}00392\ 21771\ 72648\ 22007\ 570,$$
$$\varepsilon = 0{,}00097\ 75337\ 64773\ 25984\ 896,$$
$$\zeta = 0{,}00024\ 42007\ 04724\ 92872\ 273,$$
$$\eta = 0{,}00006\ 10388\ 94539\ 49332\ 915,$$
$$\theta = 0{,}00001\ 52590\ 22251\ 27271\ 502,$$
$$\iota = 0{,}00000\ 38147\ 11827\ 44318\ 008,$$
$$\kappa = 0{,}00000\ 09536\ 75226\ 17534\ 053,$$
$$\lambda = 0{,}00000\ 02384\ 18635\ 95259\ 255,$$
$$\mu = 0{,}00000\ 00596\ 04648\ 32831\ 556,$$
$$\nu = 0{,}00000\ 00149\ 01161\ 41589\ 813,$$
$$\xi = 0{,}00000\ 00037\ 25290\ 31233\ 986,$$
$$o = 0{,}00000\ 00009\ 31322\ 57548\ 284,$$
$$\pi = 0{,}00000\ 00002\ 32830\ 64370\ 808,$$
$$\rho = 0{,}00000\ 00000\ 58207\ 66091\ 686,$$
$$\sigma = 0{,}00000\ 00000\ 14551\ 91522\ 858,$$
$$\tau = 0{,}00000\ 00000\ 03637\ 97880\ 710,$$
$$\upsilon = 0{,}00000\ 00000\ 00909\ 49470\ 177,$$
$$\varphi = 0{,}00000\ 00000\ 00227\ 37367\ 544,$$
$$\chi = 0{,}00000\ 00000\ 00056\ 84341\ 886,$$
$$\psi = 0{,}00000\ 00000\ 00014\ 21085\ 472,$$
$$\omega = 0{,}00000\ 00000\ 00003\ 55271\ 368.$$

Reliquae summae in ratione quadrupla decrescunt.

194. His ergo in subsidium vocatis erit

$$l\sin.\tfrac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3ln + 3l\pi - l8$$
$$-\tfrac{mm}{nn}\left(\alpha - \tfrac{1}{2^2}\right) - \tfrac{m^4}{2n^4}\left(\beta - \tfrac{1}{2^4}\right) - \tfrac{m^6}{3n^6}\left(\gamma - \tfrac{1}{2^6}\right) - \text{etc.},$$
$$l\cos.\tfrac{m\pi}{2n} = l(n-m) + l(n+m) - 2ln$$
$$-\tfrac{mm}{nn}(A-1) - \tfrac{m^4}{2n^4}(B-1) - \tfrac{m^6}{3n^6}(C-1) - \text{etc.};$$

quoniam igitur logarithmi $l\pi$ et $l8$ dantur, erit

Logarithmus hyperbolicus sinus anguli $\frac{m}{n}90^0$

$$
\begin{aligned}
&= lm + l(2n-m) + l(2n+m) - 3ln \\
&- \quad 0{,}93471\ 16558\ 30435\ 75411 \\
&- \frac{m^2}{n^2}\cdot 0{,}16123\ 35167\ 12056\ 60912 \\
&- \frac{m^4}{n^4}\cdot 0{,}00257\ 26010\ 53473\ 06848 \\
&- \frac{m^6}{n^6}\cdot 0{,}00009\ 03284\ 47835\ 67260 \\
&- \frac{m^8}{n^8}\cdot 0{,}00000\ 39817\ 93162\ 05502 \\
&- \frac{m^{10}}{n^{10}}\cdot 0{,}00000\ 01942\ 52954\ 65197 \\
&- \frac{m^{12}}{n^{12}}\cdot 0{,}00000\ 00100\ 13287\ 48812 \\
&- \frac{m^{14}}{n^{14}}\cdot 0{,}00000\ 00005\ 34041\ 35619 \\
&- \frac{m^{16}}{n^{16}}\cdot 0{,}00000\ 00000\ 29148\ 59659 \\
&- \frac{m^{18}}{n^{18}}\cdot 0{,}00000\ 00000\ 01617\ 97980 \\
&- \frac{m^{20}}{n^{20}}\cdot 0{,}00000\ 00000\ 00090\ 97691 \\
&- \frac{m^{22}}{n^{22}}\cdot 0{,}00000\ 00000\ 00005\ 16828 \\
&- \frac{m^{24}}{n^{24}}\cdot 0{,}00000\ 00000\ 00000\ 29608 \\
&- \frac{m^{26}}{n^{26}}\cdot 0{,}00000\ 00000\ 00000\ 01708 \\
&- \frac{m^{28}}{n^{28}}\cdot 0{,}00000\ 00000\ 00000\ 00099 \\
&- \frac{m^{30}}{n^{30}}\cdot 0{,}00000\ 00000\ 00000\ 00006.
\end{aligned}
$$

At *Logarithmus hyperbolicus cosinus anguli* $\frac{m}{n}90^0$

$$\begin{aligned}
&= l(n-m) + l(n+m) - 2ln \\
&- \tfrac{m^2}{n^2} \cdot 0{,}23370\ 05501\ 36169\ 82735 \\
&- \tfrac{m^4}{n^4} \cdot 0{,}00733\ 90158\ 02096\ 02727 \\
&- \tfrac{m^6}{n^6} \cdot 0{,}00048\ 23588\ 80314\ 04064 \\
&- \tfrac{m^8}{n^8} \cdot 0{,}00003\ 87947\ 56324\ 02983 \\
&- \tfrac{m^{10}}{n^{10}} \cdot 0{,}00000\ 34082\ 72608\ 96510 \\
&- \tfrac{m^{12}}{n^{12}} \cdot 0{,}00000\ 03143\ 08097\ 18660 \\
&- \tfrac{m^{14}}{n^{14}} \cdot 0{,}00000\ 00298\ 91502\ 74450 \\
&- \tfrac{m^{16}}{n^{16}} \cdot 0{,}00000\ 00029\ 04644\ 67239 \\
&- \tfrac{m^{18}}{n^{18}} \cdot 0{,}00000\ 00002\ 86826\ 39518 \\
&- \tfrac{m^{20}}{n^{20}} \cdot 0{,}00000\ 00000\ 28680\ 76975 \\
&- \tfrac{m^{22}}{n^{22}} \cdot 0{,}00000\ 00000\ 02896\ 97956 \\
&- \tfrac{m^{24}}{n^{24}} \cdot 0{,}00000\ 00000\ 00295\ 06025 \\
&- \tfrac{m^{26}}{n^{26}} \cdot 0{,}00000\ 00000\ 00030\ 26250 \\
&- \tfrac{m^{28}}{n^{28}} \cdot 0{,}00000\ 00000\ 00003\ 12232 \\
&- \tfrac{m^{30}}{n^{30}} \cdot 0{,}00000\ 00000\ 00000\ 32380 \\
&- \tfrac{m^{32}}{n^{32}} \cdot 0{,}00000\ 00000\ 00000\ 03373 \\
&- \tfrac{m^{34}}{n^{34}} \cdot 0{,}00000\ 00000\ 00000\ 00353 \\
&- \tfrac{m^{36}}{n^{36}} \cdot 0{,}00000\ 00000\ 00000\ 00037 \\
&- \tfrac{m^{38}}{n^{38}} \cdot 0{,}00000\ 00000\ 00000\ 00004.
\end{aligned}$$

195. Si isti sinuum et cosinuum logarithmi hyperbolici multiplicentur per 0,43429 44819 etc., prodibunt eorundem logarithmi vulgares ad radium $=1$ relati. Quoniam vero in tabulis logarithmus sinus totius statui solet $=10$, quo logarithmi tabulares sinuum et cosinuum obtineantur, post multiplicationem addi debet 10. Hinc erit

Logarithmus tabularis sinus anguli $\frac{m}{n}90^0$

$$
\begin{aligned}
&= lm + l(2n-m) + l(2n+m) - 3ln \\
&\quad + \quad 9{,}59405\ 98857\ 02190 \\
&\quad - \frac{m^2}{n^2}\cdot 0{,}07002\ 28266\ 05902 \\
&\quad - \frac{m^4}{n^4}\cdot 0{,}00111\ 72664\ 41662 \\
&\quad - \frac{m^6}{n^6}\cdot 0{,}00003\ 92291\ 46454 \\
&\quad - \frac{m^8}{n^8}\cdot 0{,}00000\ 17292\ 70798 \\
&\quad - \frac{m^{10}}{n^{10}}\cdot 0{,}00000\ 00843\ 62986 \\
&\quad - \frac{m^{12}}{n^{12}}\cdot 0{,}00000\ 00043\ 48715 \\
&\quad - \frac{m^{14}}{n^{14}}\cdot 0{,}00000\ 00002\ 31931 \\
&\quad - \frac{m^{16}}{n^{16}}\cdot 0{,}00000\ 00000\ 12659 \\
&\quad - \frac{m^{18}}{n^{18}}\cdot 0{,}00000\ 00000\ 00703 \\
&\quad - \frac{m^{20}}{n^{20}}\cdot 0{,}00000\ 00000\ 00040.
\end{aligned}
$$

Logarithmus tabularis cosinus anguli $\frac{m}{n}90^0$

$$
\begin{aligned}
&= l(n-m) + l(n+m) - 2ln \\
&+ \quad 10,00000\ 00000\ 00000 \\
&- \frac{m^2}{n^2}\cdot 0,10149\ 48593\ 41893 \\
&- \frac{m^4}{n^4}\cdot 0,00318\ 72940\ 65451 \\
&- \frac{m^6}{n^6}\cdot 0,00020\ 94858\ 00017 \\
&- \frac{m^8}{n^8}\cdot 0,00001\ 68483\ 48598 \\
&- \frac{m^{10}}{n^{10}}\cdot 0,00000\ 14801\ 93987 \\
&- \frac{m^{12}}{n^{12}}\cdot 0.00000\ 01365\ 02272 \\
&- \frac{m^{14}}{n^{14}}\cdot 0,00000\ 00129\ 81715 \\
&- \frac{m^{16}}{n^{16}}\cdot 0,00000\ 00012\ 61471 \\
&- \frac{m^{18}}{n^{18}}\cdot 0,00000\ 00001\ 24567 \\
&- \frac{m^{20}}{n^{20}}\cdot 0,00000\ 00000\ 12456 \\
&- \frac{m^{22}}{n^{22}}\cdot 0,00000\ 00000\ 01258 \\
&- \frac{m^{24}}{n^{24}}\cdot 0,00000\ 00000\ 00128 \\
&- \frac{m^{26}}{n^{26}}\cdot 0,00000\ 00000\ 00013.
\end{aligned}
$$

196. Harum ergo formularum ope inveniri possunt logarithmi sinuum et cosinuum quorumvis angulorum tam hyperbolici quam vulgares etiam ignoratis ipsis sinibus et cosinibus. Ex logarithmis autem sinuum et cosinuum per solam subtractionem inveniuntur logarithmi tangentium, cotangentium, secantium cosecantiumque, quamobrem pro his peculiaribus formulis non erit opus. Ceterum notandum est numerorum $m,\ n,\ n-m,\ n+m$ etc. logarithmos hyperbolicos accipi oportere, cum logarithmi hyperbolici sinuum cosinuumque quaeruntur, vulgares autem, cum tales ope posteriorum formularum sunt indagandi. Praeterea $m:n$ denotat rationem, quam angulus propositus habet ad angulum rectum; sicque cum sinus angulorum semirecto maiorum aequentur cosinibus angulorum semirecto minorum ac vicissim, fractio $\frac{m}{n}$ nunquam maior accipienda erit quam $\frac{1}{2}$ hancque ob rem termini illi multo magis convergent, ut semissis instituto sufficere possit.

197. Antequam hoc argumentum relinquamus, aptiorem aperiamus modum tangentes et secantes quorumvis angulorum inveniendi, quam caput praecedens suppeditat. Quanquam enim tangentes et secantes per sinus et cosinus determinantur, tamen hoc fit per divisionem, quae operatio in tantis numeris nimis est operosa. Ac tangentes quidem et cotangentes iam supra (§ 135) exhibuimus, verum illo loco rationam formularum reddere non licuit, quam huic capiti reservavimus.

198. Ex § 181 ergo primum expressionem pro tangente anguli $\frac{m\pi}{2n}$ elicimus. Cum enim sit

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}=\frac{\pi}{4mn}\,\text{tang.}\frac{m\pi}{2n},$$

erit

$$\text{tang.}\frac{m\pi}{2n}=\frac{4mn}{\pi}\left(\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}\right).$$

Cum deinde sit

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn}\cot.\frac{m\pi}{n},$$

si pro *n* scribamus *2n*, erit

$$\cot.\frac{m\pi}{2n}=\frac{2n}{m\pi}-\frac{4mn}{\pi}\left(\frac{1}{4nn-mm}+\frac{1}{16nn-mm}+\frac{1}{36nn-mm}+\text{etc.}\right).$$

Convertantur hae fractiones praeter primas, quippe quae facile in computum ducuntur, in series infinitas; erit

$$\begin{aligned}\text{tang.}\frac{m\pi}{2n}=\frac{mn}{nn-mm}.\frac{4}{\pi}&+\frac{4}{\pi}\left(\frac{m}{3^2n}+\frac{m^3}{3^4n^3}+\frac{m^5}{3^6n^5}+\text{etc.}\right)\\&+\frac{4}{\pi}\left(\frac{m}{5^2n}+\frac{m^3}{5^4n^3}+\frac{m^5}{5^6n^5}+\text{etc.}\right)\\&+\frac{4}{\pi}\left(\frac{m}{7^2n}+\frac{m^3}{7^4n^3}+\frac{m^5}{7^6n^5}+\text{etc.}\right)\\&\qquad\text{etc.},\end{aligned}$$

$$\begin{aligned}\cot.\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{2}{\pi}-\frac{mn}{4nn-mm}.\frac{4}{\pi}&-\frac{4}{\pi}\left(\frac{m}{4^2n}+\frac{m^3}{4^4n^3}+\frac{m^5}{4^6n^5}+\text{etc.}\right)\\&-\frac{4}{\pi}\left(\frac{m}{8^2n}+\frac{m^3}{8^4n^3}+\frac{m^5}{8^6n^5}+\text{etc.}\right)\\&\qquad\text{etc.}\end{aligned}$$

198a At ex valore ipsius π cognito reperitur

$$\frac{1}{\pi}=0{,}31830\ 98861\ 83790\ 67153\ 77675\ 26745\ 028724,$$

deinde hic eaedem series occurrunt, quas supra [§ 190 et 193] litteris *A*, *B*,*C*, *D* etc. et $\alpha, \beta, \gamma, \delta$ etc. indicavimus. His ergo notatis erit

$$\text{tang.}\frac{m\pi}{2n} = \frac{mn}{nn-mm}\cdot\frac{4}{\pi}$$
$$+\frac{m}{n}\cdot\frac{4}{\pi}(A-1)+\frac{m^3}{n^3}\cdot\frac{4}{\pi}(B-1)+\frac{m^5}{n^5}\cdot\frac{4}{\pi}(C-1)+\frac{m^7}{n^7}\cdot\frac{4}{\pi}(D-1)+\text{etc.}$$

Deinde erit pro cotangente

$$\text{cot.}\frac{m\pi}{2n} = \frac{n}{m}\cdot\frac{2}{\pi}-\frac{4mn}{4nn-mm}\cdot\frac{1}{\pi}$$
$$-\frac{m}{n}\cdot\frac{4}{\pi}\left(\alpha-\frac{1}{2^2}\right)-\frac{m^3}{n^3}\cdot\frac{4}{\pi}\left(\beta-\frac{1}{2^4}\right)-\frac{m^5}{n^5}\cdot\frac{4}{\pi}\left(\gamma-\frac{1}{2^6}\right)-\text{etc.},$$

atque ex his formulis natae sunt expressiones, quas supra (§ 135) pro tangente et cotangente dedimus; simul vero (§ 137) ostendimus, quomodo ex tangentibus et cotangentibus inventis per solam additionem et subtractionem secantes et cosecantes reperiantur. Harum ergo regularum ope universus canon sinuum, tangentium et secantium, eorumque logarithmorum multo facilius supputari posset, quam quidem hoc a primis conditoribus est factum.