CHAPTER X

CONCERNING THE USE OF THE FACTORS FOUND ABOVE IN DEFINING THE SUMS OF INFINITE SERIES

165. If there shall be

$$1+Az+Bz^2+Cz^3+Dz^4+\text{etc.}=(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)\ \text{etc.},$$

these factors, which shall be either for a finite or infinite number of terms, if they may actually be multiplied into each other, must produce that expression $1+Az+Bz^2+Cz^3+Dz^4+\text{etc.}$ Therefore the coefficient A will be equal to the sum of all the quantities

$$\alpha+\beta+\gamma+\delta+\varepsilon+\text{etc.}$$

Truly the coefficient B will be equal to the sum of the products from every two factors and it will be

$$B=\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta+\text{etc.}$$

Then truly the coefficient C will be equal to the sum of the products from every three factors, evidently there will be

$$C=\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\ \text{etc.}$$

And thus again D = sum of the products from every four factors, E = sum of the products from every five factors, etc., so that it agrees with common algebra.

166. Because the sum of the quantities $\alpha+\beta+\gamma+\delta+\text{etc.}$ is given by a single sum of the products from the pairs of factor, hence the sum of the squares $\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.}$ will be able to be found, evidently which is equal to the square of the sum with twice the products from the binary factors taken. In a similar manner the sum of the cubes, of the biquadratics and of the higher powers can be defined; if indeed we may put

$$\begin{aligned}
P &= \alpha + \beta + \gamma + \delta + \varepsilon + \text{etc.}\\
Q &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \text{etc.}\\
R &= \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \text{etc.}\\
S &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \text{etc.}\\
T &= \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \text{etc.}\\
V &= \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \varepsilon^6 + \text{etc.}\\
&\text{etc.,}
\end{aligned}$$

the values *P, Q, R, S, T, V* etc. will be determined in the following manner from the known *A, B, C, D* etc.:

$$\begin{aligned}
P &= A,\\
Q &= AP - 2B,\\
R &= AQ - BP + 3C,\\
S &= AR - BQ + CP - 4D,\\
T &= AS - BR + CQ - DP + 5E,\\
V &= AT - BS + CR - DQ + EP - 6F\\
&\text{etc.,}
\end{aligned}$$

the truth of which formulas from a careful examination is readily acknowledged ; yet meanwhile in the differential calculus it will be shown with the maximum rigor.

167. Therefore since above (§ 156) we have found to be

$$\begin{aligned}
\frac{e^x - e^{-x}}{2} &= x\left(1 + \frac{xx}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdots 7} + \text{etc.}\right)\\
&= x\left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right) \text{ etc.,}
\end{aligned}$$

there will be

$$1 + \frac{xx}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdots 7} + \text{etc.} = \left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right) \text{ etc.}$$

Putting $xx = \pi\pi z$ and there becomes

$$1 + \frac{\pi\pi}{1\cdot 2\cdot 3} z + \frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} z^2 + \frac{\pi^6}{1\cdot 2\cdots 7} z^3 + \text{etc.} = (1 + z)\left(1 + \frac{1}{4} z\right)\left(1 + \frac{1}{9} z\right)\left(1 + \frac{1}{16} z\right)\left(1 + \frac{1}{25} z\right) \text{ etc.}$$

Therefore with the application made of the above rule to this case there will be

$$A = \frac{\pi\pi}{6}, \quad B = \frac{\pi^4}{120}, \quad C = \frac{\pi^6}{5040}, \quad D = \frac{\pi^8}{362880} \quad \text{etc.}$$

Therefore if there may be put

$$P = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \text{etc.},$$
$$Q = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \frac{1}{36^2} + \text{etc.},$$
$$R = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \frac{1}{36^3} + \text{etc.},$$
$$S = 1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \frac{1}{36^4} + \text{etc.},$$
$$T = 1 + \frac{1}{4^5} + \frac{1}{9^5} + \frac{1}{16^5} + \frac{1}{25^5} + \frac{1}{36^5} + \text{etc.},$$
$$\text{etc.}$$

.
and the values of these letters will be determined from *A, B, C, D* etc., producing :

$$P = \frac{\pi\pi}{6},$$
$$Q = \frac{\pi^4}{90},$$
$$R = \frac{\pi^6}{945},$$
$$S = \frac{\pi^8}{9450},$$
$$T = \frac{\pi^{10}}{93555}$$
$$\text{etc.}$$

168. Therefore it is apparent the sums of all the infinite series contained in this general form

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.},$$

can be shown, as often as n should be an even number, with the aid of the semi periphery of the circle π ; for indeed the sum of the series will have always a rational ratio to π^n . But so that the value of these sums may be seen more clearly, I may add here several sums of series of this kind expressed in a more convenient manner.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=\frac{2^0}{1\cdot2\cdot3}\cdot\frac{1}{1}\pi^2,$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=\frac{2^2}{1\cdot2\cdot3\cdot4\cdot5}\cdot\frac{1}{3}\pi^4,$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=\frac{2^4}{1\cdot2\cdot3\cdots7}\cdot\frac{1}{3}\pi^6,$$
$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=\frac{2^6}{1\cdot2\cdot3\cdots9}\cdot\frac{3}{5}\pi^8,$$
$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\text{etc.}=\frac{2^8}{1\cdot2\cdot3\cdots11}\cdot\frac{5}{3}\pi^{10},$$
$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\text{etc.}=\frac{2^{10}}{1\cdot2\cdot3\cdots13}\cdot\frac{691}{105}\pi^{12},$$
$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\text{etc.}=\frac{2^{12}}{1\cdot2\cdot3\cdots15}\cdot\frac{35}{1}\pi^{14},$$
$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\text{etc.}=\frac{2^{14}}{1\cdot2\cdot3\cdots17}\cdot\frac{3617}{15}\pi^{16},$$
$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\text{etc.}=\frac{2^{16}}{1\cdot2\cdot3\cdots19}\cdot\frac{43861}{21}\pi^{18},$$
$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\text{etc.}=\frac{2^{18}}{1\cdot2\cdot3\cdots21}\cdot\frac{1222277}{55}\pi^{20},$$
$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\text{etc.}=\frac{2^{20}}{1\cdot2\cdot3\cdots23}\cdot\frac{854513}{3}\pi^{22},$$
$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\text{etc.}=\frac{2^{22}}{1\cdot2\cdot3\cdots25}\cdot\frac{1181820455}{273}\pi^{24},$$
$$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\frac{1}{4^{26}}+\frac{1}{5^{26}}+\text{etc.}=\frac{2^{24}}{1\cdot2\cdot3\cdots27}\cdot\frac{76977927}{1}\pi^{26}.$$
etc.

Hitherto I have explained those coefficients of the powers of π by another artifice, which series thus I adjoin here,

$$1,\ \frac{1}{3},\ \frac{1}{3},\ \frac{3}{5},\ \frac{5}{3},\ \frac{691}{105},\ \frac{35}{1}\ \text{etc.},$$

which is a very irregular series of fractions on being looked at initially, but which has been exceedingly useful on many occasions.

169. We may treat the equation found in § 157 in the same way, where there was

$$\frac{e^x+e^{-x}}{2}=1+\frac{xx}{1\cdot2}+\frac{x^4}{1\cdot2\cdot3\cdot4}+\frac{x^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6}+\text{etc.}$$
$$=\left(1+\frac{4xx}{\pi\pi}\right)\left(1+\frac{4xx}{9\pi\pi}\right)\left(1+\frac{4xx}{25\pi\pi}\right)\left(1+\frac{4xx}{49\pi\pi}\right)\text{ etc.}$$

Therefore on putting $xx=\frac{\pi\pi z}{4}$ it becomes

$$1+\frac{\pi\pi}{1\cdot 2\cdot 4}z+\frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 4^2}zz+\frac{\pi^6}{1\cdot 2\cdots 6\cdot 4^3}z^3+\text{etc.}$$
$$=(1+z)\left(1+\tfrac{1}{9}z\right)\left(1+\tfrac{1}{25}z\right)\left(1+\tfrac{1}{49}z\right)\text{ etc.}$$

From which on applying the preceding rule there will be

$$A=\frac{\pi\pi}{1\cdot 2\cdot 4},\quad B=\frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 4^2},\quad C=\frac{\pi^6}{1\cdot 2\cdot 3\cdots 6\cdot 4^3}\text{ etc.}$$

But if, therefore, we may put

$$P=1+\tfrac{1}{9}+\tfrac{1}{25}+\tfrac{1}{49}+\tfrac{1}{81}+\text{etc.},$$
$$Q=1+\frac{1}{9^2}+\frac{1}{25^2}+\frac{1}{49^2}+\frac{1}{81^2}+\text{etc.},$$
$$R=1+\frac{1}{9^3}+\frac{1}{25^3}+\frac{1}{49^3}+\frac{1}{81^3}+\text{etc.},$$
$$S=1+\frac{1}{9^4}+\frac{1}{25^4}+\frac{1}{49^4}+\frac{1}{81^4}+\text{etc.}$$
$$\text{etc.},$$

the following values will be found for P, Q, R, S etc.:

$$P=\frac{1}{1}\cdot\frac{\pi^2}{2^3},$$
$$Q=\frac{2}{1\cdot 2\cdot 3}\cdot\frac{\pi^4}{2^5},$$
$$R=\frac{16}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{\pi^6}{2^7},$$
$$S=\frac{272}{1\cdot 2\cdot 3\cdots 7}\cdot\frac{\pi^8}{2^9},$$
$$T=\frac{7936}{1\cdot 2\cdot 3\cdots 9}\cdot\frac{\pi^{10}}{2^{11}},$$
$$V=\frac{353792}{1\cdot 2\cdot 3\cdots 11}\cdot\frac{\pi^{12}}{2^{13}},$$
$$W=\frac{22368256}{1\cdot 2\cdot 3\cdots 13}\cdot\frac{\pi^{14}}{2^{15}}$$
$$\text{etc.}$$

170. The same sums of powers of odd numbers can be found from the preceding sums, in which all the numbers occur. For if there were

$$M=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\text{etc.},$$

on multiplying everywhere by $\frac{1}{2^n}$ it becomes

$$\frac{M}{2^n}=\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\frac{1}{8^n}+\frac{1}{10^n}+\text{etc.};$$

which series containing only even numbers if it be taken from the former, and thus the odd numbers will remain

$$M-\frac{M}{2^n}=\frac{2^n-1}{2^n}M=1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{9^n}+\text{etc.}$$

But if moreover the series $\frac{M}{2^n}$ may be taken twice from M, alternating signs will be produced and it becomes

$$M-\frac{2M}{2^n}=\frac{2^{n-1}-1}{2^{n-1}}M=1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\text{etc.}$$

Therefore these series can be summed following these precepts:

$$1\pm\frac{1}{2^n}+\frac{1}{3^n}\pm\frac{1}{4^n}+\frac{1}{5^n}\pm\frac{1}{6^n}+\frac{1}{7^n}\pm\text{etc.},$$
$$1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{9^n}+\frac{1}{11^n}+\text{etc.},$$

if indeed n shall be an even number, and the sum will be $=A\pi^n$, with the number A being a rational number.

171. Truly besides expressions shown § 164 will supply the series worthy of note in the same manner. For since there shall be

$$\cos.\tfrac{1}{2}v+\text{tang.}\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$=\left(1+\frac{v}{\pi-g}\right)\left(1-\frac{v}{\pi+g}\right)\left(1+\frac{v}{3\pi-g}\right)\left(1-\frac{v}{3\pi+g}\right)\text{ etc.},$$

If we may put $v=\frac{x}{n}\pi$ and $g=\frac{m}{n}\pi$, there will be

$$\left(1+\frac{x}{n-m}\right)\left(1-\frac{x}{n+m}\right)\left(1+\frac{x}{3n-m}\right)\left(1-\frac{x}{3n+m}\right)\left(1+\frac{x}{5n-m}\right)\left(1-\frac{x}{5n+m}\right)\text{ etc.}$$
$$=\cos.\frac{x\pi}{2n}+\text{tang.}\frac{m\pi}{2n}\sin.\frac{x\pi}{2n}$$
$$=1+\frac{\pi x}{2n}\text{tang.}\frac{m\pi}{2n}-\frac{\pi\pi xx}{2\cdot 4nn}-\frac{\pi^3x^3}{2\cdot 4\cdot 6n^3}\text{tang.}\frac{m\pi}{2n}+\frac{\pi^4x^4}{2\cdot 4\cdot 6\cdot 8n^4}+\text{etc.}$$

This infinite expression taken with § 165 will give these values

$$A = \frac{\pi}{2n}\text{tang.}\frac{m\pi}{2n},$$
$$B = \frac{-\pi\pi}{2\cdot 4nn},$$
$$C = \frac{-\pi^3}{2\cdot 4\cdot 6n^3}\text{tang.}\frac{m\pi}{2n},$$
$$D = \frac{\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},$$
$$E = \frac{\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5}\text{tang.}\frac{m\pi}{2n}$$
etc.

Then indeed there will be

$$\alpha = \frac{1}{n-m}, \quad \beta = -\frac{1}{n+m}, \quad \gamma = \frac{1}{3n-m}, \quad \delta = -\frac{1}{3n+m},$$
$$\varepsilon = \frac{1}{5n-m}, \quad \zeta = -\frac{1}{5n+m} \quad \text{etc.}$$

172. Hence therefore according to the rule of §166 the following series will arise :

$$P = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \frac{1}{5n-m} - \text{ etc.},$$
$$Q = \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \frac{1}{(5n-m)^2} + \text{ etc.},$$
$$R = \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} + \frac{1}{(3n-m)^3} - \frac{1}{(3n+m)^3} + \frac{1}{(5n-m)^3} - \text{ etc.},$$
$$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} + \frac{1}{(5n-m)^4} + \text{ etc.},$$
$$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} + \frac{1}{(5n-m)^5} - \text{ etc.},$$
$$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} + \frac{1}{(5n-m)^6} + \text{ etc.},$$

But on putting $\text{tang.}\frac{m\pi}{2n} = k$ there will be, as we have shown,

$$P = A = \tfrac{k\pi}{2n} = \tfrac{k\pi}{2n},$$
$$Q = \frac{(kk+1)\pi\pi}{4nn} = \frac{(2kk+2)\pi^2}{2\cdot 4nn},$$
$$R = \frac{(k^3+k)\pi^3}{8n^3} = \frac{(6k^3+6k)\pi^3}{2\cdot 4\cdot 6n^3},$$
$$S = \frac{(3k^4+4kk+1)\pi^4}{48n^4} = \frac{(24k^4+32kk+8)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},$$
$$T = \frac{(3k^5+5k^3+2k)\pi^5}{96n^5} = \frac{(120k^5+200k^3+80k)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5}$$

etc.

173. In the same manner in the final form § 164

$$\cos.\tfrac{1}{2}v + \cot.\tfrac{1}{2}g\ \sin.\tfrac{1}{2}v$$
$$= \left(1+\tfrac{v}{g}\right)\left(1-\tfrac{v}{2\pi-g}\right)\left(1+\tfrac{v}{2\pi+g}\right)\left(1-\tfrac{v}{4\pi-g}\right)\left(1+\tfrac{v}{4\pi+g}\right) \text{ etc.},$$

if we may put $v = \tfrac{x}{n}\pi$, $g = \tfrac{m}{n}\pi$ and $\tang.\tfrac{m\pi}{2n} = k$,so that there shall be $\cot.\tfrac{1}{2}g = \tfrac{1}{k}$, it will give

$$\cos.\tfrac{x\pi}{2n} + \tfrac{1}{k}\sin.\tfrac{x\pi}{2n}$$

$$= 1 + \frac{\pi x}{2nk} - \frac{\pi\pi xx}{2\cdot 4nn} - \frac{\pi^3x^3}{2\cdot 4\cdot 6n^3k} + \frac{\pi^4x^4}{2\cdot 4\cdot 6\cdot 8n^4} + \frac{\pi^5x^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k} - \text{etc.}$$

$$= \left(1+\tfrac{x}{m}\right)\left(1-\tfrac{x}{2n-m}\right)\left(1+\tfrac{x}{2n+m}\right)\left(1-\tfrac{x}{4n-m}\right)\left(1+\tfrac{x}{4n+m}\right) \text{ etc.}$$

Therefore on comparison with the general form put in place (§ 165) there will be

$$A = \frac{\pi}{2nk},$$
$$B = -\frac{\pi\pi}{2\cdot 4nn},$$
$$C = -\frac{\pi^3}{2\cdot 4\cdot 6n^3k},$$
$$D = \frac{\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},$$
$$E = \frac{\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k}, \text{etc.};$$

truly from the factors there will be had

$$\alpha = \tfrac{1}{m}, \quad \beta = \tfrac{-1}{2n-m}, \quad \gamma = \tfrac{1}{2n+m}, \quad \delta = \tfrac{-1}{4n-m}, \quad \beta = \tfrac{1}{4n+m} \text{ etc.}$$

174. Hence therefore according to the rule §166 the following series will be formed and the sums of these will be designated :

$$P = \tfrac{1}{m} - \tfrac{1}{2n-m} + \tfrac{1}{2n+m} - \tfrac{1}{4n-m} + \tfrac{1}{4n+m} - \text{etc.},$$
$$Q = \tfrac{1}{m^2} + \tfrac{1}{(2n-m)^2} + \tfrac{1}{(2n+m)^2} + \tfrac{1}{(4n-m)^2} + \tfrac{1}{(4n+m)^2} + \text{etc.},$$
$$R = \tfrac{1}{m^3} - \tfrac{1}{(2n-m)^3} + \tfrac{1}{(2n+m)^3} - \tfrac{1}{(4n-m)^3} + \tfrac{1}{(4n+m)^3} - \text{etc.},$$
$$S = \tfrac{1}{m^4} + \tfrac{1}{(2n-m)^4} + \tfrac{1}{(2n+m)^4} + \tfrac{1}{(4n-m)^4} + \tfrac{1}{(4n+m)^4} + \text{etc.},$$
$$T = \tfrac{1}{m^5} - \tfrac{1}{(2n-m)^5} + \tfrac{1}{(2n+m)^5} - \tfrac{1}{(4n-m)^5} + \tfrac{1}{(4n+m)^5} - \text{etc.}$$

But these sums P, Q, R, S etc. themselves thus will be had :

$$P = A = \tfrac{\pi}{2nk} \qquad = \tfrac{1\pi}{2nk},$$
$$Q = \frac{(kk+1)\pi\pi}{4nnkk} \qquad = \frac{(2+2kk)\pi^2}{2\cdot 4nnk^2},$$
$$R = \frac{(kk+1)\pi^3}{8n^3k^3} \qquad = \frac{(6+6kk)\pi^3}{2\cdot 4\cdot 6n^3k^3},$$
$$S = \frac{(k^4+4kk+3)\pi^4}{48n^4k^4} \qquad = \frac{(24+32kk+8k^4)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4k^4},$$
$$T = \frac{(2k^4+5kk+3)\pi^5}{96n^5k^5} \qquad = \frac{(120+200kk+80k^4)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k^5},$$
$$V = \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} \qquad = \frac{(720+14400kk+816k^4+96k^6)\pi^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12n^6k^6}$$
etc.

175. These general series deserve that we may derive certain individual cases, which thence will be produced, if we may determine the ratio m to n numerically. Therefore in the first case let $m = 1$ and $n = 2$;
there becomes

$$k = \text{tang.}\tfrac{\pi}{4} = \text{tang.}45^0 = 1$$

and both classes of series are equal to each other. Therefore there will be

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.},$$
$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.},$$
$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.},$$
$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.},$$
$$\frac{\pi^5}{1536} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.},$$
$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.},$$

We have elicited the first of these series now above (§ 140), those remaining, which have even powers, have been elicited in the previous manner (§ 169) ; the remaining, in which the exponents are odd numbers, occur here for the first time. Therefore it is agreed also the sums of all these series

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \text{etc.}$$

can be designated by the value of π .

176. Now let there be

$$m = 1, \quad n = 3;$$

$$k.\text{tang.}\frac{\pi}{6} = \text{tang.}30^0 = \frac{1}{\sqrt{3}}$$

and the series § 172 will change into these

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \text{etc.},$$
$$\frac{\pi\pi}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \text{etc.},$$
$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \text{etc.}$$

etc.

or

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \text{etc.},$$
$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \text{etc.},$$
$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \text{etc.}$$

etc.

In these series all the numbers divisible by three are absent ; hence the even dimensions may be deduced from those now found in this manner. Since there shall be [§ 167, 168]

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.},$$

there becomes

$$\frac{\pi\pi}{6\cdot 9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \text{etc.} = \frac{\pi\pi}{54};$$

which latter series containing all the numbers divisible by three, if taken from the former, all the numbers that will remain will not be divisible by 3 and thus there will be

$$\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.},$$

as we have found.

177. Likewise the hypothesis

$$m = 1, \quad n = 3 \quad \text{and} \quad k = \frac{1}{\sqrt{3}}$$

adapted to § 174 will present these summations

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \text{etc.},$$

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.},$$

$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^2} + \text{etc.}$$

etc.,

in the denominators of which only odd numbers occur with these excepted, which are divisible by three. The remaining even dimensions with these now known can be deduced; for since there shall be

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.},$$

there will be

$$\frac{\pi\pi}{8\cdot 9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \text{etc.} = \frac{\pi\pi}{72};$$

which series contains all the odd numbers divisible by 3 if it is taken from the above, will leave the series of odd squares not divisible by 3 and it becomes

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

178. If the series found in §§ 172 and 174 may be either added or subtracted other noteworthy series will be obtained. Clearly there will be

$$\frac{k\pi}{2n}+\frac{\pi}{2nk}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\text{etc.}=\frac{(kk+1)\pi}{2nk};$$

but there is

$$k=\text{tang.}\frac{m\pi}{2n}=\frac{\sin.\frac{m\pi}{2n}}{\cos.\frac{m\pi}{2n}} \text{ and } 1+kk=\frac{1}{\left(\cos.\frac{m\pi}{2n}\right)^2},$$

from which

$$\frac{2k}{1+kk}=2\sin.\frac{m\pi}{2n}\cos.\frac{m\pi}{2n}=\sin.\frac{m\pi}{n},$$

with which value substituted we will have

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}-\text{etc.}$$

In a like manner by subtraction there will become

$$\frac{\pi}{2nk}-\frac{k\pi}{2n}=\frac{(1-kk)\pi}{2nk}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\frac{1}{3n+m}+\text{etc.};$$

but there is

$$\frac{2k}{1-kk}=\text{tang.}2\frac{m\pi}{2n}=\text{tang.}\frac{m\pi}{n}=\frac{\sin.\frac{m\pi}{n}}{\cos.\frac{m\pi}{n}};$$

hence there will be

$$\frac{\pi\cos.\frac{m\pi}{n}}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.}$$

The series of the squares and of higher powers hence arise more easily by differentiation and hence will be deduced below.

179. Because we have now set out the cases in which $m=1$ and $n=2$ or 3, we may put

$$m=1 \quad\text{and}\quad n=4;$$

there will be

$$\sin.\frac{m\pi}{n}=\sin.\frac{\pi}{4}=\frac{1}{\sqrt{2}} \quad\text{and}\quad \cos.\frac{\pi}{4}=\frac{1}{\sqrt{2}}.$$

And thus hence there will be had

$$\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\text{etc.}$$

and

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\text{ etc.}$$

Let there be

$$m=1 \quad \text{and} \quad n=8;$$

there will be

$$\frac{m\pi}{n}=\frac{\pi}{8} \quad \text{and} \quad \sin.\frac{\pi}{8}=\sqrt{\left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)} \quad \text{and} \quad \cos.\frac{\pi}{8}=\sqrt{\left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)}$$

and

$$\frac{\cos.\frac{\pi}{8}}{\sin.\frac{\pi}{8}}=1+\sqrt{2}\ .$$

Hence thus there will be

$$\frac{\pi}{4\sqrt{\left(2-\sqrt{2}\right)}}=1+\frac{1}{7}-\frac{1}{9}-\frac{1}{15}+\frac{1}{17}+\frac{1}{23}-\text{ etc.,}$$

$$\frac{\pi}{8\sqrt{\left(2-\sqrt{2}\right)}}=1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\text{ etc.}$$

Now let there be

$$m=3 \quad \text{and} \quad n=8\ ;$$

there will be

$$\frac{m\pi}{n}=\frac{3\pi}{8} \quad \text{and} \quad \sin.\frac{3\pi}{8}=\sqrt{\left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)} \quad \text{and} \quad \cos.\frac{3\pi}{8}=\sqrt{\left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)},$$

from which

$$\frac{\cos.\frac{3\pi}{8}}{\sin.\frac{3\pi}{8}}=\frac{1}{\sqrt{2}+1}$$

and these series will be produced :

$$\frac{\pi}{4\sqrt{\left(2+\sqrt{2}\right)}}=\frac{1}{3}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{19}+\frac{1}{21}-\text{ etc.,}$$

$$\frac{\pi}{8\left(\sqrt{2}+1\right)}=\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\text{ etc.}$$

180. From these series by combination the series come about :

$$\frac{\pi\sqrt{\left(2+\sqrt{2}\right)}}{4}=1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}-\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}+\tfrac{1}{19}+\text{etc.},$$
$$\frac{\pi\sqrt{\left(2-\sqrt{2}\right)}}{4}=1-\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}+\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}-\tfrac{1}{19}-\text{etc.},$$
$$\frac{\pi\left(\sqrt{\left(4+2\sqrt{2}\right)}+\sqrt{2}-1\right)}{8}=1+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}+\tfrac{1}{19}-\text{etc.},$$
$$\frac{\pi\left(\sqrt{\left(4+2\sqrt{2}\right)}-\sqrt{2}+1\right)}{8}=1-\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}-\tfrac{1}{19}+\text{etc.},$$
$$\frac{\pi\left(\sqrt{2}+1+\sqrt{\left(4-2\sqrt{2}\right)}\right)}{8}=1+\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}+\tfrac{1}{19}+\text{etc.},$$
$$\frac{\pi\left(\sqrt{2}+1-\sqrt{\left(4-2\sqrt{2}\right)}\right)}{8}=1-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}-\tfrac{1}{19}-\text{etc.},$$

In a similar manner on putting $n=16$ and m either 1, 3, 5, or 7 it is allowed to progress further, and in this manner the sums of the series $1,\tfrac{1}{3},\tfrac{1}{5},\tfrac{1}{7},\tfrac{1}{9}$ etc., will be found in which the changes of the signs + and – may follow other rules.

181. If in the series found in § 178 two terms may be gathered into one sum, and there will be

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}+\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}+\frac{2m}{9nn-mm}-\frac{2m}{16n-mm}+\text{etc.}$$

and thus

$$\frac{1}{nn-mm}-\frac{1}{4nn-mm}+\frac{1}{9nn-mm}-\text{etc.}=\frac{\pi}{2mn\sin.\frac{m\pi}{n}}-\frac{1}{2mm}$$

Truly the other series will give

$$\frac{\pi}{n\tan.\frac{m\pi}{n}}=\frac{1}{m}-\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}-\frac{2m}{9nn-mm}-\text{etc.}$$

and hence

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn\tan.\frac{m\pi}{n}}.$$

Moreover from these joined together this arises :

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}=\frac{\pi\tan.\frac{m\pi}{2n}}{4mn}.$$

If $n = 1$ in this series and the number *m* may be put equal to some even number $2k$, on account of $\text{tang.}k\pi = 0$ there will be always, unless $k = 0$,

$$\frac{1}{1-4kk} + \frac{1}{9-4kk} + \frac{1}{25-4kk} + \frac{1}{49-4kk} + \text{etc.} = 0;$$

but if in that series $n = 2$ and *m* were some odd number

$= 2k+1$, on account of $\frac{1}{\tan.\frac{m\pi}{n}} = 0$, there will be

$$\frac{1}{4-(2k+1)^2} + \frac{1}{16-(2k+1)^2} + \frac{1}{36-(2k+1)^2} + \text{etc.} = \frac{1}{2(2k+1)^2}$$

182. The series found may be multiplied by *nn* and there becomes $\frac{m}{n} = p$; these forms will be found :

$$\frac{1}{1-pp} - \frac{1}{4-pp} + \frac{1}{9-pp} - \frac{1}{16-pp} + \text{ etc.} = \frac{\pi}{2p\sin.px} - \frac{1}{2pp},$$
$$\frac{1}{1-pp} + \frac{1}{4-pp} + \frac{1}{9-pp} + \frac{1}{16-pp} + \text{ etc.} = \frac{1}{2pp} - \frac{\pi}{2p\text{tang.}px}.$$

Let $pp = a$ and these series arise :

$$\frac{1}{1-a} - \frac{1}{4-a} + \frac{1}{9-a} - \frac{1}{16-a} + \text{ etc.} = \frac{\pi\sqrt{a}}{2a\sin.\pi\sqrt{a}} - \frac{1}{2a},$$
$$\frac{1}{1-a} + \frac{1}{4-a} + \frac{1}{9-a} + \frac{1}{16-a} + \text{ etc.} = \frac{1}{2a} - \frac{\pi\sqrt{a}}{2a\,\text{tang.}\pi\sqrt{a}}.$$

Therefore provided *a* were not a negative number nor a whole square, the sum of these series will be able to be shown by means of a circle.

183. But by the reduction of imaginary exponentials to the sines and cosines of circular arcs that I have treated above we will be able also to assign the sums of these series , if *a* shall be a negative number. For since there shall be

$$e^{x\sqrt{-1}} = \cos.x + \sqrt{-1}\cdot\sin.x \quad \text{and} \quad e^{-x\sqrt{-1}} = \cos.x - \sqrt{-1}\cdot\sin.x,$$

in turn there will be on putting $y\sqrt{-1}$ in place of *x*

$$\cos.y\sqrt{-1} = \frac{e^{-y}+e^{y}}{2} \quad \text{and} \quad \sin.y\sqrt{-1} = \frac{e^{-y}-e^{y}}{2\sqrt{-1}}.$$

But if therefore $a = -b$ and $y = \pi\sqrt{b}$, there will be

$$\cos.\pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}}}{2} \quad \text{and} \quad \sin.y\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{2\sqrt{-1}}$$

and thus

$$\text{tang}.\pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{\left(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}}\right)\sqrt{-1}}$$

Hence there will be

$$\frac{\pi\sqrt{-b}}{\sin.\pi\sqrt{-b}} = \frac{-2\pi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}} \quad \text{and} \quad \frac{\pi\sqrt{-b}}{\text{tang}.\pi\sqrt{-b}} = \frac{-\left(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}}\right)\pi\sqrt{b}}{\left(e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}\right)}$$

Therefore with these observed there will be :

$$\frac{1}{1+b} - \frac{1}{4+b} + \frac{1}{9+b} - \frac{1}{16+b} + \text{etc.} = \frac{1}{2b} - \frac{\pi\sqrt{b}}{\left(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}}\right)b},$$

$$\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \frac{1}{16+b} + \text{etc.} = \frac{\left(e^{\pi\sqrt{b}} + e^{-\pi\sqrt{b}}\right)\pi\sqrt{b}}{2b\left(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}}\right)} - \frac{1}{2b}.$$

These series likewise can be deduced from § 162 by using the same method, which I have used in this chapter. Truly because with this agreed on, the reduction of the sines and cosines of imaginary arcs to real exponential quantities may not be too hard to show, but I have been led to prefer this other explanation.

CAPUT X

DE USU FACTORUM INVENTORUM IN DEFINIENDIS SUMMIS SERIERUM INFINITARUM

165. Si fuerit

$$1+Az+Bz^2+Cz^3+Dz^4+\text{etc.}=(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)\ \text{etc.},$$

hi factores, sive sint numero finiti sive infiniti, si in se actu multiplicentur, illam expressionem $1+Az+Bz^2+Cz^3+Dz^4+\text{etc.}$ producere debent. Aequabitur ergo coefficiens A summae omnium quantitatum

$$\alpha+\beta+\gamma+\delta+\varepsilon+\text{etc.}$$

Coefficiens vero B aequalis erit summae productorum ex binis eritque

$$B=\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta+\text{etc.}$$

Tum vero coefficiens C aequabitur summae productorum ex ternis, nempe erit

$$C=\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\ \text{etc.}$$

Atque ita porro erit D = summae productorum ex quaternis, E = summae productorum ex quinis etc., id quod ex Algebra communi constat.

166. Quia summa quantitatum $\alpha+\beta+\gamma+\delta+\text{etc.}$ datur una cum summa productorum ex binis, hinc summa quadratorum $\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.}$ inveniri poterit, quippe quae aequalis est quadrato summae demptis duplicibus productis ex binis. Simili modo summa cuborum, biquadratorum et altiorum potestatum definiri potest; si enim ponamus

$$
\begin{aligned}
P &= \alpha + \beta + \gamma + \delta + \varepsilon + \text{etc.} \\
Q &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \text{etc.} \\
R &= \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \text{etc.} \\
S &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \text{etc.} \\
T &= \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \text{etc.} \\
V &= \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \varepsilon^6 + \text{etc.} \\
& \text{etc.}
\end{aligned}
$$

valores *P, Q, R, S, T, V* etc. sequenti modo ex cognitis *A, B, C, D* etc. determinabuntur:

$$
\begin{aligned}
P &= A, \\
Q &= AP - 2B, \\
R &= AQ - BP + 3C, \\
S &= AR - BQ + CP - 4D, \\
T &= AS - BR + CQ - DP + 5E, \\
V &= AT - BS + CR - DQ + EP - 6F \\
& \text{etc.},
\end{aligned}
$$

quarum formularum veritas examine instituto facile agnoscitur; interim tamen in Calculo differentiali summo cum rigore demonstrabitur.

167. Cum igitur supra (§ 156) invenerimus esse

$$
\begin{aligned}
\frac{e^x - e^{-x}}{2} &= x\left(1 + \frac{xx}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdots 7} + \text{etc.}\right) \\
&= x\left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right) \text{ etc.},
\end{aligned}
$$

erit

$$1 + \frac{xx}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdots 7} + \text{etc.} = \left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right) \text{ etc.}$$

Ponatur $xx = \pi\pi z$ eritque

$$1 + \frac{\pi\pi}{1\cdot 2\cdot 3} z + \frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} z^2 + \frac{\pi^6}{1\cdot 2\cdots 7} z^3 + \text{etc.} = (1 + z)\left(1 + \frac{1}{4} z\right)\left(1 + \frac{1}{9} z\right)\left(1 + \frac{1}{16} z\right)\left(1 + \frac{1}{25} z\right) \text{ etc.}$$

Facta ergo applicatione superioris regulae ad hunc casum erit

$$A = \frac{\pi\pi}{6}, \quad B = \frac{\pi^4}{120}, \quad C = \frac{\pi^6}{5040}, \quad D = \frac{\pi^8}{362880} \quad \text{etc.}$$

Quodsi ergo ponatur

$$P=1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\tfrac{1}{36}+\text{etc.},$$
$$Q=1+\tfrac{1}{4^2}+\tfrac{1}{9^2}+\tfrac{1}{16^2}+\tfrac{1}{25^2}+\tfrac{1}{36^2}+\text{etc.},$$
$$R=1+\tfrac{1}{4^3}+\tfrac{1}{9^3}+\tfrac{1}{16^3}+\tfrac{1}{25^3}+\tfrac{1}{36^3}+\text{etc.},$$
$$S=1+\tfrac{1}{4^4}+\tfrac{1}{9^4}+\tfrac{1}{16^4}+\tfrac{1}{25^4}+\tfrac{1}{36^4}+\text{etc.},$$
$$T=1+\tfrac{1}{4^5}+\tfrac{1}{9^5}+\tfrac{1}{16^5}+\tfrac{1}{25^5}+\tfrac{1}{36^5}+\text{etc.},$$
etc.

.

atque harum litterarum valores ex .*A*, *B*, *G*, *D* etc. determinentur, prodibit:

$$P=\tfrac{\pi\pi}{6},$$
$$Q=\tfrac{\pi^4}{90},$$
$$R=\tfrac{\pi^6}{945},$$
$$S=\tfrac{\pi^8}{9450},$$
$$T=\tfrac{\pi^{10}}{93555}$$
etc.

168. Patet ergo omnium serierum infinitarum in hac forma generali

$$1+\tfrac{1}{2^n}+\tfrac{1}{3^n}+\tfrac{1}{4^n}+\text{etc.},$$

contentarum [summas], quoties n fuerit numerus par, ope semiperipheriae circuli π exhiberi posse; habebit enim semper summa seriei ad π^n rationem rationalem. Quo autem valor harum summarum clarius perspiciatur, plures huiusmodi serierum summas commodiori modo expressas hic adiiciam.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=\frac{2^0}{1\cdot2\cdot3}\cdot\frac{1}{1}\pi^2,$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=\frac{2^2}{1\cdot2\cdot3\cdot4\cdot5}\cdot\frac{1}{3}\pi^4,$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=\frac{2^4}{1\cdot2\cdot3\cdots7}\cdot\frac{1}{3}\pi^6,$$
$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=\frac{2^6}{1\cdot2\cdot3\cdots9}\cdot\frac{3}{5}\pi^8,$$
$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\text{etc.}=\frac{2^8}{1\cdot2\cdot3\cdots11}\cdot\frac{5}{3}\pi^{10},$$
$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\text{etc.}=\frac{2^{10}}{1\cdot2\cdot3\cdots13}\cdot\frac{691}{105}\pi^{12},$$
$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\text{etc.}=\frac{2^{12}}{1\cdot2\cdot3\cdots15}\cdot\frac{35}{1}\pi^{14},$$
$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\text{etc.}=\frac{2^{14}}{1\cdot2\cdot3\cdots17}\cdot\frac{3617}{15}\pi^{16},$$
$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\text{etc.}=\frac{2^{16}}{1\cdot2\cdot3\cdots19}\cdot\frac{43861}{21}\pi^{18},$$
$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\text{etc.}=\frac{2^{18}}{1\cdot2\cdot3\cdots21}\cdot\frac{1222277}{55}\pi^{20},$$
$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\text{etc.}=\frac{2^{20}}{1\cdot2\cdot3\cdots23}\cdot\frac{854513}{3}\pi^{22},$$
$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\text{etc.}=\frac{2^{22}}{1\cdot2\cdot3\cdots25}\cdot\frac{1181820455}{273}\pi^{24},$$
$$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\frac{1}{4^{26}}+\frac{1}{5^{26}}+\text{etc.}=\frac{2^{24}}{1\cdot2\cdot3\cdots27}\cdot\frac{76977927}{1}\pi^{26}.$$

etc.

Hucusque istos potestatum ipsius π exponentes artificio alibi exponendo continuare licuit, quod ideo hic adiunxi, quod seriei fractionum primo intuitu perquam irregularis

$$1,\ \frac{1}{3},\ \frac{1}{3},\ \frac{3}{5},\ \frac{5}{3},\ \frac{691}{105},\ \frac{35}{1}\ \text{etc.}$$

in plurimis occasionibus eximius est usus.

169. Tractemus eodem modo aequationem § 157 inventam, ubi erat

$$\frac{e^x+e^{-x}}{2}=1+\frac{xx}{1\cdot2}+\frac{x^4}{1\cdot2\cdot3\cdot4}+\frac{x^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6}+\text{etc.}$$
$$=\left(1+\frac{4xx}{\pi\pi}\right)\left(1+\frac{4xx}{9\pi\pi}\right)\left(1+\frac{4xx}{25\pi\pi}\right)\left(1+\frac{4xx}{49\pi\pi}\right)\ \text{etc.}$$

Posito ergo $xx=\frac{\pi\pi z}{4}$ erit

$$1+\frac{\pi\pi}{1\cdot 2\cdot 4}z+\frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 4^2}zz+\frac{\pi^6}{1\cdot 2\cdots 6\cdot 4^3}z^3+\text{etc.}$$
$$=(1+z)\left(1+\tfrac{1}{9}z\right)\left(1+\tfrac{1}{25}z\right)\left(1+\tfrac{1}{49}z\right)\text{ etc.}$$

Unde facta applicatione erit

$$A=\frac{\pi\pi}{1\cdot 2\cdot 4},\quad B=\frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 4^2},\quad C=\frac{\pi^6}{1\cdot 2\cdot 3\cdots 6\cdot 4^3}\text{ etc.}$$

Quodsi ergo ponamus

$$P=1+\tfrac{1}{9}+\tfrac{1}{25}+\tfrac{1}{49}+\tfrac{1}{81}+\text{etc.},$$
$$Q=1+\tfrac{1}{9^2}+\tfrac{1}{25^2}+\tfrac{1}{49^2}+\tfrac{1}{81^2}+\text{etc.},$$
$$R=1+\tfrac{1}{9^3}+\tfrac{1}{25^3}+\tfrac{1}{49^3}+\tfrac{1}{81^3}+\text{etc.},$$
$$S=1+\tfrac{1}{9^4}+\tfrac{1}{25^4}+\tfrac{1}{49^4}+\tfrac{1}{81^4}+\text{etc.}$$
$$\text{etc.},$$

reperientur sequentes pro *P, Q, R, S* etc. valores:

$$P=\frac{1}{1}\cdot\frac{\pi^2}{2^3},$$
$$Q=\frac{2}{1\cdot 2\cdot 3}\cdot\frac{\pi^4}{2^5},$$
$$R=\frac{16}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{\pi^6}{2^7},$$
$$S=\frac{272}{1\cdot 2\cdot 3\cdots 7}\cdot\frac{\pi^8}{2^9},$$
$$T=\frac{7936}{1\cdot 2\cdot 3\cdots 9}\cdot\frac{\pi^{10}}{2^{11}},$$
$$V=\frac{353792}{1\cdot 2\cdot 3\cdots 11}\cdot\frac{\pi^{12}}{2^{13}},$$
$$W=\frac{22368256}{1\cdot 2\cdot 3\cdots 13}\cdot\frac{\pi^{14}}{2^{15}}$$
$$\text{etc.}$$

170. Eaedem summae potestatum numerorum imparium inveniri possunt ex summis praecedentibus, in quibus omnes numeri occurrunt. Si enim fuerit

$$M=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\text{etc.},$$

erit ubique per $\frac{1}{2^n}$ multiplicando

$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \text{etc.};$$

quae series numeros tantum pares continens si a priori subtrahatur, relinquet numeros impares eritque ideo

$$M - \frac{M}{2^n} = \frac{2^n - 1}{2^n} M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.}$$

Quodsi autem series $\frac{M}{2^n}$ bis sumta subtrahatur ab *M*, signa prodibunt alternantia eritque

$$M - \frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}} M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \text{etc.}$$

Per tradita ergo praecepta summari poterunt hae series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \text{etc.},$$
$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{etc.},$$

si quidem *n* sit numerus par, atque summa erit $= A\pi^n$ existente *A* numero rationali.

171. Praeterea vero expressiones § 164 exhibitae simili modo series notatu dignas suppeditabunt. Cum enim sit

$$\cos.\tfrac{1}{2}v + \text{tang.}\tfrac{1}{2}g \,\sin.\tfrac{1}{2}v$$
$$= \left(1 + \frac{v}{\pi - g}\right)\left(1 - \frac{v}{\pi + g}\right)\left(1 + \frac{v}{3\pi - g}\right)\left(1 - \frac{v}{3\pi + g}\right) \text{ etc.},$$

Si ponamus $v = \frac{x}{n}\pi$ et $g = \frac{m}{n}\pi$, erit

$$\left(1 + \frac{x}{n-m}\right)\left(1 - \frac{x}{n+m}\right)\left(1 + \frac{x}{3n-m}\right)\left(1 - \frac{x}{3n+m}\right)\left(1 + \frac{x}{5n-m}\right)\left(1 - \frac{x}{5n+m}\right) \text{ etc.}$$
$$= \cos.\frac{x\pi}{2n} + \text{tang.}\frac{m\pi}{2n} \sin.\frac{x\pi}{2n}$$
$$= 1 + \frac{\pi x}{2n}\text{tang.}\frac{m\pi}{2n} - \frac{\pi\pi xx}{2 \cdot 4nn} - \frac{\pi^3 x^3}{2 \cdot 4 \cdot 6n^3}\text{tang.}\frac{m\pi}{2n} + \frac{\pi^4 x^4}{2 \cdot 4 \cdot 6 \cdot 8n^4} + \text{etc.}$$

Haec expressio infinita cum § 165 collata dabit hos valores

$$A=\tfrac{\pi}{2n}\,\text{tang.}\tfrac{m\pi}{2n},$$
$$B=\tfrac{-\pi\pi}{2\cdot 4nn},$$
$$C=\tfrac{-\pi^3}{2\cdot 4\cdot 6n^3}\,\text{tang.}\tfrac{m\pi}{2n},$$
$$D=\tfrac{\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},$$
$$E=\tfrac{\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5}\,\text{tang.}\tfrac{m\pi}{2n}$$
etc.

Tum vero erit

$$\alpha=\tfrac{1}{n-m},\quad \beta=-\tfrac{1}{n+m},\quad \gamma=\tfrac{1}{3n-m},\quad \delta=-\tfrac{1}{3n+m},$$
$$\varepsilon=\tfrac{1}{5n-m},\quad \zeta=-\tfrac{1}{5n+m}\quad \text{etc.}$$

172. Hinc ergo ad normam § 166 sequentes series exorientur:

$$P=\tfrac{1}{n-m}-\tfrac{1}{n+m}+\tfrac{1}{3n-m}-\tfrac{1}{3n+m}+\tfrac{1}{5n-m}-\text{ etc.},$$
$$Q=\tfrac{1}{(n-m)^2}+\tfrac{1}{(n+m)^2}+\tfrac{1}{(3n-m)^2}+\tfrac{1}{(3n+m)^2}+\tfrac{1}{(5n-m)^2}+\text{ etc.},$$
$$R=\tfrac{1}{(n-m)^3}-\tfrac{1}{(n+m)^3}+\tfrac{1}{(3n-m)^3}-\tfrac{1}{(3n+m)^3}+\tfrac{1}{(5n-m)^3}-\text{ etc.},$$
$$S=\tfrac{1}{(n-m)^4}+\tfrac{1}{(n+m)^4}+\tfrac{1}{(3n-m)^4}+\tfrac{1}{(3n+m)^4}+\tfrac{1}{(5n-m)^4}+\text{ etc.},$$
$$T=\tfrac{1}{(n-m)^5}-\tfrac{1}{(n+m)^5}+\tfrac{1}{(3n-m)^5}-\tfrac{1}{(3n+m)^5}+\tfrac{1}{(5n-m)^5}-\text{ etc.},$$
$$V=\tfrac{1}{(n-m)^6}+\tfrac{1}{(n+m)^6}+\tfrac{1}{(3n-m)^6}+\tfrac{1}{(3n+m)^6}+\tfrac{1}{(5n-m)^6}+\text{ etc.},$$

Posito autem $\text{tang.}\tfrac{m\pi}{2n}=k$ erit, uti ostendimus,

$$P=A=\tfrac{k\pi}{2n}\qquad =\tfrac{k\pi}{2n},$$
$$Q=\frac{(kk+1)\pi\pi}{4nn}\qquad =\frac{(2kk+2)\pi^2}{2\cdot 4nn},$$
$$R=\frac{(k^3+k)\pi^3}{8n^3}\qquad =\frac{(6k^3+6k)\pi^3}{2\cdot 4\cdot 6n^3},$$
$$S=\frac{(3k^4+4kk+1)\pi^4}{48n^4}\qquad =\frac{(24k^4+32kk+8)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},$$
$$T=\frac{(3k^5+5k^3+2k)\pi^5}{96n^5}\qquad =\frac{(120k^5+200k^3+80k)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5}$$
etc.

173. Pari modo ultima forma § 164

$$\cos.\tfrac{1}{2}v + \cot.\tfrac{1}{2}g \sin.\tfrac{1}{2}v$$
$$= \left(1+\tfrac{v}{g}\right)\left(1-\tfrac{v}{2\pi-g}\right)\left(1+\tfrac{v}{2\pi+g}\right)\left(1-\tfrac{v}{4\pi-g}\right)\left(1+\tfrac{v}{4\pi+g}\right) \text{ etc.,}$$

si ponamus $v = \frac{x}{n}\pi$, $g = \frac{m}{n}\pi$ et $\tang.\frac{m\pi}{2n} = k$, ut sit $\cot.\frac{1}{2}g = \frac{1}{k}$, dabit

$$\cos.\tfrac{x\pi}{2n} + \tfrac{1}{k}\sin.\tfrac{x\pi}{2n}$$

$$= 1 + \tfrac{\pi x}{2nk} - \tfrac{\pi\pi xx}{2\cdot 4nn} - \tfrac{\pi^3 x^3}{2\cdot 4\cdot 6n^3 k} + \tfrac{\pi^4 x^4}{2\cdot 4\cdot 6\cdot 8n^4} + \tfrac{\pi^5 x^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5 k} - \text{etc.}$$

$$= \left(1+\tfrac{x}{m}\right)\left(1-\tfrac{x}{2n-m}\right)\left(1+\tfrac{x}{2n+m}\right)\left(1-\tfrac{x}{4n-m}\right)\left(1+\tfrac{x}{4n+m}\right) \text{ etc.}$$

Comparatione ergo cum forma generali (§ 165) instituta erit

$$A = \tfrac{\pi}{2nk},$$
$$B = -\tfrac{\pi\pi}{2\cdot 4nn},$$
$$C = -\tfrac{\pi^3}{2\cdot 4\cdot 6n^3 k},$$
$$D = \tfrac{\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},$$
$$E = \tfrac{\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5 k}, \text{etc.};$$

ex factoribus vero habebitur

$$\alpha = \tfrac{1}{m}, \quad \beta = \tfrac{-1}{2n-m}, \quad \gamma = \tfrac{1}{2n+m}, \quad \delta = \tfrac{-1}{4n-m}, \quad \beta = \tfrac{1}{4n+m} \text{ etc.}$$

174. Hinc ergo ad normam § 166 sequentes series formabuntur earumque summae assignabuntur

$$P=\tfrac{1}{m}-\tfrac{1}{2n-m}+\tfrac{1}{2n+m}-\tfrac{1}{4n-m}+\tfrac{1}{4n+m}-\text{etc.},$$
$$Q=\tfrac{1}{m^2}+\tfrac{1}{(2n-m)^2}+\tfrac{1}{(2n+m)^2}+\tfrac{1}{(4n-m)^2}+\tfrac{1}{(4n+m)^2}+\text{etc.},$$
$$R=\tfrac{1}{m^3}-\tfrac{1}{(2n-m)^3}+\tfrac{1}{(2n+m)^3}-\tfrac{1}{(4n-m)^3}+\tfrac{1}{(4n+m)^3}-\text{etc.},$$
$$S=\tfrac{1}{m^4}+\tfrac{1}{(2n-m)^4}+\tfrac{1}{(2n+m)^4}+\tfrac{1}{(4n-m)^4}+\tfrac{1}{(4n+m)^4}+\text{etc.},$$
$$T=\tfrac{1}{m^5}-\tfrac{1}{(2n-m)^5}+\tfrac{1}{(2n+m)^5}-\tfrac{1}{(4n-m)^5}+\tfrac{1}{(4n+m)^5}-\text{etc.}$$

Hae autem summae *P*, *Q*, *R*, *S* etc. ita se habebunt

$$P=A=\tfrac{\pi}{2nk} = \tfrac{1\pi}{2nk},$$
$$Q=\frac{(kk+1)\pi\pi}{4nnkk} = \frac{(2+2kk)\pi^2}{2\cdot 4nnk^2},$$
$$R=\frac{(kk+1)\pi^3}{8n^3k^3} = \frac{(6+6kk)\pi^3}{2\cdot 4\cdot 6n^3k^3},$$
$$S=\frac{(k^4+4kk+3)\pi^4}{48n^4k^4} = \frac{(24+32kk+8k^4)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4k^4},$$
$$T=\frac{(2k^4+5kk+3)\pi^5}{96n^5k^5} = \frac{(120+200kk+80k^4)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k^5},$$
$$V=\frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} = \frac{(720+14400kk+816k^4+96k^6)\pi^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12n^6k^6}$$
etc.

175. Series istae generales merentur, ut casus quosdam particulares inde derivemus, qui prodibunt, si rationem *m* ad *n* in numeris determinemus. Sit igitur primum
$m=1$ et $n=2$;
fiet

$$k=\text{tang.}\tfrac{\pi}{4}=\text{tang.}45^0=1$$

atque ambae serierum classes inter se congruent. Erit ergo

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.},$$
$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.},$$
$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.},$$
$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.},$$
$$\frac{\pi^5}{1536} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.},$$
$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.},$$

Harum serierum primam iam supra (§ 140) elicuimus, reliquarum illae, quae pares habent dignitates, modo ante (§ 169) sunt erutae; ceterae, in quibus exponentes sunt numeri impares, hic primum occurrunt. Constat ergo omnium quoque istarum serierum

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+13}} + \frac{1}{9^{2n+1}} - \text{etc.}$$

summas per valorem ipsius π assignari posse.

176. Sit nunc

$$m = 1, \quad n = 3;$$

$$k.\text{tang.}\frac{\pi}{6} = \text{tang.}30^0 = \frac{1}{\sqrt{3}}$$

atque series § 172 abibunt in has

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \text{etc.},$$
$$\frac{\pi\pi}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \text{etc.},$$
$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \text{etc.}$$
etc.

sive

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \text{etc.},$$
$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \text{etc.},$$
$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \text{etc.}$$
etc.

In his seriebus desunt omnes numeri per ternarium divisibiles; hinc pares dimensiones ex iam inventis deducentur hoc modo. Cum sit [§ 167, 168]

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.},$$

erit

$$\frac{\pi\pi}{6 \cdot 9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \text{etc.} = \frac{\pi\pi}{54};$$

quae posterior series continens omnes numeros per ternarium divisibiles si subtrahatur a priori, remanebunt omnes numeri non divisibiles per 3 sicque erit

$$\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.},$$

uti invenimus.

177. Eadem hypothesis

$$m = 1, \quad n = 3 \quad \text{et} \quad k = \frac{1}{\sqrt{3}}$$

ad § 174 accommodata has praebebit summationes

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \text{etc.},$$

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.},$$

$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^2} + \text{etc.}$$

etc.,

in quarum denominatoribus numeri tantum impares occurrunt exceptis iis, qui per ternarium sunt divisibiles. Ceterum pares dimensiones ex iam cognitis deduci possunt; cum enim sit

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.},$$

erit

$$\frac{\pi\pi}{8 \cdot 9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \text{etc.} = \frac{\pi\pi}{72};$$

quae series omnes numeros impares per 3 divisibiles continens si subtrahatur a superiore, relinquet seriem quadratorum numerorum imparium per 3 non divisibilium eritque

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

178. Si series in §§ 172 et 174 inventae vel addantur vel subtrahantur, obtinebuntur aliae series notatu dignae. Erit scilicet

$$\frac{k\pi}{2n}+\frac{\pi}{2nk}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\text{etc.}=\frac{(kk+1)\pi}{2nk};$$

at est

$$k=\text{tang.}\frac{m\pi}{2n}=\frac{\sin.\frac{m\pi}{2n}}{\cos.\frac{m\pi}{2n}}\quad\text{et}\quad 1+kk=\frac{1}{\left(\cos.\frac{m\pi}{2n}\right)^2},$$

unde

$$\frac{2k}{1+kk}=2\sin.\frac{m\pi}{2n}\cos.\frac{m\pi}{2n}=\sin.\frac{m\pi}{n},$$

quo valore substituto habebimus

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}-\text{etc.}$$

Simili modo per subtractionem erit

$$\frac{\pi}{2nk}-\frac{k\pi}{2n}=\frac{(1-kk)\pi}{2nk}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\frac{1}{3n+m}+\text{etc.};$$

at est

$$\frac{2k}{1-kk}=\text{tang.}2\frac{m\pi}{2n}=\text{tang.}\frac{m\pi}{n}=\frac{\sin.\frac{m\pi}{n}}{\cos.\frac{m\pi}{n}}$$

hinc erit

$$\frac{\pi\cos.\frac{m\pi}{n}}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.};$$

Series quadratorum et altiorum potestatum hinc ortae facilius per differentiationem hinc deducentur infra.

179. Quoniam casus, quibus $m=1$ et $n=2$ vel 3, iam evolvimus, ponamus

$$m=1\quad\text{et}\quad n=4;$$

erit

$$\sin.\frac{m\pi}{n}=\sin.\frac{\pi}{4}=\frac{1}{\sqrt{2}}\quad\text{et}\quad\cos.\frac{\pi}{4}=\frac{1}{\sqrt{2}}.$$

Hinc itaque habebitur

$$\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\text{etc.}$$

et

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\text{ etc.}$$

Sit

$$m=1 \text{ et } n=8;$$

erit

$$\frac{m\pi}{n}=\frac{\pi}{8} \text{ et sin.}\frac{\pi}{8}=\sqrt{\left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)} \text{ et cos.}\frac{\pi}{8}=\sqrt{\left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)}$$

et

$$\frac{\cos.\frac{\pi}{8}}{\sin.\frac{\pi}{8}}=1+\sqrt{2}$$

Hinc itaque erit

$$\frac{\pi}{4\sqrt{\left(2-\sqrt{2}\right)}}=1+\frac{1}{7}-\frac{1}{9}-\frac{1}{15}+\frac{1}{17}+\frac{1}{23}-\text{ etc.},$$

$$\frac{\pi}{8\sqrt{\left(2-\sqrt{2}\right)}}=1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\text{ etc.}$$

Sit nunc

$$m=3 \text{ et } n=8;$$

erit

$$\frac{m\pi}{n}=\frac{3\pi}{8} \text{ et sin.}\frac{3\pi}{8}=\sqrt{\left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)} \text{ et cos.}\frac{3\pi}{8}=\sqrt{\left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)},$$

unde

$$\frac{\cos.\frac{3\pi}{8}}{\sin.\frac{3\pi}{8}}=\frac{1}{\sqrt{2}+1}$$

ac prodibunt hae series

$$\frac{\pi}{4\sqrt{\left(2+\sqrt{2}\right)}}=\frac{1}{3}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{19}+\frac{1}{21}-\text{ etc.},$$

$$\frac{\pi}{8\left(\sqrt{2}+1\right)}=\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\text{ etc.}$$

180. Ex his seriebus per combinationem nascuntur

$$\frac{\pi\sqrt{(2+\sqrt{2})}}{4}=1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}-\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}+\tfrac{1}{19}+\text{etc.},$$
$$\frac{\pi\sqrt{(2-\sqrt{2})}}{4}=1-\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}+\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}-\tfrac{1}{19}-\text{etc.},$$
$$\frac{\pi\left(\sqrt{(4+2\sqrt{2})}+\sqrt{2}-1\right)}{8}=1+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}+\tfrac{1}{19}-\text{etc.},$$
$$\frac{\pi\left(\sqrt{(4+2\sqrt{2})}-\sqrt{2}+1\right)}{8}=1-\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}-\tfrac{1}{19}+\text{etc.},$$
$$\frac{\pi\left(\sqrt{2}+1+\sqrt{(4-2\sqrt{2})}\right)}{8}=1+\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}+\tfrac{1}{19}+\text{etc.},$$
$$\frac{\pi\left(\sqrt{2}+1-\sqrt{(4-2\sqrt{2})}\right)}{8}=1-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}-\tfrac{1}{19}-\text{etc.},$$

Simili modo ponendo $n=16$ et m vel 1 vel 3 vel 5 vel 7 ulterius progredi licet hocque modo summae reperientur serierum $1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\frac{1}{9}$ etc., in quibus signorum + et – vicissitudines alias leges sequantur.

181. Si in seriebus § 178 inventis bini termini in unam summam colligantur, erit

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}+\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}+\frac{2m}{9nn-mm}-\frac{2m}{16n-mm}+\text{etc.}$$

ideoque

$$\frac{1}{nn-mm}-\frac{1}{4nn-mm}+\frac{1}{9nn-mm}-\text{etc.}=\frac{\pi}{2mn\sin.\frac{m\pi}{n}}-\frac{1}{2mm}$$

Altera vero series dabit

$$\frac{\pi}{n\tan.\frac{m\pi}{n}}=\frac{1}{m}-\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}-\frac{2m}{9nn-mm}-\text{etc.}$$

hincque

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\text{etc.}=\frac{1}{2mn}-\frac{\pi}{2mn\tan.\frac{m\pi}{n}}.$$

Ex his autem coniunctis nascitur haec

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}=\frac{\pi\tan.\frac{m\pi}{2n}}{4mn}.$$

Si in hac serie sit $n=1$ et m numerus par quicunque $=2k$, ob $\tang.k\pi=0$ erit semper, nisi sit $k=0$,

$$\frac{1}{1-4kk}+\frac{1}{9-4kk}+\frac{1}{25-4kk}+\frac{1}{49-4kk}+\text{etc.}=0;$$

sin autem in illa serie fiat $n=2$ et m fuerit numerus quicunque impar

$=2k+1$, ob $\frac{1}{\tan.\frac{m\pi}{n}}=0$ erit

$$\frac{1}{4-(2k+1)^2}+\frac{1}{16-(2k+1)^2}+\frac{1}{36-(2k+1)^2}+\text{etc.}=\frac{1}{2(2k+1)^2}$$

182. Multiplicentur series inventae per nn sitque $\frac{m}{n}=p$; habebuntur istae formae

$$\frac{1}{1-pp}-\frac{1}{4-pp}+\frac{1}{9-pp}-\frac{1}{16-pp}+\text{ etc.}=\frac{\pi}{2p\sin.px}-\frac{1}{2pp},$$
$$\frac{1}{1-pp}+\frac{1}{4-pp}+\frac{1}{9-pp}+\frac{1}{16-pp}+\text{ etc.}=\frac{1}{2pp}-\frac{\pi}{2p\text{tang}.px}.$$

Sit $pp=a$ atque nascentur hae series

$$\frac{1}{1-a}-\frac{1}{4-a}+\frac{1}{9-a}-\frac{1}{16-a}+\text{ etc.}=\frac{\pi\sqrt{a}}{2a\sin.\pi\sqrt{a}}-\frac{1}{2a},$$
$$\frac{1}{1-a}+\frac{1}{4-a}+\frac{1}{9-a}+\frac{1}{16-a}+\text{ etc.}=\frac{1}{2a}-\frac{\pi\sqrt{a}}{2a\text{ tang}.\pi\sqrt{a}}.$$

Dummodo ergo a non fuerit numerus negativus nec quadratus integer, summa harum serierum per circulum exhiberi poterit.

183. Per reductionem autem exponentialium imaginariorum ad sinus et cosinus arcuum circularium supra traditam poterimus quoque summas harum serierum assignare, si a sit numerus negativus. Cum enim sit

$$e^{x\sqrt{-1}}=\cos.x+\sqrt{-1}\cdot\sin.x \text{ et } e^{-x\sqrt{-1}}=\cos.x-\sqrt{-1}\cdot\sin.x,$$

erit vicissim posito $y\sqrt{-1}$ loco x

$$\cos.y\sqrt{-1}=\frac{e^{-y}+e^{y}}{2} \text{ et } \sin.y\sqrt{-1}=\frac{e^{-y}-e^{y}}{2\sqrt{-1}}.$$

Quodsi ergo $a=-b$ et $y=\pi\sqrt{b}$, erit

$$\cos.\pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}}+e^{\pi\sqrt{b}}}{2} \quad \text{et} \quad \sin.y\sqrt{-b} = \frac{e^{-\pi\sqrt{b}}-e^{\pi\sqrt{b}}}{2\sqrt{-1}}$$

ideoque

$$\text{tang.}\pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}}-e^{\pi\sqrt{b}}}{\left(e^{-\pi\sqrt{b}}+e^{\pi\sqrt{b}}\right)\sqrt{-1}}$$

Hinc erit

$$\frac{\pi\sqrt{-b}}{\sin.\pi\sqrt{-b}} = \frac{-2\pi\sqrt{b}}{e^{-\pi\sqrt{b}}-e^{\pi\sqrt{b}}} \quad \text{et} \quad \frac{\pi\sqrt{-b}}{\text{tang.}\pi\sqrt{-b}} = \frac{-\left(e^{-\pi\sqrt{b}}+e^{\pi\sqrt{b}}\right)\pi\sqrt{b}}{\left(e^{-\pi\sqrt{b}}-e^{\pi\sqrt{b}}\right)}$$

His ergo notatis erit

$$\frac{1}{1+b}-\frac{1}{4+b}+\frac{1}{9+b}-\frac{1}{16+b}+\text{etc.} = \frac{1}{2b}-\frac{\pi\sqrt{b}}{\left(e^{\pi\sqrt{b}}-e^{-\pi\sqrt{b}}\right)b},$$

$$\frac{1}{1+b}+\frac{1}{4+b}+\frac{1}{9+b}+\frac{1}{16+b}+\text{etc.} = \frac{\left(e^{\pi\sqrt{b}}+e^{-\pi\sqrt{b}}\right)\pi\sqrt{b}}{2b\left(e^{\pi\sqrt{b}}-e^{-\pi\sqrt{b}}\right)}-\frac{1}{2b}.$$

Eaedem hae series deduci possunt ex § 162 adhibendo eandem methodum, qua in hoc capite sum usus. Quoniam vero hoc pacto reductio sinuum et cosinuum arcuum imaginariorum ad quantitates exponentiales reales non mediocriter illustratur, hanc explicationem alteri praeferendam duxi.