SUPPLEMENT IX .

TO SECTION I, BOOK II.

Concerning the Resolution of Differential Equations of the Second Order, Involving only Two Variables.

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1). A method of resolving singular differential equations of the second order.

M. S. presented to the Academy on the 19th. *Jan.* 1779.

[E677]

§. 1. If *p* and *q* were some functions of x, and this equation was proposed between the two variables x and z :

$$2p\partial z + z\partial p = \frac{\partial x}{q}\int\frac{z\partial x}{q},$$

it is evident its integral can be found easily, if that may be multiplied by *z*, so that there may be had :

$$2pz\partial z + zz\partial p = \frac{z\partial x}{q}\int\frac{z\partial x}{q}.$$

For the integral of the first part is *pzz*, truly of the latter part on putting $\int\frac{z\partial x}{q} = v$, it will change into $v\partial v$, of which the integral is $\frac{1}{2}vv + \mathrm{C}$, thus so that hence we may obtain the same integral equation $pzz = \frac{1}{2}vv + \mathrm{C}$, from which there becomes $vv = 2pzz - \mathrm{C}$, and hence

$$v = \int\frac{z\partial x}{q} = \sqrt{(2pzz - \mathrm{C})},$$

which differentiated gives :

$$\frac{z\partial x}{q} = \frac{2pz\partial z + zz\partial p}{\sqrt{(2pzz - \mathrm{C})}},$$

therefore with a division made by *z,* there will become

$$\frac{\partial x}{q} = \frac{2p\partial z + z\partial p}{\sqrt{(2pzz - \mathrm{C})}},$$

but it may not hence be apparent how the value of *z* may be expressed through *x* and its functions p and q . Moreover, so that we may obtain this goal, on putting as we have done $\int\frac{z\partial x}{q} = v$ so that there shall become $vv = 2pzz - \mathrm{C}$, we may retain the quantity

v in the calculation, and since there shall be $\frac{z\partial x}{q}=\partial v$, there will become $z=\frac{q\partial v}{\partial x}$, with which value substituted we will have

$$vv=\frac{2pqq\partial v^2}{\partial x^2}-\mathrm{C},$$

from which there is deduced:

$$\partial v=\frac{\partial x\sqrt{[vv+\mathrm{C}]}}{q\sqrt{2p}},$$

which allows a separation at once, since there shall be

$$\frac{\partial v}{\sqrt{[vv+\mathrm{C}]}}=\frac{\partial x}{q\sqrt{2p}}, \text{ and thus}$$
$$\int\frac{\partial v}{\sqrt{[vv+\mathrm{C}]}}=\int\frac{\partial x}{q\sqrt{2p}},$$

the value of which can be considered as known, since p and q are functions of x.

§. 2. Therefore we may put this integral in place :

$$\int\frac{\partial x}{q\sqrt{2p}}=l\mathrm{X},$$

so that we may have :

$$\int\frac{\partial v}{\sqrt{[vv+\mathrm{C}]}}=l\mathrm{X},$$

whereby since there may be agreed to be

$$\int\frac{\partial v}{\sqrt{[vv+\mathrm{C}]}}=l\left[v+\sqrt{(vv+\mathrm{C})}\right],$$

there will be

$$v+\sqrt{(vv+\mathrm{C})}=\mathrm{X},$$

from which there is deduced $v=\frac{\mathrm{X}^2-\mathrm{C}}{2\mathrm{X}}$, and thus v is defined by the quantity X .

§. 3. Therefore since above we have found $2pzz=vv+\mathrm{C}$,

$$2pzz=\frac{\left(\mathrm{X}^2-\mathrm{C}\right)^2}{4\mathrm{XX}}+\mathrm{C}=\frac{(\mathrm{XX}+\mathrm{C})^2}{4\mathrm{XX}},$$

and consequently there will be

$$z\sqrt{2p}=\frac{\mathrm{XX}+\mathrm{C}}{2\mathrm{X}},$$

and thus the quantity z is thus expressed by X, so that there shall be

$$z = \frac{X^2 + C}{2X\sqrt{2p}},$$

where it is required to be remembered,

$$lX = \int \frac{\partial x}{q\sqrt{2p}}, \text{ or } X = e^{\int \frac{\partial x}{q\sqrt{2p}}}.$$

§. 4. Moreover it is evident, our proposed equation, if it may be freed from the sign of integration, will change into a differential equation of the second order, the complete integral of which we have just elicited. Indeed with a multiplication made by *q,* it becomes

$$2pq\partial z + qz\partial p = \partial x \int \frac{z\partial x}{q},$$

and with the differential taken with the element ∂x constant, the following differential equation of the second order will be given :

$$\left.\begin{array}{r} 2pq\partial\partial z + 2p\partial q\partial z + z\partial q\partial p \\ +3q\partial p\partial z + qz\partial\partial p \end{array}\right\} = \frac{z\partial x^2}{q},$$

therefore of which equation, somewhat abstruse, we have found the complete integral to be

$$z = \frac{X^2 + C}{2X\sqrt{2p}}, \text{ with there being present, } X = e^{\int \frac{\partial x}{q\sqrt{2p}}},$$

thus so that this same quantity X also may involve an arbitrary constant.

§.5. But since this equation is rather complicated, it can be represented more concisely in the following manner ; since there shall be

$$\frac{q}{z}\partial . pzz = \partial x \int \frac{z\partial x}{q},$$

the differentiated equation being indicated by $\partial . \frac{q\partial . pzz}{z} = \frac{z\partial x^2}{q}$ only, from which evidently an integral quantity emerges, if it may be multiplied by $\frac{2q\partial . pzz}{z}$, as indeed we may put $\frac{q\partial . pzz}{z} = s\partial x$ for brevity, the left hand part shall be

$$2s\partial x\partial \cdot s\partial x = 2s\partial s\partial x^2,$$

and therefore its integral becomes $ss\partial x^2$: but truly on the right hand side we will have $2\partial x^2 \partial .pzz$, of which therefore the integral is :

$$2pzz\partial x^2 + C\partial x^2,$$

thus so that the integration produces for us $ss = 2pzz + C$.

§.6. Now so that we may set out this equation thoroughly, we may put as before $pzz = v$, thus so that there shall be $\frac{q\partial v}{z} = s\partial x$, and our integral found

$$ss = \frac{qq\partial v^2}{zz\partial x^2} = 2v + C,$$

which on account of $zz = \frac{v}{p}$ will change into this :

$$\frac{pqq\partial v^2}{v\partial x^2} = 2v + C,$$

from which there is elicited almost as before

$$\frac{\partial v}{\sqrt{v[2v+C]}} = \frac{\partial x}{q\sqrt{2p}},$$

which does not differ from the form found before.

§.7. In a similar manner there are also other more complicated differential equations, just as if this equation may be proposed

$$3p\partial z + z\partial p = \frac{\partial x}{q}\int \frac{zz\partial x}{q},$$

where again p and q denote some functions of x. Indeed since there shall be

$$3p\partial z + z\partial p = \frac{\partial .pz^3}{zz},$$

there will be on multiplying [the above equation] by zz :

$$\partial .pz^3 = \frac{zz\partial x}{q}\int \frac{zz\partial x}{q},$$

which on putting $\int \frac{zz\partial x}{q} = v$ will change into $\partial .pz^3 = v\partial v,$ and thus on integrating $2pz^3 = vv + C.$

§. 8. But since we have put $\int \frac{zz\partial x}{q} = v$, there will be $zz = \frac{q\partial v}{\partial x}$, and hence $z^3 = \frac{q\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}}$, from which there shall be

$$\frac{2pq\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}} = vv + \mathrm{C}.$$

Therefore with the square taken there will be

$$\frac{4ppq^3\partial v^3}{\partial x^3} = \left(vv + \mathrm{C}\right)^2,$$

and thus

$$\frac{\partial v^3}{(vv+\mathrm{C})^2} = \frac{\partial x^3}{4ppq^3},$$

the cube root of which gives :

$$\frac{\partial v}{\sqrt[3]{(vv+\mathrm{C})^2}} = \frac{\partial x}{q\sqrt[3]{4pp}}.$$

Hence therefore the quantity v is defined by x , thus so that now we may be able to regard v as a true function of x, from which there will be found :

$$z^3 = \left[\frac{q\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}}\right] = \frac{vv+\mathrm{C}}{2p}, \text{ and hence } z = \sqrt[3]{\frac{vv+\mathrm{C}}{2p}}.$$

§. 9. This same equation $\left[i.e.\ \partial.pz^3 = \frac{zz\partial x}{q}\int\frac{zz\partial x}{q},\right]$ besides will be able to be resolved in another way, since it may be represented multiplied by q thus:

$$\frac{q\partial.pz^3}{zz} = \partial x\int\frac{zz\partial x}{q}, \text{ or } \partial.\frac{q\partial.pz^3}{zz} = \frac{zz\partial x^2}{q},$$

which evidently is rendered integrable, by multiplying by $\frac{2q\partial.pz^3}{zz}$, for it gives rise to :

$$\left(\frac{q\partial.pz^3}{zz}\right)^2 = 2pz^3\partial x^2 + \mathrm{C}\partial x^2.$$

§. 10. Now there may be put $pz^3 = v$, thus so that there shall be $z^3 = \frac{v}{p}$, and $z^4 = \frac{v}{p}\sqrt[3]{\frac{v}{p}}$, with which value substituted we will have

$$\frac{ppq\partial v^2.\sqrt[3]{p}}{v\sqrt[3]{v}} = 2v\partial x^2 + \mathrm{C}\partial x^2.$$

from which it is concluded :

$$\frac{\partial v^2}{v(2v+\mathrm{C})\sqrt[3]{v}} = \frac{\partial x^2}{pqq\sqrt[3]{p}},$$

or

$$\frac{\partial v}{\sqrt{v(2v+C)\sqrt[3]{v}}}=\frac{\partial v}{v^{\frac{2}{3}}\sqrt{(2v+C)}}=\frac{\partial x}{q\sqrt[3]{pp}},$$

this equation arises simpler, on putting $v=u^3$, evidently

$$\frac{3\partial u}{\sqrt{\left(2u^3+C\right)}}=\frac{\partial x}{q\sqrt[3]{pp}}.$$

Hence it is understood, innumerable examples can be set out by these formulas.

§. 11. There is no reason, also, why much more general equations of this kind cannot be treated; and since a more general equation can be represented thus :

$$\frac{\partial .pz^m}{z^n}=\frac{\partial x}{q}\int\frac{z^n\partial x}{q},$$

which expanded out gives :

$$mpz^{m-n-1}\partial z+z^{m-n}\partial p=\frac{\partial x}{q}\int\frac{z^n\partial x}{q}.$$

But with the multiplication performed by z^n, this equation at once integral will be produced:

$$\partial .pz^m=\frac{z^n\partial x}{q}\int\frac{z^n\partial x}{q},$$

if indeed there is produced:

$$2pz^m=\left(\int\frac{z^n\partial x}{q}\right)^2+C.$$

§.12. Developing this equation further, we may put in place

$$\int\frac{z^n\partial x}{q}=v,\ \text{and there will be}\ z^n=\frac{q\partial v}{\partial x},$$

from which in the first place $2pz^m=vv+C$, and hence again :

$$(2p)^{\frac{n}{m}}.z^n=(2p)^{\frac{n}{m}}.\frac{q\partial v}{\partial x}=(vv+C)^{\frac{n}{m}}$$

which since it shall be separable at once, will give

$$\frac{\partial v}{(vv+C)^{\frac{n}{m}}} = \frac{\partial x}{q(2p)^{\frac{n}{m}}},$$

from which therefore the quantity v will be determined by x, with which found the quantity sought z will be expressed thus, so that there shall be $z^m = \frac{vv+C}{2p}$.

§. 13. We may illustrate this with by a single example demanded from the first case, by assuming surely $p = 1 + xx$ et $q = \sqrt{2}$, thus so that the proposed equation shall be

$$2\partial z(1+xx) + 2zx\partial x = \tfrac{\partial x}{2}\int z\partial x,$$

which is changed into this second order equation

$$4\partial\partial z(1+xx) + 12x\partial x\partial z + 3z\partial x^2 = 0,$$

or which therefore the integral is sought.

§. 14. Therefore we may make an application of the solution found above in §.3., where since here there shall be $p = 1 + xx$ and $q = \sqrt{2}$, there will become

$$l\mathrm{X} = \tfrac{1}{2}\int \frac{\partial x}{\sqrt{(1+xx)}} = \tfrac{1}{2} l\left[x + \sqrt{(1+xx)}\right] - \tfrac{1}{2} la,$$

from which there shall become

$$\mathrm{X} = \frac{\sqrt{\left[x+\sqrt{(1+xx)}\right]}}{\sqrt{a}},$$

therefore with this value substituted we will have

$$z = \frac{a\mathrm{C}+x+\sqrt{(1+xx)}}{2\sqrt{2a(1+xx)}\left[x+\sqrt{(1+xx)}\right]},$$

which is expressed more simply in this manner :

$$z = \frac{\left[a\mathrm{C}+x+\sqrt{(1+xx)}\right]\left[-x+\sqrt{(1+xx)}\right]}{2\sqrt{2a(1+xx)}}.$$

Therefore where two arbitrary constants are involved, and thus this complete integral will be determined algebraically. Therefore on putting $\mathrm{C} = 0$, the particular integral demanded from the first form will be

$$z=\frac{\sqrt{\left[x+\sqrt{(1+xx)}\right]}}{2\sqrt{2a(1+xx)}}.$$

§.15. Another particular integral hence can be shown, by assuming the constants thus, so that $a\mathrm{C}$ shall be indefinite, but truly $\mathrm{C}\sqrt{a}$ shall be finite, $=b$, since then there will become

$$z=\frac{a\mathrm{C}}{2\sqrt{2a(1+xx)}\left[x+\sqrt{(1+xx)}\right]}=\frac{b}{2\sqrt{2(1+xx)}\left[x+\sqrt{(1+xx)}\right]},$$

which form is reduced to this

$$z=\frac{a\left[-x+\sqrt{(1+xx)}\right]}{\sqrt{(1+xx)}}$$

2.) A new method of investigating all the cases, by which this differentio-differential equation $\partial\partial y(1-axx)-bx\partial x\partial y-cy\partial x^2=0$ may be resolved.

M. S. shown to the Academy on the 13 *January,* 1780.

[E678]

§. 16. Indeed here a method may be called into use, set out by myself and others occasionally, in which the value of y is expressed by an infinite series. Then indeed in all the cases, in which this series is interrupted somewhere, a particular integral of the proposed equation will be had ; from which indeed the complete integral will be able to be elicited without difficulty. Truly just as in this manner an endless number of integrable cases are found; yet not all become known, but an infinitude of other cases are given, which can be resolved. On account of which truly I propose this singular method, with the help of which clearly all the integral cases will be able to be elicited. But this method must be prepared, so that for any known case permitting a solution, innumerable others may be able to be deduced from that .

§. 17. Moreover at once two of the most simple cases are provided, for which the resolution succeeds, of which the one is, if $c=0$, and truly the other, if $b=a$, which two principal cases it will be necessary to set out before everything.

The first principal case in which $c=0$.

§. 18. Therefore in this case our equation will be

$$\partial\partial y(1-axx)=bx\partial x\partial y,$$

which on putting $\partial y=p\partial x$, will change into this

$$\partial p(1-axx)=bpx\partial x\text{, or}$$

$$\tfrac{\partial p}{p}=\tfrac{bx\partial x}{1-axx}$$

of which the integral is

$$lp=-\tfrac{b}{2a}l(1-axx)+l\mathrm{C},$$

and thus there will become:

$$p=\mathrm{C}(1-axx)^{-\frac{b}{2a}}=\tfrac{\partial y}{\partial x},$$

from which there is obtained:

$$y=\mathrm{C}\int\partial x(1-axx)^{-\frac{b}{2a}}:$$

where it will be pleasing to note this same value becomes algebraic whenever $-\frac{b}{2a}$ becomes a positive whole number, or $b=-2ia$ with i denoting some positive integer. Then truly the value of the integral also will become algebraic, when $-\frac{b}{2a}$ becomes either $-\frac{3}{2}$, $-\frac{5}{2}$, $-\frac{7}{2}$, etc. and thus in general $\frac{b}{2a}=2i+1$, where there cannot be $i=0$.

The other principal case in which $b=a$.

§. 19. Therefore in this other case our equation multiplied by $2\partial y$ will become

$$2\partial y\partial\partial y(1-axx)-2ax\partial x\partial y^2-2cy\partial y\partial x^2=0,$$

which is integrable at once, indeed its integral will be

$$\partial y^2(1-axx)-cyy\partial x^2=\mathrm{C}\partial x^2.$$

Therefore from this equation there will be

$$\partial y\sqrt{(1-axx)}=\partial x\sqrt{(\mathrm{C}+cyy)},$$

therefore with a separation made there will be

$$\frac{\partial x}{\sqrt{(1-axx)}}=\frac{\partial y}{\sqrt{(\mathrm{C}+cyy)}}.$$

Therefore again algebraic cases will be contained in this form, towards eliciting these we may make $a=-\alpha\alpha,\ c=\gamma\gamma$ and $\mathrm{C}=\beta\beta$; so that we may have

$$\frac{\partial x}{\sqrt{(1+\alpha\alpha xx)}}=\frac{\partial y}{\sqrt{(\beta\beta+\gamma\gamma yy)}},$$

of which the integral is

$$\tfrac{1}{\alpha}l\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]=\tfrac{1}{\gamma}l\left[\gamma y+\sqrt{(\beta\beta+\gamma\gamma yy)}\right]-\tfrac{1}{\gamma}l\Delta,$$

from which with increasing numbers there will be

$$\gamma y+\sqrt{(\beta\beta+\gamma\gamma yy)}=\Delta\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]^{\frac{\gamma}{\alpha}}.$$

Therefore with V put in place for this latter expression, there will be

$$\mathrm{V}-\gamma y=\sqrt{(\beta\beta+\gamma\gamma yy)},$$

and with squares taken, $y=\frac{\mathrm{VV}-\beta\beta}{2\gamma\mathrm{V}}$. Therefore since there shall be

$$\mathrm{V}=\Delta\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]^{\frac{\gamma}{\alpha}},$$ there will become

$$2\gamma y=\Delta\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]^{\frac{\gamma}{\alpha}}-\frac{\beta\beta}{\Delta}\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]^{-\frac{\gamma}{\alpha}},$$

where $\beta\beta=\mathrm{C}$, truly the exponent [recall $a=-\alpha\alpha,\ c=\gamma\gamma$ and $\mathrm{C}=\beta\beta$] $\frac{\gamma}{\alpha}=\sqrt{-\frac{c}{a}}$, thus whenever $\sqrt{-\frac{c}{a}}$ should be a rational number, the integral always will be rational.

§. 20. From these two principal cases set out I will examine the way of transforming the proposed equation into an infinitude of others of the same kind, thus so that an equation of this form may always be produced :

$$\partial\partial\mathrm{Y}(1-axx)-\mathrm{B}x\partial x\partial\mathrm{Y}-\mathrm{CY}\partial x^2=0,$$

which since the resolution may be admitted either for the cases $\mathrm{C}=0$ or $\mathrm{B}=a$,also this same equation proposed will be resolved for the same cases. Therefore I am now going to establish these twofold transformations.

Transformations of the former order.

§. 21. I put $y=\frac{\partial v}{\partial x}$ from which on account of

$$\partial y=\frac{\partial\partial v}{\partial x} \text{ and } \partial\partial y=\frac{\partial^3 v}{\partial x},$$

our equation will adopt this form

$$\partial^3 v(1-axx)-bx\partial x\partial\partial v-c\partial x^2\partial v=0,$$

the individual terms of which admit integration: indeed there will become :

$$\int\partial x^2\partial v=v\partial x^2,$$

$$\int x\partial x\partial\partial v=x\partial x\partial v-v\partial x^2,$$

$$\int\partial^3 v(1-axx)=\partial\partial v(1-axx)+2ax\partial x\partial v-2av\partial x^2.$$

$$\left[\begin{array}{l} i.e.\int\partial^3 v(1-axx)=\partial\partial v-a\int\partial^3 vxx=\partial\partial v-a\partial\partial vxx+2a\int\partial\partial vx\partial x \\ =\partial\partial v-a\partial\partial vxx+2a\partial vx\partial x-2a\int\partial v\partial x\partial x=\partial\partial v-a\partial\partial vxx+2ax\partial v\partial x-2av\partial x^2. \end{array}\right]$$

With these part gathered together, our equation will become

$$\partial\partial v(1-axx)-(b-2a)x\partial x\partial v-(c-b+2a)v\partial x^2=0,$$

which certainly since it will be similar to the proposed, will be integrable in these two cases $c-b+2a=0$ and $b=3a$, or as often as there were either $c=b-2a$ or $b=3a$, and with the integration put in place for each case, thus so that *v* may be expressed by *x*, then for the proposed equation itself there will be $y=\frac{\partial v}{\partial x}$; from which it is apparent, if the integrals found for *v* were algebraic, also the value of *y* to become algebraic.

§. 22. Because if further in a similar manner we may put $v=\frac{\partial v'}{\partial x}$, since by the preceding operation the letters *b* and *c* will be changed into $b-2a$ and $c-b+2a$, now this equation will arise

$$\partial\partial v'(1-axx)-(b-4a)x\partial x\partial v'-(c-2b+6a)v'\partial x^2=0,$$

which therefore will be integrable, if there were either $b=5a$ or $c=2b-6a$. And with the value found for *v*' there will become $y=\frac{\partial\partial v'}{\partial x^2}$, evidently the second differentials of v' will give *y*: and thus, if an algebraic value were produced for v', also an algebraic value will be produced for y.

§. 23. But if we may repeat the same substitution anew by putting $v'=\frac{\partial v''}{\partial x}$, for the initial letters *b* and *c* we will have now $b-6a$ and $c-3b+12a$, and the resulting equation will be

$$\partial\partial v''(1-axx)-(b-6a)x\partial x\partial v''-(c-3b+12a)v''\partial x^2=0,$$

which therefore admits a resolution, whenever there were either $b=7a$ or $c=3b-12a$, in which cases also therefore that same equation proposed by necessity shall admit a resolution, since there shall be $y=\frac{\partial^3 v''}{\partial x^3}$.

§. 24. So that if therefore we may repeat these same operations continually, to these equations perpetually, we will arrive at the same forms ; where it will suffice to be noting both the values, which we will have obtained for the letters *b* and *c* in any operation, which we may show in the following table together with the values of *y* :

	b	c	y
Operation I.	$b-2a$	$c-b+2a$	$\frac{\partial v}{\partial x}$
II.	$b-4a$	$c-2b+6a$	$\frac{\partial v'}{\partial x^2}$
III.	$b-6a$	$c-3b+12a$	$\frac{\partial v''}{\partial x^3}$
IV.	$b-8a$	$c-4b+20a$	$\frac{\partial v'''}{\partial x^4}$
–	– – –	– – – – – –	– – –
–	– – –	– – – – – –	– – –
–	– – –	– – – – – –	– – –
i	$b-2ia$	$c-ib+i(i+1)a$	$\frac{\partial^i v^{(i-1)}}{\partial x^i}$

§. 25. Hence therefore it is apparent in general, the proposed equation always can be resolved, whenever there were either $b=2ia+a$, or $c=ib-i(i+1)a$, where all the positive integers *i* are allowed to be taken, thus so that hence we may obtain two orders of innumerable cases, of which the latter will be found only by the method of series indicated initially, truly prior to this method they shall have been completely inaccessible.

Transformations of the latter order.

§. 26. Just as here we have progressed by differentiation, now we may go back by integration, and indeed in the first place we may put $y=\int z\partial x$,and the proposed equation will emerge

$$\partial z(1-axx)-bxz\partial x-c\partial x\int z\partial x=0\,,$$

which differentiated is reduced to the proposed form

$$\partial\partial z(1-axx)-(b+2a)x\partial x\partial z-(c+b)z\partial x^2=0,$$

therefore which will be allowed to be integrated following the integration principles, for the cases $c+b=0$ and $b+2a=a$, or $c=-b$ and $b=-a$.
Therefore with the integrals found there will become $y=\int z\partial x$, from which it is apparent that even if these integrals were algebraic, yet the values of y become transcending.

§: 27. In a similar manner we may put again $z=\int z'\partial x$, and since by the preceding operation in place of *b* and *c* we are able to put $b+2a$ and $c+b$, now we will come to this equation

$$\partial\partial z'(1-axx)-(b+4a)x\partial x\partial z'-(c+2b+2a)z'\partial x^2=0,$$

which therefore will be allowed to be integrated, if there were either $c+2b+2a=0$, or $b+4a=a$; or $c=-2b-2a$ and $b=-3a$. Moreover with the integrals hence found for *y*, we will have $y=\int\partial x\int z'\partial x$, which thus is reduced to the simple integral sign, so that there shall become

$$y=x\int z'\partial x-\int z'x\partial x.$$

§. 28. Again in a similar manner we may put $z'=\int z''\partial x$, and now we will be led to this equation

$$\partial\partial z''(1-axx)-(b+6a)x\partial x\partial z''-(c+3b+6a)z''\partial x^2=0,$$

which therefore will be integrable, if there were either $c+3b+6a=0$, or $b+6a=a$, that is, if there is $c=-3b-6a$ and $b=-5a$; and from these integrals there will become

$y=\int\partial x\int\partial x\int z''\partial x$, which value can be reduced from the preceding, if this may be integrated on multiplying anew by ∂x and in place of z' there may be written z'', indeed there is obtained :

$$y=\tfrac{1}{2}xx\int z''\partial x-x\int xz''\partial x+\tfrac{1}{2}\int xxz''\partial x.$$

§. 29. So that if now we may continue these operations further, the whole matter will be returned to this, so that the formulas which are going to be produced in place of b and c, may be duly formed, and likewise the values of y may be assigned, just as the following table will indicate:

	b	c	y
Operation I.	$b+2a$	$c+b$	$\int z\partial x$
II.	$b+4a$	$c+2b+2a$	$\int\partial x\int z'\partial x$
III.	$b+6a$	$c+3b+6a$	$\int\partial x\int\partial x\int z''\partial x$
IV.	$b+8a$	$c+4b+12a$	$\int\partial x\int\partial x\int\partial x\int z'''\partial x$
–	–––	––––––	–––
–	–––	––––––	–––
–	–––	––––––	–––
i	$b+2ia$	$c+ib+i(i-1)a$	$\int\partial x\int\partial x....\int z^{[i-1]}\partial x$

§. 30. It is evident enough from the preceding, how these complicated integrals may be able to be reduced to simple ones, so that we will add only the following table:

$$\int\partial x\int z'\partial x = x\int z'\partial x - \int z'x\partial x$$

$$\int\partial x\int\partial x\int z''\partial x = \tfrac{1}{2}\left(xx\int z''\partial x - 2x\int z''x\partial x + \int z''xx\partial x\right)$$

$$\int\partial x\int\partial x\int\partial x\int z'''\partial x = \tfrac{1}{6}\begin{Bmatrix} x^3\int z'''\partial x - 3xx\int z'''x\partial x \\ +3x\int z'''xx\partial x - \int z'''x^3\partial x \end{Bmatrix}$$

$$\int\partial x\int\partial x\int\partial x\int\partial x\int z^{\mathrm{IV}}\partial x = \tfrac{1}{24}\begin{Bmatrix} x^4\int z^{\mathrm{IV}}\partial x - 4x^3\int z^{\mathrm{IV}}x\partial x \\ +6xx\int z^{\mathrm{IV}}xx\partial x - 4x\int z^{\mathrm{IV}}x^3\partial x \\ +\int z^{\mathrm{IV}}x^4\partial x \end{Bmatrix}.$$

etc. etc.

§. 31. But if now we may continue these operations according to some indefinite number i , and in place of *b, c, z,* we may write B, C, Z, the resulting equation will become

$$\partial\partial\mathrm{Z}(1-axx) - \mathrm{B}x\partial x\partial\mathrm{Z} - \mathrm{CZ}\partial x^2 = 0,$$

where there will be, as we have now indicated,

$$\mathrm{B} = b + 2ia \ \text{ and } \ \mathrm{C} = c + ib + i(i-1)a;\,;$$

on account of which this equation will be admitted, as often as there were $\mathrm{C}=0$ that is $c = -ib - i(i-i)a$, or $\mathrm{B} = a$ that is $b = -(2i-1)a$; which formulas only differ from these which we have found above for the first order of transformations, in that here the letter i has taken a negative value; from which there may be added the following .....

General Conclusion.

§. 32. Here if the letter i may denote all the whole numbers either positive or negative, the proposed differential of the differential equation

$$\partial\partial y(1-axx) - \mathrm{b}x\partial x\partial y - cy\partial x^2 = 0$$

will admit to be integrated or resolved, whenever there should become

$$1^{st}.)\ 0 = ib - i(i+1)a,$$

or

$$2^{nd}.)\ b = (2i+1)a.$$

whereby it will be permitted to assert, plainly all the resolvable cases to be contained in these twofold formulas, so that evidently no case of integration may be shown allowing integration which may not be considered to be present in one of other of these two formulas, while the other opposing the method by the preceding series, of which we have made mention initially, has shown only the former integrable cases, thus so that thence an infinite number of equally resolvable cases may be excluded.

Corollary 1.

§. 33. The proposed equation may be transformed into a differential of the first order by putting $y = e^{\int u\partial x}$, and we may arrive at this equation,

$$\partial u + uu\partial u - \tfrac{bux\partial x - c\partial x}{1-axx} = 0,$$

which therefore also will be allowed to be integrated for the cases in which either $b = (2i+1)a$ or $c = ib - i(i+1)a$, with i denoting some whole number either positive or negative.

Corollary 2.

§. 34. So that if again there may be put $u = (1-axx)^n v$, by putting for the sake of brevity $n = -\tfrac{b}{2a}$, there will be come upon that equation of the kind referred to as *Riccatian* :

$$(1-axx)^n \partial v + (1-axx)^{2n} vv\partial x = \tfrac{c\partial x}{1-axx},$$

which divided by $(1-axx)^n$ will change into

$$\partial v + (1-axx)^n vv\partial x = \tfrac{c\partial x}{(1-axx)^{n+1}},$$

which therefore will be allowed to be integrated for the same cases.

Corollary 3.

§. 35. So that if we may put $a = 0$ this equation will arise:

$$\partial u + uu\partial u = bux\partial x + c\partial x,$$

which therefore will be integrable, if there were either $b = 0$ or $c = ib$, of which indeed the first case by itself is evident, since then there will be $\partial x = \frac{\partial u}{c-uu}$.
But this form will be able to be expressed more conveniently, by putting $u = \frac{1}{2}bx + v$, from which $\partial v + vv\partial v = \left(c - \frac{1}{2}b\right)\partial x + \frac{1}{4}bbxx\partial x,$
or on putting $b = 2f$, so that there may become

$$\partial v + vv\partial v = \left(c - f\right)\partial x + ffxx\partial x,$$

and this equation will be integrable, as often as there were $c = 2if$, thus so that the following equation will be integrable always :

$$\partial v + vv\partial v = \left(2i - 1\right) f\partial x + ffxx\partial x,$$

whether a positive or negative integer may be taken for i ; that is, if in the penultimate term f may be multiplied by some odd number either positive or negative, which cases therefore become more abstruse, so that the number i may be taken greater ; and thus scarcely another way may be seen to be allowed of evaluating the integrals, unless we may return to the differential equation of the second order proposed, and we may put in place the same operations which we have taught above. Yet meanwhile it is to be observed, all these cases are able also to be derived at once from that equation by continued fractions. If indeed this equation were proposed :

$$\partial v + vv\partial v = g\partial x + ffxx\partial x,$$

its value v can be expressed by a continued fraction in a twofold way.

Indeed in the first way there becomes :

$$v = fx + \cfrac{g-f}{2fx + \cfrac{g-3f}{2fx + \cfrac{g-5f}{2fx + \text{etc.}}}}$$

Truly in the other way there becomes :

$$v = -fx - \cfrac{(g-f)}{2fx + \cfrac{g+3f}{2fx + \cfrac{g+7f}{2fx + \text{etc.}}}}$$

of which the former is interrupted, whenever there were $g = (2i+1)f$, truly the latter, whenever $g = -(2i+1)f$, which are the two integrable cases found before.

[It does not seem possible to obtain the continued fraction expansion directly, but only from a series expansion of the solution of the d.e. See E71, paragraph 34. The Euler Archive indicates an English translation of this work , by B. F. Wyman and M. F. Wyman ; see also E95.]

3.) Concerned with implicated formulas for integrals, and with the expansion and transformation of these .

M. S. shown to the Academy, on the 20^{th} *of April, 1778 .*

[E679: a form of continued integration]

§. 36. The general form of such implicated formulas can be shown

$$\int p\partial x \int q\partial x \int r\partial x \int s\partial x \text{ etc.}$$

where everything following some integral sign are themselves included [to be integrated again]. Thus so that the value of this expression is required to be found starting from the last term, and on putting the integral $\int s\partial x = \mathrm{S}$ there will be

$$\int r\partial x \int s\partial x = \int \mathrm{S} r\partial x,$$

the value of which if it may be put $= \mathrm{R}$, will become

$$\int q\partial x \int r\partial x \int s\partial x = \int \mathrm{R} q\partial x,$$

which integral if it may be put $=\mathrm{Q}$, the value of this formula proposed will be $=\int \mathrm{Q}p\partial x$, where accordingly it is understood, in any integration the usual customary arbitrary constant can be introduced into the calculation.

§. 37. Here clearly it is required to be understood properly, this same expression $\int p\partial x\int q\partial x$ does not signify the product from the formula $\int p\partial x$ into the formula $\int q\partial x$, but the integral which arises , if the whole formula of the differential $p\partial x\int q\partial x$ may be integrated : for in truth if we wish to designate the product of two such integral formulas , that is accustomed to be done with the placing between of a point in this manner $\int p\partial x\cdot\int q\partial x$,where clearly the point declares the preceding integral sign must not be extended beyond this term, that this form

$$\int p\partial x\int q\partial x\cdot\int r\partial x\int s\partial x \text{ etc.}$$

expresses the product, which arises if the formula $\int p\partial x\int q\partial x$ may be multiplied by $\int r\partial x\int s\partial x.$

§. 38. Therefore here the manner of designation has received a completely different use by us, and in differential formulas it is usual to observe, where the expression of such $\partial x\partial y\partial z$ may denote the product of three differentials ∂x, ∂y and ∂z, thus so that the individual signs only may be associated with the letters following immediately: but if we wish for argument's sake to express the differential of this expression $x\partial y\partial z$, this with the interposition of the point is accustomed to become $\partial .x\partial y\partial z$, where the point indicates, the prefix ∂ to include the whole subsequent expression.

§. 39. But such implicated integral formulas arise mainly from the continued integration of liner integral equations, the form of which in general is :

$$pz+\frac{q\partial z}{\partial x}+\frac{r\partial\partial z}{\partial x^2}+\frac{s\partial^3 z}{\partial x^3}+\text{ etc.}=\mathrm{X},$$

where the letters *p, q, r, s,* etc. are given functions of the variable *x,* of which also the letter X shall be some function, truly the other variable *z* shall be only of one dimension, just as this general form shows here, referring to the third order of differentials, and thus demand three integrations, the same number as the arbitrary constants it is agreed to involve, here clearly with respect to the natural maximum number for the method of integration, which by three successive integrations would produce the desired integral.

§. 40. Evidently with such a proposed equation it will be required to know before all else the multiplier, by which that may be rendered integrable, which therefore we may suppose to be $=\partial \mathrm{P}$, and with the integration carried through this equation may be produced :

$$p'z+\frac{q'\partial z}{\partial x}+\frac{r'\partial\partial z}{\partial x^2}=\int \mathrm{X}\partial \mathrm{P},$$

which equation now is of the second order ; because if now we may put a suitable multiplier of this to be $=\partial \mathrm{P}'$, with the integration made this equation of the first order may arise, which shall be

$$p''z+\frac{q''\partial z}{\partial x}=\int \partial \mathrm{P}'\int \mathrm{X}\partial \mathrm{P},$$

from which if $\partial \mathrm{P}''$ were a suitable multiplier, the complete integral may adopt this form

$$p'''z=\int \partial \mathrm{P}''\int \partial \mathrm{P}'\int \mathrm{X}\partial \mathrm{P}.$$

And thus the quantity *z* may be expressed by the implicate integral formula.

§. 41. But with such a form for the integral found a particular difficulty is redressed here, in order that this may be set out thus, so that the formula containing the indefinite function X, which here has three prefixed integration signs, may not have a single one more before itself ; on account of which such a reduction may be able to be put in place most conveniently, to be shown here to be put in place, if indeed unless certain artifices may be used, a calculation of this kind may demand an especially troublesome operation.

§. 42. Moreover thus we may represent in general implicate formulas of this kind

$$\int \partial p\int \partial q\int \partial r\int \partial s\int \partial t \ \text{ etc.},$$

for the establishment of which we may begin with the case of two integration signs, and because there will be $\int \partial p\int \partial q=\int q\partial p$, a common reduction gives $pq-\int p\partial q$. Now again in place of *p* and *q* we may write again $\int \partial p$ and $\int \partial q$, and the expansion will be had thus :

$$\int \partial p\int \partial q=\int \partial p\cdot\int \partial q-\int \partial q\cdot\int \partial p,$$

where in general it will be pleasing to note this equality:

$$\int \partial p\int \partial q-\int \partial p\cdot\int \partial q+\int \partial q\int \partial p=0.$$

§. 43. Now we will consider the formula involving three integral signs $=\int\partial p\int\partial q\int\partial r$, and because as we have just seen there is $\int\partial q\int\partial r = qr - \int q\partial r$, our formula is split up into these parts $\int qr\partial p - \int\partial p\int q\partial r$, which latter part is reduced to this form $p\int q\partial r - \int pq\partial r$, and thus our formula will become :

$$\int qr\partial p - p\int q\partial r + \int pq\partial r.$$

Because now it is required, so that the element ∂r in the individual parts may have only a single prefixed integral sign ; we may put $q\partial p = \partial v$ so that there shall become

$$v = \int q\partial p = \int\partial p\int\partial q,$$

and there will become

$$\int qr\partial p = \int r\partial v = rv - \int v\partial r,$$

and hence there is deduced

$$\int pq\partial r - \int v\partial r = \int\partial r\left(pq - v\right) = \int\partial r\int p\partial q.$$

Now in place of the finite letters differential may again be introduced, and the value of the formula sought $\int\partial p\int\partial q\int\partial r$ will be expressed in the following manner:

$$\int\partial p\int\partial q.\int\partial r - \int\partial p.\int\partial r\int\partial q + \int\partial r\int\partial q\int\partial p\,,$$

where in the individual members the element ∂r has a single integral sign prefixed.

§. 44. Therefore between the three elements ∂p, ∂q and ∂r there is a need for the following relation to be noted

$$\int\partial p\int\partial q\int\partial r - \int\partial p\int\partial q\cdot\int\partial r + \int\partial p\cdot\int\partial r\int\partial q - \int\partial r\int\partial q\int\partial p = 0,$$

so that if moreover we may wish to pursue a similar reduction for the cases of more integral signs, we may fall into the most troublesome and tedious calculations ; yet meanwhile all this difficulty may be easily and most clearly established by the following theorem, and because it may be agreed the individual members may be resolved into two factors with the aid of points, when such a factor is lacking, we will supply its place by a one.

Theorem 1.

§. 45. For the single element ∂p this relation is obtained, quite clearly, $\int\partial p.1-1.\int\partial p=0$.

Theorem 2.

§. 46. Between the two elements ∂p and ∂q this relation will be had :

$$\int\partial p\int\partial q.1-\int\partial p.\int\partial q+1.\int\partial p\int\partial q=0.$$

Demonstration.

Towards demonstrating this, it will suffice to show the differential of this equation to be $=0$, truly because the individual members are constructed from two factors, the differentials arising separately from the former and latter factors may be considered, therefore here the differential arising from the former factors $\partial p\left(\int\partial q.1-1.\int\partial q\right)=0$, by theorem 1. But from the latter factors the differential will arise

$$-\partial q\left(\int\partial p.1-1.\int\partial p\right)=0.$$

Theorem 3.

§. 47. This relation always will be found between the three elements ∂p, ∂q and ∂r :

$$\int\partial p\int\partial q\int\partial r.1-\int\partial p\int\partial q.\int\partial r+\int\partial p.\int\partial r\int\partial q-1.\int\partial r\int\partial q\int\partial p=0.$$

Demonstration.

Here again the difference may be considered separately arising both from the former as well as from the latter factor ; moreover from the first there becomes

$$\partial p\left(\int\partial q\int\partial r\cdot 1-\int\partial q\cdot\int\partial r+1.\int\partial r\int\partial q\right),$$

the value of which evidently is reduced to zero by theorem 2, clearly if the letters p and q may be moved up by one order; then truly the differential arising from the latter factors will be

$$-\partial r\left(\int\partial p\int\partial q\cdot 1-\int\partial p\cdot\int\partial q+1.\int\partial q\int\partial p\right),$$

the value of which vanishes equally by the preceding theorem; therefore since both differentials are $=0$, also the form must be equal to zero or to a constant, but it is clear the constant itself to be involved with the integral signs.

Theorem 4.

§. 48. This relation always will be found between the four elements $\partial p,\ \partial q,\ \partial r$ and ∂s

$$\left.\begin{array}{l}\int\partial p\int\partial q\int\partial r\int\partial s\cdot 1-\int\partial p\int\partial q\int\partial r\cdot\int\partial s\\ +\int\partial p\int\partial q\cdot\int\partial s\int\partial r-\int\partial p\cdot\int\partial s\int\partial r\int\partial q\\ +1\cdot\int\partial s\int\partial r\int\partial q\int\partial p\end{array}\right\}=0.$$

Demonstration.

Differentiation of the first factor supplies the following expression

$$\partial p\left(\int\partial q\int\partial r\int\partial s\cdot 1-\int\partial q\int\partial r\cdot\int\partial s+\int\partial q\cdot\int\partial r\int\partial s-1\cdot\int\partial q\int\partial r\int\partial s\right),$$

which on account of the preceding theorem is reduced to zero. In a similar manner differentiation of the last factor presents this expression

$$-\partial s\left(\int\partial p\int\partial q\int\partial r\cdot 1-\int\partial p\int\partial q\cdot\int\partial r+\int\partial p\cdot\int\partial r\int\partial q-1\cdot\int\partial r\int\partial q\int\partial p\right),$$

which on account of theorem 3, is again $=0$.

Theorem 5.

§. 49. This relation always will be found between the five elements
$\partial p,\ \partial q,\ \partial r,\ \partial s$ and ∂t

$$\left.\begin{array}{l}\int\partial p\int\partial q\int\partial r\int\partial s\int\partial t\cdot 1-\int\partial p\int\partial q\int\partial r\int\partial s\cdot\int\partial t\\ +\int\partial p\int\partial q\int\partial r\cdot\int\partial t\int\partial s-\int\partial p\int\partial q\cdot\int\partial t\int\partial s\int\partial r\\ +\int\partial p\cdot\int\partial t\int\partial s\int\partial r\int\partial q-1\cdot\int\partial t\int\partial s\int\partial r\int\partial q\int\partial p\end{array}\right\}=0.$$

Demonstration.

The demonstration of this and of the preceding theorem has been done in exactly the same manner; and thus now most clearly it has prevailed, such relations always may be agreed to be true, even from however elements they were composed.

§. 50. So that the strength of this theorem may be seen more clearly, it will be worth the effort, that it may be illustrated by examples being determined ; therefore we may put :

$$\partial p = x^{\alpha-1}\partial x,\ \partial q = x^{\beta-1}\partial x,\ \partial r = x^{\gamma-1}\partial x,$$
$$\partial s = x^{\theta-1}\partial x,\ \partial t = x^{\varepsilon-1}\partial x,$$

and from the first theorem the identity equation arises at once :

$$\frac{x^{\alpha}}{\alpha}-\frac{x^{\alpha}}{\alpha}=0.$$

Truly the second theorem give us this equation :

$$\frac{x^{\alpha+\beta}}{\beta(\alpha+\beta)}-\frac{x^{\alpha+\beta}}{\alpha\beta}+\frac{x^{\alpha+\beta}}{\alpha(\alpha+\beta)}=0,$$

[Thus, in the first integration, the argument $x^{\beta-1}$ initially is integrated, giving $\frac{x^{\beta}}{\beta}$, which is then integrated again with the new argument $\frac{x^{\alpha+\beta-1}}{\beta}$ to give the term $\frac{x^{\alpha+\beta}}{\beta(\alpha+\beta)}$, etc.] from which on dividing by $x^{\alpha+\beta}$ this equality is produced :

$$\frac{1}{\beta(\alpha+\beta)}-\frac{1}{\alpha\beta}+\frac{1}{\alpha(\alpha+\beta)}=0,$$

the truth of which can be seen at once.

§. 51. Again these positions introduced in the third theorem produce this equation:

$$\frac{x^{\alpha+\beta+\gamma}}{\gamma(\alpha+\beta+\gamma)(\beta+\gamma)}-\frac{x^{\alpha+\beta+\gamma}}{\beta\gamma(\alpha+\beta)}+\frac{x^{\alpha+\beta+\gamma}}{\alpha\beta(\beta+\gamma)}-\frac{x^{\alpha+\beta+\gamma}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)},$$

from which on dividing by $x^{\alpha+\beta+\gamma}$ this exceptional equality is produced :

$$\frac{1}{\gamma(\alpha+\beta+\gamma)(\beta+\gamma)}-\frac{1}{\beta\gamma(\alpha+\beta)}+\frac{1}{\alpha\beta(\beta+\gamma)}-\frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)}=0,$$

§. 52. These positions again substituted into the fourth theorem give this equation

$$\left.\begin{array}{l}\frac{x^{\alpha+\beta+\gamma+\delta}}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\frac{x^{\alpha+\beta+\gamma+\delta}}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)}\\ +\frac{x^{\alpha+\beta+\gamma+\delta}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)}-\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}\\ +\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)}\end{array}\right\}=0,$$

which divided by $x^{\alpha+\beta+\gamma+\delta}$ produce this equation

$$\left.\begin{array}{l}\frac{1}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\frac{1}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)}\\ \quad+\frac{1}{\beta\gamma(\beta+\alpha)(\gamma+\delta)}-\frac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}\\ \quad+\frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)}\end{array}\right\}=0.$$

§. 53. Finally the same positions may be substituted into the fifth theorem to produce this equation

$$\left.\begin{array}{l}\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)}\\ -\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}+\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)}\\ -\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}+\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)}\\ -\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}\end{array}\right\}=0,$$

which divided by $x^{\alpha+\beta+\gamma+\delta+\varepsilon}$ give this most noteworthy equation :

$$\left.\begin{array}{l}\frac{1}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)}\\ -\frac{1}{\varepsilon\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}+\frac{1}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)}\\ -\frac{1}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}+\frac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)}\\ -\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}\end{array}\right\}=0.$$

§. 64. These therefore are the most memorable properties of this theorem, because the truth of these certainly can be investigated numerically in several ways, and thus may merit much greater attention, which another similar theorem, to which recently I have been led, certainly is able to show the demonstration of this without difficulty, which thus itself may be obtained as follows.

A Numerical Theorem.

With any four numbers taken as it pleases just as the four $\alpha,\beta,\gamma,\delta,$ so hence if just as many others may be formed in the following way

$$a=\alpha,\ b=\alpha+\beta,\ c=\alpha+\beta+\gamma \text{ and } d=\alpha+\beta+\gamma+\delta,$$

and in a similar manner also these

$$\mathrm{D}=\delta,\ \mathrm{C}=\delta+\gamma$$
$$\mathrm{B}=\delta+\gamma+\beta \text{ and } \mathrm{A}=\delta+\gamma+\beta+\alpha,$$

then there will be always:

$$\frac{1}{abcd}-\frac{1}{abc\mathrm{D}}+\frac{1}{ab\mathrm{CD}}-\frac{1}{a\mathrm{BCD}}+\frac{1}{\mathrm{ABCD}}=0.$$

Demonstration.

§. 65. The two first fractions found, on account of $\mathrm{D}-d=-c,$ give the fraction $-\frac{1}{abc\mathrm{D}}$, which jointly with the third produce $\frac{1}{ab\mathrm{CD}}$, to which the fourth fraction added gives $-\frac{1}{a\mathrm{BCD}}$, which [since $d=\mathrm{A}$] is cancelled completely by the last term .

§. 56. With the aid of the above theorem all the implicated formulas, to which the integration of linear differential equations is accustomed to lead, will be able to be resolved easily. Moreover generally it gives rise to such forms:

$$\mathrm{Z}=\int\partial q\int \mathrm{X}\partial p, \mathrm{Z}=\int\partial r\int\partial q\int \mathrm{X}\partial p,$$
$$\mathrm{Z}=\int\partial s\int\partial r\int\partial q\int \mathrm{X}\partial p, \mathrm{Z}=\int\partial t\int\partial s\int\partial r\int\partial q\int \mathrm{X}\partial p \text{ etc.}$$

where the letters p, q, r, s, t,. etc. are given functions of X, but X actually is some function of x; and thus here the whole resolution must be put in place, so that with the individual terms included the indefinite function X may be prefixed by a single integral sign: therefore this can be put in place easily with the aid of the above theorem, but only if in place of the element ∂p we may write $\mathrm{X}\partial p$, with which observed the individual reductions will be obtained in the following manner.

I. Resolution of the integral formula

$$\int\partial q\int \mathrm{X}\partial p .$$

§. 57. If we may write $\mathrm{X}\partial p$ in place of ∂p , the theorem following §.46. supplies us with this equation:

$$\int \mathrm{X}\partial p\int\partial q-\int \mathrm{X}\partial p\cdot\int\partial q+\int\partial q\int \mathrm{X}\partial p=0,$$

of which the last term is our form Z requiring to be reduced, consequently the resolution gives at once

$$Z=\int\partial q\cdot\int X\partial p-\int X\partial p\int\partial q,$$

and thus on account of $\int\partial q=q$ we will have:

$$Z=q\int X\partial p-\int Xq\partial p.$$

Corollary.

§. 68. If there were $q=p$, there will become :

$$Z=p\int X\partial p-\int Xp\partial p.$$

II. Resolution of the implicated formula

$$\int\partial r\int\partial q\int X\partial p\,.$$

§. 59. For this case we may assume theorems 3, §. 47, from which, if in place of ∂p we may write $X\partial p$, we deduce this equation :

$$\int X\partial p\int\partial q\int\partial r-\int X\partial p\int\partial q\cdot\int\partial r+\int X\partial p\cdot\int\partial r\int\partial q-\int\partial r\int\partial q\int X\partial p=0,$$

the last term of which is that form Z required to be reduced, and hence indeed there may be deduced :

$$Z=\int\partial r\int\partial q\cdot\int X\partial p-\int\partial r\cdot\int X\partial p\int\partial q+\int X\partial p\int\partial q\int\partial r,$$

which therefore reduced gives :

$$Z=\int q\partial r\int X\partial p-r\int Xq\partial p+\int X\partial p\int r\partial q.$$

Corollary.

§. 60. Therefore if here there were $q=r=p$, this resolution will be produced :

$$Z=\int\partial p\int\partial p\int X\partial p=\tfrac{1}{2}pp\int X\partial p-p\int Xp\partial p+\tfrac{1}{2}\int Xpp\partial p.$$

III. Resolution of this implicated formula

$$\int \partial s \int \partial r \int \partial q \int \mathrm{X} \partial p.$$

§. 61. For this case we may assume the theorem 4, §. 48, from which if in place of ∂p we may write $\mathrm{X}\partial p$, we deduce this equation :

$$\left.\begin{array}{l}\int \mathrm{X}\partial p \int \partial q \int \partial r \int \partial s - \int \mathrm{X}\partial p \int \partial q \int \partial r \cdot \int \partial s + \int \mathrm{X}\partial p \int \partial q \cdot \int \partial s \int \partial r \\ -\int \mathrm{X}\partial p . \int \partial s \int \partial r \int \partial q + \int \partial s \int \partial r \int \partial q \int \mathrm{X}\partial p .\end{array}\right\} = 0,$$

the final term of which is our formula Z itself being required to be reduced ;
and hence thus we gather :

$$\mathrm{Z} = \left\{\begin{array}{l}\int \partial s \int \partial r \int \partial q \cdot \int \mathrm{X}\partial p - \int \partial s \int \partial r \cdot \int \mathrm{X}\partial p \int \partial q + \int \partial s \cdot \int \mathrm{X}\partial p \int \partial q \int \partial r \\ -\int \mathrm{X}\partial p \int \partial q \int \partial r \int \partial s ,\end{array}\right.$$

which reduced therefore provides :

$$\mathrm{Z} = \int \partial s \int q\partial r \cdot \int \mathrm{X}\partial p - \int r\partial s \cdot \int \mathrm{X}q\partial p + s\int \mathrm{X}\partial p \cdot \int r\partial q - \int \mathrm{X}\partial p \int \partial q \int s\partial r .$$

Corollary.

§. 62. If there may be put $s = r = q = p$, then this equation will be resolved

$$\mathrm{Z} = \left\{\begin{array}{l}\frac{1}{6} p^3 \int \mathrm{X}\partial p - \frac{1}{2} pp \int \mathrm{X}p\partial p + \frac{1}{2} p \int \mathrm{X}pp\partial p \\ \qquad -\frac{1}{6} \int \mathrm{X}p^3 \partial p .\end{array}\right.$$

IV. Resolution of this implicated formula.

$$\mathrm{Z} = \int \partial t \int \partial s \int \partial r \int \partial q \int \mathrm{X}\partial p .$$

§: 63. For this case we may assume theorem 5, §. 49, from which if in place of ∂p there may be written $\mathrm{X}\partial p$, this equation will be produced :

$$\left.\begin{array}{l}\int \mathrm{X}\partial p \int \partial q \int \partial r \int \partial s \int \partial t - \int \mathrm{X}\partial p \int \partial q \int \partial r \int \partial s \cdot \int \partial t \\ +\int \mathrm{X}\partial p \int \partial q \int \partial r \cdot \int \partial t \int \partial s - \int \mathrm{X}\partial p \int \partial q \cdot \int \partial t \int \partial s \int \partial r \\ +\int \mathrm{X}\partial p \cdot \int \partial t \int \partial s \int \partial r \int \partial q - \int \partial t \int \partial s \int \partial r \int \partial q \int \mathrm{X}\partial p\end{array}\right\} = 0,$$

the last term of which is our form *Z* requiring to be reduced, from which therefore there is produced :

$$Z=\begin{cases}\int\partial t\int\partial s\int\partial r\int\partial q\cdot\int X\partial p-\int\partial t\int\partial s\int\partial r\cdot\int X\partial p\int\partial q\\+\int\partial t\int\partial s\cdot\int X\partial p\int\partial q\int\partial r-\int\partial t\cdot\int X\partial p\int\partial q\int\partial r\int\partial s\\+\int X\partial p\cdot\int\partial q\int\partial r\int\partial s\int\partial t,\end{cases}$$

which reduced therefore provides

$$\_Z=\begin{cases}\int\partial t\int\partial s\int q\partial r\cdot\int X\partial p-\int\partial t\int r\partial s\cdot\int Xq\partial p\\+\int s\partial t\cdot\int X\partial p\int r\partial q-t\int X\partial p\int\partial q\int s\partial r\\+\int X\partial p\cdot\int\partial q\int\partial r\int t\partial s.\end{cases}$$

Corollary.

§. 64. Here if we may assume $t=s=r=q=p$, then this resolution will be produced :

$$Z=\begin{cases}\frac{1}{24}p^4\int X\partial p-\frac{1}{6}p^3\int Xp\partial p+\frac{1}{4}pp\int Xpp\partial p\\-\frac{1}{6}p\int Xp^3\partial p+\frac{1}{24}\int Xp^4\partial p.\end{cases}$$

§. 65. So that the natures of these resolutions may be seen more clearly, because the letters *p, q, r, s, t,* may denote given functions of *x*, and thus all the expressions formed from those are to be regarded equally to be known, for the sake of brevity we may put

$$\partial p\int\partial q=\partial p';\ \partial p\int\partial q\int\partial r=\partial p'';\ \partial p\int\partial q\int\partial r\int\partial s=\partial p''';\ \partial p\int\partial q\int\partial r\int\partial s\int\partial t=\partial p'''';\text{etc.}$$

and in this manner the final resolution may be referred to thus :

$$Z=\int\partial t\int\partial s\int\partial r\int\partial q\cdot\int X\partial p-\int\partial t\int\partial s\int\partial r\cdot\int X\partial p'$$
$$+\int\partial t\int\partial s\cdot\int X\partial p''-\int\partial t\cdot\int X\partial p'''+\int X\partial p''''.$$

Because if here again we may put in place :

$$\int\partial t\int\partial s=\int s\partial t=t';\ \int\partial t\int\partial s\int\partial r=t'';\ \int\partial t\int\partial s\int\partial r\int\partial q=t''';$$

the whole resolution will be represented concisely in this manner :

$$Z = t'''\int X\partial p - t''\int X\partial p' + t'\int X\partial p'' - t\int X\partial p''' + \int X\partial p'''',$$

which representation it will be a pleasure also to be applied to the preceding resolutions.

§. 66: Therefore since the integration of the implicated formula

$$\int \partial t \int \partial s \int \partial r \int \partial q \int X\partial p$$

may be reduced to the integration of the following simple integral formulas :

$$\int X\partial p;\ \int X\partial p';\ \int X\partial p'';\ \int X\partial p''';\ \int X\partial p'''';$$

hence a somewhat unusual question arises : in what way may the quantities *q, r, s* and *t* be inferred in turn from these simple formulas, which will be established easily in the following manner. Since there shall be $\partial p' = \partial p \int \partial q$, there will become $\int \partial q = q = \frac{\partial p'}{\partial p}$. Now again there may be put $\frac{\partial p''}{\partial p} = q';\ \frac{\partial p'''}{\partial p} = q'';\ \frac{\partial p''''}{\partial p} = q''';$ etc.; from which values introduced we will have :

$$q' = \int \partial q \int \partial r;\ q'' = \int \partial q \int \partial r \int \partial s;\ q''' = \int \partial q \int \partial r \int \partial s \int \partial t;\ \text{etc.}$$

Therefore because these values $q,\ q',\ q'',\ q''',\ q''''$ have been given , from the first we gather at once $\int \partial r = \frac{\partial q'}{\partial q} = r$. But again we may put $\frac{\partial q''}{\partial q} = r';\ \frac{\partial q'''}{\partial q} = r'';$ etc. and also there will be these values given $r,\ r',\ r'',$ etc., with which substituted there will be had $r' = \int \partial r \int \partial s;\ r'' = \int \partial r \int \partial s \int \partial t;$ from which the first follows : $\int \partial s = s = \frac{\partial r'}{\partial r}$. Whereby if there may become $s' = \frac{\partial r''}{\partial r}$, also there will become $s' = \int \partial s \int \partial t$, and hence $\int \partial t = t = \frac{\partial s'}{\partial s}$. From which it is clearly understood, how these formulas may be able to be found for still more complicated cases.

§. 67. There remains, that also we may add a little from the transformation of such implicated integral formulas, so that the whole matter is able be included in the following problem.

Problem.

§. 68. *For the proposed implicated formula with three sign summation involving* $\int \partial p \int \partial q \int \partial r$ *, to investigate another similar formula*

$$\int \partial \mathrm{P} \int \partial \mathrm{Q} \int \partial \mathrm{R},$$

equal to that.

Solution.

By theorem 2 above, the proposed formula advanced has been resolved thus:

$$\int \partial q \int \partial r = \int \partial q \cdot \int \partial r - \int \partial r \int \partial q = q \int \partial r - \int q \partial r.$$

[Thus, on successive integration by parts, we can write
$\int \partial q \int \partial r = \int r \partial q = qr - \int q \partial r = \int \partial q \cdot \int \partial r - \int \partial r \int \partial q.$ etc.]

In a similar manner for the formula sought there will be

$$\int \partial \mathrm{Q} \int \partial \mathrm{R} = \mathrm{Q} \int \partial \mathrm{R} - \int \mathrm{Q} \partial \mathrm{R}.$$

therefore it is required that there shall become [on multiplying the sides of each equation by ∂p and $\partial \mathrm{P}$]

$$q \partial p \int \partial r - \partial p \int q \partial r = \mathrm{Q} \partial \mathrm{P} \int \partial \mathrm{R} - \partial \mathrm{P} \int \mathrm{Q} \partial \mathrm{R},$$

which equality shall be fulfilled, by assuming $\mathrm{P} = p$, $\mathrm{Q} = q$ and $\mathrm{R} = r$;
truly with the terms interchange we may put

$$\mathrm{Q} \partial \mathrm{P} \int \partial \mathrm{R} = -\partial p \int q \partial r \text{ and } \partial \mathrm{P} \int \mathrm{Q} \partial \mathrm{R} = -q \partial p \int \partial r,$$

and from the first equation we deduce $\mathrm{Q} \partial \mathrm{P} = -\partial p$, and thus $\partial \mathrm{P} = \frac{\partial p}{\mathrm{Q}}$, then truly $\partial \mathrm{R} = q \partial r$; in truth from the other equation we will have

$$\partial \mathrm{P} = -q \partial p \text{ and } \mathrm{Q} \partial \mathrm{R} = \partial r.$$

Therefore since there shall be $\partial \mathrm{P} = -\frac{\partial p}{\mathrm{Q}}$, there will be $\mathrm{Q} = \frac{1}{q}$, and hence again $\partial \mathrm{R} = q\partial r$, from which on account of $\mathrm{Q} = \frac{1}{q}$, there will become $\partial \mathrm{Q} = \frac{-\partial q}{qq}$. Consequently the proposed formula of the integral sought $\int \partial p \int \partial q \int \partial r$ will be equal to

$$\int q\partial p \int \frac{\partial q}{qq} \int q\partial r,$$

from which it is apparent, in place of the formula $\int \partial p \int \partial q \int \partial r$, to be possible to write this always : $\int q\partial p \int \frac{\partial q}{qq} \int q\partial r$.

Corollary I.

§. 69. Therefore when a number of integral signs themselves should be involved in turn, such as if we may have $\int \partial p \int \partial q \int \partial r \int \partial s$, this transformation in any three signs mutually following each other will be able to be put in place, from which in this formula proposed a twofold transformation will be able to be used ; clearly in the first place from the first three signs there will be produced

$$\int q\partial p \int \frac{\partial q}{qq} \int q\partial r \int \partial s,$$

but truly from the latter three this transformation used will give :

$$\int \partial p \int r\partial q \int \frac{\partial r}{rr} \int r\partial s,$$

Corollary 2.

§. 70. Hence again with the aid of the same transformation others in addition will be able to be made, just as from the final form :

$$\int \partial p \int r\partial q \int \frac{\partial r}{rr} \int r\partial s,$$

so that with the three first signs the problem may be able to be solved, if in place of $r\partial q$ we may write ∂v ,so that we may have

$$\int \partial p \int \partial v \int \frac{\partial r}{rr} \int r\partial s,$$

which may be transformed into this:

$$\int v\partial p \int \frac{\partial v}{vv} \int \frac{v\partial r}{rr} \int r\partial s,$$

all which formulas proposed are completely equal.

§. 71. So that we may illustrate this matter by an example, we may assume to be $p = x^{\alpha}$; $q = x^{\beta}$; $r = x^{\gamma}$, thus so that the proposed formula shall become :

$$\alpha\beta\gamma\int x^{\alpha-1}\partial x\int x^{\beta-1}\partial x\int x^{\gamma-1}\partial x = \frac{\alpha\beta x^{\gamma+\beta+\alpha}}{(\gamma+\beta)(\gamma+\beta+\alpha)}.$$

Now for the transformation there will be initially :

$\int q\partial r = \frac{\gamma x^{\beta+\gamma}}{\beta+\gamma}$, and thus on account of $\frac{\partial q}{qq} = \frac{\beta\partial x}{x^{\beta+1}}$, there will be

$$\int\frac{\partial q}{qq}\int q\partial r = \frac{\beta x^{\gamma}}{\beta+\gamma},$$

which multiplied by $q\partial p$ and integrated produces

$$\frac{\alpha\beta x^{\alpha+\beta+\gamma}}{(\beta+\gamma)(\alpha+\beta+\gamma)}.$$

Therefore it is apparent this transformation to be of the widest extent, and able to be applied therefore to all the implicated formulas in many diverse ways, where several integration signs may be involved in turn.

§. 72. It is by no means on this account that I am going to judge the treatment of the above resolutions applied to the summation of series of reciprocal powers, which will happen if in place of X we may assume the fraction $\frac{x}{1-x}$, then truly for the individual elements $\partial p,\ \partial q,\ \partial r,\ \partial s$, we may write $\frac{\partial x}{x}$, from which the corollaries adjoined below will be able to be called into use ; where clearly there will be $p = lx$.

§. 73. Since there shall be the infinite series

$$\mathrm{X} = x + xx + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.}$$

there will be

$$\int \mathrm{X}\partial p = \int\frac{\mathrm{X}\partial x}{x} = x + \tfrac{1}{2}xx + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \tfrac{1}{6}x^6 + \text{etc.}$$

which series is agreed to express the logarithm of the fraction $\frac{x}{1-x}$, since there is

$$\int\frac{\mathrm{X}\partial x}{x} = -l(1-x) = l\frac{1}{1-x}.$$

§. 74. This series may be multiplied again by $\frac{\partial x}{x}$ and integrated, and there will be produced :

$$\int\frac{\partial x}{x}\int\frac{X\partial x}{x}=x+\frac{1}{4}xx+\frac{1}{9}x^3+\frac{1}{16}x^4+\frac{1}{25}x^5+\text{etc.}$$

but truly the resolution of this integral formula given above in §.57, provides :

$$\int\frac{\partial x}{x}\int\frac{X\partial x}{x}=lx\int\frac{\partial x}{1-x}-\int\frac{\partial xlx}{1-x},$$

indeed which integrals thus taken are substituted, so that they may vanish on putting $x=0$; but here it may be observed especially, in the case where there is taken $x=1$, on account of $l1=0$, the sum of this series

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}$$

becomes $-\int\frac{\partial xlx}{1-x}$, the value of which at one time I was the first to find to be $\frac{\pi\pi}{6}$.

§. 75. We may multiply the above series again by $\frac{\partial x}{x}$ and by integrating we will obtain :

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^3}xx+\frac{1}{3^3}x^3+\frac{1}{4^3}x^4+\frac{1}{5^3}x^5+\text{etc.}$$

But this implicated formula thus may be resolved by §. 59 :

$$\frac{1}{2}(lx)^2\int\frac{\partial x}{1-x}-lx\int\frac{\partial xlx}{1-x}+\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x}.$$

Therefore in the case where $x=1$, the sum of the series of the reciprocals of the cubes

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}$$

will be $=\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x}$.

§ 76. In a similar manner we may multiply the above series by $\frac{\partial x}{x}$ and integrate ; then there will be produced :

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^4}xx+\frac{1}{3^4}x^3+\frac{1}{4^4}x^4+\frac{1}{5^4}x^5+\text{etc.}$$

But truly this formula implicated by §. 61, is reduced to this form

$$\frac{1}{6}(lx)^3\int\frac{\partial x}{1-x}-\frac{1}{2}(lx)^2\int\frac{\partial xlx}{1-x}+\frac{1}{2}lx\int\frac{\partial x(lx)^2}{1-x}-\frac{1}{6}\int\frac{\partial x(lx)^3}{1-x}.$$

Therefore for the case where $x=1$ the sum of the reciprocals of the biquadratics will be $\int\frac{\partial x(lx)^3}{1-x}$, the value of which at one time I have shown to be $\frac{\pi^4}{90}$.

§. 77. On putting in place again a multiplication by $\frac{\partial x}{x}$ again and with the integration performed, we will have :

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^5}xx+\frac{1}{3^5}x^3+\frac{1}{4^5}x^4+\frac{1}{5^5}x^5+\text{etc.}$$

which implicated formula is reduced by §. 63 to this form :

$$\frac{1}{24}(lx)^4\int\frac{\partial x}{1-x}-\frac{1}{6}(lx)^3\int\frac{\partial xlx}{1-x}+\frac{1}{4}(lx)^2\int\frac{\partial x(lx)^2}{1-x}-\frac{1}{6}lx\int\frac{\partial x(lx)^3}{1-x}+\frac{1}{24}\int\frac{\partial x(lx)^4}{1-x}.$$

Hence therefore in the case $x=1$ the sum of this series of reciprocal fifth powers will become $\frac{1}{24}\int\frac{\partial x(lx)^4}{1-x}$.

§. 78. We may gather together all these series for the case $x=1$, and the sum of these may be expressed in the following manner by a simple integral formula :

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.}\quad=-\int\frac{\partial x}{1-x}=\infty,$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=-\int\frac{\partial xlx}{1-x}=\frac{\pi\pi}{6},$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}=\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x},$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=-\frac{1}{6}\int\frac{\partial x(lx)^3}{1-x}=\frac{\pi^4}{90},$$

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\text{etc.}=\frac{1}{24}\int\frac{\partial x(lx)^4}{1-x},$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=-\frac{1}{120}\int\frac{\partial x(lx)^5}{1-x}=\frac{\pi^6}{945},$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\frac{1}{5^7}+\text{etc.}=\frac{1}{720}\int\frac{\partial x(lx)^6}{1-x},$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=-\frac{1}{5040}\int\frac{\partial x(lx)^7}{1-x}=\frac{\pi^8}{9450},$$

etc. etc.

§.79. Therefore in general the sum of this series

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.}$$

continued to infinity may be expressed thus :

$$\pm\frac{1}{1.2.3......(n-1)}\int\frac{\partial x(lx)^{n-1}}{1-x},$$

where the upper + sign prevails, when the exponent *n* is odd, truly the lower, when it is even. These same summations, now indeed discovered some time ago [see, e.g. E393, E463 & E464], thus here have been observed to bring about, what formerly the celebrated Lorgna was unable to express by much more implicated continued formulas ; since without doubt these same simple integral formulas may be seen to be much preferred.

SUPPLEMENTUM IX .

AD SECT. I, TOM. II.

DE RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM SECUNDI GRADUS, DUAS TANTUM VARIABILES INVOLVENTIUM.

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1). Methodus singularis resolvendi aequationes differentiales secundi gradus.

M. S. Academiae exhib. die 19 *Jan.* 1779.

[E 677]

§. 1. Si *p* et *q* fuerint functiones quaecunque ipsius *x*, atqui proposita fuit haec aequatio inter binas variabiles *x* et *z*

$$2p\partial z + z\partial p = \frac{\partial x}{q}\int\frac{z\partial x}{q},$$

evidens est ejus integrale facile inveniri posse, si ea multiplicetur per *z,* ut habeatur

$$2pz\partial z + zz\partial p = \frac{z\partial x}{q}\int\frac{z\partial x}{q}.$$

Prioris enim membri integrale est, *pzz,* posterius vero membrum posito $\int\frac{z\partial x}{q} = v$ abit in $v\partial v$, cujus integrale est $\frac{1}{2}vv + \mathrm{C}$, ita ut hinc nanciscamur istam aequationem integralem $pzz = \frac{1}{2}(vv + \mathrm{C})$, unde fit $vv = 2pzz - \mathrm{C}$, hincque

$$v = \int\frac{z\partial x}{q} = \sqrt{(2pzz - \mathrm{C})},$$

quae differentiata dat

$$\frac{z\partial x}{q} = \frac{2pz\partial z + zz\partial p}{\sqrt{(2pzz - \mathrm{C})}},$$

facto ergo divisione per *z,* erit

$$\frac{\partial x}{q} = \frac{2p\partial z + z\partial p}{\sqrt{(2pzz - \mathrm{C})}},$$

quemadmodum autem hinc valor ipsius *z* per *x,* ejusque functiones *p* et *q* exprimi queat, non liquet. Ut autem istum scopum obtineamus, posito ut fecimus $\int\frac{z\partial x}{q} = v$ ut sit $vv = 2pzz - \mathrm{C}$, retineamus quantitatem *v* in calculo, et cum sit $\frac{z\partial x}{q} = \partial v$, erit $z = \frac{q\partial v}{\partial x}$, quo valore substitute habebinius

$vv = \frac{2pqq\partial v^2}{\partial x^2} - \mathrm{C}$, unde colligitur

$$\partial v = \frac{\partial x\sqrt{[vv + \mathrm{C}]}}{q\sqrt{2p}},$$

quae sponte separationem admittit, cum sit

$$\frac{\partial v}{\sqrt{[vv + \mathrm{C}]}} = \frac{\partial x}{q\sqrt{2p}}, \text{ ideoque}$$

$$\int\frac{\partial v}{\sqrt{[vv + \mathrm{C}]}} = \int\frac{\partial x}{q\sqrt{2p}},$$

cujus valor, quoniam *p* et *q* sunt functiones ipsius *x,* tanquam cognitus spectari potest.

§. 2. Statuamus ergo hoc integrale

$$\int \frac{\partial x}{q\sqrt{2p}} = lX,$$

ut habeamus

$$\int \frac{\partial v}{\sqrt{[vv+C]}} = lX,$$

quare cum constet esse

$$\int \frac{\partial v}{\sqrt{[vv+C]}} = l\left[v+\sqrt{(vv+C)}\right],$$

erit

$$v+\sqrt{(vv+C)} = X,$$

unde colligitur $v=\frac{X^2-C}{2X}$, ideoque per quantitatem X definitur.

§. 3. Cum igitur supra invenerimus $2pzz = vv + C$,

$$2pzz = \frac{\left(X^2-C\right)^2}{4XX} + C = \frac{(XX+C)^2}{4XX},$$

consequenter erit

$$z\sqrt{2p} = \frac{XX+C}{2X},$$

sicque quantitas *z* ita per X exprimitur, ut sit

$$z = \frac{X^2+C}{2X\sqrt{2p}},$$

ubi meminisse oportet esse

$$lX = \int \frac{\partial x}{q\sqrt{2p}}, \text{ sive } X = e^{\int \frac{\partial x}{q\sqrt{2p}}}.$$

§. 4. Manifestum autem est, equationem nostram propositam, si a signo integrali liberetur, abire in aequationem differentialem secundi gradus, cujus ergo integrale completum modo elicuimus. Facta enim multiplicatione per *q* fiet

$$2pq\partial z + qz\partial p = \partial x\int \frac{z\partial x}{q},$$

et differentiatio sumto elemento ∂x constante praebebit sequentem aequationem differentialem secundi gradus

$$\left.\begin{array}{r} 2pq\partial\partial z + 2p\partial q\partial z + z\partial q\partial p \\ +3q\partial p\partial z + qz\partial\partial p \end{array}\right\} = \frac{z\partial x^2}{q},$$

cujus ergo aequationis non parum abstrusae novimus esse integrale completum

$$z=\frac{X^2+C}{2X\sqrt{2p}},\ \text{existente}\ X=e^{\int\frac{\partial x}{q\sqrt{2p}}},$$

ita ut ista quantitas X etiam constantem arbitrariam involvat.

§.5. Cum autem haec aequatio non parum sit complicata, sequenti modo concinnius repraesentari potest ; cum sit

$$\frac{q}{z}\partial.pzz=\partial x\int\frac{z\partial x}{q},$$

erit differentiationem tantum indicando

$$\partial.\frac{q\partial.pzz}{z}=\frac{z\partial x^2}{q},$$

quae manifesto integrabilis evadit, si multiplicetur per $\frac{2q\partial.pzz}{z}$, quodsi enim brevitatis gratia statuatur $\frac{q\partial.pzz}{z}=s\partial x,$ membrum sinistrum fit

$$2s\partial x\partial\cdot s\partial x=2\partial s\partial x^2,$$

ejusque ergo integrale $ss\partial x^2$: at vero ex parte dextra habebimus $2\partial x^2\partial.pzz$, cujus igitur integrale est

$$2pzz\partial x^2+C\partial x^2,$$

ita ut integratio nobis praebeat $ss=2pzz+C$.

§.6. Quo nunc hanc aequationem penitius evolvamus, statuamus ut ante $pzz=v$, ita ut sit $\frac{q\partial v}{z}=s\partial x$, eritque nostrum integrale inventum $ss=\frac{qq\partial v^2}{zz\partial x^2}=2v+C,$
quae ob $zz=\frac{v}{p}$ abit in hanc

$$\frac{pqq\partial v^2}{v\partial x^2}=2v+C,$$

unde eruitur propemodum ut ante

$$\frac{\partial v}{\sqrt{v[2v+C]}}=\frac{\partial x}{q\sqrt{2p}},$$

quae a forma ante inventa non discrepat.

§.7. Simili modo etiam aliae aequationes differentiales magis complicatae, veluti si proponatur ista aequatio

$$3p\partial z + z\partial p = \frac{\partial x}{q}\int\frac{zz\partial x}{q},,$$

ubi iterum *p* et *q* denotant functiones quascunque ipsius *x*. Cum enim sit

$$3p\partial z + z\partial z = \frac{\partial . pz^3}{zz}$$

erit per *zz* multiplicando

$$\partial . pz^3 = \frac{zz\partial x}{q}\int\frac{zz\partial x}{q},$$

quae posito $\int\frac{zz\partial x}{q} = v$ abit in $\partial . pz^3 = v\partial v,$ ideoque integrando $2pz^3 = vv + \mathrm{C}.$

§. 8. Quoniam autem posuimus $\int\frac{zz\partial x}{q} = v$, erit $zz = \frac{q\partial v}{\partial x}$, hincque $z^3 = \frac{q\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}}$, unde fit

$$\frac{2pq\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}} = vv + \mathrm{C}.$$

Sumtis ergo quadratis erit

$$\frac{4ppq^3\partial v^3}{\partial x^3} = \left(vv + \mathrm{C}\right)^2,$$

ideoque

$$\frac{\partial v^3}{\left(vv+\mathrm{C}\right)^2} = \frac{\partial x^3}{4ppq^3},$$

cujus radix cubica praebet

$$\frac{\partial v}{\sqrt[3]{\left(vv+\mathrm{C}\right)^2}} = \frac{\partial x}{q\sqrt[3]{4pp}}.$$

Hinc igitur quantitas *v* per *x* definitur, ita ut jam *v* spectare queamus tanquam veram functionem ipsius *x,* qua inventa erit

$$z^3 = \frac{vv+\mathrm{C}}{2p}, \text{ hincque } z = \sqrt[3]{\frac{vv+\mathrm{C}}{2p}}.$$

§. 9. Eadem ista aequatio adhuc alio modo resolvi poterit, quandoquidem per *q* multiplicata ita repraesentatur

$$\frac{q\partial . pz^3}{zz} = \partial x\int\frac{zz\partial x}{q}, \text{ sive } \partial . \frac{q\partial . pz^3}{zz} = \frac{zz\partial x^2}{q},$$

quae manifesto integrabilis redditur, multiplicando per $\frac{2q\partial . pz^3}{zz},$ prodit enim

$$\left(\frac{q\partial.pz^3}{zz}\right)^2 = 2pz^3\partial x^2 + C\partial x^2.$$

§. 10. Jam ponatur $pz^3 = v$, ita ut sit $z^3 = \frac{v}{p}$, et $z^4 = \frac{v}{p}\sqrt[3]{\frac{v}{p}}$, quo valore substitute habebimus

$$\frac{ppq\partial v^2.\sqrt[3]{p}}{v\sqrt[3]{v}} = 2v\partial x^2 + C\partial x^2.$$

unde concluditur

$$\frac{\partial v^2}{v(2v+C)\sqrt[3]{v}} = \frac{\partial x^2}{pqq\sqrt[3]{p}},$$

sive

$$\frac{\partial v}{\sqrt{v(2v+C)\sqrt[3]{v}}} = \frac{\partial v}{v^{\frac{2}{3}}\sqrt{(2v+C)}} = \frac{\partial x}{q\sqrt[3]{pp}},$$

haec aequatio simplicior evadit, ponendo $v = u^3$, scilicet

$$\frac{3\partial u}{\sqrt{(2u^3+C)}} = \frac{\partial x}{q\sqrt[3]{pp}}.$$

Hinc intelligitur, innumerabilia exempla per has formulas expediri posse.

§. 11. Quin etiam hujusmodi aequationes multo generaliores tractari poterunt; namque aequatio generalior ita potest repraesentari

$$\frac{\partial.pz^m}{z^n} = \frac{\partial x}{q}\int\frac{z^n\partial x}{q},$$

quae evoluta dat

$$mpz^{m-n-1}\partial z + z^{m-n}\partial p = \frac{\partial x}{q}\int\frac{z^n\partial x}{q}.$$

Facta autem multiplicatione per z^n, prodit aequatio sponte integrabilis

$$\partial.pz^m = \frac{z^n\partial x}{q}\int\frac{z^n\partial x}{q},$$

si quidem prodit

$$2pz^m = \left(\int\frac{z^n\partial x}{q}\right)^2 + C.$$

§. 12. Ad hanc aequationem ulterius evolvendam statuamus

$$\int \frac{z^n \partial x}{q} = v, \text{ eritque } z^n = \frac{q\partial v}{\partial x},$$

unde primo $2pz^m = vv + \mathrm{C}$, et hinc porro

$$(2p)^{\frac{n}{m}}.z^n = (2p)^{\frac{n}{m}}.\frac{q\partial v}{\partial x} = (vv + \mathrm{C})^{\frac{n}{m}}$$

quae cum sponte sit separabilis, dabit

$$\frac{\partial v}{(vv+\mathrm{C})^{\frac{n}{m}}} = \frac{\partial x}{q(2p)^{\frac{n}{m}}},$$

unde ergo quantitas *v* per *x* determinabitur, qua inventa ipsa quantitas quaesita *z* ita exprimetur, ut sit $z^m = \frac{vv+\mathrm{C}}{2p}$.

§. 13. Illustremus haec unico exemplo a primo casu petito, sumendo scilicet $p = 1 + xx$ et $q = \sqrt{2}$, ita ut aequatio proposita sit

$$2\partial z(1 + xx) + 2zx\partial x = \frac{\partial x}{2}\int z\partial x,$$

quae in hanc aequationem secundi gradus evolvitur

$$4\partial\partial z(1 + xx) + 12x\partial x\partial z + 3z\partial x^2 = 0,$$

cujus ergo integrale quaeritur.

§. 14. Faciamus ergo applicationem solutionis supra §.3. inventae, ubi cum hic sit $p = 1 + xx$ et $q = \sqrt{2}$, erit

$$l\mathrm{X} = \frac{1}{2}\int \frac{\partial x}{\sqrt{(1+xx)}} = \frac{1}{2}l\left[x + \sqrt{(1+xx)}\right] - \frac{1}{2}la,$$

unde fit

$$\mathrm{X} = \frac{\sqrt{\left[x+\sqrt{(1+xx)}\right]}}{\sqrt{a}},$$

hoc igitur valore substituto habebimus

$$z = \frac{a\mathrm{C}+x+\sqrt{(1+xx)}}{2\sqrt{2a(1+xx)\left[x+\sqrt{(1+xx)}\right]}},$$

quae hoc modo simplicius exprimitur

$$z=\frac{\left[a\mathrm{C}+x+\sqrt{(1+xx)}\right]\left[-x+\sqrt{(1+xx)}\right]}{2\sqrt{2a(1+xx)}},$$

Ubi ergo duae quantitates constantes arbitrariae sunt involutae, atque adeo hoc integrale completum algebraice determinetur. Posito ergo $\mathrm{C}=0$, integrale particulare erit ex prima forma petitum

$$z=\frac{\sqrt{\left[x+\sqrt{(1+xx)}\right]}}{2\sqrt{2a(1+xx)}}.$$

§.15. Aliud integrale particulare hinc exhiberi potest, constantes ita sumendo ut sit $a\mathrm{C}$ infinitum, at vero, $\mathrm{C}\sqrt{a}$ finitum $=b$, tum enim erit

$$z=\frac{a\mathrm{C}}{2\sqrt{2a(1+xx)}\left[x+\sqrt{(1+xx)}\right]}=\frac{b}{2\sqrt{2(1+xx)}\left[x+\sqrt{(1+xx)}\right]},$$

quae forma redigitur ad hanc

$$z=\frac{a\left[-x+\sqrt{(1+xx)}\right]}{\sqrt{(1+xx)}}$$

2.) Methodus nova investigandi omnes casus, quibus hanc aequationem differentio-differentialem

$$\partial\partial y\left(1-axx\right)-bx\partial x\partial y-cy\partial x^2=0$$

resolvero licet. *M. S. Academiae exhib. die* 13 *Januarii,* 1780.

[E 678]

§. 16. Hic quidem in usum vocari posset methodus a me et ab aliis jam passim exposita, qua valor ipsius *y* per seriem infinitam exprimitur. Tunc enim omnibus casibus, quibus haec series alicubi abrumpitur, habebitur integrale particulare aequationis propositae; unde quidem haud difficulter integrale completum erui poterit. Verum etsi hoc modo infiniti casus integrabilis reperiuntur; tamen non omnes innotescunt, sed dantur praeterea infiniti alii casus, qui resolutionem admittunt. Quamobrem hic methodum prorsus singularem proponam, cujus ope omnes plane casus integrabiles elici poterunt. Haec autem methodus ita est comparata, ut cognito casu quocunque resolutionem admittente, ex eo innumerabiles alii deduci queant.

§. 17. Statim autem se offerunt duo casus simplicissimi, quibus resolutio succedit, quorum alter est, si $c=0$, alter vero si $b=a$, quos ergo binos casus principales ante omnia evolvi oportet.

Casus prior principalis
quo $c=0$.

§. 18. Hoc igitur casu aequatio nostra erit

$$\partial\partial y(1-axx)=bx\partial x\partial y,$$

quae posito $\partial y=p\partial x$, abit in hanc

$$\partial p(1-axx)=bpx\partial x, \text{ sive}$$
$$\frac{\partial p}{p}=\frac{bx\partial x}{1-axx}$$

cujus integrale est

$$lp=-\tfrac{b}{2a}l(1-axx)+l\mathrm{C},$$

sicque erit

$$p=\mathrm{C}(1-axx)^{-\frac{b}{2a}}=\frac{\partial y}{\partial x},$$

unde obtinetur

$$y=\mathrm{C}\int\partial x(1-axx)^{-\frac{b}{2a}}:$$

ubi notasse juvabit istum valorem fieri algebraicum quoties fuerit $-\frac{b}{2a}$ numerus integer positivus, sive $b=-2ia$ denotante i numerum integrum quemcunque. Tum vero valor integralis etiam algebraicus evadit, quando fuerit $-\frac{b}{2a}$, sive $-\frac{3}{2}$, sive $-\frac{5}{2}$, sive $-\frac{7}{2}$, etc. ideoque in genere $\frac{b}{2a}=2i+1$, ubi esse nequit $i=0$.

Casus principalis alter
quo $b=a$.

§. 19. Hoc ergo casu aequatio nostra per $2\partial y$ multiplicata erit

$$2\partial y\partial\partial y(1-axx)-2ax\partial x\partial y^2-2cy\partial y\partial x^2=0,$$

quae sponte est integrabilis, ejus enim integrale erit

$$\partial y^2(1-axx)-cyy\partial x^2=\mathrm{C}\partial x^2.$$

Ex hac igitur aequatione erit

$$\partial y\sqrt{(1-axx)}=\partial x\sqrt{(\mathrm{C}+cyy)},$$

separatione ergo facta erit

$$\frac{\partial x}{\sqrt{(1-axx)}}=\frac{\partial y}{\sqrt{(\mathrm{C}+cyy)}}.$$

In hac ergo forma iterum continentur casus algebraici, ad quos eruendos faciamus $a=-\alpha\alpha,\ c=\gamma\gamma$ et $\mathrm{C}=\beta\beta$ *;* ut habeamus

$$\frac{\partial x}{\sqrt{(1+\alpha\alpha xx)}}=\frac{\partial y}{\sqrt{(\beta\beta+\gamma\gamma yy)}},$$

cujus integrale est

$$\tfrac{1}{\alpha}l\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]=\tfrac{1}{\gamma}l\left[\gamma y+\sqrt{(\beta\beta+\gamma\gamma yy)}\right]-\tfrac{\gamma}{1}l\Delta,$$

unde ad numeros ascendendo erit

$$\gamma y+\sqrt{(\beta\beta+\gamma\gamma yy)}=\Delta\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]^{\frac{\gamma}{\alpha}}.$$

Posito ergo V pro hac expressione posteriore erit

$$\mathrm{V}-\gamma y=\sqrt{(\beta\beta+\gamma\gamma yy)},$$

et sumtis quadratis $y=\frac{\mathrm{VV}-\beta\beta}{2\gamma\mathrm{V}}$. Cum igitur sit

$$\mathrm{V}=\Delta\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]^{\frac{\gamma}{\alpha}},\text{erit}$$

$$2\gamma y=\Delta\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]^{\frac{\gamma}{\alpha}}-\frac{\beta\beta}{\Delta}\left[\alpha x+\sqrt{(1+\alpha\alpha xx)}\right]^{-\frac{\gamma}{\alpha}},$$

ubi est $\beta\beta=\mathrm{C}$, exponens vero $\frac{\gamma}{\alpha}=\sqrt{\frac{c}{a}}$, sicque, quoties $\sqrt{\frac{c}{a}}$ fuerit numerus rationalis, integrale semper erit algebraicum.

§. 20. His duobus casibus principalibus expeditis duplicem tradam viam aequationem propositam in infinitas alias ejusdem generis transformandi; ita ut semper aequatio hujus formae

$$\partial\partial\mathrm{Y}(1-axx)-\mathrm{B}x\partial x\partial\mathrm{Y}-\mathrm{CY}\partial x^2=0$$

prodeat, quae cum resolutionem admittat casibus vel $\mathrm{C}=0$ vel $\mathrm{B}=a$, iisdem casibus etiam ipsa aequatio proposita erit resolubilis. Duplices igitur hasce transformationes jam sum expositurus.

Transformationes prioris ordinis.

§. 21 Statuo $y = \frac{\partial v}{\partial x}$ unde ob

$$\partial y = \frac{\partial\partial v}{\partial x} \text{ et } \partial\partial y = \frac{\partial^3 v}{\partial x},$$

aequatio nostra induet hanc formam

$$\partial^3 v(1-axx) - bx\partial x\partial\partial v - c\partial x^2 \partial v = 0,$$

cujus singuli termini integrationem admittunt: erit enim

$$\int \partial x^2 \partial v = v\partial x^2,$$

$$\int x\partial x\partial\partial v = x\partial x\partial v - v\partial x^2,$$

$$\int \partial^3 v(1-axx) = \partial\partial v(1-axx) + 2ax\partial x\partial v - 2av\partial x^2.$$

His partibus colligendis, aequatio nostra erit

$$\partial\partial v(1-axx) - (b-2a)x\partial x\partial v - (c-b+2a)v\partial x^2 = 0,$$

quae cum propositae prorsus sit similis, integrabilis erit his duobus casibus $c-b+2a=0$ et $b=3a$, sive quoties fuerit $c=b-2a$ vel $b=3a$, atque integratione pro utroque casu instituta, ita ut *v* exprimatur per *x*, tum pro ipsa aequatione proposita erit $y = \frac{\partial v}{\partial x}$; unde patet, si integralia pro *v* inventa fuerint algebraica, fore quoque valorem ipsius *y* algebraicum.

§. 22. Quod si ulterius simili modo statuamus $v = \frac{\partial v'}{\partial x}$, quoniam per operationem praecedentem litterae *b* et *c* transibunt in $b-2a$ et $c-b+2a$, nunc ista aequatio proveniet

$$\partial\partial v'(1-axx) - (b-4a)x\partial x\partial v' - (c-2b+6a)v'\partial x^2 = 0,$$

quae ergo integrabilis erit, si fuerit vel $b=5a$ vel $c=2b-6a$. Atque inventis valoribus pro *v'* fiet $y = \frac{\partial\partial v'}{\partial x^2}$, scilicet differentialia secunda ipsius *v'* dabunt *y*: sicque, si pro *v'* valor algebraicus prodierit, etiam *y* adipiscetur valorem algebraicum.

§. 23. Quod si eandem substitutionem denuo repetamus ponendo $v' = \frac{\partial v''}{\partial x}$, pro litteris initialibus *b* et *c* jam habebimus $b - 6a$ et $c - 3b + 12a$, et aequatio resultans erit

$$\partial\partial v''(1 - axx) - (b - 6a)x\partial x\partial v'' - (c - 3b + 12a)v''\partial x^2 = 0,$$

quae ergo resolutionem admittet; quoties fuerit vel $b = 7a$ vel $c = 3b - 12a$, quibus ergo casibus etiam ipsa aequatio proposita resolutionem admittat necesse est, cum sit $y = \frac{\partial^3 v''}{\partial x^3}$.

§. 24. Quod si ergo easdem has operationes continuo repetamus, perpetuo ad aequationes, ejusdem formae perveniemus; ubi notasse sufficiet ambos valores, quos pro litteris *b* et *c* in qualibet operatione obtinuerimus, quos una cum valoribus ipsius *y* in sequenti tabula ob oculos ponamus

	b	*c*	*y*
Operatio I.	$b - 2a$	$c - b + 2a$	$\frac{\partial v}{\partial x}$
II.	$b - 4a$	$c - 2b + 6a$	$\frac{\partial v'}{\partial x^2}$
III.	$b - 6a$	$c - 3b + 12a$	$\frac{\partial v''}{\partial x^3}$
IV.	$b - 8a$	$c - 4b + 20a$	$\frac{\partial v'''}{\partial x^4}$
–	– – –	– – – – – –	– – –
–	– – –	– – – – – –	– – –
–	– – –	– – – – – –	– – –
i	$b - 2ia$	$c - ib + i(i+1)a$	$\frac{\partial^i v^{(i-1)}}{\partial x^i}$

§. 25. Hinc igitur in genere patet, aequationem propositam semper resolutionem admittere, quoties fuerit vel $b = 2ia + a$, vel $c = ib - i(i+1)a$, ubi pro *i* omnes numeros integros positives accipere licet, ita ut hinc duos ordines innumerabilium casuum integrabilium nanciscamur, quorum posteriores tantum per methodum serierum initio indicatam reperiuntur, priores vero huic methodo prorsus sint inaccessi.

Transformationes posterioris ordinis.

§. 26. Quemadmodum hic per differentialia sumus progressi, nunc per integralia regrediamur, ac primo quidem ponamus $y = \int z\partial x$, et aequatio proposita evadet

$$\partial z(1 - axx) - bxz\partial x - c\partial x\int z\partial x = 0,$$

quae differentiata ad formam propositam reducitur

$$\partial\partial z(1-axx)-(b+2a)x\partial x\partial z-(c+b)z\partial x^2=0,$$

quae ergo secundum casus principales integrationem admittet, casibus $c+b=0$ et $b+2a=a$, sive $c=-b$ et $b=-a$.
Integralibus igitur inventis erit $y=\int z\partial x$, unde patet etiamsi haec integralia fuerint algebraica, tamen valores ipsius y fieri transcendentes.

§: 27. Simili modo statuamus porro $z=\int z'\partial x$, et quia per praecedentem operationem loco b et c adepti sumus $b+2a$ et $c+b$, nunc perveniemus ad hanc aequationem

$$\partial\partial z'(1-axx)-(b+4a)x\partial x\partial z'-(c+2b+2a)z'\partial x^2=0,$$

quae ergo integrationem admittet, si fuerit vel $c+2b+2a=0$, vel $b+4a=a$, sive $c=-2b-2a$ et $b=-\ 3a$. Integralibus autem hinc inventis pro y habebimus $y=\int\partial x\int z'\partial x$, quae ita ad signum integrale simplex reducitur, ut sit

$$y=x\int z'\partial x-\int z'x\partial x.$$

§. 28. Simili modo statuamus porro $z'=\int z''\partial x$, atque nunc deducemur ad hanc aequationem

$$\partial\partial z''(1-axx)-(b+6a)x\partial x\partial z''-(c+3b+6a)z''\partial x^2=0,$$

quae igitur integrabilis erit, si fuerit vel $c+3b+6a=0$, vel $b+6a=a$, hoc est si $c=-3b-6a$ et $b=-5a$; atque ex his integralibus fiet $y=\int\partial x\int\partial x\int z''\partial x$, qui valor ex praecedente reduci potest, si is per ∂x multiplicatus denuo integretur et loco *z'* s«ribatur *z''*, obtinetur enim

$$y=\tfrac{1}{2}xx\int z''\partial x-x\int xz''\partial x+\tfrac{1}{2}\int xxz''\partial x.$$

§. 29. Quod si jam has operationes ulterius continuemus, totum negotium huc redibit, ut formulae quae loco *b* et *c* sunt prodituræ, rite formentur, simulque valores ipsius *y* assignentur, quemadmodum sequens tabula indicabit

	b	c	y
Operatio I.	$b+2a$	$c+b$	$\int z\partial x$

II.	$b+4a$	$c+2b+2a$	$\int \partial x \int z'\partial x$
III.	$b+6a$	$c+3b+6a$	$\int \partial x \int \partial x \int z''\partial x$
IV.	$b+8a$	$c+4b+12a$	$\int \partial x \int \partial x \int \partial x \int z'''\partial x$
–	– – –	– – – – – –	– – –
–	– – –	– – – – – –	– – –
–	– – –	– – – – – –	– – –
i	$b+2ia$	$c+ib+i(i-1)a$	$\int \partial x \int \partial x .... \int z^{[i-1]}\partial x$

§. 30. Ex antecedentibus satis manifestum est, quomodo integralia ista complicata ad simplicia reduci queant, unde tantum sequentem tabulam subjungemus

$$\int \partial x \int z'\partial x = x\int z'\partial x - \int z' x\partial x$$

$$\int \partial x \int \partial x \int z''\partial x = \tfrac{1}{2}\left(xx\int z''\partial x - 2x\int z'' x\partial x + \int z'' xx\partial x \right)$$

$$\int \partial x \int \partial x \int \partial x \int z'''\partial x = \tfrac{1}{6}\left\{ \begin{array}{l} x^3\int z'''\partial x - 3xx\int z''' x\partial x \\ +3x\int z''' xx\partial x - \int z''' x^3\partial x \end{array} \right\}$$

$$\int \partial x \int \partial x \int \partial x \int \partial x \int z^{\mathrm{IV}}\partial x = \tfrac{1}{24}\left\{ \begin{array}{l} x^4\int z^{\mathrm{IV}}\partial x - 4x^3\int z^{\mathrm{IV}} x\partial x \\ +6xx\int z^{\mathrm{IV}} xx\partial x - 4x\int z^{\mathrm{IV}} x^3\partial x \\ +\int z^{\mathrm{IV}} x^4\partial x \end{array} \right\}.$$

etc. etc.

§. 31. Quod si jam has operationes secundum numerum indefinitum *i* continuemus, et loco *b, c, z,* scribamus B, C, Z, aequatio resultans erit

$$\partial\partial \mathrm{Z}(1-axx) - \mathrm{B}x\partial x\partial \mathrm{Z} - \mathrm{CZ}\partial x^2 = 0,$$

ubi erit, uti jam indicavimus

$$\mathrm{B} = b + 2ia \;\text{ et }\; \mathrm{C} = c + ib + i(i-1)a;;$$

quamobrem haec aequatio integrationem admittet, quoties fuerit vel $\mathrm{C}=0$ hoc est $c=-ib-i(i-i)a$, vel $\mathrm{B}=a$ hoc est $b=-(2i-1)a$; quae formulae ab illis quas supra pro priori transformationum ordine invenimus, tantum in hoc discrepant, quod hic littera *i* valorem negativum accepit; unde adjungatur sequens

Conclusio generalis.

§. 32. Si littera *i* hic denotet omnes numeros integros sive positives sive negativus, aequatio proposita differentio-differentialis

$$\partial\partial y(1-axx)-bx\partial x\partial y-cy\partial x^2=0$$

semper integrationem sive resolutionem admittet, quoties fuerit,

$$1°.)\ 0=ib-i(i+1)a,\text{ vel}$$
$$2°.)\ b=(2i+1)a:$$

ubi asseverare licet, omnes plane casus resolubiles in hac duplici forma contineri, ita ut nullus plane casus integrationem admittens exhiberi queat, qui non in alterutra harum duarum formularum comprehendatur, dum contra methodus per series praecedens, cuius initio mentionem fecimus, tantum casus integrabiles priores ostendit, ita ut, inde infinitus numerus casuum pariter resolubilium inde excludatur.

Corollarium 1.

§. 33. Transformetur aequatio proposita differentialem primi gradus ponendo $y=e^{\int u\partial x}$, ac pervenimus ad hanc aequationem,

$$\partial u+uu\partial u-\tfrac{bux\partial x-c\partial x}{1-axx}=0,$$

quae ergo etiam integrationem admittet casibus quibus vel $b=(2i+1)a$ vel $c=ib-i(i+1)a$, denotante *i* numerum quemcunque integrum sive positivum sive negativum.

Corollarium 2.

§. 34. Quod si porro ponatur $u=(1-axx)^n v$, posito brevitatis gratia $n=-\tfrac{b}{2a}$, pervenietur, ad hanc aequationem ad genus *Riccatianum* referendam

$$(1-axx)^n\partial v+(1-axx)^{2n}vv\partial x=\tfrac{c\partial x}{1-axx},$$

quae per $(1-axx)^n$ divisa abit in hanc

$$\partial v+\left(1-axx\right)^{n}vv\partial x=\frac{c\partial x}{\left(1-axx\right)^{n+1}},$$

quae ergo iisdem casibus integrationem admittet,

Corollarium 3.

§. 35. Quod si sumamus $a=0$ orietur ista aequatio

$$\partial u+uu\partial u=bux\partial x+c\partial x,$$

quae ergo integrabilis erit, si fuerit vel $b=0$ vel $c=ib$, quorum quidem prior casus per se est manifestus, quia tum erit $\partial x=\frac{\partial u}{c-uu}$.
Haec forma autem commodius exprimi poterit; ponendo

$$u=\tfrac{1}{2}bx+v,\ \text{unde}\ \partial v+vv\partial v=\left(c-\tfrac{1}{2}b\right)\partial x+\tfrac{1}{4}bbxx\partial x,$$

sive ponendo $b=2f$, ut fiat

$$\partial v+vv\partial v=\left(c-f\right)\partial x+ffxx\partial x,$$

eritque haec aequatio integrabilis, quoties fuerit $c=2if$, ita ut sequens aequatio semper integrationem admittat

$$\partial v+vv\partial v=\left(2i-1\right)f\partial x+ffxx\partial x,$$

quicunque numerus integer sive positivus sive negativus pro i accipiatur; hoc est, si in penultimo termino f multiplicetur per numerum imparem quemcunque sive positivum sive negativum, qui casus eo erunt abstrusiores, quo maior accipiatur numerus i; atque adeo vix alia via patere videtur integralia eruendi, nisi ut ad aequationem differentialem secundi gradus propositam regrediamur atque easdem operationes instituamus quas supra docuimus. Interim tamen observavi, omnes istos casus etiam immediate ex ipsa aequatione per fractiones continuas derivari posse. Si enim proposita fuerit haec aequatio

$$\partial v+vv\partial v=g\partial x+ffxx\partial x,$$

valor ipsius v duplici modo per fractionem continuam exprimi potest.
Est enim priori modo

$$v=fx+\cfrac{g-f}{2fx+\cfrac{g-3f}{2fx+\cfrac{g-5f}{2fx+\text{etc.}}}}$$

Altero vero modo est

$$v = -fx - \cfrac{(g-f)}{2fx + \cfrac{g+3f}{2fx + \cfrac{g+7f}{2fx + \text{etc.}}}}$$

quarum prior abrumpitur, quoties fuerit $g=(2i+1)f$, posterior vero, quoties fuerit $g=-(2i+1)f$ qui sunt ipsi casus integrabiles ante inventi.

3.) De formulis integralibus implicatis, earumque evolutione et transformatione.
M. S. Academiae exhib. die 20 *Aprilis* 1778 .

[E 679]

§. 36.Talium formulatum implicatarum forma generalis exhiberi potest

$$\int p\partial x\int q\partial x\int r\partial x\int s\partial x \text{ etc.}$$

ubi quovis signum integrale omnia sequentia in se complectitur. Ita ad valorem hiujus expressionis inveniendum a fine est incipiendum, positoque integrali $\int s\partial x=\mathrm{S}$ erit

$$\int r\partial x\int s\partial x=\int \mathrm{S}r\partial x,$$

cujus valor si ponatur $=\mathrm{R}$, erit

$$\int q\partial x\int r\partial x\int s\partial x=\int \mathrm{R}q\partial x,$$

quod integrale si ponatur $=\mathrm{Q}$, valor ipsius formulae propositae erit $=\int \mathrm{Q}p\partial x$, ubi per se intelligitur, in qualibet integratione more solito constantem arbitiariam in calculum introduci posse.

§. 37. Hic scilicet probe tenendum est, istam expressionem $\int p\partial x\int q\partial x$ non significare productum ex formula $\int p\partial x$ in formulam $\int q\partial x$, sed integrale quod oritur, si tota formula differentialis $p\cdot\partial x\int q\cdot\partial x$ integretur : at vera si velimus productum talium duarum formularum integralium designare, id interpositione puncti fieri solet hoc modo

$\int p\partial x\cdot\int q\partial x$, ubi scilicet punctum declarat praecedentia signa integralia non ultra hunc terminum extendi debere, ita haec forma

$$\int p\partial x\int q\partial x\cdot\int r\partial x\int s\partial x \text{ etc.}$$

exprimit productum, quod oritur si formula $\int p\partial x\int q\partial x$ multiplicetur per $\int r\partial x\int s\partial x.$

§. 38. Hic igitur signandi modus nos prorsus contrarius usu est receptus, atque in formulis differentialibus observari solet, ubi talis expressio $\partial x\partial y\partial z$ denotat productum trium differentialium ∂x, ∂y et ∂z, ita ut singula signa differentiationis tantum litteras immediate sequentes afficiant: at si velimus verbi gratia differentiale hujus expressionis $x\partial y\partial z$ exprimere, hoc interpositione puncti fieri solet $\partial.x\partial y\partial z$, ubi punctum significat, praefixum ∂ complecti totam expressionem sequentem.

§. 39. Tales autem formulae integrales implicatae potissimmn nascuntur ex continua integratione aequationum integralium linearium, quarum forma in genere est

$$pz+\frac{q\partial z}{\partial x}+\frac{r\partial\partial z}{\partial x^2}+\frac{s\partial^3 z}{\partial x^3}+\text{ etc.}=\text{X},$$

ubi litterae *p, q, r, s,* etc. sunt functiones datae variabilis *x,* cujus etiam functio quaecunque sit littera X, altera vera variabilis *z* ubique unam tantum tenet dimensionem, prouti haec forma generalis hic exhibetur, ad ordinem tertium differentialium refertur, ideoque ternas integrationes postulat, totidemque constantes arbitrarias involvero est censenda, hic scilicet ad methodum integrandi maxime naturalem respicio, quae per ternas integrationes successivas integrale desideratum producat.

§. 40. Tali scilicet aequatione proposita ante omnia nosse oportet multiplicatorem, quo ea reddatur integrabilis, quem ergo supponamus esse $=\partial\text{P}$, atque integratione peracta prodeat ista aequatio

$$p'z+\frac{q'\partial z}{\partial x}+\frac{r'\partial\partial z}{\partial x^2}=\int \text{X}\partial\text{P},$$

quae aequatio jam est ordinis secundi; quodsi jam ponamus hujus multiplicatorem idoneum esse $=\partial\text{P}'$, facta integratione oriatur haec aequatio primi ordinis, quae sit

$$p''z+\frac{q''\partial z}{\partial x}=\int \partial\text{P}'\int \text{X}\partial\text{P},$$

pro qua si $\partial\text{P}''$ fuerit multiplicator idoneus, completum integrale induct hanc formam

$$p'''z = \int \partial P'' \int \partial P' \int X \partial P.$$

Sicque quantitas z exprimetur per formulam integralem implicatam.

§. 41. Tali autem forma pro integrali inventa praecipuum negotium huc redit, ut ea ita evolvatur, ut formula continens functionem indefinitam X, quae hic terna signa integralia habet praefixa, plus unico ante se non habeat; quamobrem quemadmodum talis reductio commodissime institui queat, hic ostendere constitui, siquidem nisi certa artificia adhibeantur, hujusmodi operatio calculos maxime molestos postularet.

§. 42. In genere autem hujusmodi formulas implicatas ita repraesentemus

$$\int \partial p \int \partial q \int \partial r \int \partial s \int \partial t \text{ etc.}$$

pro cujus evolutione a casu duorum signorum integralium inchoemus, et quia erit $\int \partial p \int \partial q = \int q \partial p$, reductio vulgaris dat $pq - \int p \partial q$. Jam loco p et q iterum scribamus $\int \partial p$ et $\int \partial q$, atque evolutio ita se habebit

$$\int \partial p \int \partial q = \int \partial p \cdot \int \partial q - \int \partial q \cdot \int \partial p,$$

ubi in genere hanc aequalitatem notasse juvabit

$$\int \partial p \int \partial q - \int \partial p \cdot \int \partial q + \int \partial q \int \partial p = 0.$$

§. 43. Consideremus nunc formulam tria signa integralia involventem $= \int \partial p \int \partial q \int \partial r$, et quia ut modo vidimus est $\int \partial q \int \partial r = qr - \int q \partial r$, nostra formula in has partes discerpitur $\int qr \partial p - \int \partial p \int q \partial r$, quae posterior pars reducitur ad hanc formam $p \int q \partial r - \int pq \partial r$, sicque formula nostra erit

$$\int qr \partial p - p \int q \partial r + \int pq \partial r.$$

Quoniam nunc requiritur, ut elementum ∂r in singulis partibus unicum tantum signum integrale habeat praefixum ; ponamus $q \partial p = \partial v$ ut sit

$$v = \int q \partial p = \int \partial p \int \partial q,$$

eritque

$$\int qr\partial p = \int r\partial v = rv - \int v\partial r,$$

hincque colligitur

$$\int pq\partial r - \int v\partial r = \int \partial r\left(pq - v\right) = \int \partial r \int p\partial q.$$

Jam loco litterarum finitarum differentialia rursus introducentur, atque valor quaesitus formulae $\int \partial p \int \partial q \int \partial r$ sequenti modo exprimetur

$$\int \partial p \int \partial q . \int \partial r - \int \partial p . \int \partial r \int \partial q + \int \partial r \int \partial q \int \partial p\,,,$$

ubi in singulis membris elemento ∂r unicum signum integrale est praefixum.

§. 44. Inter terna igitur elementa $\partial p,\ \partial q$ et ∂r sequentem relationem notari operae erit pretium

$$\int \partial p \int \partial q \int \partial r - \int \partial p \int \partial q \cdot \int \partial r + \int \partial p \cdot \int \partial r \int \partial q - \int \partial r \int \partial q \int \partial p = 0,$$

quodsi autem similem reductionem pro casibus plurium signorum integralium exsequi vellemus, in calculos molestissimos ac taediosissimos delaberemur; interim tamen totum hoc negotium per sequentia theoremata facillime et planissime expedietur, et quoniam singula membra ope puncti in duos factores resolvi convenit, ubi talis factor deest, ejus locum unitate supplebimus.

Theorema 1.

§. 45. Pro unico elemento ∂p haec relatio habetur $\int \partial p . 1 - 1 . \int \partial p = 0,$ maxime obvia.

Theorema 2.

§. 46. Inter bina elementa ∂p et ∂q semper locum habebit haec relatio

$$\int \partial p \int \partial q . 1 - \int \partial p . \int \partial q + 1 . \int \partial p \int \partial q = 0.$$

Demonstratio.

Ad hoc demonstrandum sufficiet ostendisse, differentiale hujus aequationis esse $= 0$, quoniam vero singula membra binis constant factoribus, seorsim considerentur differentialia ex factoribus prioribus et posterioribus oriunda, hic igitur ex factoribus prioribus oritur differentiale $\partial p\left(\int \partial q.1-1.\int \partial q\right)=0$ per theorema 1. At ex factoribus posterioribus oritur differentiale

$$-\partial q\left(\int \partial p.1-1.\int \partial p\right)=0.$$

Theorema 3.

§. 47. Inter terna elementa ∂p, ∂q et ∂r semper haec relatio locum habet

$$\int \partial p\int \partial q\int \partial r.1-\int \partial p\int \partial q.\int \partial r+\int \partial p.\int \partial r\int \partial q-1.\int \partial r\int \partial q\int \partial p=0.$$

Demonstratio.

Hic iterum seorsim perpendantur differentialia tam ex prioribus quam ex posterioribus factoribus oriunda ; ex prioribus autem oritur

$$\partial p\left(\int \partial q\int \partial r\cdot 1-\int \partial q\cdot \int \partial r+1.\int \partial r\int \partial q\right),$$

cujus valor manifesto ad nihilum redigitur per theorema 2, si scilicet litterae *p* et *q* uno gradu promoveantur; tum vero differentiale ex factoribus posterioribus ortum est

$$-\partial r\left(\int \partial p\int \partial q\cdot 1-\int \partial p\cdot \int \partial q+1.\int \partial q\int \partial p\right),$$

cujus valor pariter per theorema praecedens evanescit; quoniam igitur ambo differentialia sunt $=0$, etiam ipsa forma nihilo vel etiam constanti aequalis esse debet, evidens autem est constantem sponte involvi in signis integralibus.

Theorema 4.

§. 48. Inter quaterna elementa ∂p, ∂q, ∂r et ∂s semper ista relatio locum habet

$$\left.\begin{array}{l} \int \partial p\int \partial q\int \partial r\int \partial s\cdot 1-\int \partial p\int \partial q\int \partial r\cdot \int \partial s \\ +\int \partial p\int \partial q\cdot \int \partial s\int \partial r-\int \partial p\cdot \int \partial s\int \partial r\int \partial q \\ +1\cdot \int \partial s\int \partial r\int \partial q\int \partial p \end{array}\right\}=0.$$

Demonstratio.

Differentiatio factorum priorum suppeditat sequentem expressionem

$$\partial p\left(\int\partial q\int\partial r\int\partial s\cdot 1-\int\partial q\int\partial r\cdot\int\partial s+\int\partial q\cdot\int\partial r\int\partial s-1\cdot\int\partial q\int\partial r\int\partial s\right),$$

quae ob theorema praecedens ad nihilum reducitur. Simili modo differentiatio factorum posteriorum praebet hanc expressionem

$$-\partial s\left(\int\partial p\int\partial q\int\partial r\cdot 1-\int\partial p\int\partial q\cdot\int\partial r+\int\partial p\cdot\int\partial r\int\partial q-1\cdot\int\partial r\int\partial q\int\partial p\right),$$

quae ob theorema 3. iterum est $=0$.

Theorema 5.

§. 49. Inter quina elementa $\partial p,\ \partial q,\ \partial r,\ \partial s$ et ∂t semper haec relatio locum habet

$$\left.\begin{array}{l}\int\partial p\int\partial q\int\partial r\int\partial s\int\partial t\cdot 1-\int\partial p\int\partial q\int\partial r\int\partial s\cdot\int\partial t\\ +\int\partial p\int\partial q\int\partial r\cdot\int\partial t\int\partial s-\int\partial p\int\partial q\cdot\int\partial t\int\partial s\int\partial r\\ +\int\partial p\cdot\int\partial t\int\partial s\int\partial r\int\partial q-1\cdot\int\partial t\int\partial s\int\partial r\int\partial q\int\partial p\end{array}\right\}=0.$$

Demonstratio.

Hujus theorematis demonstratio prorsus eodem modo se habet ac theorematum praecedentium; sicque clarissime jam est evictum tales, relationes perpetuo veritati esse consentaneas, quotcunque etiam elementis fuerint composita.

§. 50. Quo vis horum theorematum clarius perspiciatur, operae pretium erit, ea per exempla determinata illustrasse;
ponamus igitur esse:

$$\partial p=x^{\alpha-1}\partial x,\ \partial q=\ x^{\beta-1}\partial x,\ \partial\ r=x^{\gamma-1}\partial x,$$
$$\partial s=x^{\theta-1}\partial x,\ \partial t=x^{\varepsilon-1}\partial x,$$

atque ex theoremate primo statim aequatio identica nascitur

$$\frac{x^{\alpha}}{\alpha}-\frac{x^{\alpha}}{\alpha}=0.$$

Verum theorema secundum nobis praebet hanc aequationem

$$\frac{x^{\alpha+\beta}}{\beta(\alpha+\beta)}-\frac{x^{\alpha+\beta}}{\alpha\beta}+\frac{x^{\alpha+\beta}}{\alpha(\alpha+\beta)}=0,$$

unde per $x^{\alpha+\beta}$ dividendo prodit haec aequalitas

$$\frac{1}{\beta(\alpha+\beta)}-\frac{1}{\alpha\beta}+\frac{1}{\alpha(\alpha+\beta)}=0,$$

cujus veritas satis facile in oculos incurrit.

§. 51. Hae porro positiones in theoremate tertio introductae producent hanc aequationem

$$\frac{x^{\alpha+\beta+\gamma}}{\gamma(\alpha+\beta+\gamma)(\beta+\gamma)}-\frac{x^{\alpha+\beta+\gamma}}{\beta\gamma(\alpha+\beta)}+\frac{x^{\alpha+\beta+\gamma}}{\alpha\beta(\beta+\gamma)}-\frac{x^{\alpha+\beta+\gamma}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)},$$

unde per $x^{\alpha+\beta+\gamma}$ dividendo prodit haec egregia aequalitas

$$\frac{1}{\gamma(\alpha+\beta+\gamma)(\beta+\gamma)}-\frac{1}{\beta\gamma(\alpha+\beta)}+\frac{1}{\alpha\beta(\beta+\gamma)}-\frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)}=0,$$

§. 52. Hae positiones iterum in theoremate quarto substitutae dant hanc aequationem

$$\left.\begin{aligned}\frac{x^{\alpha+\beta+\gamma+\delta}}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\frac{x^{\alpha+\beta+\gamma+\delta}}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)}\\+\frac{x^{\alpha+\beta+\gamma+\delta}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)}-\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}\\+\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)}\end{aligned}\right\}=0,$$

quae per $x^{\alpha+\beta+\gamma+\delta}$ divisa producit hanc aequationem

$$\left.\begin{aligned}\frac{1}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\frac{1}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)}\\+\frac{1}{\beta\gamma(\beta+\alpha)(\gamma+\delta)}-\frac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}\\+\frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)}\end{aligned}\right\}=0.$$

§. 53. Denique eaedem positiones in theoremate quinto substitutae producunt hanc aequationem

$$\left.\begin{array}{l}
\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)} \\
-\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}+\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)} \\
-\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}+\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)} \\
-\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}
\end{array}\right\}=0,$$

quae per $x^{\alpha+\beta+\gamma+\delta+\varepsilon}$ divisa dat hanc aequationem maxime notatu dignam

$$\left.\begin{array}{l}
\frac{1}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)} \\
-\frac{1}{\varepsilon\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}+\frac{1}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)} \\
-\frac{1}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}+\frac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)} \\
-\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}
\end{array}\right\}=0.$$

§. 64. Haec theorematis eo magis sunt memorabilia, quod eorum veritas non nisi per plures ambages in numeris explorari potest, ideoque multo majorem attentionem merentur, quam aliud simile theorema, ad quod nuper sum perductus, quippe cujus demonstratio haud difficulter exhiberi potest, quod ita se habet.

Theorema numericum.

Sumtis pro lubitu quotcunque numeris veluti quatuor $\alpha,\beta,\gamma,\delta,$ si hinc totidem alii sequenti modo formentur

$$a=\alpha,\ b=\alpha+\beta,\ c=\alpha+\beta+\gamma \text{ et } d=\alpha+\beta+\gamma+\delta,$$

similique modo etiam isti

$$\mathrm{D}=\delta,\ \mathrm{C}=\delta+\gamma$$
$$\mathrm{B}=\delta+\gamma+\beta \text{ et } \mathrm{A}=\delta+\gamma+\beta+\alpha,$$

tam semper erit

$$\frac{1}{abcd}-\frac{1}{abc\mathrm{D}}+\frac{1}{ab\mathrm{CD}}-\frac{1}{a\mathrm{BCD}}+\frac{1}{\mathrm{ABCD}}=0.$$

Demonstratio.

§. 65. Binae fractiones priores inventae, ob $\mathrm{D}-d=-c,$ dant fractionem $-\frac{1}{abc\mathrm{D}}$, quae cum tertia conjuncta producit $\frac{1}{ab\mathrm{CD}}$, cui quarta fractio juncta dat $-\frac{1}{a\mathrm{BCD}}$, quae [ob $d=\mathrm{A}$] a termino ultimo penitus destruitur.

§. 56. Ope superiorum theorematum omnes formulae integrales implicatae, ad quas integratio aequationum linearum perducere solet, facile resolvi poterunt. Pervenitur autem plerumque ad tales formas :

$$Z=\int \partial q\int X\partial p, Z=\int \partial r\int \partial q\int X\partial p,$$
$$Z=\int \partial s\int \partial r\int \partial q\int X\partial p, Z=\int \partial t\int \partial s\int \partial r\int \partial q\int X\partial p \text{ etc.}$$

ubi litterae *p, q, r, s, t,*. etc. sunt functiones datae ipsius X, at vera X functio quaecunque ipsius *x*; atque hic tota resolutio ita institui debet, ut in singulis membris functio haec indefinita X unicum tantum signum integrale habeat praefixum: hoc igitur, ope superiorum theorematum, facile praestari poterit, si modo ibi loco elementi ∂p scribamus $X\partial p$ quo observato singulae reductiones sequenti modo se habebunt.

I. Resolutio formulae integralis

$$\int \partial q\int X\partial p .$$

§. 57. Si loco ∂p scribamus $X\partial p$ theorema secundum §.46. nobis suppeditat hanc aequationem:

$$\int X\partial p\int \partial q-\int X\partial p\cdot\int \partial q+\int \partial q\int X\partial p=0,$$

cujus postremum membrum est ipsa nostra forma reducenda Z, consequenter resolutio statim dat

$$Z=\int \partial q\cdot\int X\partial p-\int X\partial p\int \partial q,$$

ideoque ob $\int \partial q=q$ habebimus

$$Z=q\int X\partial p-\int Xq\partial p.$$

Corollarium.

§. 68. Si fuerit $q=p$, erit

$$Z=p\int X\partial p-\int Xp\partial p.$$

II. Resolutio formulae implicatae

$$\int \partial r\int \partial q\int X\partial p .$$

§. 59. Pro hoc casu sumamus theorema 3. §. 47. unde, si loco ∂p scribatur $X\partial p$, deducimus hanc aequationem

$$\int X\partial p\int \partial q\int \partial r-\int X\partial p\int \partial q.\int \partial r+\int X\partial p.\int \partial r\int \partial q-\int \partial r\int \partial q\int X\partial p=0,$$

cujus postremum membrum est ipsa forma reducenda Z, hincque adeoque colligitur

$$Z=\int \partial r\int \partial q\int X\partial p-\int \partial r\cdot\int X\partial p\int \partial q+\int X\partial p\int \partial q\int \partial r,$$

quae ergo reducta dat

$$Z=\int q\partial r\int X\partial p-r\int Xq\partial p+\int X\partial p\int r\partial q.$$

Corollarium.

§. 60. Si ergo hic fuerit $q=r=p$, prodibit ista resolutio :

$$Z=\int \partial p\int \partial p\int X\partial p=\tfrac{1}{2}pp\int X\partial p-p\int Xp\partial p+\tfrac{1}{2}\int Xpp\partial p.$$

III. Resolutio hujus formulae implicatae
$\int \partial s\int \partial r\int \partial q\int X\partial p.$

§. 61. Pro hoc casu sumamus theorema 4, §. 48, unde si loco ∂p scribatur $X\partial p$ deducimus hanc aequationem

$$\left.\begin{aligned}&\int X\partial p\int \partial q\int \partial r\int \partial s-\int X\partial p\int \partial q\int \partial r\cdot\int \partial s+\int X\partial p\int \partial q\cdot\int \partial s\int \partial r\\&-\int X\partial p.\int \partial s\int \partial r\int \partial q+\int \partial s\int \partial r\int \partial q\int X\partial p.\end{aligned}\right\}=0,$$

cujus postremum membrum est ipsa nostra formula reducenda Z; hincque adeo colligimus

$$Z=\begin{cases}\int \partial s\int \partial r\int \partial q\cdot\int X\partial p-\int \partial s\int \partial r\cdot\int X\partial p\int \partial q+\int \partial s\cdot\int X\partial p\int \partial q\int \partial r\\-\int X\partial p\int \partial q\int \partial r\int \partial s,\end{cases}$$

quae ergo reducta praebet

$$Z=\int \partial s\int q\partial r\cdot\int X\partial p-\int r\partial s\cdot\int Xq\partial p+s\int X\partial p\cdot\int r\partial q-\int X\partial p\int \partial q\int s\partial r.$$

Corollarium.

§. 62. Si ponatur $s=r=q=p$, tum prodibit ista resolutio

$$Z=\begin{cases}\frac{1}{6}p^3\int X\partial p-\frac{1}{2}pp\int Xp\partial p+\frac{1}{2}p\int Xpp\partial p\\ \qquad -\frac{1}{6}\int Xp^3\partial p.\end{cases}$$

IV. Resolutio hujus formulae implicatae.

$$Z=\int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p.$$

§: 63. Pro hoc casu sumamus theorema 5, §. 49, unde si loco ∂p scribatur $X\partial p$, prodibit ista aequatio

$$\left.\begin{matrix}\int X\partial p\int\partial q\int\partial r\int\partial s\int\partial t-\int X\partial p\int\partial q\int\partial r\int\partial s\cdot\int\partial t\\ +\int X\partial p\int\partial q\int\partial r\cdot\int\partial t\int\partial s-\int X\partial p\int\partial q\cdot\int\partial t\int\partial s\int\partial r\\ +\int X\partial p\cdot\int\partial t\int\partial s\int\partial r\int\partial q-\int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p\end{matrix}\right\}=0,$$

cujus postremum membrum est ipsa nostra forma reducenda Z, unde ergo prodit

$$Z=\begin{cases}\int\partial t\int\partial s\int\partial r\int\partial q\cdot\int X\partial p-\int\partial t\int\partial s\int\partial r\cdot\int X\partial p\int\partial q\\ +\int\partial t\int\partial s\cdot\int X\partial p\int\partial q\int\partial r-\int\partial t\cdot\int X\partial p\int\partial q\int\partial r\int\partial s\\ +\int X\partial p\cdot\int\partial q\int\partial r\int\partial s\int\partial t,\end{cases}$$

quae ergo reducta praebet

$$\_Z=\begin{cases}\int\partial t\int\partial s\int q\partial r\cdot\int X\partial p-\int\partial t\int r\partial s\cdot\int Xq\partial p\\ +\int s\partial t\cdot\int X\partial p\int r\partial q-t\int X\partial p\int\partial q\int s\partial r\\ +\int X\partial p\cdot\int\partial q\int\partial r\int t\partial s.\end{cases}$$

Corollarium.

§. 64. Si hic sumatur $t=s=r=q=p$, tum prodibit ista resolutio

$$Z=\begin{cases}\frac{1}{24}p^4\int X\partial p-\frac{1}{6}p^3\int Xp\partial p+\frac{1}{4}pp\int Xpp\partial p\\ -\frac{1}{6}p\int Xp^3\partial p+\frac{1}{24}\int Xp^4\partial p.\end{cases}$$

§. 65. Quo indoles harum resolutionum clarius perspiciatur, quoniam litterae *p, q, r, s, t,* functiones datas ipsius *x* denotant, ideoque omnes expressiones ex iis formatae pariter ut cognitae spectari possunt, statuamus brevitatis gratia

$$\partial p\int\partial q=\partial p';\ \partial p\int\partial q\int\partial r=\partial p'';\ \partial p\int\partial q\int\partial r\int\partial s=\partial p''';\ \partial p\int\partial q\int\partial r\int\partial s\int\partial t=\partial p'''';\text{etc.}$$

hocque modo postrema resolutio ita referetur

$$\begin{aligned}Z=&\int\partial t\int\partial s\int\partial r\int\partial q\cdot\int X\partial p-\int\partial t\int\partial s\int\partial r\cdot\int X\partial p'\\&+\int\partial t\int\partial s\cdot\int X\partial p''-\int\partial t\cdot\int X\partial p'''+\int X\partial p''''.\end{aligned}$$

Quod si hic porro statuamus

$$\int\partial t\int\partial s=\int s\partial t=t';\ \int\partial t\int\partial s\int\partial r=t'';\ \int\partial t\int\partial s\int\partial r\int\partial q=t''';$$

tota resolutio hoc modo conncinne repraesentabitur

$$Z=t'''\int X\partial p-t''\int X\partial p'\ +t'\int X\partial p''-t\int X\partial p'''+\int X\partial p'''',$$

quam repraesentationem etiam ad praecedentes resolutiones accommodasse juvabit.

§. 66: Cum igitur integratio formulae implicatae

$$\int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p$$

reducatur ad integrationem sequentium formularum integralium simplicium:

$$\int X\partial p;\ \int X\partial p';\ \int X\partial p'';\ \int X\partial p''';\ \int X\partial p'''';$$

quaestio hinc oritur non parum curiosa: quemadmodum ex his formulis simplicibus vicissim quantitates *q, r, s* et *t* concludi queant? quod sequenti modo facile praestabitur. Cum sit $\partial p'=\partial p\int\partial q$, erit $\int\partial q=q=\frac{\partial p'}{\partial p}$. Ponatur nunc porro $\frac{\partial p''}{\partial p}=q';\ \frac{\partial p'''}{\partial p}=q'';\ \frac{\partial p''''}{\partial p}=q''';$ etc. quibus valoribus introductis habebimus

$$q'=\int\partial q\int\partial r;\ q''=\int\partial q\int\partial r\int\partial s;\ q'''=\int\partial q\int\partial r\int\partial s\int\partial t;\ \text{etc.}$$

Quoniam igitur hi valores $q,\ q',\ q'',\ q''',\ q''''$ sunt dati, ex prima statim colligimus $\int\partial r=\frac{\partial q'}{\partial q}=r$. Ponamus autem porro $\frac{\partial q''}{\partial q}=r';\ \frac{\partial q'''}{\partial q}=r'';$ etc. eruntque etiam hi valores,

r, r', r'', etc. dati, quibus substitutis habebitur $r' = \int \partial r \int \partial s$; $r'' = \int \partial r \int \partial s \int \partial t$; ex quarum prima sequitur $\int \partial s = s = \frac{\partial r'}{\partial r}$. Quare si porro fiat $s' = \frac{\partial r''}{\partial r}$, erit quoque $s' = \int \partial s \int \partial t$, hincque $\int \partial t = t = \frac{\partial s'}{\partial s}$. Ex his clare intelligitur, quomodo hae formulae inveniri queant pro casibus adhuc magis complicatis.

§. 67. Superest ut etiam de transformatione talium formularum integralium implicatarum pauca adjicamus, quod totum negotium sequenti problemate includi potest.

Problema.

§. 68. *Proposita formula implicata terna signa summatoria involuente* $\int \partial p \int \partial q \int \partial r$, *investigare aliam similem formulam* $\int \partial \mathrm{P} \int \partial \mathrm{Q} \int \partial \mathrm{R}$, *illi aequalem.*

Solutio.

Per theorema 2. supra allatum formula proposita ita est resoluta

$$\int \partial q \int \partial r = \int \partial q \cdot \int \partial r - \int \partial r \int \partial q = q \int \partial r - \int q \partial r.$$

Simili modo pro formula quaesita erit

$$\int \partial \mathrm{Q} \int \partial \mathrm{R} = \mathrm{Q} \int \partial \mathrm{R} - \int \mathrm{Q} \partial \mathrm{R}.$$

requiritur igitur ut sit

$$q \partial p \int \partial r - \partial p \int q \partial r = \mathrm{Q} \partial \mathrm{P} \int \partial \mathrm{R} - \partial \mathrm{P} \int \mathrm{Q} \partial \mathrm{R},$$

quae aequalitas adimpleretur, sumendo $\mathrm{P} = p$, $\mathrm{Q} = q$ et $\mathrm{R} = r$;
verum permutandis membris statuamus

$$\mathrm{Q} \partial \mathrm{P} \int \partial \mathrm{R} = -\partial p \int q \partial r \quad \text{et} \quad \partial \mathrm{P} \int \mathrm{Q} \partial \mathrm{R} = -q \partial p \int \partial r,$$

atque ex priore aequatione deducimus $\mathrm{Q} \partial p = -\partial p$, ideoque $\partial \mathrm{P} = \frac{\partial p}{\mathrm{Q}}$, tum vero $\partial \mathrm{R} = q \partial r$; ex altera vera aequatione habemus $\partial \mathrm{P} = -q \partial p$ et $\mathrm{Q} \partial \mathrm{R} = \partial r$. Cum igitur esset $\partial \mathrm{P} = -\frac{\partial p}{\mathrm{Q}}$, erit $\mathrm{Q} = \frac{1}{q}$, hincque porro $\partial \mathrm{R} = q \partial r$, unde ob $\mathrm{Q} = \frac{1}{q}$, erit $\partial \mathrm{Q} = \frac{-\partial q}{qq}$. Consequenter formula integralis quaesita proposita $\int \partial p \int \partial q \int \partial r$ aequalis erit

$$\int q \partial p \int \frac{\partial q}{qq} \int q \partial r,$$

unde patet perpetuo loco formulae $\int \partial p \int \partial q \int \partial r$ scribi posse istam: $\int q\partial p \int \frac{\partial q}{qq} \int q\partial r$.

Corollarium I.

§. 69. Quando igitur plura signa integralia sibi in vicem fuerint involuta, veluti si habeamus $\int \partial p \int \partial q \int \partial r \int \partial s$, ista transformatio in quibusvis ternis signis se mutua sequentibus institui poterit, unde in hac formula proposita duplex transformatio adhiberi poterit; prior scilicet in ternis signis prioribus praebebit

$$\int q\partial p \int \frac{\partial q}{qq} \int q\partial r \int \partial s \,,$$

at vera in ternis posterioribus haec transformatio adhibita dabit

$$\int \partial p \int r\partial q \int \frac{\partial r}{rr} \int r\partial s \,,$$

Corollarium 2.

§. 70. Hinc porro ope ejusdem transformationis aliae insuper fieri possunt, veluti ex postrema forma

$$\int \partial p \int r\partial q \int \frac{\partial r}{rr} \int r\partial s \,,$$

ut in ternis prioribus signis res expediri queat, loco $r\partial q$ scribamus ∂v ,ut habeamus

$$\int \partial p \int \partial v \int \frac{\partial r}{rr} \int r\partial s \,,$$

quae transformatur in hanc

$$\int v\partial p \int \frac{\partial v}{vv} \int \frac{v\partial r}{rr} \int r\partial s \,,$$

quae omnes formulae ipsi propositae sunt prorsus aequales.

§. 71. Ut rem exemplo illustremus, sumamus esse $p = x^{\alpha}$; $q = x^{\beta}$; $r = x^{\gamma}$, ita ut formula proposita sit

$$\alpha\beta\gamma \int x^{\alpha-1}\partial x \int x^{\beta-1}\partial x \int x^{\gamma-1}\partial x = \frac{\alpha\beta x^{\alpha+\beta+\gamma}}{(\gamma+\beta)(\gamma+\beta+\alpha)}.$$

Jam pro transformatione erit primo

$\int q\partial r = \frac{\gamma x^{\beta+\gamma}}{\beta+\gamma}$, ideoque ob $\frac{\partial q}{qq} = \frac{\beta\partial x}{x^{\beta+1}}$, erit

$$\int \frac{\partial q}{qq}\int q\partial r = \frac{\beta x^{\gamma}}{\beta+\gamma},$$

quod ductum in $q\partial p$ et integratum producit

$$\frac{\alpha\beta x^{\alpha+\beta+\gamma}}{(\beta+\gamma)(\alpha+\beta+\gamma)}.$$

Patet igitur hanc transformationem latissime patere, atque ad omnes formulas implicatas accommodari posse eo pluribus diversis modis, quo plura signa integralia invicem involvantur.

§. 72. Hand abs re fore judico resolutiones supra traditas ad summationem serierum potestatum reciprocarum applicare, quod fiet si loco X sumamus fractionem $\frac{x}{1-x}$, tum vero pro singulis elementis ∂p, ∂q, ∂r, ∂s, scribamus $\frac{\partial x}{x}$, unde corollaria subnexa in usum vocari poterunt; ubi scilicet erit $p = lx$.

§. 73. Cum sit per seriem infinitam

$$\mathrm{X} = x + xx + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.}$$

erit

$$\int \mathrm{X}\partial p = \int \frac{\mathrm{X}\partial x}{x} = x + \tfrac{1}{2}xx + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \tfrac{1}{6}x^6 + \text{etc.}$$

quam seriem constat exprimere logarithmum fractionis $\frac{x}{1-x}$, quandoquidem est

$$\int \frac{\mathrm{X}\partial x}{x} = -l(1-x) = l\frac{1}{1-x}.$$

§. 74. Multiplicetur haec series porro per $\frac{\partial x}{x}$ et integretur, prodibitque

$$\int \frac{\partial x}{x}\int \frac{\mathrm{X}\partial x}{x} = x + \tfrac{1}{4}xx + \tfrac{1}{9}x^3 + \tfrac{1}{16}x^4 + \tfrac{1}{25}x^5 + \text{etc.}$$

at vera hujus formulae integralis resolutio supra §.57. data praebet

$$\int \frac{\partial x}{x}\int \frac{\mathrm{X}\partial x}{x} = lx\int \frac{\partial x}{1-x} - \int \frac{\partial x lx}{1-x},$$

quae quidem integralia ita accipi supponuntur, ut posito $x = 0$ evanescant; hic autem imprimis notetur, casu quo sumitur $x = 1$, ob $l1 = 0$, hujus seriei

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}$$

summam fore $-\int\frac{\partial x lx}{1-x}$, cujus valorem olim primus inveni esse $\frac{\pi\pi}{6}$.

§. 75. Ducamus superiorem seriem denuo in $\frac{\partial x}{x}$ et integrando obtinebimus:

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^3}xx+\frac{1}{3^3}x^3+\frac{1}{4^3}x^4+\frac{1}{5^3}x^5+\text{etc.}$$

Formula autem haec implicata per §. 59. ita resolutur

$$\frac{1}{2}(lx)^2\int\frac{\partial x}{1-x}-lx\int\frac{\partial x lx}{1-x}+\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x}.$$

Casu igitur quo $x=1$, summa seriei reciprocae cuborum

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}$$

erit $=\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x}$.

§ 76. Simili modo superiorem seriem per $\frac{\partial x}{x}$ multiplicemus et integremus; tum prodibit

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^4}xx+\frac{1}{3^4}x^3+\frac{1}{4^4}x^4+\frac{1}{5^4}x^5+\text{etc.}$$

At vero haec formula implicata per §. 61, reducitur ad hanc formam

$$\frac{1}{6}(lx)^3\int\frac{\partial x}{1-x}-\frac{1}{2}(lx)^2\int\frac{\partial x lx}{1-x}+\frac{1}{2}lx\int\frac{\partial x(lx)^2}{1-x}-\frac{1}{6}\int\frac{\partial x(lx)^3}{1-x}.$$

Pro casu ergo quo $x=1$ hujus seriei reciprocae biquadratorum summa erit $\int\frac{\partial x(lx)^3}{1-x}$, cujus valorem olim ostendi esse $\frac{\pi^4}{90}$.

§. 77. Multiplicatione denuo per $\frac{\partial x}{x}$ instituta et integratione peracta habebimus:

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^5}xx+\frac{1}{3^5}x^3+\frac{1}{4^5}x^4+\frac{1}{5^5}x^5+\text{etc.}$$

quae formula implicata per §. 63. reducitur ad hanc formam

$$\frac{1}{24}(lx)^4\int\frac{\partial x}{1-x}-\frac{1}{6}(lx)^3\int\frac{\partial xlx}{1-x}+\frac{1}{4}(lx)^2\int\frac{\partial x(lx)^2}{1-x}-\frac{1}{6}lx\int\frac{\partial x(lx)^3}{1-x}+\frac{1}{24}\int\frac{\partial x(lx)^4}{1-x}.$$

Hinc ergo casu $x=1$ hujus seriei reciprocae potestatum quintarum summa erit $\frac{1}{24}\int\frac{\partial x(lx)^4}{1-x}$.

§. 78. Colligamus omnes istas series pro casu $x=1$, earumque summae sequenti modo per formulam integralem simplicem exprimetur :

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.}\quad=-\int\frac{\partial x}{1-x}=\infty,$$
$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=-\int\frac{\partial xlx}{1-x}=\frac{\pi\pi}{6},$$
$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}=\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x},$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=-\frac{1}{6}\int\frac{\partial x(lx)^3}{1-x}=\frac{\pi^4}{90},$$
$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\text{etc.}=\frac{1}{24}\int\frac{\partial x(lx)^4}{1-x},$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=-\frac{1}{120}\int\frac{\partial x(lx)^5}{1-x}=\frac{\pi^6}{945},$$
$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\frac{1}{5^7}+\text{etc.}=\frac{1}{720}\int\frac{\partial x(lx)^6}{1-x},$$
$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=-\frac{1}{5040}\int\frac{\partial x(lx)^7}{1-x}=\frac{\pi^8}{9450},$$
$$\text{etc.}\qquad\qquad\text{etc.}$$

§.79. In genere igitur hujus seriei

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.}$$

in infinitum continuatae summa ita exprimetur

$$\pm\frac{1}{1.2.3......(n-1)}\int\frac{\partial x(lx)^{n-1}}{1-x},$$

ubi signum superius + valet, quando exponens *n* est impar, inferius vero, quando est par. Istas summationes, jam pridem quidem repertas, ideo hic afferre visum est, quod non ita pridem Celeberr. Lorgna easdem has summationes per formulas continuo magis implicatas expressas exhibuit; cum sine dubio istae formulae integrales simplices longe praeferendae videantur.