2) A more succinct method of preparing the transcending quantities required to be found for the integration of

$$\int \frac{P\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}.$$

M. S. shown to the Academy, 3rd Nov. 1777
Foundations of the integral calculus 4, 1794, p. 504-524

[E676]

In Chapter VI Sect. II, Book I, of my *Foundations of Integral Calculus*, I examined the preparations of the most conspicuous transcending quantities for the integration, which I had deduced according to a completely indirect method. Therefore, thus not long afterwards, a most ingenious method was thought out for finding the same preparations for integration by the most illustrious *Lagrange*, by which the whole argument was able to be treated much more succinctly and elegantly, than was possible by me at the time, from which the following Geometrical Supplements will hardly disappoint.

Hypothesis 1.

§. 80. Here the value of the integral formula thus assumed,

$$\int \frac{\partial z}{\sqrt{(\alpha+\beta z+\gamma zz+\delta z^3+\varepsilon z^4)}},$$

always will be indicated by the symbol $\Pi : z$, thus so that it may vanish on putting $z = 0$. Moreover, for the sake of brevity, there may be put

$$\alpha+\beta z+\gamma zz+\delta z^3+\varepsilon z^4 = Z,$$

thus so that there shall become:

$$\Pi : z = \int \frac{\partial z}{\sqrt{Z}}$$

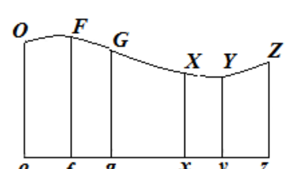

Fig. 1

Truly moreover a curve OZ of this kind may be considered to be raised above the axis *oz* (Fig. 1) , and an individual arc of which *OZ* corresponding to the abscissa $oz = z$ may be expressed by the formula $\Pi : z = \int \frac{\partial z}{\sqrt{Z}}$; and this curve itself will be endowed with this conspicuous property, so that for any arc FG taken on that as it pleases, from some other point X an arc XY always shall be able to be cut off equal to that arc *FG* , the solution of which will be supplied by the demonstration of the following problem.

PROBLEM 1

§. 81. *If on the curve in the manner described there may be put in place some arc FG, innumerable other arcs XY to be assigned to the same curve geometrically, which shall be equal to that same arc FG.*

SOLUTION

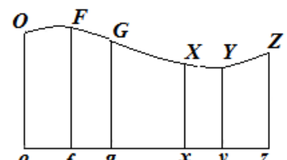

Fig. 1

With the applied lines F*f* and G*g* drawn from the points F and G to the axis *oz* the abscissas may be called $of = f$ and $og = g$ and the arcs will be $OF = \Pi : f$ and $OG = \Pi : g$, from which the length of the proposed arc FG will be $\Pi : g - \Pi : f$. In a similar manner for any arc sought XY the abscissas may be called $ox = x$ and $oy = y$ and the arcs will be $OX = \Pi : x$ and $OY = \Pi : y$, and thus the arc $XY = \Pi : y - \Pi : x$; which since it must become equal to the arc FG, this equation will be had :

$$\Pi : y - \Pi : x = \Pi : g - \Pi : f \ ,$$

which it will be required to satisfy.

§. 82. Since the points F and G may be considered as fixed , while the points X and Y are able to be varied through the whole curve, differentiation will present us with this equation $\partial\Pi : y - \partial\Pi : x = 0$. Whereby, since there shall be by hypothesis :

$$\Pi : x = \int\frac{\partial x}{\sqrt{X}} \ \text{ et } \ \Pi : y = \int\frac{\partial y}{\sqrt{Y}}$$

with there being present:

$$X = \alpha + \beta x + \gamma xx + \delta x^{3} + \varepsilon x^{4} \ \text{ and } \ Y = \alpha + \beta y + \gamma yy + \delta y^{3} + \varepsilon y^{4},$$

the solution of the problem has led us to this differential equation:

$$\frac{\partial y}{\sqrt{Y}} - \frac{\partial x}{\sqrt{X}} = 0.$$

§.83. Now here calling into help the method of the illustrious *Lagrange* we may put in place

$$\frac{\partial x}{\sqrt{X}} = \partial t$$

and there will be $\frac{\partial y}{\sqrt{Y}}=\partial t$. Evidently here we introduce a new element ∂t into the calculation, which in the following differentiations may be treated as constant; then truly we will have

$$\frac{\partial x}{\partial t}=\sqrt{X} \text{ and } \frac{\partial y}{\partial t}=\sqrt{Y}.$$

So that if therefore again we may put

$$y+x=p \text{ and } y-x=q,$$

hence we will have

$$\frac{\partial p}{\partial t}=\sqrt{Y}+\sqrt{X} \text{ and } \frac{\partial q}{\partial t}=\sqrt{Y}-\sqrt{X},$$

of which the product of the formulas will give

$$\frac{\partial p\partial q}{\partial t^2}=Y-X.$$

Therefore with the value substituted in place of Y and X there will be

$$\frac{\partial p\partial q}{\partial t^2}=\beta(y-x)+\gamma\left(y^2-x^2\right)+\delta\left(y^3-x^3\right)+\varepsilon\left(y^4-x^4\right).$$

Whereby, since there shall be

$$y=\frac{p+q}{2} \text{ and } x=\frac{p-q}{2},$$

there will become:

$$y-x=q,\ y^2-x^2=pq,\ y^3-x^3=\tfrac{1}{4}q(3pp+qq)$$

and

$$y^4-x^4=\tfrac{1}{2}pq(pp+qq),$$

with which substituted and with division made by q there will be had :

$$\frac{\partial p\partial q}{q\partial t^2}=\beta+\gamma p+\tfrac{1}{4}\delta p(3pp+qq)+\tfrac{1}{2}p(pp+qq),$$

of which equation the greatest use will be in the following calculation.

§.84. Now with the squares taken the first equations will give :

$$\frac{\partial x^2}{\partial t^2}=\mathrm{X}\ \text{ et }\ \frac{\partial y^2}{\partial t^2}=\mathrm{Y}$$

which may be differentiated again, where finally we may put for brevity,

$$\partial\mathrm{X}=\mathrm{X}'\partial x\ \text{ et }\ \partial\mathrm{Y}=\mathrm{Y}'\partial y$$

and hence we will obtain

$$\frac{2\partial\partial x}{\partial t^2}=\mathrm{X}'\ \text{ et }\ \frac{2\partial\partial y}{\partial t^2}=\mathrm{Y}'$$

with which added there will become

$$\frac{2\partial\partial p}{\partial t^2}=\mathrm{X}'+\mathrm{Y}'.$$

Therefore since there shall be

$$\mathrm{X}'=\beta+2\gamma x+3\delta xx+4\varepsilon x^3\ \text{ and }\ \mathrm{Y}'=\beta+2\gamma y+3\delta y^2+4\varepsilon y^3,$$

there will become

$$\frac{2\partial\partial p}{\partial t^2}=2\beta+2\gamma\left(x+y\right)+3\delta\left(x^2+y^2\right)+4\varepsilon\left(x^3+y^3\right).$$

Therefore by introducing as before the letters p and q there will become

$$x+y=p,\ \ x^2+y^2=\tfrac{1}{2}\left(pp+qq\right),\ \ x^3+y^3=\tfrac{1}{4}p\left(pp+3qq\right),$$

and thus this same equation will adopt this form

$$\frac{2\partial\partial p}{\partial t^2}=2\beta+2\gamma p+\tfrac{3}{2}\delta\left(pp+qq\right)+\varepsilon p\left(pp+3qq\right).$$

§.85. Now from this latter equation the preceding taken twice may be subtracted and there will remain

$$\frac{2\partial\partial p}{\partial t^2}-\frac{2\partial p\partial q}{q\partial t^2}=\delta qq+2\varepsilon pqq.$$

Hence on dividing by qq we will have

$$\frac{1}{\partial t^2}\left(\frac{2\partial\partial p}{qq}-\frac{2\partial p\partial q}{q^3}\right)=\delta+2\varepsilon p.$$

and each part of which clearly admits integration, if it may be multiplied by the element ∂p. Indeed with this done the equation of the integral will be

$$\frac{\partial p^2}{qq\partial t^2}=C+\delta p+\varepsilon pp.$$

§.86. But in the first place we see that there is $\frac{\partial p}{\partial t}=\sqrt{X}+\sqrt{Y}$ and hence immediately we come upon this algebraic integral equation

$$\frac{\left(\sqrt{X}+\sqrt{Y}\right)^2}{qq}=C+\delta p+\varepsilon pp.$$

Whereby since there shall be $p=x+y$ and $q=y-x$, this equation expanded out will become

$$\frac{X+Y+2\sqrt{X}\sqrt{Y}}{(y-x)^2}=C+\delta(x+y)+\varepsilon(x+y)^2,$$

where the constant introduced by the integration according to the nature of the problem, so that, while the point X is incident on the point F, the point Y may fall on the point G, or so that on making $x=f$ there becomes $y=g$.

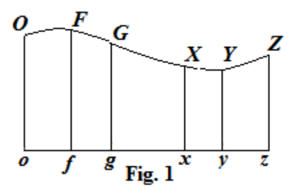

Fig. 1

§.87. Now since there shall be

$$X+Y=2\alpha+\beta(x+y)+\gamma\left(x^2+y^2\right)+\delta\left(x^3+y^3\right)+\varepsilon\left(x^4+y^4\right),$$

if above we transfer the terms $\delta(x+y)+\varepsilon(x+y)^2$ to the other side, we will arrive at this equation :

$$\frac{2\alpha+\beta(x+y)+\gamma\left(x^2+y^2\right)+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{XY}}{(y-x)^2}=C.$$

But above we may subtract γ from each side, and in place of $C-\gamma$ we may write Δ and in this manner our equation will be reduced to this neat enough form :

$$\frac{2\alpha+\beta(x+y)+2\gamma xy+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{XY}}{(y-x)^2}=\Delta.$$

§.88. Now thus because Δ must be determined, so that on assuming $x = f$, there may become $y = g$, if following the analogy we may put

$$\alpha+\beta f+\gamma ff+\delta f^3+\varepsilon f^4=\mathrm{F} \text{ and } \alpha+\beta g+\gamma gg+\delta g^3+\varepsilon g^4 =\mathrm{G},$$

this same constant Δ will be expressed thus :

$$\Delta=\frac{2\alpha+\beta(f+g)+2\gamma fg+\delta fg(f+g)+2\varepsilon ffgg+2\sqrt{\mathrm{FG}}}{(g-f)^2}.$$

Therefore if , with this equation found, some value as it pleases may be attributed to *x* itself , then the value of *y* will be able to be elicited, thus so that the other term X of the arc sought XY shall be able to be assumed arbitrarily. Truly it is readily apparent that determination precipitates an extremely troublesome calculation, since indeed the equation found from taking the square must be freed from the irrationality $\sqrt{\mathrm{XY}}$. But the investigation will be able to be supported by the following calculation.

§.89. Since this formula

$$2\alpha+\beta(x+y)+\gamma\left(x^2+y^2\right)+\delta xy(x+y)+2\varepsilon xxyy$$

essentially inters into the calculation, in its place for the sake of brevity we may write this symbol $[x, y]$, therefore its value will be known, even if other values may be accepted in place of x and y. Therefore the equation found will be able to be referred to in this manner :

$$\frac{[x,y]+2\sqrt{\mathrm{XY}}}{(y-x)^2}=\frac{[f,g]+2\sqrt{\mathrm{FG}}}{(g-f)^2}.$$

which equation therefore expresses the relation between the two ordinates *x* and *y*, in order that the problem is satisfied, that is, so that there becomes

$$\Pi:y-\Pi:x=\Pi:g-\Pi:f.$$

Whereby since hence also there may follow :

$$\Pi:y-\Pi:g=\Pi:x-\Pi:f,$$

this equation hence arises

$$\frac{[g,y]+2\sqrt{\mathrm{GY}}}{(y-g)^2}=\frac{[f,x]+2\sqrt{\mathrm{FX}}}{(x-f)^2}.$$

§.90. Now from this equation taken jointly with the earlier one, it will be possible to eliminate the formula of the root $\sqrt{Y}$ and thus the equation will be had involving only the letter *y*, from which its value can be defined without difficulty. But this calculation requiring to be put in place will appear to arrive only at a square equation, thus so that, thus so that both values may be found for Y , just as the nature of the problem demands, while with the point X assumed the other point Y will be able to fall on the right as well as on the left. Hence moreover, we shall not tarry further on the calculation, since here chiefly it is proposed : the solution of this whole problem to be returned by a direct method from the beginning.

HYPOTHESIS 2 .

§.91. With the curve OZ established above the axis *oz* (Fig. 2) described in the first hypothesis, another curve $\mathfrak{O}\mathfrak{Z}$ may be considered described above the same axis may be prepared thus, so that for the abscissa $oz = z$ there may correspond the arc $\mathfrak{O}\mathfrak{Z} = \Phi : z$, thus so that there shall be

$$\Phi : z = \int \frac{\partial z\left(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}+z^3+\text{etc.}\right)}{\sqrt{Z}},$$

with this equally assumed of the integral, so that it may vanish on putting $z = 0$, with there being as before,

$$Z = \alpha + \beta z + \gamma zz + \delta z^3 + \varepsilon z^4.$$

Just as for the numerator we may put for brevity,
$\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \text{etc.} = \mathfrak{Z}$, thus so that there shall be :

$$\Phi : z = \int \frac{\mathfrak{Z}\partial z}{\sqrt{Z}}$$

§.92. Now this same curve described by this ratio will be endowed with this conspicuous property, so that, if in the former curve the arcs FG and XY were cut equal to each other, with the same applied lines produced into the new curve the difference of the arcs of the curve $\mathfrak{F}\mathfrak{G}$ and $\mathfrak{X}\mathfrak{Y}$ cut in this manner may be able to be assigned either algebraically, or perhaps by logarithms or circular arcs, concerning the truth of which the solution of the following problem will demonstrate.

PROBLEM 2

§.93. *If two equal arcs* FG *and* XY *were described on the curve following the first hypothesis and with these on the curve just described there may correspond the arcs* $\mathfrak{FG}$ *and* $\mathfrak{XY}$, *with which clearly the same abscissas on the axis may agree, to investigate the difference between these two arcs.*

SOLUTION

Therefore because here the difference is sought between the arcs $\mathfrak{FG}$ and $\mathfrak{XY}$, that may be put $=\mathrm{V}$, which therefore will be considered as a function of x and y, if indeed we may consider the points $\mathfrak{F}$ and $\mathfrak{G}$ as fixed. Therefore since there shall be

$$\text{arc } \mathfrak{FG}=\varPhi:g-\varPhi:f \text{ , and the arc } \mathfrak{XY}=\varPhi:y-\varPhi:x\,,$$

we will have

$$\varPhi:y-\varPhi:x=\varPhi:g-\varPhi:f+\mathrm{V}\,,$$

from which on differentiating we will have : $\frac{\mathfrak{Y}\partial y}{\sqrt{\mathrm{Y}}}-\frac{\mathfrak{X}\partial x}{\sqrt{\mathrm{X}}}=\partial\mathrm{V},$

because we consider the letters f and g to be constants.

§.94. Now we may put, as has been done above,

$$\frac{\partial x}{\sqrt{\mathrm{X}}}-\frac{\partial y}{\sqrt{\mathrm{Y}}}=\partial t$$

and this equation adopts this same form

$$\left(\mathfrak{Y}-\mathfrak{X}\right)\partial t=\partial\mathrm{V}.$$

Truly in the solution of the first problem deduced we have come to this final equation

$$\frac{\partial p^{2}}{qq\partial t^{2}}=\mathrm{C}+\delta p+\varepsilon pp.$$

from which there becomes:

$$\frac{\partial p}{q\partial t}=\sqrt{\left(\mathrm{C}+\delta p+\varepsilon pp\right)}=\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)},$$

and hence we deduce

$$\partial t = \frac{\partial p}{q\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}},$$

where there is $p = x + y$ and $q = y - x$. Therefore with this value introduced, the differential equation requiring to be resolved is :

$$\partial \mathrm{V} = \frac{(\mathfrak{Y}-\mathfrak{X})\partial p}{q\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}},$$

where there is

$$\mathfrak{X} = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx + \mathfrak{D}x^3 + \text{etc.}$$

and in a similar manner:

$$\mathfrak{Y} = \mathfrak{A} + \mathfrak{B}y + \mathfrak{C}yy + \mathfrak{D}y^3 + \text{etc.},$$

to be continued as long as it pleases.

§.95. But if now we may substitute these values, we will have

$$\mathfrak{Y} - \mathfrak{X} = \mathfrak{B}(y-x) + \mathfrak{C}\left(y^2-x^2\right) + \mathfrak{D}\left(y^3-x^3\right) + \mathfrak{E}\left(y^4-x^4\right) + \text{etc.},$$

from which, if in place of x and y we may introduce the quantities *p* and *q*, on account of $x = \frac{p-q}{2}$ and $y = \frac{p+q}{2}$, the following values will arise :

$$y - x = q,\ y^2 - x^2 = pq,\ y^3 - x^3 = \tfrac{1}{4}q(3pp+qq),$$
$$y^4 - x^4 = \tfrac{1}{2}pq(pp+qq),\ y^5 - x^5 = \tfrac{1}{16}q\left(5p^4 + 10ppqq + q^4\right) \text{ etc.}$$

§.96. Truly the quantity V is determined by the following integral formulas according to the number of the letters $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc.

$$\partial \mathrm{V} = \mathfrak{B}\int \frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \mathfrak{C}\int \frac{p\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$
$$+ \tfrac{1}{4}\mathfrak{D}\int \frac{(3pp+qq)\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \tfrac{1}{2}\mathfrak{E}\int \frac{p(pp+qq)\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$
$$+ \tfrac{1}{16}\mathfrak{F}\int \frac{\left(5p^4+10ppqq+q^4\right)\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} \text{ etc.}$$

Now the first two of these formulas are able to be shown absolutely, either algebraically, which arises, if $\varepsilon = 0$, or by logarithms, if the value of ε were positive, or by circular arcs, if the value of ε were negative. Truly the remaining formulas require a relation between *p* and *q*, which we will investigate next. Yet here it may be observed only even powers of *q* enter into these formulas.

§.97. But here the letter Δ may designate the same constant value, which we have now defined above, which was

$$\Delta = \frac{2\alpha+\beta(f+g)+2\gamma fg+\delta fg(f+g)+2\varepsilon ffgg+2\sqrt{FG}}{(g-f)^2}.$$

Truly besides since there must be:

$$\Phi : y - \Phi : x = \Phi : g - \Phi : f + V,$$

it is clear in the case, where $x = f$ and $y = g$, there must become $V = 0$; on account of which these integrals found for V thus must be taken, so that on putting $p = f + g$ and $q = g - f$ the value of V may vanish.

Analysis for investigating the relation between *p* and *q*.

§.98. Now since we have found a finite equation between *x* and *y*, from which likewise on putting $y = \frac{p+q}{2}$ and $x = \frac{p-q}{2}$, a relation between the letters *p* and *q* may be able to be derived ; in truth this demands an exceedingly tedious calculation, on account of which we may undertake another way of deducing this relation from differential formulas. Indeed since there shall be

$$\frac{\partial p}{\partial q} = \frac{\partial y+\partial x}{\partial y-\partial x},$$

on account of the proportion

$$\partial x : \partial y = \sqrt{X} : \sqrt{Y}$$

there will be

$$\frac{\partial p}{\partial q} = \frac{\sqrt{Y}+\sqrt{X}}{\sqrt{Y}-\sqrt{X}};$$

but above we have found to be

$$\frac{\sqrt{Y}+\sqrt{X}}{q} = \sqrt{\left(\Delta + \gamma + \delta p + \varepsilon pp\right)},$$

where Δ may denote that same constant, just as we have found before.

§.99. Therefore now the fraction found for $\frac{\partial p}{\partial q}$ may be multiplied above and below by
$\sqrt{\mathrm{Y}}+\sqrt{\mathrm{X}}$,
and since there shall be

$$\left(\sqrt{\mathrm{Y}}+\sqrt{\mathrm{X}}\right)^2 = qq\left(\Delta+\gamma+\delta p+\varepsilon pp\right),$$

we will have this equation

$$\frac{\partial p}{\partial q}=\frac{qq(\Delta+\gamma+\delta p+\varepsilon pp)}{Y-X},$$

the denominator of which we have set out now above in § 83, where we have found to be

$$\mathrm{Y}-\mathrm{X}=\beta q+\gamma pq+\tfrac{1}{4}\delta q\left(3pp+qq\right)+\tfrac{1}{2}\varepsilon pq\left(pp+qq\right);$$

with which value substituted there will be

$$\frac{\partial p}{\partial q}=\frac{q(\Delta+\gamma+\delta p+\varepsilon pp)}{\beta+\gamma p+\frac{1}{4}\delta(3pp+qq)+\frac{1}{2}\varepsilon p(pp+qq)},$$

which is reduced to this form

$$2q\partial q=\frac{\left(2\beta+2\gamma p+\frac{1}{2}\delta(3pp+qq)+\varepsilon p(pp+qq)\right)\partial p}{\Delta+\gamma+\delta p+\varepsilon pp}.$$

§.100. We may transfer the terms, which contain *qq*, from the right to the left hand side, so that we may obtain this equation

$$2q\partial q-\frac{qq\partial p\left(\frac{1}{2}\delta+\varepsilon p\right)}{\Delta+\gamma+\delta p+\varepsilon pp}=\frac{\left(2\beta+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3\right)\partial p}{\Delta+\gamma+\delta p+\varepsilon pp}.$$

The left hand member of this equation can be rendered integrable, if it may be multiplied by a certain function of *p*, which shall be $=\Pi$, when there would be

$$\frac{\partial\Pi}{\Pi}=-\frac{\partial p\left(\frac{1}{2}\delta+\varepsilon p\right)}{\Delta+\gamma+\delta p+\varepsilon pp},$$

which equation integrated gives

$$l\Pi=-\tfrac{1}{2}l\left(\Delta+\gamma+\delta p+\varepsilon pp\right).$$

And thus this will be the multiplier

$$\Pi = \frac{1}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}};$$

while moreover the integral sought will be

$$\frac{qq}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} = \int \frac{\left(2\beta+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3\right)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}}.$$

§.101. This last integral evidently contains the form

$$\frac{pp}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}},$$

certainly its differential is

$$\frac{\left(2\Delta p+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3\right)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}};$$

whereby the integral can be represented thus

$$\frac{qq}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} = \frac{pp}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \int \frac{(2\beta-2\Delta p)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}},$$

which latter integral may be put

$$= \frac{m+np}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}};$$

of which indeed the differential is

$$\frac{\left((\Delta+\gamma)n-\frac{1}{2}\delta m+\left(\frac{1}{2}\delta n-\varepsilon m\right)p\right)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}}$$

and thus there must become

$$(\Delta+\gamma)n-\tfrac{1}{2}\delta m = 2\beta \text{ and } \tfrac{1}{2}\delta n-\varepsilon m = -2\Delta$$

from which the values may be deduced

$$m=\frac{4\beta\delta+8\Delta\Delta+8\Delta\gamma}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta}\quad\text{and}\quad n=\frac{8\beta\varepsilon+4\Delta\delta}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta},$$

of which we may retain the letters *m* and *n* of the fraction in place in the calculation ; consequently with a constant added thus the equation of the integral thus will itself be had :

$$qq=pp+np+m+\mathrm{C}\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}\ .$$

§.102. But this constant must be defined thus, so that on putting $p=f+g$ there becomes $q=g-f$, from which quantity that constant thus will be determined

$$\mathrm{C}=\frac{4fg+n(f+g)+m}{\sqrt{\left(\Delta+\gamma+\delta(f+g)+\varepsilon(f+g)^2\right)}}.$$

Therefore with this value found it will be easy to assign values not only of *qq*, but also of its even powers $q^4,\ q^6,\ q^8$ etc., which we require. And hence it is understood for finding the value of V other formulas do not occur, except for those which involve the root quantity $\sqrt{\left(\Delta+\gamma+\delta(f+g)+\varepsilon(f+g)^2\right)}$, of which therefore the integration, unless it may be able to be performed algebraically, always can be expedited by logarithms or circular arcs. Moreover it is evident in the case, where $\varepsilon=0$, all the integrals can be expressed algebraically.

§.103. So that therefore if for the first curve OZ there were

$$\Pi{:}z=\int\frac{\partial z}{\sqrt{\left(\alpha+\beta z+\gamma z^2+\delta z^3\right)}},$$

truly for the other curve

$$\Phi{:}z=\int\frac{\partial z\left(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\text{etc.}\right)}{\sqrt{\left(\alpha+\beta z+\gamma z^2+\delta z^3\right)}},$$

while with equal arcs FG and XY assumed in the first curve, the arcs $\mathfrak{FG}$ and $\mathfrak{XY}$ will correspond to these on the other curve , the difference of which will be able always to be assigned geometrically. Meanwhile it will always be able to happen, that the difference V may become zero, which indeed always happens on supposing $x=f$.

§.104. Truly in addition another most remarkable case is given, because that difference V will be able to be expressed algebraically, which evidently always will happen, when both in the denominator as well as in the numerator only even powers of *z* occur, that is, if for the first curve there were :

$$\Pi{:}z = \int \frac{\partial z}{\sqrt{\left(\alpha+\gamma zz+ez^4\right)}},$$

and indeed for the other curve :

$$\Phi{:}z = \int \frac{\partial z\left(\mathfrak{A}+\mathfrak{C}zz+\mathfrak{E}z^4+\mathfrak{G}z^6+\text{etc.}\right)}{\sqrt{\left(\alpha+\gamma zz+ez^4\right)}}.$$

For in these cases, if in the first curve equal arcs FG and XY may be considered, while in the other curve the difference of the corresponding arcs $\mathfrak{FG}$ and $\mathfrak{XY}$ always will be able to be shown algebraically or geometrically, also for however many terms the numerator may be continued $\mathfrak{A}+\mathfrak{C}zz+\mathfrak{E}z^4+\text{etc.}$, and here it is the case, which I have treated further formerly both in the *Integral Calculus,* as well as elsewhere [E261].

§.105. Towards showing this, since we have both $\delta=0$ as well as $\beta=0$, in the first place there will be :

$$qq = pp+m+\mathrm{C}\sqrt{\left(\Delta+\gamma+\varepsilon pp\right)},$$

thus so that here only even powers of p may occur; then moreover the integral formulas for the German letters $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{G}$ etc. will themselves be had in the following way:
For the letter $\mathfrak{C}$ [*i.e.* C]

$$\int \frac{p\partial p}{\sqrt{\left(\Delta+\gamma+\varepsilon pp\right)}},$$

which by itself is absolutely integrable.
For the letter $\mathfrak{E}$ [*i.e.* E]

$$\int \frac{p\left(pp+qq\right)\partial p}{\sqrt{\left(\Delta+\gamma+\varepsilon pp\right)}},$$

which with the value substituted in place of *qq* will adopt this form

$$\int \frac{p(2pp+m)\partial p}{\sqrt{\left(\Delta+\gamma+\varepsilon pp\right)}}+\mathrm{C}\int p\partial p,$$

where the integration is clear, which also arises usually for the following formulas associated with the letters $\mathfrak{G}$ etc. [*i.e.* G]. For it is evident, if there may be put $\sqrt{(\Delta+\gamma+\varepsilon pp)}=s$, to become

$$pp=\frac{ss-\Delta-\gamma}{s} \text{ and } p\partial p=\frac{s\partial s}{s}$$

and thus

$$\frac{p\partial p}{\sqrt{(\Delta+\gamma+\varepsilon pp)}}=\frac{\partial s}{\varepsilon},$$

by which substitution all the formulas become rational and whole.

§.106. But since this latter case now shall be treated at enough length, and illustrated by several examples taken from the rectification of the ellipse and the hyperbola , the first case, where there was only $\varepsilon=0$, with that more attention worthy, because, a far indeed as I know, at this stage it has not been observed by anyone, therefore its development according to this new method is taken to be especially noteworthy. But since these have been deduced from the relation between p and q, thus also a most elegant relation will be able to be elicited between these quantities $p=x+y$ and $u=xy$, as we are going to adjoin below.

Analysis for investigating the relation between p and u.

§.107. Here initially we may enquire equally into the relation between ∂p and ∂u, and since there shall be

$$\frac{\partial p}{\partial u}=\frac{\partial x+\partial y}{y\partial x+x\partial y},$$

on account of $\partial x:\partial y=\sqrt{X}:\sqrt{Y}$ there will be

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

and with the squares taken

$$\frac{\partial p^2}{\partial u^2}=\frac{X+Y+2\sqrt{X}\sqrt{Y}}{yyX+xxY+2xy\sqrt{X}\sqrt{Y}}.$$

But above we have seen to be

$$\left(\sqrt{Y}+\sqrt{X}\right)^2=qq\left(\Delta+\gamma+\delta p+\varepsilon pp\right),$$

with there being $q=y-x$. But we may use for the denominator the relation found in § 87 :

$$\Delta=\frac{2\alpha+\beta(x+y)+2\gamma xy+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{\mathrm{XY}}}{(y-x)^2},$$

from which there becomes:

$$2\sqrt{\mathrm{XY}}=\Delta qq-2\alpha-\beta p-2\gamma u-\delta pu-2\varepsilon uu,$$

with which value substituted our equation will become

$$\frac{\partial p^2}{\partial u^2}=\frac{qq(\Delta+\gamma+\delta p+\varepsilon pp)}{yy\mathrm{X}+xx\mathrm{Y}+\Delta qqu-2\alpha u-\beta pu-2\gamma uu-\delta puu-2\varepsilon u^3}.$$

§.108. But here with the values substituted in place of X and Y, we will have initially :

$$yy\mathrm{X}+xx\mathrm{Y}=\alpha(xx+yy)+\beta xy(x+y)+2\gamma xxyy+\delta xxyy(x+y)+\varepsilon xxyy\left(xx+yy\right),$$

which on account of $x+y=p,\ xy=u$ and $xx+yy=pp-2u$, there will be

$$yy\mathrm{X}+xx\mathrm{Y}=\alpha\left(pp-2u\right)+\beta pu+2\gamma uu+\delta puu+\varepsilon uu(pp-2u),$$

from which the whole denominator will be found to become :

$$\alpha\left(pp-4u\right)+\varepsilon uu\left(pp-4u\right)+\Delta qqu;$$

whereby, since there shall be $pp-4u=qq$, our fraction will be

$$\frac{\partial p^2}{\partial u^2}=\frac{\Delta+\gamma+\delta p+\varepsilon pp}{\Delta u+\alpha+\varepsilon uu},$$

from which this separate equation follows :

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}};$$

from which there is deduced, this

Memorable Theorem.

§.109. *If this differential equation may be held between the two variables x and y*

$$\frac{\partial x}{\sqrt{\left(\alpha+\beta x+\gamma xx+\delta x^3+\varepsilon x^4\right)}}=\frac{\partial y}{\sqrt{\left(\alpha+\beta y+\gamma yy+\delta y^3+\varepsilon y^4\right)}},$$

then on putting $x+y=p$ and $xy=u$, this differential equation will always occur between these two variables p and u

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}},$$

where indeed Δ is an arbitrary constant introduced into the latter equation, truly counter wise the first equation contains the arbitrary constant β not occurring in the second equation.

§.110. But the integration of the latter equation is performed at once. Indeed if we may multiply each side by $\sqrt{\varepsilon}$, the integral is expressed by logarithms thus :

$$l\left(p\sqrt{\varepsilon}+\tfrac{\delta}{2\sqrt{\varepsilon}}+\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}\right)$$
$$=l\left(u\sqrt{\varepsilon}+\tfrac{\Delta}{2\sqrt{\varepsilon}}+\sqrt{(\alpha+\Delta u+\varepsilon uu)}\right)+l\Gamma$$

and the integral thus will be expressed algebraically :

$$\varepsilon p+\tfrac{1}{2}\delta+\sqrt{\varepsilon}\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}=\Gamma\left(\varepsilon u+\tfrac{1}{2}\Delta+\sqrt{\varepsilon}\,(\alpha+\Delta u+\varepsilon uu)\right).$$

Where this same constant Γ is defined easily from the condition, so that on putting $x=f$ there must become $y=g$, that is, so that on putting $p=f+g$ there becomes $u=fg$, certainly from which condition the former constant Δ now has been defined.

§.111. So that hence it may now be possible to define more easily either p in terms of u or u in terms of p, there may be observed to be :

$$\frac{1}{\varepsilon p+\frac{1}{2}\delta+\sqrt{\varepsilon}\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\varepsilon p+\frac{1}{2}\delta-\sqrt{\varepsilon}\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}{\frac{1}{4}\delta\delta-\varepsilon(\Delta+\gamma)}$$

and

$$\frac{1}{\varepsilon u+\frac{1}{2}\Delta+\sqrt{\varepsilon}\left(\alpha+\Delta p+\varepsilon uu\right)}=\frac{\varepsilon u+\frac{1}{2}\Delta-\sqrt{\varepsilon}\left(\alpha+\Delta u+\varepsilon uu\right)}{\frac{1}{4}\Delta\Delta-\alpha\varepsilon}$$

Hence by inversion the following equation therefore will result :

$$\frac{\varepsilon p+\frac{1}{2}\delta-\sqrt{\varepsilon}\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}{\frac{1}{4}\delta\delta-\varepsilon\left(\Delta+\gamma\right)}=\frac{1}{\Gamma}\cdot\frac{\varepsilon u+\frac{1}{2}\Delta-\sqrt{\varepsilon}\left(\alpha+\Delta u+\varepsilon uu\right)}{\frac{1}{4}\Delta\Delta-\varepsilon\left(\Delta+\gamma\right)}$$

or

$$\varepsilon p+\tfrac{1}{2}\delta-\sqrt{\varepsilon}\left(\Delta+\gamma+\delta p+\varepsilon pp\right)$$
$$=\frac{\frac{1}{4}\delta\delta-\varepsilon\left(\Delta+\gamma\right)}{\Gamma\left(\frac{1}{4}\Delta\Delta-\varepsilon\left(\Delta+\gamma\right)\right)}\cdot\left(\varepsilon u+\tfrac{1}{2}\Delta-\sqrt{\varepsilon}\left(\alpha+\Delta u+\varepsilon uu\right)\right)$$

from which two equations without any other trouble either *p* will be able to be expressed in terms of *u* or *u* in terms of *p*.

§.112. Therefore in this way in place of the variable *p* required for finding the quantity V it may be easy to introduce the variable *u*, if indeed in place of the formula

$$\frac{\partial p}{\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}},$$

the formula may be substituted equal to this

$$\frac{\partial u}{\sqrt{\left(\alpha+\Delta u+\varepsilon uu\right)}}.$$

Truly in this way those cases, in which the quantity V can become algebraic, may appear to be not so easy ; yet meanwhile in this way we will be certain all the algebraic integrals must succeed both for the cases in which $\varepsilon=0$, as well as where $\beta=0$, $\delta=0$ and in the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc., where even powers only occur. Instead of ending at this point, we will investigate another relation between the quantities *p* and *u* , the consideration of which may be seen promise a significant increase in the integration of equations.

Another analysis for the investigation of the relation between *p* and *u*

§.113. Since there shall be, as we have seen before,

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

we may multiply above and below by $\sqrt{X}+\sqrt{Y}$,so that the numerator may become

$$\left(\sqrt{X}+\sqrt{Y}\right)^2 = qq\left(\Delta+\gamma+\delta p+\varepsilon pp\right);$$

but then the denominator will be produced :

$$yX + xY + \left(x+y\right)\sqrt{XY},$$

where the rational part of the denominator gives :

$$\alpha\left(x+y\right)+2\beta xy+\gamma xy\left(x+y\right)+\delta xy\left(xx+yy\right)+\varepsilon xy\left(x^3+y^3\right),$$

which expression, on account of $x+y=p,\ y-x=q$ and $xy=u$, will be changed into

$$\alpha p+2\beta u+\gamma pu+\delta u(pp-2u)+\varepsilon pu(pp-3u).$$

Then as we have seen to be

$$2\sqrt{XY}=\Delta qq-2\alpha-\beta p-2\gamma u-\delta pu-2\varepsilon uu,$$

which multiplied by $\frac{1}{2}p$ and added to the above gives

$$\tfrac{1}{2}\Delta pqq-\tfrac{1}{2}\beta\left(pp-4u\right)+\tfrac{1}{2}\delta u\left(pp-4u\right)+\varepsilon pu\left(pp-4u\right),$$

whereby the denominator on account of $pp-4u=qq$ will adopt this form

$$\tfrac{1}{2}\Delta pqq-\tfrac{1}{2}\beta qq+\tfrac{1}{2}\delta uqq+\varepsilon puqq;$$

hence the equation will be

$$\frac{\partial p}{\partial u}=\frac{\Delta+\gamma+\delta p+\varepsilon pp}{\frac{1}{2}\Delta p-\frac{1}{2}\beta+\frac{1}{2}\delta u+\varepsilon pu},$$

from which there is deduced :

$$\partial p\left(\tfrac{1}{2}\Delta p-\tfrac{1}{2}\beta+\tfrac{1}{2}\delta u+\varepsilon pu\right)=\partial u\left(\Delta+\gamma+\delta p+\varepsilon pp\right),$$

which therefore surely is integrable ; that which thus may thence be apparent, because the other variable u never rises beyond the first dimension.

§.114. Truly at this stage the relation between p and u can be investigated in another way ; evidently the first equation found :

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

if it may be multiplied above and below by $\sqrt{Y}-\sqrt{X}$, will give

$$\frac{\partial p}{\partial u}=\frac{Y-X}{-yX+xY+\sqrt{XY}\left(y-x\right)}$$

Therefore we will have now for the numerator :

$$\beta q+\gamma pq+\delta q(pp-u)+\varepsilon pq\left(pp-2u\right).$$

Truly the rational part for the denominator will be

$$-\alpha q+\gamma qu+\delta pqu+\varepsilon qu\left(pp-u\right),$$

truly the irrational part

$$\tfrac{1}{2}\Delta q^{3}-\alpha q-\tfrac{1}{2}\beta pq-\gamma qu-\tfrac{1}{2}\delta pqu-\varepsilon quu,$$

from which the whole denominator is made :

$$\tfrac{1}{2}\Delta q^{3}-2\alpha q-\tfrac{1}{2}\beta pq+\tfrac{1}{2}\delta pqu+\varepsilon qu\left(pp-2u\right),$$

from which this differential equation follows

$$\frac{\partial p}{\partial u}=\frac{\beta+\gamma p+\delta(pp-u)+\varepsilon p(pp-2u)}{\frac{1}{2}\Delta(pp-4u)-2\alpha-\frac{1}{2}\beta p+\frac{1}{2}\delta pu+\varepsilon u(pp-2u)},$$

which arranged in order thus becomes

$$\begin{aligned}&\partial p\left(\Delta\left(pp-4u\right)-4\alpha-\beta p+\delta pu+2\varepsilon u\left(pp-2u\right)\right)\\&=2\partial u\left(\beta+\gamma p+\delta(pp-u)+\varepsilon p\left(pp-2u\right)\right),\end{aligned}$$

which thus has now been prepared, so that no way may be seen by which its integration may be established, even if we may actually be able to show its integral.

§.115. It is possible to define a relation between *p* and *u* in another way, if we may multiply the latter part of the equation

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

above and below by $y\sqrt{X}-x\sqrt{Y}$, so that there may be produced :

$$\frac{\partial p}{\partial u}=\frac{y\sqrt{X}-x\sqrt{Y}+(y-x)\sqrt{XY}}{yyX-xxY}.$$

Now indeed the denominator appears

$$\alpha pq+\beta qu-\delta quu-\varepsilon pquu .$$

Moreover for the numerator, rational part gives

$$\alpha q-\gamma qu-\delta pqu-\varepsilon qu\left(pp-u\right)$$

and the irrational part

$$\tfrac{1}{2}\Delta q^{3}-\alpha q-\tfrac{1}{2}\beta pq-\gamma qu-\tfrac{1}{2}\delta pqu-\varepsilon quu;$$

therefore the whole numerator will become:

$$\tfrac{1}{2}\Delta q^{3}-\tfrac{1}{2}\beta pq-2\gamma qu-\tfrac{3}{2}\delta pqu-\varepsilon qupp$$

and thus

$$\frac{\partial p}{\partial u}=\frac{\frac{1}{2}\Delta(pp-4u)-\frac{1}{2}\beta p-2\gamma u-\frac{3}{2}\delta pu-\varepsilon ppu}{\alpha p+\beta u-\delta uu-\varepsilon puu}$$

or

$$2\partial p\left(\alpha p+\beta u-\delta uu-\varepsilon puu\right)=\partial u\left(\Delta\left(pp-4u\right)-\beta p-4\gamma u-3\delta pu-2\varepsilon ppu\right).$$

But here it does not appear clear, how this equation must be multiplied in order to become integrable, but from which there is no doubt, why this may not be able to be brought about, extending the limits of analysis with this being considered.

2) Methodus succinctior comparationes quantitatum transcendentium in forma $\int\frac{P\partial z}{\sqrt{\left(A+2Bz+Czz\ +2Dz^{3}+Ez^{4}\right)}}$ contentarum inveniendi

M. S. Academiae exhib. die 3 Novembris 1777
Institutiones calculi integralis 4, 1794, p. 504-524
[E676]

In Capite VI Sect. II *Institutionum* mearum *Calculi Integralis* Tom. I insignes tradidi comparationes inter quantitates maxime transcendentes, ad quas deductus eram methodo penitus indirecta. Postquam igitur non ita pridem illustris *de la Grange* methodum maxime ingeniosam excogitasset easdem comparationes inveniendi, totum hoc argumentum multo succinctius et elegantius tractari poterit, quam mihi quidem tum temporis licebat, unde sequentia Supplementa Geometris haud displicebunt.

Hypothesis 1.

§. 80. Denotet hic perpetuo character $\Pi : z$ valorem formulae integralis

$$\int\frac{\partial z}{\sqrt{\left(\alpha+\beta z+\gamma zz\ +\delta z^{3}+\varepsilon z^{4}\right)}}$$

ita sumtae, ut evanescat posito $z=0$. Ponatur autem brevitatis gratia

$$\alpha+\beta z+\gamma zz+\delta z^{3}+\varepsilon z^{4}=Z,$$

ita ut sit

$$\Pi : z=\int\frac{\partial z}{\sqrt{Z}}$$

Tum vero concipiatur super axe *oz* (Fig. 1) exstructa eiusmodi curva OZ, cuius singuli arcus *OZ* abscissis $oz=z$ respondentes exprimantur per formulam $\Pi : z=\int\frac{\partial z}{\sqrt{Z}}$; atque haec curva ista insigni proprietate erit praedita, ut sumto in ea pro lubitu arcu quocunque FG a quovis alio puncto X semper arcus XY illi arcui *FG* aequalis geometrice abscindi possit, cuius demonstrationem solutio sequentis problematis suppeditabit.

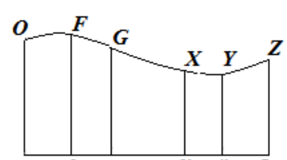

Fig. 1

PROBLEMA 1

§. 81. *Si in curva modo descripta proponatur arcus quicunque FG, innumerabiles* *alios arcus XY in eadem curva geometrice assignare, qui singuli eidem arcui FG sint aequales.*

SOLUTIO

Ductis ex punctis F et G ad axem *oz* applicatis F*f* et G*g* vocentur abscissae $of = f$ et $og = g$ eruntque arcus $OF = \Pi : f$ et $OG = \Pi : g$, unde longitudo arcus propositi FG erit $\Pi : g - \Pi : f$. Simili modo pro quovis arcu quaesito XY vocentur abscissae $ox = x$ et $oy = y$ eruntque arcus $OX = \Pi : x$ et $OY = \Pi : y$ ideoque arcus $XY = \Pi : y - \Pi : x$; qui cum aequalis esse debeat arcui FG, habebitur ista aequatio

$$\Pi : y - \Pi : x = \Pi : g - \Pi : f,$$

cui satisfieri oportet.

§. 82. Quoniam puncta F et G considerantur ut fixa, dum puncta X et Y per totam curvam variari possunt, differentiatio nobis praebebit hanc aequationem $\partial\Pi : y - \partial\Pi : x = 0$. Quare, cum sit per hypothesin

$$\Pi : x = \int \frac{\partial x}{\sqrt{X}} \text{ et } \Pi : y = \int \frac{\partial y}{\sqrt{Y}}$$

existente

$$X = \alpha + \beta x + \gamma xx + \delta x^3 + \varepsilon x^4 \text{ et } Y = \alpha + \beta y + \gamma yy + \delta y^3 + \varepsilon y^4,$$

solutio problematis perducta est ad hanc aequationem differentialem

$$\frac{\partial y}{\sqrt{Y}} - \frac{\partial x}{\sqrt{X}} = 0.$$

§.83. Hic iam methodum ill. DE LA GRANGE in subsidium vocantes statuamus

$$\frac{\partial x}{\sqrt{X}} = \partial t$$

eritque $\frac{\partial y}{\sqrt{Y}} = \partial t$. Hic scilicet novum elementum ∂t in calculum introducimus, quod in sequentibus differentiationibus ut constans tractetur; tum igitur habebimus

$$\frac{\partial x}{\partial t} = \sqrt{X} \text{ et } \frac{\partial y}{\partial t} = \sqrt{Y}.$$

Quodsi ergo porro statuamus

$$y + x = p \text{ et } y - x = q,$$

habebimus hinc

$$\frac{\partial p}{\partial t} = \sqrt{Y} + \sqrt{X} \text{ et } \frac{\partial q}{\partial t} = \sqrt{Y} - \sqrt{X},$$

quarum formularum productum praebet

$$\frac{\partial p\partial q}{\partial t^2}=\mathrm{Y}-\mathrm{X}.$$

Valoribus ergo loco Y et X substitutis erit

$$\frac{\partial p\partial q}{\partial t^2}=\beta\left(y-x\right)+\gamma\left(y^2-x^2\right)+\delta\left(y^3-x^3\right)+\varepsilon\left(y^4-x^4\right).$$

Quare, cum sit

$$y=\tfrac{p+q}{2}\ \text{ et }\ x=\tfrac{p-q}{2},$$

erit

$$y-x=q,\ y^2-x^2=pq,\ y^3-x^3=\tfrac{1}{4}q\left(3pp+qq\right)$$

et

$$y^4-x^4=\tfrac{1}{2}pq\left(pp+qq\right),$$

quibus substitutis factaque divisione per q habebitur

$$\frac{\partial p\partial q}{q\partial t^2}=\beta+\gamma p+\tfrac{1}{4}\delta p\left(3pp+qq\right)+\tfrac{1}{2}p\left(pp+qq\right),$$

cuius aequationis plurimus erit usus in sequenti calculo.

§.84. Iam sumtis quadratis primae aequationes dabunt

$$\frac{\partial x^2}{\partial t^2}=\mathrm{X}\ \text{ et }\ \frac{\partial y^2}{\partial t^2}=\mathrm{Y}$$

quae denuo differentientur, quem infinem ponamus brevitatis gratia

$$\partial\mathrm{X}=\mathrm{X}'\partial x\ \text{ et }\ \partial\mathrm{Y}=\mathrm{Y}'\partial y$$

atque hinc nanciscemur

$$\frac{2\partial\partial x}{\partial t^2}=\mathrm{X}'\ \text{ et }\ \frac{2\partial\partial y}{\partial t^2}=\mathrm{Y}'$$

quibus additis erit

$$\frac{2\partial\partial p}{\partial t^2}=\mathrm{X}'+\mathrm{Y}'.$$

Cum igitur sit

$$\mathrm{X}'=\beta+2\gamma x+3\delta xx+4\varepsilon x^3\ \text{ et }\ \mathrm{Y}'=\beta+2\gamma y+3\delta y^2+4\varepsilon y^3,$$

erit

$$\frac{2\partial\partial p}{\partial t^2}=2\beta+2\gamma\left(x+y\right)+3\delta\left(x^2+y^2\right)+4\varepsilon\left(x^3+y^3\right).$$

Introducendo igitur litteras p et q ut ante fiet

$$x+y=p,\ \ x^2+y^2=\tfrac{1}{2}\left(pp+qq\right),\ \ x^3+y^3=\tfrac{1}{4}p\left(pp+3qq\right),$$

sicque ista aequatio hanc induet formam

$$\frac{2\partial\partial p}{\partial t^2}=2\beta+2\gamma p+\tfrac{3}{2}\delta\left(pp+qq\right)+\varepsilon p\left(pp+3qq\right).$$

§.85. Ab hac iam postrema aequatione subtrahatur praecedens bis sumta ac remanebit

$$\frac{2\partial\partial p}{\partial t^2}-\frac{2\partial p\partial q}{q\partial t^2}=\delta qq+2\varepsilon pqq.$$

Hinc per qq dividendo habebimus

$$\frac{1}{\partial t^2}\left(\frac{2\partial\partial p}{qq}-\frac{2\partial p\partial q}{q^3}\right)=\delta+2\varepsilon p.$$

cuius utrumque membrum manifesto integrationem admittit, si ducatur in elementum ∂p. Hoc enim facto aequatio integralis erit

$$\frac{\partial p^2}{qq\partial t^2}=\mathrm{C}+\delta p+\varepsilon pp.$$

§.86. Initio autem vidimus esse $\frac{\partial p}{\partial t}=\sqrt{\mathrm{X}}+\sqrt{\mathrm{Y}}$ hincque statim pervenimus ad aequationem integralem algebraicam hanc

$$\frac{\left(\sqrt{\mathrm{X}}+\sqrt{\mathrm{Y}}\right)^2}{qq}=\mathrm{C}+\delta p+\varepsilon pp.$$

Quare cum sit $p=x+y$ et $q=y-x$, haec aequatio evoluta fiet

$$\frac{\mathrm{X+Y}+2\sqrt{\mathrm{X}}\sqrt{\mathrm{Y}}}{(y-x)^2}=\mathrm{C}+\delta\left(x+y\right)+\varepsilon\left(x+y\right)^2,$$

ubi constantem per integrationem ingressam secundum indolem problematis ita definiri oportet, ut, dum punctum X incidit in punctum F, punctum Y in ipsum punctum G cadat, sive ut facto $x=f$ fiat $y=g$.

§.87. Cum iam sit

$$X+Y=2\alpha+\beta(x+y)+\gamma\left(x^2+y^2\right)+\delta\left(x^3+y^3\right)+\varepsilon\left(x^4+y^4\right),$$

si terminos $\delta(x+y)+\varepsilon(x+y)^2$ in alteram partem transferimus, perveniemus ad hanc aequationem

$$\frac{2\alpha+\beta(x+y)+\gamma\left(x^2+y^2\right)+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{XY}}{(y-x)^2}=C.$$

Subtrahamus autem insuper utrinque γ et loco $C-\gamma$ scribamus Δ hocque modo nostra aequatio reducetur ad hanc formam satis concinnam

$$\frac{2\alpha+\beta(x+y)+2\gamma xy+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{XY}}{(y-x)^2}=\Delta.$$

§.88. Quia nunc Δ ita determinari debet, ut sumto $x=f$ fiat $y=g$, si secundum analogiam statuamus

$$\alpha+\beta f+\gamma ff+\delta f^3+\varepsilon f^4=F \text{ et } \alpha+\beta g+\gamma gg+\delta g^3+\varepsilon g^4=G,$$

erit ista constans Δ ita expressa

$$\Delta=\frac{2\alpha+\beta(f+g)+2\gamma fg+\delta fg(f+g)+2\varepsilon ffgg+2\sqrt{FG}}{(g-f)^2}.$$

Hac igitur aequatione inventa si ipsi *x* pro lubitu tribuatur valor quicunque, inde elici poterit valor ipsius *y,* ita ut alter terminus X arcus quaesiti XY pro arbitrio assumi possit. Verum facile patet istam determinationem in calculos perquam molestos praecipitare, quandoquidem aequatio inventa quadratis sumendis ab irrationalitate $\sqrt{XY}$ liberari deberet. Sequenti autem modo ista investigatio sublevari poterit.

§.89. Quoniam ista formula

$$2\alpha+\beta(x+y)+\gamma\left(x^2+y^2\right)+\delta xy(x+y)+2\varepsilon xxyy$$

essentialiter in calculum ingreditur, eius loco brevitatis gratia scribamus hunc characterem $[x,\ y]$, cuius ergo valor erit cognitus, etiam si loco *x* et *y* aliae litterae accipiantur. Hoc igitur modo aequatio inventa ita referri poterit

$$\frac{[x,y]+2\sqrt{XY}}{(y-x)^{2}}=\frac{[f,g]+2\sqrt{FG}}{(g-f)^{2}}.$$

quae ergo aequatio exprimit relationem inter binas ordinatas *x* et *y*, ut problemati satisfiat, hoc est, ut fiat

$$\Pi:y-\Pi:x=\Pi:g-\Pi:f.$$

Quare cum hinc etiam sequatur

$$\Pi:y-\Pi:g=\Pi:x-\Pi:f\ ,$$

aequatio hinc ista exsurget

$$\frac{[g,y]+2\sqrt{GY}}{(y-g)^{2}}=\frac{[f,x]+2\sqrt{FX}}{(x-f)^{2}}.$$

§.90. Ex hac iam aequatione cum priore coniuncta facile eliminari poterit formula radicalis $\sqrt{Y}$ sicque aequatio habebitur tantum letteram *y* tanquam incognitam involvens, unde eius valor haud difficulter definiri potest. Calculum autem hunc instituenti patebit tantum ad aequationem quadraticam perveniri, ita ut bini valores pro puncto Y reperiantur, quemadmodum rei natura postulat, dum sumto puncto X alterum punctum Y tam dextrorsum quam sinistrorsum cadere poterit. Hinc autem calculo fusius non immoramur, quandoquidem hic potissimum est propositum totam huius problematis solutionem per methodum directam a priori repetere.

HYPOTHESIS 2

§.91. Constituta super axe *oz* (Fig. 2) curva OZ in priori hypothesi descripta concipiatur super eodem axe alia curva insuper descripta $\mathfrak{O}\mathfrak{Z}$ ita comparata, ut abscissae $oz = z$ respondeat arcus $\mathfrak{O}\mathfrak{Z} = \Phi : z$, ita ut sit

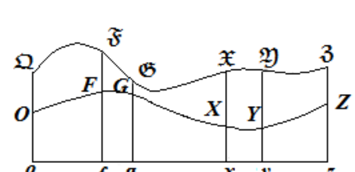

Fig. 2

$$\Phi:z=\int\frac{\partial z\left(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}+z^{3}+\text{etc.}\right)}{\sqrt{Z}}$$

integrali hoc pariter ita sumto, ut evanescat posito $z=0$, existente ut ante

$$Z=\alpha+\beta z+\gamma zz+\delta z^3+\varepsilon z^4.$$

Pro numeratore autem ponamus brevitatis gratia

$\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\text{etc.}=\mathfrak{Z}$, ita ut sit
$$\Phi:z=\int\frac{\mathfrak{Z}\partial z}{\sqrt{Z}}$$

§.92. Ista iam curva hac ratione descripta hac insigni proprietate erit praedita, ut, si in priore curva rescissi fuerint arcus FG et XY inter se aequales, productis iisdem applicatis in nova curva arcuum hoc modo rescissorum $\mathfrak{FG}$ et $\mathfrak{XY}$ differentia vel algebraice vel saltem per logarithmos et arcus circulares assignari possit, cuius rei veritatem solutio sequentis problematis demonstrabit.

PROBLEMA 2

§.93. *Si in curva secundum primam hypothesin descripta abscissi fuerint duo arcus aequales* FG *et* XY *iisque in curva modo descripta respondeant arcus* $\mathfrak{FG}$ et $\mathfrak{XY}$, quibus *scilicet eaedem abscissae in axe conveniant, differentiam inter hos binos arcus investigare.*

SOLUTIO

Quia igitur hic quaeritur differentia inter arcus $\mathfrak{FG}$ et $\mathfrak{XY}$, ponatur ea $=\mathrm{V}$, quae ergo spectari poterit tanquam certa functio ipsarum *x* et *y*, si quidem puncta $\mathfrak{F}$ et $\mathfrak{G}$ tanquam fixa consideramus. Cum igitur sit

$$\text{arcus}\,\mathfrak{FG}=\Phi:g-\Phi:f\ \text{ et arcus }\ \mathfrak{XY}=\Phi:y-\Phi:x,$$

habebimus
$$\Phi:y-\Phi:x=\Phi:g-\Phi:f+\mathrm{V},$$

unde differentiando habebimus
$$\frac{\mathfrak{Y}\partial y}{\sqrt{Y}}-\frac{\mathfrak{X}\partial x}{\sqrt{X}}=\partial\mathrm{V},$$

quia litteras *f* et *g* pro constantibus habemus.

§.94. Ponamus nunc, ut supra factum est,

$$\frac{\partial x}{\sqrt{X}}-\frac{\partial y}{\sqrt{Y}}=\partial t$$

et haec aequatio induet istam formam

$$(\mathfrak{Y}-\mathfrak{X})\partial t=\partial \mathrm{V}.$$

Verum in solutione primi problematis deducti fuimus ad hanc aequationem finalem

$$\frac{\partial p^2}{qq\partial t^2}=\mathrm{C}+\delta p+\varepsilon pp.$$

unde fit

$$\frac{\partial p}{q\partial t}=\sqrt{(\mathrm{C}+\delta p+\varepsilon pp)}=\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)},$$

atque hinc colligimus

$$\partial t=\frac{\partial p}{q\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}},$$

ubi est $p=x+y$ et $q=y-x$. Hoc ergo valore inducto aequatio differentialis resolvenda est

$$\partial \mathrm{V}=\frac{(\mathfrak{Y}-\mathfrak{X})\partial p}{q\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}},$$

ubi est

$$\mathfrak{X}=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx+\mathfrak{D}x^3+\text{etc.}$$

similique modo

$$\mathfrak{Y}=\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}yy+\mathfrak{D}y^3+\text{etc.},$$

quousque libuerit continuando.

§.95. Quodsi iam hos valores substituamus, habebimus

$$\mathfrak{Y}-\mathfrak{X}=\mathfrak{B}(y-x)+\mathfrak{C}\left(y^2-x^2\right)+\mathfrak{D}\left(y^3-x^3\right)+\mathfrak{E}\left(y^4-x^4\right)+\text{etc.},$$

unde, si loco *x* et *y* introducamus quantitates *p* et *q*, ob $x=\frac{p-q}{2}$ et $y=\frac{p+q}{2}$, orientur sequentes valores

$$y-x=q,\ y^2-x^2=pq,\ y^3-x^3=\tfrac{1}{4}q(3pp+qq),$$
$$y^4-x^4=\tfrac{1}{2}pq(pp+qq),\ y^5-x^5=\tfrac{1}{16}q\left(5p^4+10ppqq+q^4\right)\ \text{etc.}$$

§.96. Quantitas ergo V per sequentes formulas integrales secundum numerum litterarum $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. determinatur

$$\partial \mathrm{V} = \mathfrak{B}\int \frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \mathfrak{C}\int \frac{p\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$
$$+\frac{1}{4}\mathfrak{D}\int \frac{(3pp+qq)\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \frac{1}{2}\mathfrak{E}\int \frac{p(pp+qq)\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$
$$+\frac{1}{16}\mathfrak{F}\int \frac{\left(5p^4+10ppqq+q^4\right)\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} \text{ etc.}$$

Quarum formularum duae priores iam absolute exhiberi possunt, sive algebraice, quod evenit, si $\varepsilon = 0$, sive per logarithmos, si valor ipsius ε fuerit positivus, sive per arcus circulares, si valor ipsius ε fuerit negativus. Reliquae vero formulae exigunt relationem inter *p* et *q,* quam deinceps investigabimus. Hic tantum notetur potestates solas pares ipsius *q* in has formulas ingredi.

§.97. Hic autem littera Δ eundem valorem constantem designat, quem supra iam definivimus, qui erat

$$\Delta = \frac{2\alpha+\beta(f+g)+2\gamma fg+\delta fg(f+g)+2\varepsilon ffgg+2\sqrt{\mathrm{FG}}}{(g-f)^2}.$$

Praeterea vero cum esse debeat

$$\Phi : y - \Phi : x = \Phi : g - \Phi : f + \mathrm{V},$$

evidens est casu, quo $x = f$ et $y = g$, fieri debere $\mathrm{V} = 0$; quamobrem formulae illae integrales pro V inventae ita capi debebunt, ut posito $p = f + g$ et $q = g - f$ valor ipsius V evanescat.

Analysis pro investiganda relatione inter *p* et *q*.

§.98. Quia iam invenimus aequationem finitam inter *x* et *y,* ex ea quoque ponendo $y = \frac{p+q}{2}$ et $x = \frac{p-q}{2}$, relatio inter litteras *p* et *q* derivari posset; verum hoc calculos nimis taediosos postularet, quamobrem aliam viam ineamus istam relationem ex formulis differentialibus deducendi. Cum enim sit

$$\frac{\partial p}{\partial q} = \frac{\partial y+\partial x}{\partial y-\partial x},$$

ob proportionem

$$\partial x : \partial y = \sqrt{\mathrm{X}} : \sqrt{\mathrm{Y}}$$

erit

$$\frac{\partial p}{\partial q}=\frac{\sqrt{Y}+\sqrt{X}}{\sqrt{Y}-\sqrt{X}}\ ;$$

supra autem invenimus esse

$$\frac{\sqrt{Y}+\sqrt{X}}{q}=\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)},$$

ubi Δ eandem denotat constantem, quam modo ante definivimus.

§.99. Nunc igitur fractio pro $\frac{\partial p}{\partial q}$ inventa supra et infra multiplicetur per $\sqrt{Y}+\sqrt{X}$, et cum sit

$$\left(\sqrt{Y}+\sqrt{X}\right)^2=qq\left(\Delta+\gamma+\delta p+\varepsilon pp\right),$$

habebimus hanc aequationem

$$\frac{\partial p}{\partial q}=\frac{qq\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}{Y-X},$$

cuius denominatorem iam supra § 83 evolvimus, ubi invenimus esse

$$Y-X=\beta q+\gamma pq+\tfrac{1}{4}\delta q\left(3pp+qq\right)+\tfrac{1}{2}\varepsilon pq\left(pp+qq\right);$$

quo valore substituto erit

$$\frac{\partial p}{\partial q}=\frac{q\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}{\beta+\gamma p+\frac{1}{4}\delta\left(3pp+qq\right)+\frac{1}{2}\varepsilon p\left(pp+qq\right)},$$

quae reducitur ad hanc formam

$$2q\partial q=\frac{\left(2\beta+2\gamma p+\frac{1}{2}\delta\left(3pp+qq\right)+\varepsilon p\left(pp+qq\right)\right)\partial p}{\Delta+\gamma+\delta p+\varepsilon pp}.$$

§.100. Transferamus terminos, qui continent *qq*, a dextra in sinistram partem, ut obtineamus hanc aequationem

$$2q\partial q-\frac{qq\partial p\left(\frac{1}{2}\delta+\varepsilon p\right)}{\Delta+\gamma+\delta p+\varepsilon pp}=\frac{\left(2\beta+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3\right)\partial p}{\Delta+\gamma+\delta p+\varepsilon pp}.$$

Membrum huius aequationis sinistrum integrabile reddi potest, si per certam functionem ipsius *p,* quae sit $=\Pi$, multiplicetur, quando fuerit

$$\frac{\partial\Pi}{\Pi}=-\frac{\partial p\left(\frac{1}{2}\delta+\varepsilon p\right)}{\Delta+\gamma+\delta p+\varepsilon pp},$$

quae aequatio integrata dat

$$l\Pi=-\tfrac{1}{2}l\left(\Delta+\gamma+\delta p+\varepsilon pp\right).$$

Sicque erit multiplicator iste

$$\Pi=\frac{1}{\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}};$$

tum autem integrale quaesitum erit

$$\frac{qq}{\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}}=\int\frac{\left(2\beta+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3\right)\partial p}{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)^{\frac{3}{2}}}.$$

§.101. Hoc postremum integrale manifesto continet formam

$$\frac{pp}{\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}},$$

quippe cuius differentiale est

$$\frac{\left(2\Delta p+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3\right)\partial p}{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)^{\frac{3}{2}}};$$

quare integrale ita potest repraesentari

$$\frac{qq}{\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}}=\frac{pp}{\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}}+\int\frac{(2\beta-2\Delta p)\partial p}{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)^{\frac{3}{2}}},$$

quod postremum integrale statuatur

$$=\frac{m+np}{\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}};$$

huius enim differentiale est

$$\frac{\left((\Delta+\gamma)n-\frac{1}{2}\delta m+\left(\frac{1}{2}\delta n-\varepsilon m\right)p\right)\partial p}{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)^{\frac{3}{2}}}$$

ideoque fieri debet

$$\left(\Delta+\gamma\right)n-\tfrac{1}{2}\delta m=2\beta \text{ et } \tfrac{1}{2}\delta n-\varepsilon m=-2\Delta$$

unde deducuntur valores

$$m=\frac{4\beta\delta+8\Delta\Delta+8\Delta\gamma}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta} \text{ et } n=\frac{8\beta\varepsilon+4\Delta\delta}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta},$$

quarum fractionum loco in calculo retineamus litteras *m* et *n;* consequenter adiecta constante aequatio integralis ita se habebit

$$qq=pp+np+m+C\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)} .$$

§.102. Ista autem constans ita definiri debet, ut posito $p=f+g$ fiat $q=g-f$, ex quo quantitas illa constans ita determinabitur

$$C=\frac{4fg+n(f+g)+m}{\sqrt{\left(\Delta+\gamma+\delta(f+g)+\varepsilon(f+g)^2\right)}}.$$

Hoc ergo valore invento facile assignari poterunt valores non solum ipsius *qq,* sed etiam eius potestatum parium $q^4,\ q^6,\ q^8$ etc., quibus indigemus. Atque hinc intelligitur pro inveniendo valore ipsius V alias formulas integrales non occurrere, nisi quae involvant quantitatem radicalem $\sqrt{\left(\Delta+\gamma+\delta(f+g)+\varepsilon(f+g)^2\right)}$, quarum ergo integratio, nisi algebraice institui queat, semper per logarithmos et arcus circulares expediri poterit. Evidens autem est casu, quo $\varepsilon=0$, omnia integralia algebraice exprimi posse.

§.103. Quodsi ergo pro priori curva OZ fuerit

$$\Pi{:}z=\int\frac{\partial z}{\sqrt{\left(\alpha+\beta z+\gamma z^2+\delta z^3\right)}},$$

pro altera vero curva

$$\varPhi{:}z=\int\frac{\partial z\left(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\text{etc.}\right)}{\sqrt{\left(\alpha+\beta z+\gamma z^2+\delta z^3\right)}},$$

tum sumtis in priori curva arcubus aequalibus FG et XY iis in altera curva respondebunt arcus $\mathfrak{FG}$ et $\mathfrak{XY}$, quorum differentia semper geometrice assignari poterit. Interdum etiam fieri potest, ut differentia V in nihilum abeat, id quod quidem semper evenit sumto $x=f$.

§.104. Praeterea vero etiam datur alius casus maxime memorabilis, quod differentia illa V algebraice exprimi poterit, qui scilicet semper locum habebit, quando tam in denominatore quam in numeratore tantum potestates pares ipsius z occurrunt, hoc est, si fuerit pro curva priore

$$\Pi{:}z = \int \frac{\partial z}{\sqrt{\left(\alpha+\gamma zz+ez^4\right)}},$$

pro altera vero curva

$$\Phi{:}z = \int \frac{\partial z\left(\mathfrak{A}+\mathfrak{C}zz+\mathfrak{E}z^4+\mathfrak{G}z^6+\text{etc.}\right)}{\sqrt{\left(\alpha+\gamma zz+ez^4\right)}}.$$

His enim casibus si in priore curva arcus aequales FG et XY abscindantur, tum arcuum in altera curva respondentium $\mathfrak{FG}$ et $\mathfrak{XY}$ differentia semper algebraice seu geometrice exhiberi poterit, ad quotcunque terminos etiam numerator $\mathfrak{A}+\mathfrak{C}zz+\mathfrak{E}z^4+\text{etc.}$ continuetur, atque hic est casus, quem olim tam in *Calculo integrali* quam alibi fusius pertractavi.

§.105. Ad hoc ostendendum, quia habemus tam $\delta=0$ quam $\beta=0$, prima erit

$$qq = pp+m+\text{C}\sqrt{\left(\Delta+\gamma+\varepsilon pp\right)},$$

ita ut hic tantum potestates pares ipsius p occurrant; tum autem pro litteris germanicis $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{G}$ etc. formulae integrandae sequenti modo se habebunt:
Pro littera $\mathfrak{C}$

$$\int \frac{p\partial p}{\sqrt{(\Delta+\gamma+\varepsilon pp)}},$$

quae per se est absolute integrabilis.
Pro littera $\mathfrak{E}$

$$\int \frac{p(pp+qq)\partial p}{\sqrt{(\Delta+\gamma+\varepsilon pp)}},$$

quae loco qq substituto valore induet hanc formam

$$\int \frac{p(2pp+m)\partial p}{\sqrt{(\Delta+\gamma+\varepsilon pp)}}+\text{C}\int p\partial p,$$

ubi integratio est manifesta, quod etiam usu venit pro sequentibus formulis litteris $\mathfrak{G}$ etc. affectis. Evidens enim est, si ponatur $\sqrt{\left(\Delta+\gamma+\varepsilon pp\right)}=s$, fieri

$$pp=\tfrac{ss-\Delta-\gamma}{s} \quad \text{et} \quad p\partial p=\tfrac{s\partial s}{s}$$

ideoque

$$\frac{p\partial p}{\sqrt{(\Delta+\gamma+\varepsilon pp)}}=\frac{\partial s}{\varepsilon},$$

qua substitutione omnes formulae integrandae fiunt rationales et integrae.

§.106. Cum autem iste posterior casus iam satis prolixe sit tractatus ac pluribus exemplis a rectificatione ellipsis et hyperbolae desumtis illustratus, casus prior, quo tantum erat $\varepsilon=0$, eo maiore attentione est dignus, quod, quantum equidem scio, a nemine adhuc est observatus, cuius ergo evolutio novae huic methodo unice accepta est referenda. Quemadmodum autem haec deducta sunt ex relatione inter *p* et *q*, ita etiam relatio elegantissima erui potest inter has quantitates $p=x+y$ et $u=xy$, quam hic subiungamus.

Analysis pro investiganda relatione inter *p* et *u*.

§.107. Hic pariter primo in relationem inter ∂p et ∂u inquiramus, et cum sit

$$\frac{\partial p}{\partial u}=\frac{\partial x+\partial y}{y\partial x+x\partial y},$$

ob $\partial x:\partial y=\sqrt{X}:\sqrt{Y}$ erit

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

et sumtis quadratis

$$\frac{\partial p^2}{\partial u^2}=\frac{X+Y+2\sqrt{X}\sqrt{Y}}{yyX+xxY+2xy\sqrt{X}\sqrt{Y}}.$$

Supra autem vidimus esse

$$\left(\sqrt{Y}+\sqrt{X}\right)^2=qq\left(\Delta+\gamma+\delta p+\varepsilon pp\right),$$

existente $q=y-x$. Pro denominatore autem utamur relatione § 87 inventa

$$\Delta=\frac{2\alpha+\beta(x+y)+2\gamma xy+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{XY}}{(y-x)^2},$$

unde fit

$$2\sqrt{XY}=\Delta qq-2\alpha-\beta p-2\gamma u-\delta pu-2\varepsilon uu,$$

quo valore substituto aequatio nostra erit

$$\frac{\partial p^2}{\partial u^2}=\frac{qq(\Delta+\gamma+\delta p+\varepsilon pp)}{yy\mathrm{X}+xx\mathrm{Y}+\Delta qqu-2\alpha u-\beta pu-2\gamma uu-\delta puu-2\varepsilon u^3}.$$

§.108. Hic autem substitutis loco X et Y valoribus habebimus primo

$$yy\mathrm{X}+xx\mathrm{Y}=\alpha(xx+yy)+\beta xy(x+y)+2\gamma xxyy+\delta xxyy(x+y)+\varepsilon xxyy\left(xx+yy\right),$$
quae ob $x+y=p,\ xy=u$ et $xx+yy=pp-2u$ erit

$$yy\mathrm{X}+xx\mathrm{Y}=\alpha\left(pp-2u\right)+\beta pu+2\gamma uu+\delta puu+\varepsilon uu(pp-2u),$$

unde totus denominator reperietur fore

$$\alpha\left(pp-4u\right)+\varepsilon uu\left(pp-4u\right)+\Delta qqu;$$

quare, cum sit $pp-4u=qq$, nostra fractio erit

$$\frac{\partial p^2}{\partial u^2}=\frac{\Delta+\gamma+\delta p+\varepsilon pp}{\Delta u+\alpha+\varepsilon uu},$$

unde sequitur haec aequatio separata

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}};$$

unde deducitur hoc

THEOREMA MEMORABILE

§.109. *Si inter binas variabiles x et y habeatur haec aequatio differentialis*

$$\frac{\partial x}{\sqrt{\left(\alpha+\beta x+\gamma xx+\delta x^3+\varepsilon x^4\right)}}=\frac{\partial y}{\sqrt{\left(\alpha+\beta y+\gamma yy+\delta y^3+\varepsilon y^4\right)}},$$

tum posito $x+y=p$ *et* $xy=u$ *inter has variabiles p et u semper locum habebit haec aequatio differentialis*

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}},$$

ubi Δ *quidem est constans arbitraria in aequationem posteriorem ingressa, contra vero etiam prior aequatio continet constantem arbitrariam* β *in altera non occurrentem.*

§.110. Aequationis autem posterioris integratio in promptu est. Si enim utrinque multiplicemus per $\sqrt{\varepsilon}$, integrale per logarithmos ita exprimitur

$$l\left(p\sqrt{\varepsilon}+\tfrac{\delta}{2\sqrt{\varepsilon}}+\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}\right)$$
$$=l\left(u\sqrt{\varepsilon}+\tfrac{\Delta}{2\sqrt{\varepsilon}}+\sqrt{\left(\alpha+\Delta u+\varepsilon uu\right)}\right)+l\Gamma$$

ideoque integrale ita algebraice exprimetur

$$\varepsilon p+\tfrac{1}{2}\delta+\sqrt{\varepsilon}\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}=\Gamma\left(\varepsilon u+\tfrac{1}{2}\Delta+\sqrt{\varepsilon}\left(\alpha+\Delta u+\varepsilon uu\right)\right).$$

Ubi constans ista Γ facile definitur ex conditione, quod posito $x=f$ fieri debet $y=g$, hoc est, ut posito $p=f+g$ fiat $u=fg$, quippe ex qua conditione constans prior Δ iam est definita.

§.111. Quo hinc iam facilius sive *p* per *u* sive *u* per *p* definiri possit, notetur esse

$$\frac{1}{\varepsilon p+\frac{1}{2}\delta+\sqrt{\varepsilon}\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}}=\frac{\varepsilon p+\frac{1}{2}\delta-\sqrt{\varepsilon}\sqrt{\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}}{\frac{1}{4}\delta\delta-\varepsilon\left(\Delta+\gamma\right)}$$

et

$$\frac{1}{\varepsilon u+\frac{1}{2}\Delta+\sqrt{\varepsilon}\left(\alpha+\Delta p+\varepsilon uu\right)}=\frac{\varepsilon u+\frac{1}{2}\Delta-\sqrt{\varepsilon}\left(\alpha+\Delta u+\varepsilon uu\right)}{\frac{1}{4}\Delta\Delta-\alpha\varepsilon}$$

Hinc igitur per inversionem sequens aequatio resultabit

$$\frac{\varepsilon p+\frac{1}{2}\delta-\sqrt{\varepsilon}\left(\Delta+\gamma+\delta p+\varepsilon pp\right)}{\frac{1}{4}\delta\delta-\varepsilon\left(\Delta+\gamma\right)}=\frac{1}{\Gamma}\cdot\frac{\varepsilon u+\frac{1}{2}\Delta-\sqrt{\varepsilon}\left(\alpha+\Delta u+\varepsilon uu\right)}{\frac{1}{4}\Delta\Delta-\varepsilon\left(\Delta+\gamma\right)}$$

sive

$$\varepsilon p+\tfrac{1}{2}\delta-\sqrt{\varepsilon}\left(\Delta+\gamma+\delta p+\varepsilon pp\right)$$
$$=\frac{\frac{1}{4}\delta\delta-\varepsilon\left(\Delta+\gamma\right)}{\Gamma\left(\frac{1}{4}\Delta\Delta-\varepsilon\left(\Delta+\gamma\right)\right)}\cdot\left(\varepsilon u+\tfrac{1}{2}\Delta-\sqrt{\varepsilon}\left(\alpha+\Delta u+\varepsilon uu\right)\right)$$

ex quibus duabus aequationibus sine alio negotio sive *p* per *u* sive *u* per *p* exprimi poterit.

§.112. Hoc igitur modo loco variabilis *p* pro invenienda quantitate V facile introduci posset variabilis *u*, si quidem loco formulae

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$

substituatur formula ipsi aequalis

$$\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}}.$$

Verum hoc modo casus illi, quibus quantitas *V* fieri potest algebraica, non tam facile patescent; interim tamen etiam hoc modo certi erimus tam casibus, quibus $\varepsilon=0$, quam, quo $\beta=0$, $\delta=0$ et in serie $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. tantum potestates pares occurrunt, omnes integrationes algebraice succedere debere. Coronidis loco adhuc aliam relationem inter quantitates *p* et *u* investigemus, cuius contemplatio insigne incrementum in integratione aequationum polliceri videtur.

Alia analysis pro investigatione relationis inter *p* et *u*

§.113. Cum sit, ut ante vidimus,

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

multiplicemus supra et infra per $\sqrt{X}+\sqrt{Y}$, ut numerator evadat

$$\left(\sqrt{X}+\sqrt{Y}\right)^{2}=qq\left(\Delta+\gamma+\delta p+\varepsilon pp\right);$$

tum autem denominator prodibit

$$yX+xY+\left(x+y\right)\sqrt{XY},$$

ubi denominatoris pars rationalis dat

$$\alpha\left(x+y\right)+2\beta xy+\gamma xy\left(x+y\right)+\delta xy\left(xx+yy\right)+\varepsilon xy\left(x^{3}+y^{3}\right),$$

quae expressio ob $x+y=p,\ y-x=q$ et $xy=u$ abit in

$$\alpha p+2\beta u+\gamma pu+\delta u(pp-2u)+\varepsilon pu(pp-3u).$$

Deinde ante vidimus esse

$$2\sqrt{XY} = \Delta qq - 2\alpha - \beta p - 2\gamma u - \delta pu - 2\varepsilon uu,$$

quod ductum in $\frac{1}{2}p$ et superiori additum praebet

$$\tfrac{1}{2}\Delta pqq - \tfrac{1}{2}\beta\left(pp-4u\right) + \tfrac{1}{2}\delta u\left(pp-4u\right) + \varepsilon pu\left(pp-4u\right),$$

quare denominator ob $pp-4u = qq$ induet hanc formam

$$\tfrac{1}{2}\Delta pqq - \tfrac{1}{2}\beta qq + \tfrac{1}{2}\delta uqq + \varepsilon puqq;$$

hinc aequatio erit

$$\frac{\partial p}{\partial u} = \frac{\Delta+\gamma+\delta p+\varepsilon pp}{\frac{1}{2}\Delta p-\frac{1}{2}\beta+\frac{1}{2}\delta u+\varepsilon pu},$$

unde deducitur

$$\partial p\left(\tfrac{1}{2}\Delta p - \tfrac{1}{2}\beta + \tfrac{1}{2}\delta u + \varepsilon pu\right) = \partial u\left(\Delta + \gamma + \delta p + \varepsilon pp\right),$$

quae ergo certe est integrabilis; id quod adeo inde patet, quod altera variabilis *u* nusquam ultra primam dimensionem exsurgit.

§.114. Verum adhuc alia modo relatio inter *p* et *u* investigari potest; scilicet aequatio prima inventa

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

si supra et infra multiplicetur per $\sqrt{Y}-\sqrt{X}$, dabit

$$\frac{\partial p}{\partial u} = \frac{Y-X}{-yX+xY+\sqrt{XY}\left(y-x\right)}$$

Nunc igitur pro numeratore habebimus

$$\beta q + \gamma pq + \delta q(pp-u) + \varepsilon pq\left(pp-2u\right).$$

Pro denominatore vero pars rationalis erit

$$-\alpha q + \gamma qu + \delta pqu + \varepsilon qu\left(pp-u\right),$$

pars vero irrationalis

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq - \gamma qu - \tfrac{1}{2}\delta pqu - \varepsilon quu,$$

unde totus denominator conficitur

$$\tfrac{1}{2}\Delta q^3 - 2\alpha q - \tfrac{1}{2}\beta pq + \tfrac{1}{2}\delta pqu + \varepsilon qu\left(pp-2u\right),$$

unde sequitur haec aequatio differentialis

$$\frac{\partial p}{\partial u} = \frac{\beta+\gamma p+\delta(pp-u)+\varepsilon p(pp-2u)}{\frac{1}{2}\Delta(pp-4u)-2\alpha-\frac{1}{2}\beta p+\frac{1}{2}\delta pu+\varepsilon u(pp-2u)},$$

quae in ordinem redacta ita se habebit

$$\partial p\left(\Delta\left(pp-4u\right)-4\alpha-\beta p+\delta pu+2\varepsilon u\left(pp-2u\right)\right)$$
$$=2\partial u\left(\beta+\gamma p+\delta(pp-u)+\varepsilon p\left(pp-2u\right)\right),$$

quae iam ita est comparata, ut nulla via eius integrationem instituendi perspici queat, etiamsi eius integrale revera exhibere queamus.

§.115. Alio insuper modo relationem inter p et u definire licet, si aequationis

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

posterius membrum supra et infra multiplicemus per $y\sqrt{X}-x\sqrt{Y}$, ut prodeat

$$\frac{\partial p}{\partial u} = \frac{y\sqrt{X}-x\sqrt{Y}+(y-x)\sqrt{XY}}{yyX-xxY}$$

Nunc enim denominator evadet

$$\alpha pq + \beta qu - \delta quu - \varepsilon pquu .$$

Pro numeratore autem pars rationalis praebet

$$\alpha q - \gamma qu - \delta pqu - \varepsilon qu\left(pp-u\right)$$

et pars irrationalis

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq - \gamma qu - \tfrac{1}{2}\delta pqu - \varepsilon quu;$$

totus igitur numerator erit

$$\tfrac{1}{2}\Delta q^3-\tfrac{1}{2}\beta pq\alpha q-2\gamma qu-\tfrac{3}{2}\delta pqu-\varepsilon qupp$$

ideoque

$$\frac{\partial p}{\partial u}=\frac{\frac{1}{2}\Delta(pp-4u)-\frac{1}{2}\beta p-2\gamma u-\frac{3}{2}\delta pu-\varepsilon ppu}{\alpha p+\beta u-\delta uu-\varepsilon puu}$$

sive

$$2\partial p\left(\alpha p+\beta u-\delta uu-\varepsilon puu\right)=\partial u\left(\Delta\left(pp-4u\right)-\beta p-4\gamma u-3\delta pu-2\varepsilon ppu\right).$$

Hic autem penitus non patet, quomodo multiplicator hanc aequationem integrabilem reddens investigari debeat, unde nullum est dubium, quin ista contemplatio haud parum ad limites analyseos prolatandos conferre possit.