SUPPLEMENT VIII.

TO BOOK I. SECT. II. CHAP. VI.

CONCERNING THE PREPARATION OF TRANSCENDING QUANTITIES CONTAINED IN THE FORM

$$\int \frac{P\partial z}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}}.$$

Elucidations on the most elegant method, which has been used by the most illustrious Lagrange, in integrating the differential equation

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$$

[E506]

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§.1. After I had exerted myself at length, and after much searching in vain, scrutinizing the differential equation

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}},$$

first and foremost for a direct method by which the integral might be produced by an easy and obvious way; I was completely astonished, when such a method was reported to me by the most illustrious *Lagrange*, published in the fourth volume of the *Turin Miscellanies*, with the aid of which, for the case where

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$

and

$$Y = A + By + Cyy + Dy^3 + Ey^4,$$

thus he had elucidated completely most successfully this algebraic integral of the proposed differential equation,

$$\frac{\sqrt{X}+\sqrt{Y}}{x-y} = \sqrt{\left(\Delta + D(x+y) + E(x+y)^2\right)},$$

where Δ may denote some arbitrary constant quantity introduced into the integration.

§. 2. But I have admired more from that same outstanding discovery, as indeed I had always thought it would be necessary to search for such a method by looking for a suitable factor, by which the proposed equation would be rendered integrable, since generally the whole method of integrating may be considered to consist either to be from the separation of the variables, or by a suitable multiplication, even if likewise in certain cases that may be able to be produced from a differentiation, just as has been shown by me and others by numerous examples. But it is observed that the *Lagrangian* way itself can duly be considered to be important according to this third way.

§. 3. Moreover though it is easy to add a little to this method found by the illustrious *Lagrange*, yet with regard to this issue, much more hard work will involved in order that it may be considered to have been used accurately, and to be more applicable for analytical use; if indeed the whole affair may be seen capable of being set out much easier and simpler ; on account of which for the given work, I am going to set out here further what I have considered may be agreed to be worth the most attention, with regard to this argument.

§.4. But since this integral was found by the most illustrious Lagrange from these same forms, which I gave some time ago, most often it will be in disagreement and go far beyond the simplicity, thus before everything it is considered to enquire, how it may satisfy the differential equation. To this end I put for brevity $\sqrt{X} + \sqrt{Y} = V$ so that I may have

$$\frac{V}{x-y} = \sqrt{\left(\Delta + D(x+y) + E(x+y)^2\right)},$$

which equation it will be required to differentiate thus, so that the arbitrary constant *Δ* may depart from the differential. Therefore with the squares taken, there will become :

$$\frac{V^2}{(x-y)^2} = \Delta + D(x+y) + E(x+y)^2,$$

which differentiated gives :

$$\frac{2V\partial V}{(x-y)^2} - \frac{2VV(\partial x - \partial y)}{(x-y)^3} - D(\partial x + \partial y) - 2E(x+y)(\partial x + \partial y) = 0.$$

§.5. Now so that the calculation may be returned more clearly, we may investigate the parts themselves associated either with ∂x or with ∂y. Therefore for the element ∂x, if y may be regarded as constant, there will be

$$\partial V = \frac{X'\partial x}{2\sqrt{X}},$$

from which the individual parts thus themselves will be had:

$$\partial x\left(\frac{VX'}{(x-y)^2\sqrt{X}}-\frac{2VV}{(x-y)^3}-D-2E(x+y)\right),$$

where there may be observed to be $V=\sqrt{X}+\sqrt{Y}$, and hence

$$VV\sqrt{X}=(X+Y)\sqrt{X}+2X\sqrt{Y},$$

from which here terms of two kinds occur, while they have been associated either with $\sqrt{X}$ or with $\sqrt{Y}$. But there are two terms present associated with $\sqrt{Y}$, which are :

$$-\frac{4X\sqrt{Y}}{(x-y)^3}+\frac{X'\sqrt{Y}}{(x-y)^2},$$

which therefore must be taken together

$$\frac{\sqrt{Y}}{(x-y)^3}\left(X'(x-y)-4X\right),$$

which form on account of

$$X=A+Bx+Cxx+Dx^3+Ex^4$$

and hence

$$X'=Bx+2Cx+3Dxx+4Ex^3$$

will give

$$X'(x-y)-4X=-4A-B(3x+y)-2C(xx+xy)-D\left(x^3+3xxy\right)-4Ex^3y.$$

But the terms associated with $\sqrt{X}$ are

$$\frac{\sqrt{X}}{(x-y)^3}\left(X'(x-y)-2(X+Y)-D(x-y)^3-2E(x+y)(x-y)^3\right).$$

Therefore since there shall be :

$$X+Y=2A+B(x+y)+C\left(x^2+y^2\right)+D\left(x^3+y^3\right)+E\left(x^4+y^4\right),$$

with the substitution made this latter factor will be

$$-4A-B(x+3y)-2C(xy+yy)-D\left(3xyy+y^3\right)-4Exy^3,$$

which form differs from this former, only because the letters x and y have been interchanged.

§.6. So that therefore if we may put for brevity,

$$M = 4A + B(3x + y) + 2C(xx + xy) + D(x^3 + 3xxy) + 4Ex^3 y,$$
$$N = 4A + B(x + 3y) + 2C(yy + xy) + D(y^3 + 3xyy) + 4Exy^3,$$

hence the part associated with the element ∂x will be expressed thus:

$$-\frac{\partial x}{(x-y)^3\sqrt{X}}\left(M\sqrt{Y} + N\sqrt{X}\right).$$

§.7. In a similar manner on account of $\partial V = \frac{Y'\partial Y}{2\sqrt{Y}}$, [when x is constant] the parts associated with the element ∂y are:

$$\frac{\partial y}{\sqrt{Y}}\left(\frac{YY'}{(x-y)^2} + \frac{2VV\sqrt{Y}}{(x-y)^3} - D\sqrt{Y} - 2E(x+y)\sqrt{Y}\right)$$

Now this form on account of

$$V = \sqrt{X} + \sqrt{Y} \text{ and } VV\sqrt{Y} = (X + Y)\sqrt{Y} + 2Y\sqrt{X},$$

will contain the following terms associated with $\sqrt{X}$

$$\frac{\sqrt{X}}{(x-y)^3}\left[Y'(x-y) + 4Y\right],$$

which form arises from the former preceding calculation, if the letters x and y and likewise the signs may be interchanged ; from which it is apparent this expression bears the value + N. Moreover, the remaining terms associated with $\sqrt{Y}$ are :

$$\frac{\sqrt{Y}}{(x-y)^3}\left(Y'(x-y) + 2(X+Y) - D(x-y)^3 - 2E(x+y)(x-y)^3\right).$$

This form again arises from the form of the previous calculation from the interchange of the letters and of the sign ; which therefore since that shall be $-N$, this will be + M. Therefore in this manner the parts containing the element ∂y will be

$$\frac{+\partial y}{(x-y)^3\sqrt{Y}}\left[N\sqrt{X} + M\sqrt{Y}\right].$$

§.8. Therefore with these parts being joined together the differential equation arising from the *Lagrangian* form will be :

$$\left(\frac{\partial y}{\surd Y}-\frac{\partial x}{\surd Y}\right)\left[\frac{N\surd X+M\surd Y}{(x-y)^3}\right]=0,$$

which divided by the common factor gives rise to that same differential equation proposed $\frac{\partial x}{\surd x}=\frac{\partial y}{\surd Y}$; from which likewise it is apparent the integral equation itself to have been shown correctly and thus the value of the letter Δ to be left completely at our choice.

§.9. But before we may apply the *Lagrangian* method to this same differential equation $\frac{\partial x}{\surd x}=\frac{\partial y}{\surd Y}$ in any acceptable extension, we may proceed from a simpler case, where this rational equation is proposed thus :

$$\frac{\partial x}{a+2bx+cxx}=\frac{\partial y}{a+2by+cyy}.$$

Analysis for the integration of the differential equation

$$\frac{\partial x}{a+2bx+cxx}=\frac{\partial y}{a+2by+cyy}.$$

§.10. For the sake of brevity we may put $a+2bx+cxx=X$ and $a+2by+cyy=Y$, so that there may be able to become $\frac{\partial x}{X}=\frac{\partial y}{Y}$, which formulas, since they must be equal to each other, each may be designated by the same element ∂t, thus so that we may obtain these two formulas

$$\frac{\partial x}{\partial t}=X \text{ and } \frac{\partial y}{\partial t}=Y.$$

But if now therefore we may put

$$x-y=q, \text{ there will become } \frac{\partial q}{\partial t}=X\text{-}Y=2bq+cq(x+y),$$

from which on dividing by *q* there will become $\frac{\partial q}{q\partial t}=2b+c(x+y)$.

§.11. Now we may differentiate the first equation, with the element ∂t assumed constant and by making

$$\partial X=X'\partial x \text{ and } \partial Y=Y'\partial y$$

these two equations will arise

$$\frac{\partial\partial x}{\partial x\partial t}=X' \text{ and } \frac{\partial\partial y}{\partial y\partial t}=Y',$$

which added in turn produce

$$\frac{\partial\partial x}{\partial x\partial t} + \frac{\partial\partial y}{\partial y\partial t} = X' + Y'.$$

Whereby, since there shall be

$$X' = 2b + 2cx \text{ and } Y' = 2b + 2cy,$$

there will become

$$\frac{1}{\partial t}\left(\frac{\partial\partial x}{\partial x} + \frac{\partial\partial y}{\partial y}\right) = 4b + 2c(x + y).$$

§.12. Therefore since here the latter value is twice as great as the preceding $\frac{\partial q}{q\partial t}$, in this way we have deduced the sum for this equation :

$$\frac{\partial\partial x}{\partial x} + \frac{\partial\partial y}{\partial y} = \frac{2\partial q}{q},$$

which integrated gives $l\partial x + l\partial y = 2lq + \text{const.}$, and hence in terms of numbers there will be

$$\partial x\partial y = Cqq\partial t^2,$$

thus so that there shall be

$$C = \frac{\partial x\partial y}{qq\partial t^2},$$

Whereby, since there shall be

$$\frac{\partial x}{\partial t} = X \text{ and } \frac{\partial y}{\partial t} = Y,$$

the equation of the integral will be

$$\frac{XY}{(x-y)^2} = C,$$

which therefore is not only algebraic, but also complete.

§.13. Therefore if this differential equation were proposed

$$\frac{dx}{a+2bx+cxx} = \frac{dy}{a+2by+cyy},$$

its complete integral will be expressed thus :

$$\frac{(a+2bx+cxx)(a+2by+cyy)}{(x-y)^2} = C,$$

which with $bb - ac$ being added to each side will adopt this form

$$\frac{aa+2ab(x+y)+2acxy+bb(x+y)^2+2bcxy(x+y)+ccxxyy}{(x-y)^2}=\Delta\Delta ,$$

and thus with the root extracted it will have this form

$$\frac{a+b(x+y)+cxy}{x-y}=\Delta ,$$

which without doubt is the most simple, since both y can be expressed easily by x and x by y, since there shall be

$$y=\frac{(\Delta-b)x-a}{\Delta+b+cx} \text{ and } x=\frac{a+(\Delta+b)y}{\Delta-b-cy}.$$

§.14. The calculation, which we have used here, will be shown readily to be considered in terms of the forms X and Y, and it is not allowed to progress beyond the square. For if we may attribute the extra term ∂x^3 to X above, and the term ∂y^3 to Y itself, there will be produced for the first form :

$$\frac{\text{X-Y}}{x-y}=2b+c(x+y)+d(xx+xy+yy)=\frac{\partial q}{q\partial t},$$

moreover, for the other form, there is

$$\text{X}'+\text{Y}'=4b+2c(x+y)+3d(xx+yy)=\frac{\partial\partial x}{\partial x\partial t}+\frac{\partial\partial y}{\partial y\partial t}.$$

Whereby, hence if we may take away twice the preceding equation, there is found :

$$\frac{\partial\partial x}{\partial x\partial t}+\frac{\partial\partial y}{\partial y\partial t}-\frac{2\partial q}{q\partial t}=d(x-y)^2 ,$$

which equation may not be integrated further.

§.15. But it can be shown easily such a differential equation, in which it is produced beyond the square, in no manner is able to be integrated further algebraically. For if such a case may be proposed here $\frac{\partial x}{1+x^3}=\frac{\partial y}{1+y^3}$, is it observed on both sides with an integral to involve partially logarithms and partially circular arcs and thus to involve diverse transcendent quantities, which cannot be compared with each other in any manner. Clearly comparisons of this kind only can have a place with these, when transcending quantities of a single kind occur on both sides.

Analysis for the integration of the differential equation

$$\frac{\partial x}{a+2bx+cxx}+\frac{\partial y}{a+2by+cyy}=0.$$

§.16. Because if here as before we may put

$$\frac{\partial x}{a+2bx+cxx}=\partial t,$$

there must be put in place

$$\frac{\partial y}{a+2by+cyy}=-\partial t;$$

but truly if we may wish to establish the calculation in a similar manner as before, clearly we can make no progress [i.e. above we had each equation equal to ∂t, which may be regarded as constant; now one is the negative of the other]. But after I have considered all the difficulties with care, I fall upon the artifice, by which this case may be able to be extricated, thus so that hence I may not be considered to be ignoring the advancement that may be brought to me from the *Lagrangian* method.

§.17. Therefore since I have these two equations

$$\frac{\partial x}{\partial t}=\mathrm{X} \text{ and } \frac{\partial y}{\partial t}=-\mathrm{Y},$$

hence I form that same new equation

$$\frac{y\partial x+x\partial y}{\partial t}=y\mathrm{X}-x\mathrm{Y}.$$

Now I make $xy=u$, so that I may have

$$\frac{\partial u}{\partial t}=a(y-x)+cxy(x-y),$$

from which on placing $x-y=q$ there will become $\frac{\partial u}{\partial t}=q\left(cu-a\right),$ which equation divided by $cu-a$ and multiplied by c gives

$$\frac{c\partial u}{\partial t\left(cu-a\right)}=cq,$$

and in this way a sum arises from the difference of logarithms.

§.18. Truly thereafter as before we may differentiate the principal equations, and we will obtain

$$\frac{\partial\partial x}{\partial t\partial x} = X' \text{ and } \frac{\partial\partial y}{\partial t\partial y} = -Y',$$

which in turn added give

$$\frac{1}{\partial t}\left(\frac{\partial\partial x}{\partial x} + \frac{\partial\partial y}{\partial y}\right) = X' - Y' = 2cq;$$

whereby if we may subtract twice the preceding equation, there will remain :

$$\frac{1}{\partial t}\left(\frac{\partial\partial x}{\partial x} + \frac{\partial\partial y}{\partial y} - \frac{2c\partial u}{cu-a}\right) = 0,$$

from which on multiplying by ∂t and integrating we obtain $l\partial x + l\partial y - 2l\left(cu - a\right) = lC$ and thus $\frac{\partial x\partial y}{(cu-a)^2} = C\partial t^2$.
[Recall that ∂t is regarded as a constant differential.]
Therefore since there shall be $\partial x = X\partial t$ and $dy = -Y\partial t$, our equation of the integral will be $-\frac{XY}{(cu-a)^2} = C$.

§.19. Therefore by this analysis we have deduced for this equation the integral of the proposed equation

$$\frac{(a+2bx+cxx)(a+2by+cyy)}{(a-cxy)^2} = C$$

which equation, if one may be subtracted from each side, is reduced to this form :

$$\frac{2ab(x+y)+ac(x+y)^2+4bbxy+2bcxy(x+y)}{(a-cxy)^2} = C.$$

§.20. We will illustrate this integration by an example, by putting $a=1$, $b=0$ and $c=1$, thus so that this differential equation shall be proposed

$$\frac{dx}{1+xx} + \frac{dy}{1+yy} = 0,$$

the integral of which we know to be $\text{Atang.}x + \text{Atang.}y = \text{Atang.}\frac{x+y}{1-xy} = C$, and thus we know to be $\frac{x+y}{1-xy} = C$. But truly our last formula gives for this case

$$\frac{(x+y)^2}{(1-xy)^2} = C \text{ and therefore } \frac{x+y}{1-xy} = C,$$

which agrees precisely.

§.21. We may consider the case also, where $a=1$, $b=\frac{1}{2}$ and $c=1$, thus so that this equation is proposed

$$\frac{\partial x}{1+x+xx}+\frac{\partial y}{1+y+yy}=0,$$

of which the integral is

$$\frac{2}{\sqrt{3}}\,\text{Atang.}\frac{x\sqrt{3}}{2+x}+\frac{2}{\sqrt{3}}\,\text{Atang.}\frac{y\sqrt{3}}{2+y}=\text{C},$$

from which it follows to become

$$\text{Atang.}\frac{2(x+y+xy)\sqrt{3}}{4+2(x+y)-2xy}=\text{C}$$

and thus also $\frac{x+y+xy}{2+x+y-xy}=\text{C}.$ But truly the form of the integral found for this case will give

$$\frac{x+y+(x+y)^2+xy+xy(x+y)}{(1-xy)^2}=\text{C},$$

which resolved into factors gives

$$\frac{(1+x+y)(x+y+xy)}{(1-xy)^2}=\text{C}.$$

Truly the first equation found $\frac{x+y+xy}{2+x+y-xy}=\text{C}$ inverted gives $\frac{2+x+y-xy}{x+y+xy}=\text{C}$ and with one subtracted, $\frac{1-xy}{x+y+xy}=\text{C}$, and this multiplied by the preceding product gives $\frac{1+x+y}{1-xy}=\text{C}.$

§.22. Therefore we may observe, whether these latter equations may agreed between themselves, and because the constants are able to be separated amongst themselves on both sides, thus we may refer to both the equations

$$\frac{1-xy}{x+y+xy}=\alpha \quad\text{and}\quad \frac{1+x+y}{1-xy}=\beta;$$

from which since there shall be $\frac{1}{\alpha}=\frac{x+y+xy}{1-xy}$, there is evidently to become $\beta-\frac{1}{\alpha}=1$, ex from which the most favorable agreement is shown between the two formulas.

From these examples it is understood the general equation found above can be represented by factors in this manner

$$\frac{(2b+c(x+y))(a(x+y)+2bxy)}{(a-cxy)^2}.$$

Furthermore the consideration of these formulas will hardly be able to suppress unfavorable speculations.

§.23. But that form of the following integral found :

$$\frac{(2b+c(x+y))(a(x+y)+2bxy)}{(a-cxy)^2} = C$$

can be reduced at once to the most simple form ; if indeed we may put its factors :

$$\frac{2b+c(x+y)}{a-cxy} = P \text{ and } \frac{a(x+y)+2bxy}{a-cxy} = Q,$$

so that there must be $PQ = C$, there will be $aP - cQ = \frac{2ab-2bcxy}{a-cxy} = 2b,$ from which there shall become

$$Q = \frac{aP-2b}{c},$$

and thus this form $\frac{aPP-2bP}{c}$ must be equal to a constant quantity; from which is apparent, also that same quantity P must be equal to a constant, thus so that now our integral equation shall be

$$\frac{2b+c(x+y)}{a-cxy} = C \text{ or also } \frac{a(x+y)+2bxy}{a-cxy} = C.$$

Another easier solution of the same equation

$$\frac{\partial x}{a+2bx+cxx} + \frac{\partial y}{a+2by+cyy} = 0.$$

§.24. With the last reduction discovered properly considered it is possible to arrive at once from the initial to the simplest form of the integral, and thus not to be necessary to rise to differentials of the second order. If indeed as before we may put $x + y = p,\ x - y = q$ et $xy = u,$ from the formulas

$$\frac{\partial x}{\partial t} = X \text{ and } \frac{\partial y}{\partial t} = -Y$$

we deduce at once

$$\frac{\partial p}{\partial t} = X - Y = 2bq + cpq, \text{from which there becomes } \frac{\partial p}{2b+cp} = q\partial t.$$

§.25. Again truly there will be :

$$\frac{y\partial x + x\partial y}{\partial t} = \frac{\partial u}{\partial t} = y\mathrm{X} - x\mathrm{Y} = -aq + cqu,$$

from which there becomes $\frac{\partial u}{cu-a} = q\partial t$, on account of which we deduce this equation at once : $\frac{\partial p}{2b+cp} = \frac{\partial u}{cu-a}$, the integration of which gives

$$l(2b+cp) = l\left(cu-a\right) + l\mathrm{C};$$

from which this algebraic equation is deduced $\frac{2b+cp}{cu-a} = \mathrm{C}$, which with the letters x and y restored, gives $\frac{2b+c(x+y)}{cxy-a} = \mathrm{C}$, which is the simplest form of the equation of the integral desired. Here especially noteworthy it comes to mind, as the first case cannot be resolved by this reasoning.

§.26. But from the form of the integral found others are derived easily; just as if we may add $\frac{2b}{a}$, this form will arise $\frac{a(x+y)+2bxy}{cxy-a} = \mathrm{C}$, which divided by the preceding supplies a new form again, evidently $\frac{2b+c(x+y)}{a(x+y)+2bxy} = \mathrm{C}$; how which forms may be satisfactory will deserve the effort to be shown. And indeed the latter form, differentiated, will become

$$\frac{-2ab(\partial x+\partial y)-4bb(y\partial x+x\partial y)-2bc(yy\partial x+xx\partial y)}{\left(a(x+y)+2bxy\right)^2},$$

which rendered in order provides

$$\partial x(2ab+4bby+2bcyy) + \partial y(2ab+4bbx+2bcxx) = 0.$$

This divided by $2b$ and separated gives

$$\frac{\partial x}{a+2bx+cxx} + \frac{\partial y}{a+2by+cyy} = 0,$$

which is the same itself proposed.

Analysis for the integration of the equation

$$\frac{\partial x}{\sqrt{(\mathrm{A}+\mathrm{B}x+\mathrm{C}xx)}} = \frac{\partial y}{\sqrt{(\mathrm{A}+\mathrm{B}y+\mathrm{C}yy)}}$$

§.27. With the new element ∂t introduced, henceforth on being regarded as constant, these two equations arise

$$\frac{\partial x}{\partial t}=\sqrt{X} \text{ and } \frac{\partial y}{\partial t}=\sqrt{Y} ,$$

where we may assign initial values to the letters X and Y . Moreover we will see according to the method which we will use here, the terms associated with the letters D and E must be omitted.
Therefore with the squares taken, there will be

$$\frac{\partial x^2}{\partial t^2}=X \text{ and } \frac{\partial y^2}{\partial t^2}=Y.$$

§.28. Now we may differentiate these same formulas, and on putting in place, as there is accustomed to become, $\partial X = X'\partial x$ and $\partial Y = Y'\partial y$ we will obtain these equations

$$\frac{2\partial\partial x}{\partial t^2}=X' \text{ et } \frac{2\partial\partial y}{\partial t^2}=Y'$$

and on putting $x+y=p$ there will become $\frac{2\partial\partial p}{\partial t^2}=X'+Y'$. Now since there shall be

$$X'=B+2Cx+3Dxx+4Ex^3 \text{ and } Y'=B+2Cy+3Dyy+4Ey^3$$

there will be

$$X'+Y'=2B+2Cp+3D(xx+yy)+4E\left(x^3+y^3\right)=\frac{2\partial\partial p}{\partial t^2},$$

which equation evidently will be allowed to be integrated, if there were both $D=0$ and $E=0$, just as we have assumed. Therefore on being multiplied by ∂p and integrating we obtain

$$\frac{dp^2}{dt^2}=\Delta+2Bp+Cpp$$

and on extracting the root

$$\frac{dp}{dt}=\sqrt{(\Delta+2Bp+Cpp)}.$$

Therefore since there shall be $\frac{\partial p}{\partial t}=\sqrt{X}+\sqrt{Y}$ *,* the equation of the integral, which we have arrived at, will be

$$\sqrt{X}+\sqrt{Y}=\sqrt{\left(\Delta+2B(x+y)+C(x+y)^2\right)},$$

which thus is algebraic ; where it may be observed to be

$$X=A+Bx+Cxx \text{ and } Y=A+By+Cyy.$$

§.29. Therefore we may take the squares and our equation of the integral will be

$$2A+B(x+y)+C(x^2+y^2)+2\sqrt{XY}=\Delta+2B(x+y)+C(x+y)^2$$

or

$$2A-B(x+y)-2Cxy+2\sqrt{XY}=\Delta,$$

which finally freed from irrationality, on putting $\Delta-2A=\Gamma$, will give

$$4XY=4AA+4AB(x+y)+4AC(xx+yy)+4BBxy+4BCxy(x+y)$$
$$+4CCxxyy=\Gamma^2+2\Gamma B(x+y)+4\Gamma Cxy+BB(x+y)^2+4BCxy(x+y)+4CCxxyy$$

or

$$(4AA-\Gamma^2)+2B(2A-\Gamma)(x+y)+4(BB-\Gamma C)xy$$
$$+4AC(xx+yy)-B^2(x+y)^2=0.$$

§.30. But if now we may compare this rational equation with the canonical formula, of which I had made use some time ago for evaluating integrations of this kind, which was

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0,$$

while evidently in place of $(x+y)^2$ we may write $(xx+yy)+2xy$, we will find to be:

$$\alpha=4AA-\Gamma^2,\ \beta=B(2A-\Gamma),\ \gamma=4AC-B^2,\ \delta=BB-2\Gamma C.$$

§.31. Truly in addition we will be able to integrate the above differential equation proposed by another way by introducing the letter $q=x-y$; then indeed we will have

$$\frac{2\partial\partial q}{\partial t^2}=X'-Y'$$

But truly there will become

$$X'-Y'=2Cq+3Dq(x+y)+4Eq(xx+xy+yy),$$

where again it is apparent there must be put in place both $D=0$ and $E=0$, so that the integration may succeed on being multiplied by ∂q. Moreover, with this noted the integral will become

$$\frac{\partial q^2}{\partial t^2}=\text{Const.}+Cqq\ \text{ and thus }\ \frac{\partial q}{\partial t}=\sqrt{(\Delta+Cqq)}.$$

§.32. Therefore since there shall be $\frac{\partial q}{\partial t} = \sqrt{X} - \sqrt{Y}$, thus from this integral there will be the expression

$$\sqrt{X} - \sqrt{Y} = \sqrt{(\Delta + Cqq)},$$

which equation with the square taken will turn into this :

$$2A + B(x+y) + C(xx+yy) - 2\sqrt{XY} = \Delta + C(x-y)^2$$

or

$$2A + B(x+y) + 2Cxy - 2\sqrt{XY} = \Delta,$$

from which there becomes

$$2\sqrt{XY} = 2A - \Delta + B(x+y) + 2Cxy,$$

where if there may be put $2A - \Delta = -\Gamma$, the equation in short does not differ from the one found above.

§.33. But if the proposed equation were

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}} + \frac{\partial y}{\sqrt{(A+By+Cyy)}} = 0,$$

the integrals found before will refer to this case, only if in place of $\sqrt{Y}$ there may be written $-\sqrt{Y}$; from which it is apparent according to this case this equation to be had

$$\sqrt{X} - \sqrt{Y} = \sqrt{\left(\Delta + 2B(x+y) + C(x+y)^2\right)}$$

or also

$$\sqrt{X} + \sqrt{Y} = \sqrt{\left(\Delta + C(x-y)^2\right)}$$

§.34. This singular case occurs, when the formulas $A + Bx + Cxx$ are squares. Indeed if there shall be

$$X = (a+bx)^2 \quad \text{and} \quad Y = (a+by)^2$$

and thus

$$A = aa, \; B = 2ab, \; C = bb;$$

then indeed the first form of the integral will be

$$b(x-y)=\sqrt{\left(\Delta+4ab(x+y)+bb(x+y)^2\right)}$$

and with the square taken,

$$-4bbxy=\Delta+4ab(x+y)$$

and thus

$$\Delta=a(x+y)+bxy,$$

of which the equation of the differential is

$$a(\partial x+\partial y)+b(x\partial y+y\partial v)=0 \text{ and thus } \partial x(a+by)+dy(a+bx)=0.$$

But if the other formula may be used, there will be

$$2a+b(x+y)=\sqrt{\left(\Delta+bb(x-y)^2\right)},$$

from which with the squares taken and with $\Delta-4aa=\Gamma$ put in place, there arises as before :

$$\Gamma=a(x+y)+bxy.$$

The Analysis
For integrating the equation:

$$\frac{dx}{\sqrt{X}}=\frac{dy}{\sqrt{Y}}$$

with there being

$$X=A+Bx+Cxx+Dx^3+Ex^4$$

and

$$Y=A+By+Cyy+Dy^3+Ey^4.$$

§.35. Again with the element ∂t introduced, so that there shall be

$$\frac{\partial x}{\partial t}=\sqrt{X} \text{ and } \frac{\partial y}{\partial t}=\sqrt{Y}$$

and thus with the squares taken

$$\frac{\partial x^2}{\partial t^2}=\mathrm{X} \text{ and } \frac{\partial y^2}{\partial t^2}=\mathrm{Y},$$

we may put in place $x+y=p$ and $x-y=q$, and because hence there arises

$$\partial x^2-\partial y^2=\partial p\partial q, \text{ erit},$$

$$\frac{\partial p\partial q}{\partial t^2}=\mathrm{X}-\mathrm{Y}=\mathrm{B}(x-y)+\mathrm{C}\left(x^2-y^2\right)$$
$$+\mathrm{D}\left(x^3-y^3\right)+\mathrm{E}\left(x^4-y^4\right).$$

§.36. Therefore, since there is $x=\frac{p+q}{2}$ and $y=\frac{p-q}{2}$, with these values introduced there will be found

$$\mathrm{X}-\mathrm{Y}=\mathrm{B}q+\mathrm{C}pq+\tfrac{1}{4}\mathrm{D}q(3pp+qq)+\tfrac{1}{2}\mathrm{E}pq(pp+qq),$$

from which on being divided by *q,* there arises

$$\frac{\partial p\partial q}{q\partial t^2}=\mathrm{B}+\mathrm{C}p+\tfrac{1}{4}\mathrm{D}(3pp+qq)+\tfrac{1}{2}\mathrm{E}p(pp+qq).$$

§.37. Now also we will differentiate the quadratic formulas shown at first, and by putting in place as before

$$\partial\mathrm{X}=\mathrm{X}'\partial x \text{ and } \partial\mathrm{Y}=\mathrm{Y}'\partial x$$

we will have

$$\frac{2\partial\partial x}{\partial t^2}=\mathrm{X}' \text{ and } \frac{2\partial\partial y}{\partial t^2}=\mathrm{Y}'$$

and hence by adding

$$\frac{2\partial\partial p}{\partial t^2}=\mathrm{X}'+\mathrm{Y}'.$$

Truly since there shall be

$$\mathrm{X}'=\mathrm{B}+2\mathrm{C}x+3\mathrm{D}xx+4\mathrm{E}x^3 \text{ and } \mathrm{Y}'=\mathrm{B}+2\mathrm{C}y+3\mathrm{D}yy+4\mathrm{E}y^3,$$

there will become

$$\mathrm{X}'+\mathrm{Y}'=2\mathrm{B}+2\mathrm{C}p+\tfrac{3}{2}\mathrm{D}(pp+qq)+\mathrm{E}p(pp+3qq),$$

thus so that with this value substituted there may become

$$\frac{\partial\partial p}{\partial t^2}=\mathrm{B}+\mathrm{C}p+\tfrac{3}{4}\mathrm{D}(pp+qq)+\tfrac{1}{2}\mathrm{E}p(pp+3qq),$$

from which equation, if we may subtract the former for $\frac{\partial p\partial q}{q\partial t^2}$, the following will remain

$$\frac{\partial\partial p}{\partial t^2}-\frac{\partial p\partial q}{q\partial t^2}=\tfrac{1}{2}\mathrm{D}qq+\mathrm{E}pqq.$$

§.38. Now this equation divided by qq will give rise to that same

$$\frac{1}{\partial t^2}\left(\frac{\partial\partial p}{qq}-\frac{\partial p\partial q}{q^3}\right)=\tfrac{1}{2}\mathrm{D}+\mathrm{E}p,$$

which multiplied by $2\partial p$ evidently becomes integrable ; for there arises

$$\frac{\partial p^2}{qq\partial t^2}=\Delta+\mathrm{D}p+\mathrm{E}pp,$$

from which with the root extracted there is deduced :

$$\frac{\partial p}{q\partial t}=\sqrt{(\Delta+\mathrm{D}p+\mathrm{E}pp)}.$$

Therefore since we will have put $p=x+y$ and $q=x-y$, there will become

$$\frac{\partial p}{\partial t}=\sqrt{\mathrm{X}}+\sqrt{\mathrm{Y}}\,,$$

from which this algebraic equation of the integral arises :

$$\frac{\sqrt{\mathrm{X}}+\sqrt{\mathrm{Y}}}{x-y}=\sqrt{\left(\Delta+\mathrm{D}(x+y)+\mathrm{E}(x+y)^2\right)},$$

which is that same form found by the illustrious *Lagrange*.

§.39. We may evolve this form further, and with the square taken there will be :

$$\frac{\mathrm{X}+\mathrm{Y}+2\sqrt{\mathrm{XY}}}{(x-y)^2}=\Delta+\mathrm{D}(x+y)+\mathrm{E}(x+y)^2,$$

Truly there becomes

$$\mathrm{X}+\mathrm{Y}=2\mathrm{A}+\mathrm{B}(x+y)+\mathrm{C}(xx+yy)+\mathrm{D}\left(x^3+y^3\right)+\mathrm{E}\left(x^4+y^4\right);$$

from which if we may take away

$$\left(\mathrm{D}(x+y)+\mathrm{E}(x+y)^2\right)(x-y)^2,$$

there will remain:

$$2A + B(x + y) + C\left(x^2 + y^2\right) + Dxy(x + y) + 2Exxyy,$$

with which substituted, the equation of the integral will become :

$$\frac{2A+B(x+y)+C\left(x^2+y^2\right)+Dxy(x+y)+2Ex^2y^2+2\sqrt{XY}}{(x-y)^2} = \Delta.$$

§.40. This equation can be returned somewhat more concisely by subtracting C from each side C and by setting $\Delta - C = \Gamma$*;* indeed with this done there will be had :

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy+2\sqrt{XY}}{(x-y)^2} = \Gamma,$$

from which we deduce

$$2\sqrt{XY} = \Gamma(x-y)^2 - 2A - B(x+y) - 2Cxy - Dxy(x+y) - 2Exxyy,$$

or by putting

$$2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy = V,$$

our equation will become :

$$2\sqrt{XY} = \Gamma(x-y)^2 - V,$$

which with squares taken will change into this

$$4XY = \Gamma^2(x-y)^4 - 2\Gamma V(x-y)^2 + VV$$

or

$$4XY - VV = \Gamma^2(x-y)^4 - 2\Gamma V(x-y)^2.$$

§.41. Now with the substitution made there will become

$$\begin{aligned}4XY = {} & 4A^2 + 4AB(x+y) + 4AC(xx+yy) + 4AD\left(x^3+y^3\right) + 4AE\left(x^4+y^4\right)\\ & +4BBxy + 4BCxy(x+y) + 4BDxy(xx+yy) + 4BExy\left(x^3+y^3\right)\\ & +4CCxxyy + 4CDxxyy(x+y) + 4CExxyy(xx+yy)\\ & +4DDx^3y^3 + 4DEx^3y^3(x+y) + 4EEx^4y^4.\end{aligned}$$

But truly there may be gathered to become :

$$\begin{aligned}&VV=4AA+4AB(x+y)+8ACxy+4ADxy(x+y)+8AExxyy\\&+BB(x+y)^2+4BCxy(x+y)+2BDxy(x+y)^2+4BE(x+y)xxyy\\&+4CCxxyy+4CD(x+y)xxyy+8CEx^3y^3\\&+DDxxyy(x+y)^2+4DEx^3y^3(x+y)+4EEx^4y^4.\end{aligned}$$

§.42. But if now we may subtract the latter formula from the former, and we may place the analogous terms in order, we will find :

$$\begin{aligned}4XY-VV&=4AC(x-y)^2+4AD(x+y)(x-y)^2(x-y)^2\\&+4AE(x+y)^2(x-y)^2-B^2(x-y)^2+2BDxy(x-y)^2+4BExy(x+y)(x-y)^2,\\&+4CExxyy(x-y)^2-DDxxyy(x-y)^2.\end{aligned}$$

which expression has the common factor $(x-y)^2$, therefore if we divide by which, we will arrive at this more concise equation:

$$\begin{aligned}&4AC+4AD(x+y)+4AE(x+y)^2-BB+2BDxy+4BExy(x+y)\\&+(4CE-DD)xxyy=\Gamma\Gamma(x-y)^2-4\Gamma A-2\Gamma B(x+y)-4\Gamma Cxy-2\Gamma Dxy(x+y)-4\Gamma Exxyy.\end{aligned}$$

§.43. Now we may transfer all the terms to the left hand side and we may write $(xx+yy)+2xy$ in place of $(x+y)^2$, while truly $(xx+yy)-2xy$ in place of $(x-y)^2$, with which done my corresponding canonical equation arises

$$0=\begin{cases}4AC+4AD(x+y)+4AE(x^2+y^2)+2BDxy+4BExy(x+y)+4CExxyy\\-BB+2\Gamma B(x+y)-\Gamma\Gamma(x^2+y^2)+8AExy+2\Gamma Dxy(x+y)-DDxxyy\\+4\Gamma A\qquad\qquad+2\Gamma^2xy\qquad\qquad+4\Gamma Exxyy\\\qquad\qquad+4\Gamma Cxy\end{cases}$$

§.44. Hence therefore the Greek letters $\alpha,\ \beta,\ \gamma,\ \delta$ for the canonical equation may be determined by the Latin letters A, B, C, D, E together with the constant Γ in the following manner :

$$\alpha = 4\mathrm{AC} + 4\Gamma\mathrm{A} - \mathrm{BB}$$
$$\beta = 2\mathrm{AD} + \Gamma\mathrm{B}$$
$$\gamma = 4\mathrm{AE} - \Gamma\Gamma$$
$$\delta = \mathrm{BD} + 4\mathrm{AE} + \Gamma^2 + 2\Gamma\mathrm{C}$$
$$\varepsilon = 2\mathrm{BE} + \Gamma\mathrm{D}$$
$$\zeta = 4\mathrm{CE} + 4\Gamma\mathrm{E} - \mathrm{DD},$$

thus so that the canonical equation, which I used at one time, shall be :

$$a + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0.$$

§.45. But it is apparent this integral equation leads more widely to rationality than the proposed differential equation $\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0$; indeed likewise it includes the integral of this equation: $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$. Clearly this equation includes the two factors, of which the one satisfies the other. Moreover from its origins it is apparent that this equation to be a product from these factors

$$\Delta(x-y)^2 - \mathrm{V} + 2\sqrt{\mathrm{XY}} \text{ and } \Delta(x-y)^2 - \mathrm{V} - 2\sqrt{\mathrm{XY}}.$$

§.46. Now above we have observed how in this way differential equations of this same can be shown too :

$$\frac{\mathrm{M}\sqrt{\mathrm{Y}} + \mathrm{N}\sqrt{\mathrm{X}}}{(x-y)^3} = \mathrm{C}$$

(see § 8 and the preceding.) with their being :

$$\mathrm{M} = 4\mathrm{A} + \mathrm{B}(3x+y) + 2\mathrm{C}x(x+y) + \mathrm{D}xx(x+3y) + 4\mathrm{E}x^3y,$$
$$\mathrm{N} = 4\mathrm{A} + \mathrm{B}(3y+x) + 2\mathrm{C}y(x+y) + \mathrm{D}yy(y+3x) + 4\mathrm{E}xy^3,$$

where it help to be noted :

$$\mathrm{M} + \mathrm{N} = 8\mathrm{A} + 4\mathrm{B}(x+y) + 2\mathrm{C}(x+y)^2 + \mathrm{D}(x+y)^3 + 4\mathrm{E}xy(xx+yy),$$
$$\mathrm{M} - \mathrm{N} = 2\mathrm{B}(x-y) + 2\mathrm{C}(x+y)(x-y) + \mathrm{D}(x-y)\left(x^2+4xy+y^2\right)$$
$$+ 4\mathrm{E}xy(x+y)(x-y).$$

Yet meanwhile it is not easy to understand, how this form agrees with the other found, while yet we can be certain of the agreement.

§.47. From these, which at this point have been brought forwards, it is clear the same integral equation can be shown in innumerable ways, just as the arbitrary constant is represented in one way or another ; from which a certain rule can be established between most, according to which we may wish to express that arbitrary constant in whatever case . To this end, this rule itself may be observed, so that the integral may always be taken thus, so that on putting $y=0$ there may become $x=k$ and hence the following rule of composition $\mathrm{X}=\mathrm{K}$ with there being

$$\mathrm{K}=\mathrm{A}+\mathrm{B}k+\mathrm{C}kk+\mathrm{D}k^{3}+\mathrm{E}k^{4}.$$

Indeed with this rule observed all the integrals, however diverse they may be seen, will be able to be produced in perfect agreement. Therefore in this way which we have now found, we may include by the following theorems.

THEOREM 1

§.48. *Thus, if this differential equation*

$$\frac{\partial x}{a+bx+cxx}-\frac{\partial y}{a+by+cyy}=0$$

may be integrated, so that on putting $y=0$ *there becomes* $x=k$, *thus the integral will itself be found:*

$$\frac{2a+b(x+y)+2cxy}{x-y}=\frac{2a+bk}{k}.$$

THEOREM 2

§.49. *Thus, if this differential equation*

$$\frac{\partial x}{a+bx+cxx}-\frac{\partial y}{a+by+cyy}=0$$

may be integrated, so that on putting $y=0$ *there becomes* $x=k$ *,the above integral has been found in three ways; indeed there will be*

$$\text{I.}\quad \frac{b+c(x+y)}{cxy-a}=-\frac{b+ck}{a}$$

$$\text{II.}\quad \frac{a(x+y)+bxy}{cxy-a}=-k$$

$$\text{III.}\quad \frac{b+c(x+y)}{a(x+y)+bxy}=\frac{b+ck}{ak}.$$

THEOREM 3

§.50. *Thus, if this differential equation*

$$\frac{\partial x}{\sqrt{(\mathrm{A}+\mathrm{B}x+\mathrm{C}xx)}}-\frac{\partial y}{\sqrt{(\mathrm{A}+\mathrm{B}y+\mathrm{C}yy)}}=0,$$

so that on putting $y=0$ *there becomes* $x=k$, *the integral will be*

$$-\mathrm{B}(x+y)-2\mathrm{C}xy+2\sqrt{(\mathrm{A}+\mathrm{B}x+\mathrm{C}xx)}\sqrt{(\mathrm{A}+\mathrm{B}y+\mathrm{C}yy)}$$
$$=\mathrm{B}k+2\sqrt{\mathrm{A}(\mathrm{A}+\mathrm{B}k+\mathrm{C}kk)}$$

or

$$\mathrm{B}(k-x-y)-2\mathrm{C}xy=2\sqrt{\mathrm{A}(\mathrm{A}+\mathrm{B}k+\mathrm{C}kk)}$$
$$-2\sqrt{(\mathrm{A}+\mathrm{B}x+Cxx)}\sqrt{(\mathrm{A}+\mathrm{B}y+\mathrm{C}yy)}.$$

COROLLARY

§.51. Hence therefore it is apparent, if the differential equation proposed were this :

$$\frac{\partial x}{\sqrt{(\mathrm{A}+\mathrm{B}x+\mathrm{C}xx)}}+\frac{\partial y}{\sqrt{(\mathrm{A}+\mathrm{B}y+\mathrm{C}yy)}}=0$$

and this may be integrated thus, so that on putting $y=0$ there may become $x=k$, with the integral to become :

$$\mathrm{B}(k-x-y)-2\mathrm{C}xy$$
$$=2\sqrt{(\mathrm{A}+\mathrm{B}x+\mathrm{C}xx)}\sqrt{(\mathrm{A}+\mathrm{B}y+\mathrm{C}yy)}-2\sqrt{\mathrm{A}(\mathrm{A}+\mathrm{B}k+\mathrm{C}kk)}.$$

THEOREM 4

§.52. *If for the sake of brevity by putting*

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$
$$Y = A + By + Cyy + Dy^3 + Ey^4$$
$$K = A + Bk + Ckk + Dk^3 + Ek^4$$

this differential equation may be proposed

$$\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0,$$

which thus may be integrated, so that on putting $y = 0$ *there may become* $x = k$ *, its integral thus will be the expression:*

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy + 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk + 2\sqrt{AK}}{kk}.$$

But if the equation proposed were

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

its integral will be

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy - 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk - 2\sqrt{AK}}{kk}.$$

COROLLARY 1

§.53. But if here we may put $D = 0$ and $E = 0$, the case of the third theorem arises according to the equation:

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}} - \frac{\partial y}{\sqrt{(A+By+Cyy)}} = 0,$$

of which the integral hence will be

$$\frac{2A + B(x+y) + 2Cxy + 2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2} = \frac{2A + Bk + 2\sqrt{A(A+Bk+Ckk)}}{kk};$$

which form if it may be compared with the above, the irrational formulas will be able to be eliminated.

But since from before there is

$$2\sqrt{XY} = 2\sqrt{A\left(A + Bk + Ckk\right)} - B\left(k - x - y\right) + 2Cxy,$$

this will be the final integral

$$\frac{2A+B(2x+2y-k)+4Cxy+2\sqrt{A\left(A+Bk+Ckk\right)}}{(x-y)^2} = \frac{2A+Bk+2\sqrt{A\left(A+Bk+Ckk\right)}}{kk},$$

from which the canonical equation can be deduced at once :

$$\alpha + 2\beta\left(x + y\right) + \gamma\left(xx + yy\right) + 2\delta xy = 0.$$

COROLLARY 2

§.54. Now we may put $A = 0$ and $B = 0$, so that there shall become

$$X = xx\left(C + Dx + Exx\right),\ Y = yy\left(C + Dy + Eyy\right),\ \text{and}\ K = kk\left(C + Dk + Ekk\right);$$

the differential equation being integrated will become

$$\frac{\partial x}{x\sqrt{\left(A+Bx+Cxx\right)}} - \frac{\partial y}{y\sqrt{\left(A+By+Cyy\right)}} = 0$$

therefore the integral of which will become :

$$\frac{xy\left(2C+D\left(x+y\right)+2Exy\right)+2xy\sqrt{\left(C+Dx+Exx\right)\left(C+Dy+Eyy\right)}}{(x-y)^2} = \Delta,$$

and here the constant Δ will not be allowed to be defined by k ; for on putting $y = 0$ there is an inconsistency involved. Yet all the same, the integration is especially worth being mentioned.

COROLLARY 3

§.55. But if in this latter integration we may write $\frac{1}{x}$ and $\frac{1}{y}$ in place of x and y, the first differential equation will be

$$\frac{\partial y}{\sqrt{\left(Cyy+Dy+E\right)}} - \frac{\partial x}{\sqrt{\left(Cxx+Dx+E\right)}} = 0\ ;$$

while truly the following integral will adopt the form

$$\frac{2Cxy+D(x+y)+2E+2\sqrt{(Cxx+Dx+E)(Cyy+Dy+E)}}{(y-x)^2}$$
$$=\Delta=\frac{Dk+2E+2\sqrt{E(Ckk+Dk+E)}}{kk}.$$

If therefore here in place of the letters E, D, C we may write A, B, C*,* the differential equation treated above will be produced

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}-\frac{\partial y}{\sqrt{(A+By+Cyy)}}=0,$$

of which therefore the integral will be

$$\frac{2A+B(x+y)+2Cxy+2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2}$$
$$=\frac{Bk+2A+2\sqrt{A(A+Bk+Ckk)}}{kk},$$

which agrees precisely with that given in Corol.1.

COROLLARY 4

§.56. We may now also consider the case, where the formula

$$A+Bx+Cxx+Dx^3+Ex^4$$

becomes a square, which shall be $(a+bx+cxx)^2$, thus so that now we may have

$$A=aa,\ B=2ab,\ C=bb+2ac,\ D=2bc,\ E=cc,$$

then truly

$$\sqrt{X}=a+bx+cxx,\ \sqrt{Y}=a+by+cyy,\ \sqrt{K}=a+bk+ckk$$

and the differential equation from the earlier case will be

$$\frac{\partial x}{a+bx+cxx}-\frac{\partial y}{a+by+cyy}=0,$$

the integral of which therefore will be

$$\left\{\begin{matrix} 2aa+2ab(x+y)+2(bb+2ac)xy+2bcxy(x+y) \\ +2ccxxyy+2(a+bx+cxx)(a+by+cyy) \end{matrix}\right\} : (x-y)^2 = \Delta,$$

which is reduced to

$$\frac{aa+ab(x+y)+(bb+2ac)xy+bcxy(x+y)+ccxxyy)}{(x-y)^2} = \frac{aa+abk}{kk}.$$

But if now we may add $\frac{1}{4}bb$ to each side, there will be produced

$$\frac{\left(a+\frac{1}{2}b(x+y)+cxy\right)^2}{(x-y)^2} = \frac{\left(a+\frac{1}{2}bk\right)^2}{kk},$$

from which with the root extracted the form of the integral assigned in the first theorem is obtained.

§.57. But if in this manner we may wish to set out the case of the other equation

$$\frac{\partial x}{a+bx+cxx} - \frac{\partial y}{a+by+cyy} = 0,$$

we arrive at this equation

$$\frac{2aa+2ab(x+y)+2(bb+2ac)xy+2bcxy(x+y)+2ccxxyy}{(x-y)^2}$$
$$-\frac{2(a+bx+cxx)(a+by+cyy)}{(x-y)^2} = \Delta,$$

which expanded out gives $\Delta = 2ac$, and this equation evidently is absurd and it indicates nothing about the integral sought, an account of which of the greatest relevance will be sought.

A Conspicuous Paradox.

§.58. *Since the integral of this differential equation*

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

in general shall be

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy-2\sqrt{XY}}{(x-y)^2} = \Delta$$

but in the case, where

$$\sqrt{X}=a+bx+cxx \text{ and } \sqrt{Y}=a+by+cyy$$

may be put in place, an absurd equation thence may arise, an explanation of this conspicuous difficulty is sought, and especially the way the true value of this integral hence is required to be investigated.

The Resolution of the Paradox.

§.59. Evidently just as in the analysis of this kind formulas are accustomed to occur, which will be seen to be indeterminate in certain cases and plainly according to that to be of no significance, which thus usually happens here likewise, but by far by another way, where neither by fractions of which the numerator or denominator likewise vanish, nor an infinity may arise from the differentiation between the two, which example therefore is the more noteworthy, because I do not recall a similar case having presented itself to me at any time. This same singular phenomenon without doubt prevails, when both the formulas X and Y avoid being squares, towards which being resolved it is therefore necessary to have recourse to a similar artifice, where the formulas X and Y are not themselves equal to squares, but they are assumed to differ from these by an indefinitely small amount.

§.60. Therefore we may establish

$$X=(a+bx+cxx)^2+\alpha \text{ and } Y=(a+by+cyy)^2+\alpha,$$

thus so that for the capital letters A, B, C, D, E there may become $A=aa+\alpha,\ B=2ab,\ C=2ac+bb,\ D=2bc,\ E=cc$, where α denotes an infinitely small quantity finally required to be put equal to zero. Hence therefore for the sake of brevity we may put

$$a+bx+cxx=R \text{ and } a+by+cyy=S,$$

there will be

$$\sqrt{X}=R+\frac{\alpha}{2R} \text{ and } \sqrt{Y}=S+\frac{\alpha}{2S}.$$

§.61. Therefore now we may consider the form of the integral first found, which was

$$\frac{\sqrt{X}-\sqrt{Y}}{x-y}=\sqrt{\left(\Delta+D(x+y)+E(x+y)^2\right)},$$

for which therefore we will have

$$\sqrt{X}-\sqrt{Y}=R-S-\frac{\alpha(R-S)}{2RS}.$$

Because truly here there will become $R - S = b(x-y) + c(xx-yy)$, there will become

$$\frac{R-S}{x-y} = b + c(x+y).$$

But truly for the sake of brevity $x + y = p$ there will be

$$\frac{R-S}{x-y} = b + cp,$$

from which our equation will be

$$b + cp - \frac{\alpha(b+cp)}{2RS} = \sqrt{(\Delta + 2bcp + ccpp)}.$$

§.62. Now squares are assumed on each side and our equation adopts the form $bb - \frac{\alpha}{RS}(b+cp)^2 = \Delta.$ Clearly the higher powers of α are omitted everywhere here. Therefore here the account of our paradox can be seen clearly, because on putting $\alpha = 0$ there arises $bb = \Delta$; from which, so that Δ may remain an arbitrary constant, it is evident the difference between *bb* and Δ also must be put to be infinitely small ; on account of which we may put $\Delta = bb - \alpha\Gamma$ and this completely determined equation will be obtained, $\frac{(b+cp)^2}{RS} = \Gamma$ or

$$\left(b + c(x+y)\right)^2 = \Gamma(a + bx + cxx)(a + by + cyy),$$

which form may not differ much from the formula found above.

§.63. Indeed this form is more complicated than the solutions §. 24 et seq. found, but by the following artifice it will be able to be rendered according to the simplest form. Since this fraction $\frac{RS}{(b+cp)^2}$ let that $= F$, so that there must be $F(cp+b)^2 = RS$, and just as here we have put $x + y = p$, again we may put $xy = u$ and there will become

$$RS = aa + abp + ac(pp - 2u) + bbu + bcpu + ccuu$$

and the second power of *p* equation now put in place will be

$$F(cp+b)^2 = acpp + abp + bcpu + bbu + aa - 2acu + ccuu;$$

where in the first place we may divide both sides by $cp+b$, in as much as it can be done, and there will be found :

$$F(cp+b) = ap + bu + \frac{(a-cu)^2}{cp+b}.$$

Now we may divide again by $cp+b$, as far as it can be done, and there will become :

$$F = \frac{a}{c} - \frac{b}{c}\cdot\frac{a-cu}{cp+b} + \frac{(a-cu)^2}{(cp+b)^2}.$$

§.64. With this form found, if we may put in place

$$\frac{a-cu}{cp+b} = V,$$

there will become

$$F = \frac{a}{c} - \frac{b}{c}\cdot V + VV.$$

Therefore since this expression must be equal to a constant quantity, it is evident that same quantity V must be constant, thus so that now our integral shall be reduced to the form

$$\frac{a-cu}{cp+b} = \frac{a-cxy}{c(x+y)+b} = \text{Const.}$$

We may subtract $\frac{a}{b}$ on both sides and there will become

$$\frac{cxy+a(x+y)}{b+c(x+y)} = \text{Const.},$$

which form divided by the first produces this

$$\frac{a(x+y)+cxy}{cxy-a} = \text{Const.},$$

which forms agree with the ones shown above.

THEOREM 5

§.65. *If in general this designated ratio may be used, so that there shall become*

$$Z = A + Bz + Czz + Dz^2 + Ez^4,$$

and the value of this integral formula $\int \frac{\partial z}{\sqrt{Z}}$ *be assumed thus, so that it may vanish on putting* $z = 0$, *this may be designated by the symbol* $\Pi : z$, *then, so that there may become* $\Pi : k = \Pi : x \pm \Pi : y$ *, it is necessary that this relation may be put in place between the quantities k, x, y :*

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk \mp 2\sqrt{AK}}{kk},$$

the account of which is evident from the above developments.

Indeed since k may denote a constant quantity, there will become

$$\partial.\Pi : x \pm \partial.\Pi : y = 0 \text{ or } \frac{\partial x}{\sqrt{X}} \pm \frac{\partial y}{\sqrt{Y}} = 0,$$

the integral of which may be expressed in the manner we have seen above :

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} = \Delta.$$

Whereby since there must be $\Pi : x \pm \Pi : y = \Pi : k$, it is evident on putting $y = 0$ there must become $\Pi : x = \Pi : k$, and thus $x = k$, from which the indefinite constant Δ in short is defined in the same manner, as has been shown.

COROLLARY 1

§.66. Hence if the formula $\Pi : z$ may express the arc of some curved line corresponding to the abscissa or applied line Z, on this curve all the arcs may be able to be compared with each other in the same way, since the third arc $\Pi : k$ from two proposed arcs $\Pi : x$ and $\Pi : y$ will always be able to shown to be equal to the sum or difference of these arcs.

COROLLARY 2

§.67. Thus if in this formula $\Pi : k = \Pi : x \pm \Pi : y$ there may be put $y = x$, there will be put $\Pi : k = 2\Pi : x$ and thus the arc found equal to twice the other. But if we may make $y = x$ in our formula, both the numerator and the denominator will change into zero. But so that we may elicit its true value, we may use the first equation found in §. 38

$$\frac{\sqrt{X}+\sqrt{Y}}{x-y} = \sqrt{\left(\Delta + D(x+y) + E(x+y)^2\right)},$$

and now on the left hand side *y* may be considered as constant ; truly we may attribute an infinitely small different value to *x,* or what amounts to the same, in place of the numerator and denominator the difference of these may be substituted with *x* alone assumed to be variable and in this manner, for the case $y = x$, the left hand member will become $\frac{X'}{2\sqrt{X}}$, where there becomes :

$$X' = B + 2Cx + 3Dxx + 4Ex^3.$$

Therefore now with the squares taken, there will be had

$$\frac{X'X'}{4X} = \Delta + 2Dx + 4Exx$$

with Δ present as before $= \frac{2A+Bk-2\sqrt{AK}}{kk}$.

COROLLARY 3

§.68. Truly without these ambiguities the arc doubled can be deduced from the other formula

$$\Pi : k = \Pi : x \pm \Pi : y$$

by putting $y = k$, if indeed hence there becomes $\Pi : x = 2\Pi : k$, for which case the relation between *x* and *k* will be expressed by this equation

$$\frac{2A+B(k+x)+2Ckx+Dkx(k+x)+2Ekkxx+2\sqrt{KY}}{(x-k)^2} = \frac{2A+Bk+2\sqrt{AK}}{kk}.$$

Moreover it is readily apparent, how hence it may be able to progress to the triple, quadruple, and to any multiple of the arcs, which argument at one time I have treated further.

THEOREM 6

§.69. *If in the above forms found there may be put both* $B=0$ *as well as* $D=0$ *, so that there shall be*

$$X=A+Cxx+Ex^4,\ Y=A+Cyy+Ey^4,\ \textit{and}\ K=A+Ckk+Ek^4,$$

then if this equation $\frac{\partial x}{\sqrt{X}}\pm\frac{\partial y}{\sqrt{Y}}=0$ *may be integrated thus, so that on putting* $y=0$ *there may become* $x=k$ *, then the integral will become :*

$$\frac{A+Cxy+2Exxyy\mp2\sqrt{XY}}{(x-y)^2}=\frac{A\mp\sqrt{AK}}{kk}.$$

COROLLARY 1

§.70. Here this same case deserves to be noted at this stage, to be able to be deduced from another general formula, if indeed evidently there may be assumed $A=0$ and $E=0$, then indeed this differential equation is produced

$$\frac{\partial x}{\sqrt{(Bx+Cxx+Dx^3)}}\pm\frac{\partial y}{\sqrt{(By+Cyy+Dy^3)}}=0,$$

the integral of which therefore will become

$$\frac{B(x+y)+2Cxy+Dxy(x+y)\mp2\sqrt{(Bx+Cxx+Dx^3)(By+Cyy+Dy^3)}}{(x-y)^2}=\frac{Bk}{kk}=\frac{B}{k},$$

just as the value of the constant vanishes everywhere. Now in these formulas we may write *xx* and *yy* in place of *x* and *y,* but truly in place of the letters B and D, we may write A and E and the differential equation will become

$$\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}}\pm\frac{\partial y}{\sqrt{(A+Cyy+Ey^3)}}=0,$$

of which the integral also may be expressed in this manner

$$\frac{A(xx+yy)+2Cxxyy+Exxyy(xx+yy)\mp2xy\sqrt{XY}}{(xx-yy)^2}=\frac{A}{kk}.$$

COROLLARY 2

§.71. Behold therefore we have arrived at this ratio for another form of the integral no less noteworthy than the first and thus now from the combination of these the formula of the root $\sqrt{XY}$ will be able to be eliminated, since indeed from the latter there becomes

$$\mp 2\sqrt{XY}=\frac{A(xx-yy)^2}{kkxy}-\frac{A(xx+yy)}{xy}-2Cxy-Exy(xx+yy),$$

which value substituted into the former leads to this rational equation :

$$2A+2Cxy+2Exxyy+\frac{A(xx-yy)^2}{kkxy}-\frac{A(xx+yy)}{xy}-2Cxy-Exy(xx+yy)$$
$$=\frac{2A(x-y)^2}{kk}\mp\frac{2(x-y)^2\sqrt{AK}}{kk},$$

which reduced again and on being divided by $(x-y)^2$ may be recalled to this form :

$$\frac{2A\mp 2\sqrt{AK}}{kk}=\frac{A(x+y)^2}{kkxy}-Exy-\frac{A}{xy}$$

or to this :

$$\frac{A}{kk}(xx+yy-kk)-Exxyy\pm\frac{2xy\sqrt{AK}}{kk}=0,$$

which agrees precisely with the canonical equation, which I have used at one time, as it were :

$$0=\alpha+\gamma(xx+yy)+2\delta xy+\zeta xxyy,$$

if indeed there were

$$\alpha=-A,\ \ \gamma=+\frac{A}{kk},\ \ 2\delta=\pm\frac{2\sqrt{AK}}{kk},\ \ \zeta=-E.$$

COROLLARY 3

§.72. By this latter method, with which here we have made use for integrating this equation, can be applied to a much more general equation, where they may rise to powers as far as the sixth dimension in the formulas with the roots. And if we may put only $A=0$, so that the equation shall become

$$\frac{\partial x}{\sqrt{x\left(B+Cx+Dxx+Ex^3\right)}}\pm\frac{\partial y}{\sqrt{y\left(B+Cy+Dyy+Ey^3\right)}}=0,$$

its integral is

$$\frac{B(x+y)+2Cxy+Dxy(x+y)+2Exxyy}{(x-y)^2} \mp \frac{2\sqrt{xy(B+Cx+Dxx+Ex^3)(B+Cy+Dyy+Ey^3)}}{(x-y)^2} = \frac{B}{k}.$$

But if now here we may write *xx* and *yy* in place of *x* and *y* , the differential equation will become

$$\frac{\partial x}{\sqrt{(B+Cxx+Dx^4+Ex^6)}} \pm \frac{\partial y}{\sqrt{(B+Cyy+Dy^4+Ey^6)}} = 0,$$

of which the integral therefore is

$$\frac{B(xx+yy)+2Cxxyy+Dxxyy(xx+yy)+2Ex^4y^4}{(xx-yy)^2} \mp \frac{2xy\sqrt{(B+Cxx+Dx^4+Ex^6)(B+Cyy+Dy^4+Ey^6)}}{(xx-yy)^2} = \frac{B}{kk}.$$

But now we will show, how with the help of to method of the illustrious *De La Grange* the same integral may be able to be obtained.

Analysis for the integration of the differential equation

$$\frac{\partial x}{\sqrt{X}} \pm \frac{\partial y}{\sqrt{Y}} = 0$$

with X and Y given by

$$X = B + Cxx + Dx^4 + Ex^6 \text{ and } Y = B+Cyy + Dy^4 + Ey^6$$

§.73. Therefore on putting $\frac{\partial x}{\sqrt{X}} = \partial t$ there will be $\frac{\partial y}{\sqrt{Y}} = \mp\partial t$, and hence with the squares taken

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y.$$

Hence these equations may be formed

$$\frac{xx\partial x^2}{\partial t^2} = xxX \text{ and } \frac{yy\partial y^2}{\partial t^2} = yyY.$$

Now two new variable *p* and *q* may be introduced, so that there shall be

$$xx + yy = 2p \text{ and } xx - yy = 2q ,$$

from which there becomes

$$x\partial x + y\partial y = \partial p,\ x\partial x - y\partial y = \partial q \quad \text{and hence} \quad xx\partial x^2 - yy\partial y^2 = \partial p\partial q;$$

on account of which we will have

$$\frac{\partial p \partial q}{\partial t^2} = xxX - yyY,$$

which equation may be divided by $xx - yy = 2q$, and there will be produced

$$\frac{\partial p \partial q}{2q\partial t^2} = \frac{xxX - yyY}{xx - yy},$$

which formula with the variable X and Y substituted will give

$$\frac{\partial p \partial q}{2q\partial t^2} == B + 2Cp + D(3pp + qq) + 4E\left(p^3 + pqq\right).$$

§.74. Now the equations for $\frac{\partial x^2}{\partial t^2}$ and $\frac{\partial y^2}{\partial t^2}$ differentiated again will give

$$\frac{2\partial x \partial x}{\partial t^2} = X' \text{ and } \frac{2\partial\partial y}{\partial t^2} = Y'.$$

From the former there becomes $\frac{2\partial x \partial x}{\partial t^2} = xX'$, to which there may be added $\frac{2\partial x^2}{\partial t^2} = 2X$, so that there may be produced

$$\frac{2\left(x\partial\partial x + \partial x^2\right)}{\partial t^2} = \frac{2\partial . x\partial x}{\partial t^2} = xX' + 2X.$$

In a similar manner there will be:

$$\frac{2\partial . y\partial y}{\partial t^2} = yY' + 2Y.$$

which two equations added in turn will give :

$$\frac{2\partial . \partial p}{\partial t^2} = \frac{2\partial\partial p}{\partial t^2} = xX' + yY' + 2(X + Y).$$

But with the values substituted and with the substitution made with respect of the letters *p* and *q* there is found :

$$2X + 2Y = 4B + 4Cp + 4D(pp + qq) + 4Ep(pp + 3qq).$$

From which on account of

$$xX' = 2Cxx + 4Dx^4 + 6Ex^6 \text{ and } yY' = 2Cyy + 4Dy^4 + 6Ey^6$$

there will be

$$xX' + yY' = 4Cp + 8D(pp+qq) + 12Ep(pp+3qq),$$

from which taken together there becomes

$$\frac{2\partial\partial p}{\partial t^2} = 4B + 8Cp + 12D(pp+qq) + 16Ep(pp+3qq).$$

§.75. From this formula the above $\frac{\partial p\partial q}{2q\partial t^2}$ taken four times may subtracted and there will remain

$$\frac{2\partial\partial p}{\partial t^2} - \frac{2\partial p\partial q}{q\partial t^2} = 8Dqq + 32Epqq.$$

Now both sides may be multiplied by $\frac{\partial p}{qq}$ and there will be produced

$$\frac{1}{\partial t^2}\left(\frac{2\partial p\partial\partial p}{qq} - \frac{2\partial p^2\partial q}{q^3}\right) = 8D\partial p + 32Ep\partial p,$$

the integral of which at once itself presents this expression

$$\frac{\partial p^2}{qq\partial t^2} = 4\Delta + 8Dp + 16Epp,$$

and thus with the root extracted : $\frac{\partial p}{q\partial t} = 2\sqrt{(\Delta + 2Dp + 4Epp)}$.

§.76. Now since there shall be

$$\frac{\partial p}{\partial t} = x\sqrt{X} \mp y\sqrt{Y} \text{ and } 2q = xx - yy,$$

with which substitution made this equation will arise

$$\frac{x\sqrt{X} \mp y\sqrt{Y}}{xx-yy} = \sqrt{\left(\Delta + D(xx+yy) + E(xx+yy)^2\right)},$$

which with the squares taken it will be reduced to this same form

$$\frac{xxX + yyY \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \Delta + D(xx+yy) + E(xx+yy)^2.$$

Therefore there is:

$$xxX+yyY=B(xx+yy)+C\left(x^4+y^4\right)+D\left(x^6+y^6\right)+E\left(x^8+y^8\right)$$

and hence this equation may be come upon

$$\frac{B(xx+yy)+C\left(x^4+y^4\right)+Dxxyy(xx+yy)+2Ex^4y^4\mp 2xy\sqrt{XY}}{(xx-yy)^2}=\Delta.$$

§.77. Now we may take as the above constant Δ thus, so that on putting $y=0$ there may become

$$x=k \text{ and } X=K=B+Ckk+Dk^4+Ek^6,$$

and the equation of the integral will adopt this form

$$\frac{B(xx+yy)+C\left(x^4+y^4\right)+Dxxyy(xx+yy)+2Ex^4y^4\mp 2xy\sqrt{XY}}{(xx-yy)^2}=\frac{B+Ckk}{kk},$$

which comes out somewhat simpler, if we may subtract C from both sides; indeed there will become :

$$\frac{B(xx+yy)+Cxxyy+Dxxyy(xx+yy)+2Ex^4y^4\mp 2xy\sqrt{XY}}{\left(x^2-y^2\right)^2}=\frac{B}{kk},$$

which agrees precisely with the integral I have shown above in § 72.

§.78. Here the noteworthy case presents itself , while $B=0$; but then the differential equation this itself will be had:

$$\frac{\partial x}{x\sqrt{\left(C+Dxx+Ex^4\right)}}\pm\frac{\partial y}{y\sqrt{\left(C+Dyy+Ey^4\right)}}=0,$$

of which therefore the integral expressed by the constant Δ will become :

$$\frac{C\left(x^4+y^4\right)+Dxxyy(xx+yy)+2Ex^4y^4\mp 2xy\sqrt{XY}}{(xx-yy)^2}=\Delta.$$

But in this case the integration cannot thus be determined, so that on putting $y=0$ there may become $x=k$, because the integral of the second part in this case evidently becomes infinite; on account of which the integration may be agreed to be determined by another way, just as on putting $y=b$ there may become $x=a$; then moreover there will become this constant

$$\Delta=\frac{C\left(a^4+b^4\right)+Daabb(aa+bb)+2Ea^4b^4\mp2ab\sqrt{AB}}{(aa-bb)^2}.$$

with

$$A=C+Daa+Ea^4 \text{ and } B=C+Dbb+Eb^4$$

present.

§.79. Who would wish to compare her the usual analytical process with the method, by which the illustrious Lagrange has used in the *Turnin Miscellany.* Book. IV, will observe that to be much easier and to produce the desired goal and much more conveniently to whatever case it can be applied. Moreover the illustrious man has introduced the formula $\frac{\partial t}{T}$ into the calculation, in place of which I have made use of the simple element ∂t, and henceforth he has considered the quantity T as a function of the letters *p* and *q*, which in place has demanded a difficult enough calculation, while it has been allowed by our method to investigate the same integrations far more neatly. But nevertheless there is no doubt, why that kind of analysis may not promise a significant advance, yet while it is not apparent, how that may be able to be applied to other integrations besides those cases, which we have treated here and which formerly I had derived from the canonical equation.

SUPPLEMENTUM VIII.

AD TOM. I. SECT. II. CAP. VI.

DE COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA

$$\int \frac{P\partial z}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}}$$

CONTENTARUM.

Dilucidationes super methodo elegantissima, qua illustris *de la Grange* usus est, in integranda aequatione differentiali

$$\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$$

[Commentatio 506 indicis ENESTROEMIANI]

Acta academiae scientiarum Petropolitanae 1778: I (1780), p. 20-57

§.1. Postquam diu et multum in perscrutanda aequatione differentiali

$$\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$$

desudassem atque imprimis in methodum *directam,* quae via facili ac plana ad eius integrale perduceret, nequicquam inquisivissem, penitus obstupui, cum mihi nunciaretur in volumine quarto Miscellaneorum Taurinensium ab Illustri De La Grange talem methodum esse expositam, cuius ope pro casu, quo

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$

et

$$Y = A + By + Cyy + Dy^3 + Ey^4,$$

propositae aequationis differentialis hoc integrale algebraicum atque adeo completum felicissimo successu elicuit

$$\frac{\sqrt{X}+\sqrt{Y}}{x-y} = \sqrt{\left(\Delta + D(x+y) \ + E(x+y)^2 \ \right)},$$

ubi *d* denotat quantitatem constantem arbitrariam per integrationem ingressam.

§. 2. Istud autem egregium inventum eo magis sum admiratus, quod equidem semper putaveram talem methodum in investigando idoneo factore, quo aequatio proposita integrabilis redderetur, quaeri oportere, cum vulgo omnis methodus integrandi vel in separatione variabilium vel in idoneo multiplicatore contineri videatur, etiamsi certis casibus quoque ipsa differentiatio ad integrale perducere queat, quemadmodum tam a me ipso quam ab aliis per plurima exempla est ostensum. Ad hanc autem tertiam viam illa ipsa methodus GRANGIANA rite referri posse videtur.

§. 3. Quanquam autem facile est inventis aliquid addere, tamen in re tam ardua plurimum intererit hanc methodum ab Illustri LA GRANGE adhibitam accuratius perpendisse atque ad usum analyticum magis accommodasse; siquidem totum negotium multo facilius ac simplicius expediri posse videtur; quamobrem, quae de hoc argumento, quod merito maximi momenti est censendum, sum meditatus, hic data opera fusius sum expositurus.

§.4. Quoniam autem hoc integrale ab Illustri LA GRANGE inventum ab iis formis, quas ipse olim dederam, plurimum discrepat ac simplicitate non mediocriter antecellit, ante omnia visum est scitari, quomodo aequationi differentiali satisfaciat. Hunc in finem pono brevitatis gratia $\sqrt{X} + \sqrt{Y} = V$ ut habeam

$$\frac{V}{x-y} = \sqrt{\left(\Delta + D(x+y) + E(x+y)^2\right)},$$

quam aequationem ita differentiare oportet, ut constans arbitraria Δ ex differentiali excedat. Sumtis igitur quadratis erit

$$\frac{V^2}{(x-y)^2} = \Delta + D(x+y) + E(x+y)^2,$$

quae differentiata dat

$$\frac{2V\partial V}{(x-y)^2} - \frac{2VV(\partial x - \partial y)}{(x-y)^3} - D(\partial x + \partial y) - 2E(x+y)(\partial x + \partial y) = 0.$$

§.5. Quo nunc calculus planior reddatur, seorsim partes vel per ∂x vel per ∂y affectas investigemus. Pro elemento igitur ∂x, si y ut constans spectetur, erit

$$\partial V = \frac{X'\partial x}{2\sqrt{X}},$$

unde singulae partes ita se habebunt

$$\partial x\left(\frac{VX'}{(x-y)^2\sqrt{X}} - \frac{2VV}{(x-y)^3} - D - 2E(x+y)\right),$$

ubi notetur esse $V = \sqrt{X} + \sqrt{Y}$ hincque

$$VV\sqrt{X} = (X+Y)\sqrt{X} + 2X\sqrt{Y},$$

unde hic duplicis generis termini occurrunt, dum vel per $\sqrt{X}$ vel per $\sqrt{Y}$ sunt affecti. Duo autem termini adsunt $\sqrt{Y}$ affecti, qui sunt

$$-\frac{4X\sqrt{Y}}{(x-y)^3} + \frac{X'\sqrt{Y}}{(x-y)^2},$$

qui ergo iunctim sumti dabunt

$$\frac{\sqrt{Y}}{(x-y)^3}\left(X'(x-y) - 4X\right),$$

quae forma ob

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$

hincque

$$X' = Bx + 2Cx + 3Dxx + 4Ex^3$$

dabit

$$X'(x-y) - 4X = -4A - B(3x+y) - 2C(xx+xy) - D\left(x^3 + 3xxy\right) - 4Ex^3y.$$

Termini autem per $\sqrt{X}$ affecti sunt

$$\frac{\sqrt{X}}{(x-y)^3}\left(X'(x-y) - 2(X+Y) - D(x-y)^3 - 2E(x+y)(x-y)^3\right).$$

Cum igitur sit

$$X + Y = 2A + B(x+y) + C\left(x^2+y^2\right) + D\left(x^3+y^3\right) + E\left(x^4+y^4\right),$$

facta substitutione iste postremus factor erit

$$-4A - B(x+3y) - 2C(xy+yy) - D\left(3xyy + y^3\right) - 4Exy^3,$$

quae forma a praecedente hoc tantum discrepat, quod litterae x et y sunt permutatae.

§.6. Quod si ergo brevitatis gratia ponamus

$$M = 4A + B(3x+y) + 2C(xx+xy) + D(x^3+3xxy) + 4Ex^3y,$$
$$N = 4A + B(x+3y) + 2C(yy+xy) + D(y^3+3xyy) + 4Exy^3,$$

hinc pars elemento ∂x affecta ita erit expressa

$$-\frac{\partial x}{(x-y)^3\sqrt{X}}\left(M\sqrt{Y} + N\sqrt{X}\right).$$

§.7. Simili modo ob $\partial V=\frac{Y'\partial Y}{2\sqrt{Y}}$, partes elemento ∂y affectae erunt

$$\frac{\partial y}{\sqrt{Y}}\left(\frac{YY'}{(x-y)^2}+\frac{2VV\sqrt{Y}}{(x-y)^3}-D\sqrt{Y}-2E(x+y)\sqrt{Y}\right)$$

Haec iam forma ob

$$V=\sqrt{X}+\sqrt{Y} \text{ et } VV\sqrt{Y}=(X+Y)\sqrt{Y}+2Y\sqrt{X},$$

continebit sequentes terminos per $\sqrt{X}$ affectos

$$\frac{\sqrt{X}}{(x-y)^3}\left[Y'(x-y)+4Y\right],$$

quae forma ex priore praecedentis calculi oritur, si litterae *x* et *y* permutentur simulque signa; unde patet hanc expressionem praebere valorem + N. Reliqui autem termini per $\sqrt{Y}$ effecti erunt

$$\frac{\sqrt{Y}}{(x-y)^3}\left(Y'(x-y)+2(X+Y)-D(x-y)^3-2E(x+y)(x-y)^3\right).$$

Haec forma iterum ex permutatione litterarum et signorum ex forma praecedentis calculi oritur; quae ergo cum esset −N, haec erit + M. Hoc igitur modo partes elementum ∂y continentes erunt

$$\frac{+\partial y}{(x-y)^3\sqrt{Y}}\left[N\sqrt{X}+M\sqrt{Y}\right].$$

§.8. Coniungendis igitur his membris aequatio differentialis ex forma *Grangiana* orta erit

$$\left(\frac{\partial y}{\sqrt{Y}}-\frac{\partial x}{\sqrt{Y}}\right)\left[\frac{N\sqrt{X}+M\sqrt{Y}}{(x-y)^3}\right]=0,$$

quae per factorem communem divisa praebet ipsam aequationem differentialem propositam $\frac{\partial x}{\sqrt{X}}=\frac{\partial y}{\sqrt{Y}}$; unde simul patet aequationem integralem exhibitam recte se habere atque adeo valorem litterae Δ arbitrio nostro penitus relinqui.

§.9. Antequam autem methodum *Grangianam* ad ipsam aequationem differentialem

$\frac{\partial x}{\sqrt{X}}=\frac{\partial y}{\sqrt{Y}}$ in omni extensione acceptam applicemus, a casu simpliciore inchoemus, quo aequatio adeo rationalis proponitur haec

$$\frac{\partial x}{a+2bx+cxx}=\frac{\partial y}{a+2by+cyy}.$$

Analysis pro integratione aequationis differentialis

$$\frac{\partial x}{a+2bx+cxx}=\frac{\partial y}{a+2by+cyy}.$$

§.10. Ponamus brevitatis gratia $a+2bx+cxx=X$ et $a+2by+cyy=Y$, ut fieri debeat $\frac{\partial x}{X}=\frac{\partial y}{Y}$, quae formulae cum inter se debeant esse aequales, utraque per idem elementum ∂t designetur, ita ut nanciscamur has duas formulas $\frac{\partial x}{\partial t}=X$ et $\frac{\partial y}{\partial t}=Y$. Quodsi ergo iam statuamus

$$x-y=q, \text{erit } \frac{\partial q}{\partial t}=X\text{-}Y=2bq+cq(x+y),$$

unde per q dividendo erit $\frac{\partial q}{q\partial t}=2b+c(x+y)$.

§.11. Nunc primas formulas differentiemus, sumto elemento ∂t constante et facto

$$\partial X=X'\partial x \text{ et } \partial Y=Y'\partial y$$

orientur hae duae aequationes

$$\frac{\partial\partial x}{\partial x\partial t}=X' \text{ et } \frac{\partial\partial y}{\partial y\partial t}=Y',$$

quae invicem additae praebent

$$\frac{\partial\partial x}{\partial x\partial t}+\frac{\partial\partial y}{\partial y\partial t}=X'+Y'.$$

Quare, cum sit

$$X'=2b+2cx \text{ et } Y'=2b+2cy,$$

erit

$$\frac{1}{\partial t}\left(\frac{\partial\partial x}{\partial x}+\frac{\partial\partial y}{\partial y}\right)=4b+2c(x+y).$$

§.12. Quoniam igitur hic postremus valor duplo maior est praecedente $\frac{\partial q}{q\partial t}$, hoc modo deducti sumus ad hanc aequationem

$$\frac{\partial\partial x}{\partial x}+\frac{\partial\partial y}{\partial y}=\frac{2\partial q}{q},$$

quae integrata dat $l\partial x+l\partial y=2lq+\text{const.}$, hincque in numeris erit

$$\partial x \partial y = Cqq\partial t^2,$$

ita ut sit

$$C = \frac{\partial x \partial y}{qq\partial t^2},$$

Quare, cum sit

$$\frac{\partial x}{\partial t} = \text{X} \text{ et } \frac{\partial y}{\partial t} = \text{Y},$$

aequatio integralis erit

$$\frac{\text{XY}}{(x-y)^2} = \text{C},$$

quae ergo non solum est algebraica, sed etiam completa.

§.13. Si igitur proposita fuerit haec aequatio differentialis

$$\frac{dx}{a+2bx+cxx} = \frac{dy}{a+2by+cyy},$$

eius integrale completum ita erit expressum

$$\frac{(a+2bx+cxx)(a+2by+cyy)}{(x-y)^2} = \text{C},$$

quae utrinque addendo $bb-ac$ induet hanc formam

$$\frac{aa+2ab(x+y)+2acxy+bb(x+y)^2+2bcxy(x+y)+ccxxyy}{(x-y)^2} = \Delta\Delta,$$

sicque extracta radice integrale hanc formam habebit

$$\frac{a+b(x+y)+cxy}{x-y} = \Delta,$$

quae sine dubio est simplicissima, quandoquidem tam *y* per *x* quam *x* per *y* facillime exprimi potest, cum sit

$$y = \frac{(\Delta-b)x-a}{\Delta+b+cx} \text{ et } x = \frac{a+(\Delta+b)y}{\Delta-b-cy}.$$

§.14. Calculum, quo hic usi sumus, perpendenti facile patebit in his formis X et Y non ultra quadrata progredi licere. Si enim ipsi X insuper tribuamus terminum ∂x^3 et ipsi Y terminum ∂y^3, pro priore forma prodit

$$\frac{X-Y}{x-y} = 2b + c\left(x+y\right) + d\left(xx+xy+yy\right) = \frac{\partial q}{q\partial t},$$

pro altera autem forma est

$$X' + Y' = 4b + 2c(x+y) + 3d(xx+yy) = \frac{\partial\partial x}{\partial x\partial t} + \frac{\partial\partial y}{\partial y\partial t}.$$

Quare si hinc duplum praecedentis auferamus, colligitur

$$\frac{\partial\partial x}{\partial x\partial t} + \frac{\partial\partial y}{\partial y\partial t} - \frac{2\partial q}{q\partial t} = d\left(x-y\right)^2,$$

quam aequationem non amplius integrare licet.

§.15. Facile autem ostendi potest talem aequationem differentialem, in qua ultra quadratum proceditur, nullo amplius modo algebraice integrari posse. Si enim tantum hic casus proponeretur $\frac{\partial x}{1+x^3} = \frac{\partial y}{1+y^3}$, notum est utrinque integrale partim logarithmos partim arcus circulares involvere ideoque quantitates transcendentes diversos, quae nullo modo inter se comparari possunt. Huiusmodi scilicet comparationes iis tantum casibus locum habere possunt, quando utrinque unius generis tantum quantitates transcendentes occurrunt.

Analysis pro integratione aequationis differentialis

$$\frac{\partial x}{a+2bx+cxx} + \frac{\partial y}{a+2by+cyy} = 0$$

§.16. Quodsi hic ut ante ponamus

$$\frac{\partial x}{a+2bx+cxx} = \partial t,$$

statui debebit

$$\frac{\partial y}{a+2by+cyy} = -\partial t;$$

at vero si calculum simili modo quo ante instituere velimus, nihil plane proficimus. Postquam autem omnes difficultates probe perpendissem, tandem in artificium incidi, quo hunc casum expedire licuit, ita ut hinc non contemnendum incrementum methodo *Grangianae* attulisse mihi videar.

§.17. Quoniam igitur has duas habeo aequationes

$$\frac{\partial x}{\partial t} = X \;\text{ et }\; \frac{\partial y}{\partial t} = -Y,$$

hinc formo istam novam aequationem

$$\frac{y\partial x + x\partial y}{\partial t} = yX - xY.$$

Iam facio $xy = u$, ut habeam

$$\frac{\partial u}{\partial t} = a(y-x) + cxy(x-y),$$

unde posito $x - y = q$ erit $\frac{\partial u}{\partial t} = q\left(cu - a\right)$, quae aequatio per $cu - a$ divisa ductaque in c praebet

$$\frac{c\partial u}{\partial t\left(cu-a\right)} = cq,$$

hocque modo nacti sumus differentiale logarithmicum.

§.18. Dein vero aequationes principales ut ante differentiemus et obtinebimus

$$\frac{\partial\partial x}{\partial t\partial x} = X' \text{ et } \frac{\partial\partial y}{\partial t\partial y} = -Y',$$

quae invicem additae dant

$$\frac{1}{\partial t}\left(\frac{\partial\partial x}{\partial x} + \frac{\partial\partial y}{\partial y}\right) = X' - Y' = 2cq;$$

quare si hine duplum praecedentis aequationis subtrahamus, remanebit

$$\frac{1}{\partial t}\left(\frac{\partial\partial x}{\partial x} + \frac{\partial\partial y}{\partial y} - \frac{2c\partial u}{cu-a}\right) = 0,$$

unde per ∂t multiplicando et integrando nanciscimur $ldx + ldy - 2l\left(cu - a\right) = lC$ ideoque $\frac{\partial x\partial y}{\left(cu-a\right)^2} = C\partial t^2$. Cum igitur sit $\partial x = X\partial t$ et $dy = -Y\partial t$, aequatio integralis nostra erit $-\frac{XY}{\left(cu-a\right)^2} = C.$

§.19. Per hanc ergo analysin deducti sumus ad hanc aequationem integralem aequationis propositae

$$\frac{\left(a+2bx+cxx\right)\left(a+2by+cyy\right)}{\left(a-cxy\right)^2} = C$$

quae aequatio, si utrinque unitas subtrahatur, reducitur ad hanc formam

$$\frac{2ab(x+y) + ac(x+y)^2 + 4bbxy + 2bcxy(x+y)}{\left(a-cxy\right)^2} = C.$$

§.20. Illustremus hanc integrationem exemplo, ponendo $a=1,\ b=0$ et $c=1$, ita ut proposita sit haec aequatio differentialis

$$\frac{dx}{1+xx}+\frac{dy}{1+yy}=0,$$

cuius integrale novimus esse $\text{Atang.}x+\text{Atang.}y=\text{A tang.}\frac{x+y}{1-xy}=\text{C},$ sicque novimus esse $\frac{x+y}{1-xy}=\text{C}$. At vero nostra postrema formula dat pro hoc casu

$$\frac{(x+y)^2}{(1-xy)^2}=\text{C ideoque }\frac{x+y}{1-xy}=\text{C},$$

quod egregie convenit.

§.21. Consideremus etiam casum, quo $a=1,\ b=\frac{1}{2}$ et $c=1$, ita ut proponatur haec aequatio

$$\frac{\partial x}{1+x+xx}+\frac{\partial y}{1+y+yy}=0,$$

cuius integrale est

$$\frac{2}{\sqrt{3}}\text{Atang. }\frac{x\sqrt{3}}{2+x}+\frac{2}{\sqrt{3}}\text{Atang.}\frac{y\sqrt{3}}{2+y}=\text{C},$$

unde sequitur fore

$$\text{Atang.}\frac{2(x+y+xy)\sqrt{3}}{4+2(x+y)-2xy}=\text{C}$$

ideoque etiam $\frac{x+y+xy}{2+x+y-xy}=\text{C}.$ At vero forma integralis inventa pro hoc casu dabit

$$\frac{x+y+(x+y)^2+xy+xy(x+y)}{(1-xy)^2}=\text{C},$$

quae in factores resoluta dat

$$\frac{(1+x+y)(x+y+xy)}{(1-xy)^2}=\text{C}.$$

Prior vero aequatio $\frac{x+y+xy}{2+x+y-xy}=\text{C}$ inversa praebet $\frac{2+x+y-xy}{x+y+xy}=\text{C}$ et unitate subtracta $\frac{1-xy}{x+y+xy}=\text{C}$ atque haec in praecedentem ducta dat $\frac{1+x+y}{1-xy}=\text{C}.$

§.22. Videamus igitur, utrum hae posteriores aequationes inter se conveniant, et quia constantes utrinque inter se discrepare possunt, ambas aequationes ita referamus

$$\frac{1-xy}{x+y+xy}=\alpha\ \text{ et }\ \frac{1+x+y}{1-xy}=\beta;$$

unde cum sit $\frac{1}{\alpha} = \frac{x+y+xy}{1-xy}$, evidens est fore $\beta - \frac{1}{\alpha} = 1$, ex quo pulcherrimus consensus inter ambas formulas elucet.

Ex his exemplis intelligitur aequationem generalem supra inventam hoc modo per factores repraesentari posse

$$\frac{(2b+c(x+y))(a(x+y)+2bxy)}{(a-cxy)^2}.$$

Ceterum consideratio harum formularum haud iniucundas speculationes suppeditare poterit.

§.23. Sequenti autem modo forma illa integralis inventa

$$\frac{(2b+c(x+y))(a(x+y)+2bxy)}{(a-cxy)^2} = \mathrm{C}$$

statim ad formam simplicissimam reduci potest; si enim eius factores statuamus

$$\frac{2b+c(x+y)}{a-cxy} = \mathrm{P} \quad \text{et} \quad \frac{a(x+y)+2bxy}{a-cxy} = \mathrm{Q},$$

ut esse debeat $\mathrm{PQ} = \mathrm{C}$, erit $a\mathrm{P} - c\mathrm{Q} = \frac{2ab-2bcxy}{a-cxy} = 2b,$ unde fit

$$\mathrm{Q} = \frac{a\mathrm{P}-2b}{c},$$

sicque quantitati constanti aequari debet haec forma $\frac{a\mathrm{PP}-2b\mathrm{P}}{c}$; ex quo patet, etiam ipsam quantitatem P constanti aequari debere, ita ut iam aequatio nostra integralis sit

$$\frac{2b+c(x+y)}{a-cxy} = \mathrm{C} \quad \text{vel etiam} \quad \frac{a(x+y)+2bxy}{a-cxy} = \mathrm{C}.$$

Alia solutio facillima ejusdem aequationis

$$\frac{\partial x}{a+2bx+cxx} + \frac{\partial y}{a+2by+cyy} = 0.$$

§.24. Postrema reductione probe perpensa, comperui statim ab initio ad formam integralis simplicissimam perveniri posse atque adeo non necesse esse ad differentialia secunda ascendere. Si enim ut ante ponamus $x + y = p,\ x - y = q$ et $xy = u,$ ex formulis

$$\frac{\partial x}{\partial t} = \mathrm{X} \quad \text{et} \quad \frac{\partial y}{\partial t} = -\mathrm{Y}$$

statim deducimus

$$\frac{\partial p}{\partial t} = X - Y = 2bq + cpq, \text{unde fit } \frac{\partial p}{2b+cp} = q\partial t.$$

§.25. Porro vero erit

$$\frac{y\partial x + x\partial y}{\partial t} = \frac{\partial u}{\partial t} = yX - xY = -aq + cqu,$$

unde fit $\frac{\partial u}{cu-a} = q\partial t$, quam ob rem hinc statim colligimus hanc aequationem $\frac{\partial p}{2b+cp} = \frac{\partial u}{cu-a}$, cuius integratio praebet

$$l(2b+cp) = l(cu-a) + lC;$$

unde deducitur haec aequatio algebraica $\frac{2b+cp}{cu-a} = C$, quae restitutis litteris x et y, dat $\frac{2b+c(x+y)}{cxy-a} = C$, quae est forma simplicissima aequationis integralis desideratae. Hic imprimis notatu dignum occurrit, quod casum primum hac ratione resolvere non licet.

§.26. Ex forma autem integrali inventa facile aliae derivantur; veluti si addamus $\frac{2b}{a}$ orietur haec forma $\frac{a(x+y)+2bxy}{cxy-a} = C$, quae per praecedentem divisa denuo novam formam suppeditat, scilicet $\frac{2b+c(x+y)}{a(x+y)+2bxy} = C$; quae formae quomodo satisfaciant, operae pretium erit ostendisse. Et quidem postrema forma, differentiata, erit

$$\frac{-2ab(\partial x+\partial y)-4bb(y\partial x+x\partial y)-2bc(yy\partial x+xx\partial y)}{\left(a(x+y)+2bxy\right)^2},$$

quae in ordinem redacta praebet

$$\partial x(2ab + 4bby + 2bcyy) + \partial y(2ab + 4bbx + 2bcxx) = 0.$$

Haec per $2b$ divisa et separata dat

$$\frac{\partial x}{a+2bx+cxx} + \frac{\partial y}{a+2by+cyy} = 0,$$

quae est ipsa proposita.

Analysis pro integratione aequationis

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}} = \frac{\partial y}{\sqrt{(A+By+Cyy)}}$$

§.27. Introducto novo elemento ∂t, deinceps pro constanti habendo, oriuntur hae duae aequationes

$$\frac{\partial x}{\partial t} = \sqrt{X} \text{ et } \frac{\partial y}{\partial t} = \sqrt{Y} ,$$

ubi litteris X et Y valores initio assignatos tribuamus. Videbimus autem pro methodo, qua hic utemur, terminos litteris D et E affectos omitti debere.
Sumtis ergo quadratis erit

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y .$$

§.28. Nunc istas formulas differentiemus positoque, ut fieri solet, $\partial X = X'\partial x$ et $\partial Y = Y'\partial y$ nanciscemur has aequationes

$$\frac{2\partial\partial x}{\partial t^2} = X' \text{ et } \frac{2\partial\partial y}{\partial t^2} = Y'$$

ac posito $x + y = p$ fiet $\frac{2\partial\partial p}{\partial t^2} = X' + Y'$. Cum iam sit

$$X' = B + 2Cx + 3Dxx + 4Ex^3 \text{ et } Y' = B + 2Cy + 3Dyy + 4Ey^3$$

erit

$$X' + Y' = 2B + 2Cp + 3D(xx + yy) + 4E(x^3 + y^3) = \frac{2\partial\partial p}{\partial t^2},$$

quae aequatio manifesto integrationem admittet, si fuerit et $D = 0$ et $E = 0$, quemadmodum assumsimus. Multiplicando igitur per ∂p et integrando nanciscimur

$$\frac{dp^2}{dt^2} = \Delta + 2Bp + Cpp$$

et radicem extrahendo

$$\frac{dp}{dt} = \sqrt{(\Delta + 2Bp + Cpp)} .$$

Cum igitur sit $\frac{\partial p}{\partial t} = \sqrt{X} + \sqrt{Y}$, aequatio integralis, quam sumus adepti, erit

$$\sqrt{X} + \sqrt{Y} = \sqrt{\left(\Delta + 2B(x + y) + C(x + y)^2\right)},$$

quae adeo est algebraica; ubi notetur esse

$$X = A + Bx + Cxx \text{ et } Y = A + By + Cyy.$$

§.29. Sumamus igitur quadrata et nostra aequatio integralis erit

$$2A + B(x+y) + C(x^2+y^2) + 2\sqrt{XY} = \Delta + 2B(x+y) + C(x+y)^2$$

sive

$$2A - B(x+y) - 2Cxy + 2\sqrt{XY} = \Delta,$$

quae penitus ab irrationalitate liberata posito $\Delta - 2A = \Gamma$ praebebit

$$4XY = 4AA + 4AB(x+y) + 4AC(xx+yy) + 4BBxy + 4BCxy(x+y)$$
$$+4CCxxyy = \Gamma^2 + 2\Gamma B(x+y) + 4\Gamma Cxy + BB(x+y)^2 + 4BCxy(x+y) + 4CCxxyy$$

sive

$$(4AA - \Gamma^2) + 2B(2A - \Gamma)(x+y) + 4(BB - \Gamma C)xy$$
$$+4AC(xx+yy) - B^2(x+y)^2 = 0.$$

§.30. Quodsi iam hanc aequationem rationalem cum formula *canonica,* qua olim sumusus ad huiusmodi integrationes expediendas, comparemus, quae erat

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy = 0,$$

dum scilicet loco $(x+y)^2$ scribamus $(xx+yy)+2xy$ reperiemus fore

$$\alpha = 4AA - \Gamma^2,\ \beta = B(2A - \Gamma),\ \gamma = 4AC - B^2,\ \delta = BB - 2\Gamma C.$$

§.31. Alio vero insuper modo eandem aequationem differentialmn propositam integrare poterimus introducendo litteram $q = x - y$ *;* tum enim habebimus

$$\frac{2\partial\partial q}{\partial t^2} = X' - Y'$$

At vero erit

$$X' - Y' = 2Cq + 3Dq(x+y) + 4Eq(xx+xy+yy),$$

ubi iterum patet statui debere tam $D = 0$ quam $E = 0$, ut integratio multiplicando per ∂q succedat. Hoc autem notato erit integrale

$$\frac{\partial q^2}{\partial t^2} = \text{Const.} + Cqq \ \text{ ideoque } \ \frac{\partial q}{\partial t} = \sqrt{(\Delta + Cqq)}.$$

§.32. Cum igitur sit $\frac{\partial q}{\partial t} = \sqrt{X} - \sqrt{Y}$, hoc integrale ita erit expressum

$$\sqrt{X} - \sqrt{Y} = \sqrt{(\Delta + Cqq)},$$

quae aequatio sumtis quadratis abit in hanc

$$2A + B(x+y) + C(xx+yy) - 2\sqrt{XY} = \Delta + C(x-y)^2$$

sive

$$2A + B(x+y) + 2Cxy - 2\sqrt{XY} = \Delta,$$

unde fit

$$2\sqrt{XY} = 2A - \Delta + B(x+y) + 2Cxy,$$

ubi si ponatur $2A - \Delta = -\Gamma$, aequatio ab ante inventa prorsus non discrepat.

§.33. Quodsi autem proposita fuisset aequatio

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}} + \frac{\partial y}{\sqrt{(A+By+Cyy)}} = 0,$$

integralia ante inventa ad hunc casu $\sqrt{Y}$ referentur, si modo loco $-\sqrt{Y}$ scribatur ; unde patet pro hoc casu haberi hanc aequationem

$$\sqrt{X} - \sqrt{Y} = \sqrt{\left(\Delta + 2B(x+y) + C(x+y)^2\right)}$$

vel etiam

$$\sqrt{X} + \sqrt{Y} = \sqrt{\left(\Delta + C(x-y)^2\right)}$$

§.34. Hic singularis casus occurrit, quando formulae $A + Bx + Cxx$ sunt quadrata. Sit enim

$$X = (a+bx)^2 \quad \text{et} \quad Y = (a+by)^2$$

ideoque

$$A = aa,\ B = 2ab,\ C = bb;$$

tum enim prior forma integralis erit

$$b(x-y) = \sqrt{\left(\Delta + 4ab(x+y) + bb(x+y)^2\right)}$$

sumtisque quadratis

$$-4bbxy = \Delta + 4ab(x+y)$$

ideoque

$$\Delta = a(x+y) + bxy,$$

cuius aequationis differentiale est

$$a(\partial x + \partial y) + b(x\partial y + y\partial v) = 0 \text{ ideoque } \partial x(a+by) + dy(a+bx) = 0.$$

Sin autem altera formula utatur, erit

$$2a + b(x+y) = \sqrt{\left(\Delta + bb(x-y)^2\right)},$$

unde quadratis sumtis positoque $\Delta - 4aa = \Gamma$ prodit ut ante

$$\Gamma = a(x+y) + bxy.$$

Analysis
Pro integrande aequatione
$$\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$$

existente

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$
et
$$Y = A + By + Cyy + Dy^3 + Ey^4.$$

§.35. Introducto iterum elemento ∂t, ut sit

$$\frac{\partial x}{\partial t} = \sqrt{X} \text{ et } \frac{\partial y}{\partial t} = \sqrt{Y}$$

ideoque sumtis quadratis

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y,$$

statuamus $x + y = p$ et $x - y = q$, et quia hinc prodit

$$\partial x^2 - \partial y^2 = \partial p \partial q, \text{ erit},$$

$$\frac{\partial p\partial q}{\partial t^2} = X - Y = B(x-y) + C\left(x^2-y^2\right) + D\left(x^3-y^3\right) + E\left(x^4-y^4\right).$$

§.36. Quoniam igitur est $x = \frac{p+q}{2}$ et $y = \frac{p-q}{2}$ his valoribus introductis reperietur

$$X - Y = Bq + Cpq + \tfrac{1}{4}Dq(3pp+qq) + \tfrac{1}{2}Epq(pp+qq),$$

unde per q dividendo oritur

$$\frac{\partial p\partial q}{q\partial t^2} = B + Cp + \tfrac{1}{4}D(3pp+qq) + \tfrac{1}{2}Ep(pp+qq).$$

§.37. Nunc etiam formulas quadratas primo exhibitas differentiemus et statuendo ut ante

$$\partial X = X'\partial x \quad \text{et} \quad \partial Y = Y'\partial x$$

habebimus

$$\frac{2\partial\partial x}{\partial t^2} = X' \quad \text{et} \quad \frac{2\partial\partial y}{\partial t^2} = Y'$$

hincque addendo

$$\frac{2\partial\partial p}{\partial t^2} = X' + Y'.$$

Cum vero sit

$$X' = B + 2Cx + 3Dxx + 4Ex^3 \quad \text{et} \quad Y' = B + 2Cy + 3Dyy + 4Ey^3,$$

erit

$$X' + Y' = 2B + 2Cp + \tfrac{3}{2}D(pp+qq) + Ep(pp+3qq),$$

ita ut substituto hoc valore fiat

$$\frac{\partial\partial p}{\partial t^2} = B + Cp + \tfrac{3}{4}D(pp+qq) + \tfrac{1}{2}Ep(pp+3qq),$$

a qua aequatione $\frac{\partial p\partial q}{q\partial t^2}$ si priorem pro subtrahamus, remanebit sequens

$$\frac{\partial\partial p}{\partial t^2} - \frac{\partial p\partial q}{q\partial t^2} = \tfrac{1}{2}Dqq + Epqq.$$

§.38. Haec iam aequatio per qq divisa producit istam

$$\frac{1}{\partial t^2}\left(\frac{\partial\partial p}{qq} - \frac{\partial p \partial q}{q^3}\right) = \frac{1}{2}\mathrm{D} + \mathrm{E}p,$$

quae ducta in $2\partial p$ manifesto fit integrabilis; prodit enim

$$\frac{\partial p^2}{qq\partial t^2} = \Delta + \mathrm{D}p + \mathrm{E}pp,$$

ex qua radice extracta colligitur

$$\frac{\partial p}{q\partial t} = \sqrt{(\Delta + \mathrm{D}p + \mathrm{E}pp)}.$$

Cum igitur posuerimus $p = x + y$ et $q = x - y$, erit

$$\frac{\partial p}{\partial t} = \sqrt{\mathrm{X}} + \sqrt{\mathrm{Y}}\ ,$$

unde resultat haec aequatio integralis algebraica

$$\frac{\sqrt{\mathrm{X}}+\sqrt{\mathrm{Y}}}{x-y} = \sqrt{\left(\Delta + \mathrm{D}(x+y) + \mathrm{E}(x+y)^2\right)},$$

quae est ipsa forma ab illustri *Lagrange* inventa.

§.39. Evolvamus ulterius hanc formam ac sumtis quadratis erit

$$\frac{\mathrm{X}+\mathrm{Y}+2\sqrt{\mathrm{XY}}}{(x-y)^2} = \Delta + \mathrm{D}(x+y) + \mathrm{E}(x+y)^2,$$

Est vero

$$\mathrm{X}+\mathrm{Y} = 2\mathrm{A} + \mathrm{B}(x+y) + \mathrm{C}(xx+yy) + \mathrm{D}\left(x^3+y^3\right) + \mathrm{E}\left(x^4+y^4\right);$$

unde si auferamus

$$\left(\mathrm{D}(x+y) + \mathrm{E}(x+y)^2\right)(x-y)^2,$$

remanebit

$$2\mathrm{A} + \mathrm{B}(x+y) + \mathrm{C}\left(x^2+y^2\right) + \mathrm{D}xy(x+y) + 2\mathrm{E}xxyy,$$

quo substituto aequatio integralis erit

$$\frac{2\mathrm{A}+\mathrm{B}(x+y)+\mathrm{C}\left(x^2+y^2\right)+\mathrm{D}xy(x+y)+2\mathrm{E}x^2y^2+2\sqrt{\mathrm{XY}}}{(x-y)^2} = \Delta.$$

§.40. Haec aequatio aliquanto concinnior reddi potest subtrahendo utrinque C et statuendo $\Delta - C = \Gamma$; habebitur enim hoc facto

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy+2\sqrt{XY}}{(x-y)^2}=\Gamma,$$

unde deducimus

$$2\sqrt{XY}=\Gamma(x-y)^2-2A-B(x+y)-2Cxy-Dxy(x+y)-2Exxyy,$$

sive ponendo

$$2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy=V$$

aequatio nostra erit

$$2\sqrt{XY}=\Gamma(x-y)^2-V$$

quae sumtis quadratis abit in hanc

$$4XY=\Gamma^2(x-y)^4-2\Gamma V(x-y)^2+VV$$

sive

$$4XY-VV=\Gamma^2(x-y)^4-2\Gamma V(x-y)^2.$$

§.41. Facta nunc substitutione erit

$$\begin{aligned}4XY&=4A^2+4AB(x+y)+4AC(xx+yy)+4AD\left(x^3+y^3\right)+4AE\left(x^4+y^4\right)\\&+4BBxy+4BCxy(x+y)+4BDxy(xx+yy)+4BExy\left(x^3+y^3\right)\\&+4CCxxyy+4CDxxyy(x+y)+4CExxyy(xx+yy)\\&+4DDx^3y^3+4DEx^3y^3(x+y)+4EEx^4y^4.\end{aligned}$$

At vero porro colligitur fore

$$\begin{aligned}VV&=4AA+4AB(x+y)+8ACxy+4ADxy(x+y)+8AExxyy\\&+BB(x+y)^2+4BCxy(x+y)+2BDxy(x+y)^2+4BE(x+y)xxyy\\&+4CCxxyy+4CD(x+y)xxyy+8CEx^3y^3\\&+DDxxyy(x+y)^2+4DEx^3y^3(x+y)+4EEx^4y^4.\end{aligned}$$

§.42. Quodsi iam posteriorem formuIam a priore subtrahamus et singulos terminos ordine analogos disponamus, reperiemus

$$\begin{aligned}&4XY - VV = 4AC(x-y)^2 + 4AD(x+y)(x-y)^2(x-y)^2\\ &+4AE(x+y)^2(x-y)^2 - B^2(x-y)^2 + 2BDxy(x-y)^2 + 4BExy(x+y)(x-y)^2,\\ &+4CExxyy(x-y)^2 - DDxxyy(x-y)^2.\end{aligned}$$

quae expressio factorem habet communem $(x-y)^2$, per quem ergo si dividamus, perveniemus ad hanc aequationem concinniorem

$$\begin{aligned}&4AC + 4AD(x+y) + 4AE(x+y)^2 - BB + 2BDxy + 4BExy(x+y)\\ &+(4CE - DD)xxyy = \Gamma\Gamma(x-y)^2 - 4\Gamma A - 2\Gamma B(x+y) - 4\Gamma Cxy - 2\Gamma Dxy(x+y) - 4\Gamma Exxyy.\end{aligned}$$

§.43. Transferamus nunc omnes terminos ad partem sinistram et loco $(x+y)^2$ scribamus $(xx+yy)+2xy$, tum vero $(xx+yy)-2xy$ loco $(x-y)^2$, quo facto talis oritur aequatio meae canonicae respondens

$$0 = \begin{cases} 4AC + 4AD(x+y) + 4AE\left(x^2+y^2\right) + 2BDxy + 4BExy(x+y) + 4CExxyy\\ -BB + 2\Gamma B(x+y) - \Gamma\Gamma\left(x^2+y^2\right) + 8AExy + 2\Gamma Dxy(x+y) - DDxxyy\\ +4\Gamma A \qquad\qquad\qquad\qquad + 2\Gamma^2 xy \qquad\qquad + 4\Gamma Exxyy\\ \qquad\qquad\qquad\qquad\qquad\quad + 4\Gamma Cxy \end{cases}$$

§.44. Hinc ergo pro aequatione canonica litterae graecae $\alpha,\ \beta,\ \gamma,\ \delta$ etc. per latinas A, B, C, D, E una cum constante Γ sequenti modo determinantur

$$\begin{aligned}&\alpha = 4AC + 4\Gamma A - BB\\ &\beta = 2AD + \Gamma B\\ &\gamma = 4AE - \Gamma\Gamma\\ &\delta = BD + 4AE + \Gamma^2 + 2\Gamma C\\ &\varepsilon = 2BE + \Gamma D\\ &\zeta = 4CE + 4\Gamma E - DD,\end{aligned}$$

ita ut aequatio canonica, qua olim sumusus, sit

$$a + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0.$$

§.45. Haec autem aequatio integralis ad rationalitatem perducta latius patet quam aequatio proposita differentialis $\frac{dx}{\sqrt{X}}-\frac{dy}{\sqrt{Y}}=0$; simul enim complectitur integrale huius $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$. Scilicet haec aequatio complectitur duos factores, quorum alteruter alterutri satisfacit. Ex genesi autem patet hanc aequationem esse productum ex his factoribus

$$\Delta(x-y)^2-\mathrm{V}+2\sqrt{\mathrm{XY}} \ \text{ et } \Delta(x-y)^2-\mathrm{V}-2\sqrt{\mathrm{XY}}..$$

§.46. Supra iam observavimus eiusdem aequationis differentialis integrale hoc quoque modo exhiberi posse

$$\frac{\mathrm{M}\sqrt{\mathrm{Y}}+\mathrm{N}\sqrt{\mathrm{X}}}{(x-y)^3}=\mathrm{C}$$

(vide § 8 et praec.) existente

$$\mathrm{M}=4\mathrm{A}+\mathrm{B}(3x+y)+2\mathrm{C}x(x+y)+\mathrm{D}xx(x+3y)+4\mathrm{E}x^3y,$$
$$\mathrm{N}=4\mathrm{A}+\mathrm{B}(3y+x)+2\mathrm{C}y(x+y)+\mathrm{D}yy(y+3x)+4\mathrm{E}xy^3,$$

ubi notasse iuvabit esse

$$\mathrm{M}+\mathrm{N}=8\mathrm{A}+4\mathrm{B}(x+y)+2\mathrm{C}(x+y)^2+\mathrm{D}(x+y)^3+4\mathrm{E}xy(xx+yy),$$
$$\mathrm{M}-\mathrm{N}=2\mathrm{B}(x-y)+2\mathrm{C}(x+y)(x-y)+\mathrm{D}(x-y)(x^2+4xy+y^2)$$
$$+4\mathrm{E}xy(x+y)(x-y).$$

Interim tamen haud facile intelligitur, quomodo haec forma cum ante inventa consentiat, dum tamen de consensu certi esse possumus.

§.47. Ex iis, quae hactenus sunt allata, satis liquet eandem aequationem integralem innumeris modis exhiberi posse, prout constans arbitraria alio atque alio modo repraesentatur; unde plurimum intererit certam legem stabilire, secundum quam quovis casu constantem illam arbitrariam exprimere velimus. Hunc in finem ista regula observetur, ut perpetuo integralia ita capiantur, ut posito $y=0$ fiat $x=k$ hincque secundum legem compositionis $\mathrm{X}=\mathrm{K}$ existente

$$\mathrm{K}=\mathrm{A}+\mathrm{B}k+\mathrm{C}kk+\mathrm{D}k^3+\mathrm{E}k^4.$$

Hac enim lege observata omnia integralia, utcunque diversa videantur, ad perfectum consensum perduci poterunt. Hoc igitur modo quae hactenus invenimus, sequentibus theorematibus complectamur.

THEOREMA 1

§.48. *Si haec aequatio differentialis*

$$\frac{\partial x}{a+bx+cxx} - \frac{\partial y}{a+by+cyy} = 0$$

ita integretur, ut posito $y = 0$ *fiat* $x = k$*, integrale ita se habebit*

$$\frac{2a+b(x+y)+2cxy}{x-y} = \frac{2a+bk}{k}.$$

THEOREMA 2

§.49. *Si haec aequatio differentialis*

$$\frac{\partial x}{a+bx+cxx} - \frac{\partial y}{a+by+cyy} = 0$$

ita integretur, ut posito $y = 0$ *fiat* $x = k$*, integrale supra triplici modo est inventum; erit enim*

$$\text{I.} \quad \frac{b+c(x+y)}{cxy-a} = -\frac{b+ck}{a}$$

$$\text{II.} \quad \frac{a(x+y)+bxy}{cxy-a} = -k$$

$$\text{III.} \quad \frac{b+c(x+y)}{a(x+y)+bxy} = \frac{b+ck}{ak}.$$

THEOREMA 3

§.50. *Si haec aequatio differentialis*

$$\frac{\partial x}{\sqrt{(\mathrm{A}+\mathrm{B}x+\mathrm{C}xx)}} - \frac{\partial y}{\sqrt{(\mathrm{A}+\mathrm{B}y+\mathrm{C}yy)}} = 0$$

ita integretur, ut posito $y = 0$ *fiat* $x = k$*, integrale erit*

$$-B(x+y) - 2Cxy + 2\sqrt{(A+Bx+Cxx)}\sqrt{(A+By+Cyy)}$$
$$= Bk + 2\sqrt{A(A+Bk+Ckk)}$$

sive

$$B(k-x-y) - 2Cxy = 2\sqrt{A(A+Bk+Ckk)} - 2\sqrt{(A+Bx+Cxx)}\sqrt{(A+By+Cyy)}.$$

COROLLARIUM

§.51. Hinc ergo patet, si aequatio differentialis proposita fuerit ista

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}} + \frac{\partial y}{\sqrt{(A+By+Cyy)}} = 0$$

eaque integretur ita, ut posito $y = 0$ fiat $x = k$, integrale fore

$$B(k-x-y) - 2Cxy$$
$$= 2\sqrt{(A+Bx+Cxx)}\sqrt{(A+By+Cyy)} - 2\sqrt{A(A+Bk+Ckk)}.$$

THEOREM 4

§.52. *Si posito brevitatis gratia*

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$
$$Y = A + By + Cyy + Dy^3 + Ey^4$$
$$K = A + Bk + Ckk + Dk^3 + Ek^4$$

haec proponetur aequatio differentialis

$$\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0,$$

quae ita integrari debeat, ut posito $y = 0$ *fiat* $x = k$, *eius integrale ita erit expressum*

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy + 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk + 2\sqrt{AK}}{kk}.$$

Sin autem aequatio proposita fuerit

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

eius integrale erit

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy-2\sqrt{XY}}{(x-y)^2} = \frac{2A+Bk-2\sqrt{AK}}{kk}.$$

COROLLARIUM 1

§.53. Quodsi hic ponamus $D=0$ et $E=0$, casus oritur theorematis tertii pro aequatione

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}} - \frac{\partial y}{\sqrt{(A+By+Cyy)}} = 0$$

cuius ergo integrale hinc erit

$$\frac{2A+B(x+y)+2Cxy+2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2} = \frac{2A+Bk+2\sqrt{A(A+Bk+Ckk)}}{kk};$$

quae forma si cum superiori comparetur, formulae irrationales eliminari poterunt. Quoniam enim ex priore est

$$2\sqrt{XY} = 2\sqrt{A(A+Bk+Ckk)} - B(k-x-y) + 2Cxy,$$

erit hoc integrale postremum

$$\frac{2A+B(2x+2y-k)+4Cxy+2\sqrt{A(A+Bk+Ckk)}}{(x-y)^2} = \frac{2A+Bk+2\sqrt{A(A+Bk+Ckk)}}{kk},$$

unde statim deduci potest aequatio canonica

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy = 0.$$

COROLLARIUM 2

§.54. Ponamus nunc esse $A=0$ et $B=0$, ut sit

$$X = xx(C+Dx+Exx) \text{ et } Y = yy(C+Dy+Eyy) \text{ et } K = kk(C+Dk+Ekk);$$

aequatio differentialis integranda fiet

$$\frac{\partial x}{x\sqrt{(A+Bx+Cxx)}} - \frac{\partial y}{y\sqrt{(A+By+Cyy)}} = 0$$

cuius ergo integrale erit

$$\frac{xy\left(2C+D(x+y)+2Exy\right)+2xy\sqrt{(C+Dx+Exx)(C+Dy+Eyy)}}{(x-y)^2}=\Delta,$$

atque hic constantem Δ per k definire non licebit; positio enim $y=0$ incongruum iam involvit. Interim tamen et haec integratio maxime est memoratu digna.

COROLLARIUM 3

§.55. Quodsi autem in hac postrema integratione loco x et y scribamus $\frac{1}{x}$ et $\frac{1}{y}$, primo aequatio differentialis erit

$$\frac{\partial y}{\sqrt{(Cyy+Dy+E)}}-\frac{\partial x}{\sqrt{(Cxx+Dx+E)}}=0\ ;$$

tum vero integrale sequentem induet formam

$$\frac{2Cxy+D(x+y)+2E+2\sqrt{(Cxx+Dx+E)(Cyy+Dy+E)}}{(y-x)^2}$$
$$=\Delta=\frac{Dk+2E+2\sqrt{E(Ckk+Dk+E)}}{kk}.$$

Si igitur hic loco literarum E, D, C scribamus A, B, C, prodibit aequatio differentialis supra tractata

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}-\frac{\partial y}{\sqrt{(A+By+Cyy)}}=0,$$

cuius ergo integrale erit

$$\frac{2A+B(x+y)+2Cxy+2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2}$$
$$=\frac{Bk+2A+2\sqrt{A(A+Bk+Ckk)}}{kk},$$

quae egregie convenit cum ea in coroll.1 data.

COROLLARIUM 4

§.56. Contemplemur nunc etiam casum, quo formula

$$A+Bx+Cxx+Dx^3+Ex^4$$

fit quadratum, quod sit $(a+bx+cxx)^2$, ita ut iam habeamus

$$\mathrm{A}=aa,\ \mathrm{B}=2ab,\ \mathrm{C}=bb+2ac,\ \mathrm{D}=2bc,\ \mathrm{E}=cc,$$

tum vero

$$\sqrt{\mathrm{X}}=a+bx+cxx,\ \sqrt{\mathrm{Y}}=a+by+cyy,\ \sqrt{\mathrm{K}}=a+bk+ckk$$

atque aequatio differentialis pro priore casu erit

$$\frac{\partial x}{a+bx+cxx}-\frac{\partial y}{a+by+cyy}=0,$$

cuius ergo integrale erit

$$\left\{\begin{array}{l}2aa+2ab(x+y)+2(bb+2ac)xy+2bcxy(x+y)\\+2ccxxyy+2(a+bx+cxx)(a+by+cyy)\end{array}\right\}:(x-y)^2=\Delta,$$

quae reducitur ad

$$\frac{aa+ab(x+y)+(bb+2ac)xy+bcxy(x+y)+ccxxyy)}{(x-y)^2}=\frac{aa+abk}{kk}.$$

Quodsi iam utrinque addamus $\frac{1}{4}bb$, prodibit

$$\frac{\left(a+\frac{1}{2}b(x+y)+cxy\right)^2}{(x-y)^2}=\frac{\left(a+\frac{1}{2}bk\right)^2}{kk},$$

unde extracta radice obtinetur forma integralis in theoremate primo assignata.

§.57. Sin autem hoc modo alterum casum aequationis

$$\frac{\partial x}{a+bx+cxx}-\frac{\partial y}{a+by+cyy}=0,$$

evolvere velimus, pervenimus ad hanc aequationem

$$\frac{2aa+2ab(x+y)+2(bb+2ac)xy+2bcxy(x+y)+2ccxxyy}{(x-y)^2}$$
$$-\frac{2(a+bx+cxx)(a+by+cyy)}{(x-y)^2}=\Delta,$$

quae evoluta praebet $\Delta = 2ac$, haecque aequatio manifesto est absurda et nihil circa integrale quaesitum declarat, cuius rationem maximi momenti erit perscrutari.

Insigne Paradoxon

§.58. *Cum huius aequationis differentialis*

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

integrale in genere inventum sit

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy - 2\sqrt{XY}}{(x-y)^2} = \Delta$$

casu autem, quo statuitur

$$\sqrt{X} = a + bx + cxx \text{ et } \sqrt{Y} = a + by + cyy,$$

aequatio absurda inde oriatur, quaeritur enodatio huius insignis difficultatis ac praecipue modus hinc verum integralis valorem investigandi.

Enodatio Paraxoxi

§.59. Quemadmodum scilicet in Analysi eiusmodi formulae occurrere solent, quae certis casibus indeterminatae atque ad eo nihil plane significare videntur, ita hic simile quid usu venit, sed longe alio modo, cum neque ad fractionem, cuius numerator et denominator simul evanescunt, neque ad differentiam inter duo infinita perveniatur, quod exemplum eo magis est notatu dignum, quod non memini similem casum mihi unquam se obtulisse. Istud singulare phaenomenon se nimirum exerit, quando ambae formulae X et Y evadunt quadrata, ad quod ergo resolvendum ad simile artificium recurri oportet, quo formulae X et Y non ipsis quadratis aequales, sed ab iis infinite parum discrepare assumuntur.

§.60. Statuamus igitur

$$X = (a + bx + cxx)^2 + \alpha \text{ et } Y = (a + by + cyy)^2 + \alpha,$$

ita ut pro litteris maiusculis A, B, C, D, E fiat
$A = aa + \alpha,\ B = 2ab,\ C = 2ac + bb,\ D = 2bc,\ E = cc$, ubi α denotat quantitatem infinite parvam deinceps nihilo aequalem ponendam. Hinc ergo si brevitatis gratia ponamus

$$a + bx + cxx = R \text{ et } a + by + cyy = S,$$

erit

$$\sqrt{X} = R + \frac{\alpha}{2R} \text{ et } \sqrt{Y} = S + \frac{\alpha}{2S}.$$

§.61. Nunc igitur consideremus formam integralis prima inventam, quae erat

$$\frac{\sqrt{X}-\sqrt{Y}}{x-y} = \sqrt{\left(\Delta + D(x+y) + E(x+y)^2\right)},$$

pro qua igitur habebimus

$$\sqrt{X} - \sqrt{Y} = R - S - \frac{\alpha(R-S)}{2RS}.$$

Quia vero hic erit $R - S = b(x-y) + c(xx-yy)$, fiet

$$\frac{R-S}{x-y} = b + c(x+y).$$

At posito brevitatis gratia $x + y = p$ erit

$$\frac{R-S}{x-y} = b + cp,$$

unde aequatio nostra erit

$$b + cp - \frac{\alpha(b+cp)}{2RS} = \sqrt{(\Delta + 2bcp + ccpp)}.$$

§.62. Sumantur nunc utrinque quadrata et aequatio nostra sequentem induet formam $bb - \frac{\alpha}{RS}(b+cp)^2 = \Delta.$ Alteriores scilicet potestates ipsius α hic ubique praetermittuntur. Hic ergo ratio nostri paradoxi manifesto in oculos incidit, quia posito $\alpha = 0$ oritur $bb = \varDelta$ unde, ut Δ maneat constans arbitraria, evidens est differentiam inter *bb* et Δ etiam infinite parvam statui debere; quamobrem ponamus $\Delta = bb - \alpha\Gamma$ ac obtinebitur ista aequatio penitus determinata $\frac{(b+cp)^2}{RS} = \Gamma$ sive

$$\left(b + c(x+y)\right)^2 = \Gamma(a + bx + cxx)(a + by + cyy),$$

quae forma non multum discrepat a formula supra inventa.

§.63. Haec quidem forma magis est complicata quam solutiones § 24 et seqq. inventae, sequenti autem artificio ad formam simplicissimam redigi poterit. Cum haec fractio $\frac{RS}{(b+cp)^2}$ debeat esse quantitas constans, sit ea $= F$, ut esse debeat $F(cp+b)^2 = RS$, et quemadmodum hic posuimus $x + y = p$, ponamus porro $xy = u$ fietque

$$RS = aa + abp + ac(pp - 2u) + bbu + bcpu + ccuu$$

atque aequatio iam secundum potestates ipsius *p* disposita erit

$$F(cp+b)^2 = acpp + abp + bcpu + bbu + aa - 2acu + ccuu;$$

ubi primo utrinque dividamus, quatenus fieri potest, per $cp+b$, ac reperietur

$$F\left(cp+b\right) = ap + bu + \frac{(a-cu)^2}{cp+b}.$$

Dividamus nunc porro per $cp+b$, quatenus fieri potest, ac fiet

$$F = \frac{a}{c} - \frac{b}{c} \cdot \frac{a-cu}{cp+b} + \frac{(a-cu)^2}{(cp+b)^2}.$$

§.64. Hac forma inventa si statuamus

$$\frac{a-cu}{cp+b} = V,$$

erit

$$F = \frac{a}{c} - \frac{b}{c} \cdot V + VV.$$

Cum igitur ista expressio aequari debeat quantitati constanti, evidens est ipsam quantitatum V constantem esse debere, ita ut iam nostrum integrale reductum sit ad hanc formam

$$\frac{a-cu}{cp+b} = \frac{a-cxy}{c(x+y)+b} = \text{Const.}$$

Subtrahamus utrinque $\frac{a}{b}$ fietque

$$\frac{cxy+a(x+y)}{b+c(x+y)} = \text{Const.}$$

quae forma per priorem divisa producit hanc

$$\frac{a(x+y)+cxy}{cxy-a} = \text{Const.},$$

quae formae conveniunt cum supra exhibitis.

THEOREMA 5

§.65. *Si in genere haec ratio designandi adhibeatur, ut sit*

$$Z=A+Bz+Czz+Dz^2+Ez^4,$$

atque valor huius formulae integralis $\int\frac{\partial z}{\sqrt{Z}}$ *ita sumtus, ut evanescat posito* $z=0$, *designetur hoc charactere* $\Pi:z$, *tum, ut fiat* $\Pi:k=\Pi:x\pm\Pi:y$ *, necesse est, ut inter quantitates k, x, y ista relatio substitat*

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy\mp2\sqrt{XY}}{(x-y)^2}=\frac{2A+Bk\mp2\sqrt{AK}}{kk},$$

cuius ratio ex superioribus est manifesta.

Cum enim *k* denotet quantitatem constantem, erit

$$\partial.\Pi:x\pm\partial.\Pi:y=0 \text{ sive } \frac{\partial x}{\sqrt{X}}\pm\frac{\partial y}{\sqrt{Y}}=0,$$

cuius integrale modo ante vidimus ita exprimi

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy\mp2\sqrt{XY}}{(x-y)^2}=\Delta.$$

Quare cum esse debeat $\Pi:x\pm\Pi:y=\Pi:k$, manifestum est posito $y=0$ fieri debere $\Pi:x=\Pi:k$, ideoque $x=k$, unde constans indefinita Δ eodem prorsus modo definitur, uti est exhibita.

COROLLARIUM 1

§.66. Hinc si formula $\Pi:z$ exprimat arcum cuiuspiam lineae curvae abscissae sive applicatae *Z* respondentem, in hac curva omnes arcus eodem modo inter se comparare licebit, quo arcus circulares inter se comparantur, quandoquidem propositis duobus arcubus $\Pi:x$ et $\Pi:y$ tertius arcus $\Pi:k$ semper exhiberi poterit vel summae vel differentiae eorum arcuum aequalis.

COROLLARIUM 2

§.67. Ita si in hac forma $\Pi:k=\Pi:x\pm\Pi:y$ statuatur $y=x$, prodibit $\Pi:k=2\Pi:x$ sicque arcus reperitur duplo alterius aequalis. At vero si in nostra forma faciamus $y=x$, tam numerator quam denominator in nihilum abeunt. Ut autem eius verum valorem eruamus, utamur aequatione primum § 38 inventa

$$\frac{\sqrt{X}+\sqrt{Y}}{x-y} = \sqrt{\left(\Delta + D(x+y) + E(x+y)^2\right)},$$

et iam in membro sinistro spectetur y ut constans; ipsi x vero valorem tribuamus infinite parum discrepantem sive, quod eodem redit, loco numeratoris et denominatoris eorum differentialia substituantur sumta sola x variabili hocque modo pro casu $y = x$ membrum sinistrum evadit $\frac{X'}{2\sqrt{X}}$, ubi est

$$X' = B + 2Cx + 3Dxx + 4Ex^3.$$

Nunc ergo sumtis quadratis habebitur

$$\frac{X'X'}{4X} = \Delta + 2Dx + 4Exx$$

existente Δ ut ante $= \frac{2A+Bk-2\sqrt{AK}}{kk}$.

COROLLARIUM 3

§.68. Verum sine his ambagibus duplicatio arcus ex altera forma

$$\Pi : k = \Pi : x \pm \Pi : y$$

deduci potest ponendo $y = k$, siquidem hinc fit $\Pi : x = 2\Pi : k$, pro quo ergo casu relatio inter x et k hac aequatione exprimetur

$$\frac{2A+B(k+x)+2Ckx+Dkx(k+x)+2Ekkxx+2\sqrt{KY}}{(x-k)^2} = \frac{2A+Bk+2\sqrt{AK}}{kk}.$$

Facile autem patet, quomodo hinc ad triplicationem, quadruplicationem et quamlibet multiplicationem arcuum progredi debeat, quod argumentum olim fusius sum tractatus.

THEOREMA 6

§.69. *Si in formis supra inventis ponatur tam* $B = 0$ *quam* $D = 0$, *ut sit*

$$X = A + Cxx + Ex^4 \;\; et \;\; Y = A + Cyy + Ey^4 \;\; et \;\; K = A + Ckk + Ek^4,$$

tum si ista aequatio $\frac{\partial x}{\sqrt{X}} \pm \frac{\partial y}{\sqrt{Y}} = 0$ *ita integretur, ut posito* $y = 0$ *fiat* $x = k$ *,tum integralis erit*

$$\frac{A+Cxy+2Exxyy\mp2\sqrt{XY}}{(x-y)^2} = \frac{A\mp\sqrt{AK}}{kk}.$$

COROLLARIUM 1

§.70. Hic notari meretur istum casum adhuc alio modo ex forma generali deduci posse, si scilicet sumatur $A=0$ et $E=0$, tum enim prodit ista aequatio differentialis

$$\frac{\partial x}{\sqrt{\left(\mathrm{B}x+\mathrm{C}xx+\mathrm{D}x^3\right)}}\pm\frac{\partial y}{\sqrt{\left(\mathrm{B}y+\mathrm{C}yy+\mathrm{D}y^3\right)}}=0,$$

cuius ergo integrale erit

$$\frac{\mathrm{B}(x+y)+2\mathrm{C}xy+\mathrm{D}xy(x+y)\mp2\sqrt{\left(\mathrm{B}x+\mathrm{C}xx+\mathrm{D}x^3\right)\left(\mathrm{B}y+\mathrm{C}yy+\mathrm{D}y^3\right)}}{(x-y)^2}=\frac{\mathrm{B}k}{kk}=\frac{\mathrm{B}}{k},$$

ubi valor constantis admodum simplex evasit. Nunc in his formulis loco *x* et *y* scribamus *xx* et *yy,* at vero loco litterarum B et D scribamus A et E fietque aequatio differentialis

$$\frac{\partial x}{\sqrt{\left(\mathrm{A}+\mathrm{C}xx+\mathrm{E}x^4\right)}}\pm\frac{\partial y}{\sqrt{\left(\mathrm{A}+\mathrm{C}yy+\mathrm{E}y^3\right)}}=0,$$

cuius ergo integrale etiam hoc modo exprimetur

$$\frac{\mathrm{A}(xx+yy)+2\mathrm{C}xxyy+\mathrm{E}xxyy(xx+yy)\mp2xy\sqrt{\mathrm{XY}}}{(xx-yy)^2}=\frac{\mathrm{A}}{kk}.$$

COROLLARIUM 2

§.71. Ecce ergo hac ratione pervenimus ad aliam integralis formam non minus notabilem priore atque adeo nunc ex earum combinatione formula radicalis $\sqrt{\mathrm{XY}}$ eliminari poterit, quandoquidem ex posteriore fit

$$\mp2\sqrt{\mathrm{XY}}=\frac{\mathrm{A}(xx-yy)^2}{kkxy}-\frac{\mathrm{A}(xx+yy)}{xy}-2Cxy-Exy(xx+yy),$$

qui valor in priore substitutus conducit ad hanc aequationem rationalem

$$2\mathrm{A}+2\mathrm{C}xy+2\mathrm{E}xxyy+\frac{\mathrm{A}(xx-yy)^2}{kkxy}-\frac{\mathrm{A}(xx+yy)}{xy}-2\mathrm{C}xy-\mathrm{E}xy(xx+yy)$$
$$=\frac{2\mathrm{A}(x-y)^2}{kk}\mp\frac{2(x-y)^2\sqrt{\mathrm{AK}}}{kk},$$

quae porro reducta et per $(x-y)^2$ divisa revocatur ad hanc formam

$$\frac{2A\mp2\sqrt{AK}}{kk}=\frac{A(x+y)^2}{kkxy}-Exy-\frac{A}{xy}$$

sive ad hanc

$$\frac{A}{kk}(xx+yy-kk)-Exxyy\pm\frac{2xy\sqrt{AK}}{kk}=0,$$

quae egregie convenit cum aequatione canonica, qua olim sumusus, scilicet

$$0=\alpha+\gamma(xx+yy)+2\delta xy+\zeta xxyy,$$

si quidem est

$$\alpha=-A,\ \ \gamma=+\frac{A}{kk},\ \ 2\delta=\pm\frac{2\sqrt{AK}}{kk},\ \ \zeta=-E.$$

COROLLARIUM 3

§.72. Methodo posteriore, qua hic usi sumus ad hanc aequationem integrandam, aequatio multo generalior tractari poterit, ubi in formulis radicalibus potestates usque ad sextam dimensionem assurgunt. Namque si tantum statuamus $A=0$, ut sit aequatio

$$\frac{\partial x}{\sqrt{x(B+Cx+Dxx+Ex^3)}}\pm\frac{\partial y}{\sqrt{y(B+Cy+Dyy+Ey^3)}}=0,$$

eius integrale est

$$\frac{B(x+y)+2Cxy+Dxy(x+y)+2Exxyy}{(x-y)^2}\mp\frac{2\sqrt{xy(B+Cx+Dxx+Ex^3)(B+Cy+Dyy+Ey^3)}}{(x-y)^2}=\frac{B}{k}.$$

Quod si iam hic loco *x* et *y* scribamus *xx* et *yy,* aequatio differentialis fiet

$$\frac{\partial x}{\sqrt{(B+Cxx+Dx^4+Ex^6)}}\pm\frac{\partial y}{\sqrt{(B+Cyy+Dy^4+Ey^6)}}=0,$$

cuius ergo integrale erit

$$\frac{B(xx+yy)+2Cxxyy+Dxxyy(xx+yy)+2Ex^4y^4}{(xx-yy)^2}\mp\frac{2xy\sqrt{(B+Cxx+Dx^4+Ex^6)(B+Cyy+Dy^4+Ey^6)}}{(xx-yy)^2}=\frac{B}{kk}.$$

Nunc autem ostendamus, quomodo ope methodi Illustris *De La Grange* idem integrale impetrari queat.

Analysis Pro Integratione Aequationis Differentialis

$$\frac{\partial x}{\sqrt{X}}\pm\frac{\partial y}{\sqrt{Y}}=0$$

Existente

$$X=B+Cxx+Dx^4+Ex^6 \quad \text{et} \quad Y=B+Cyy+Dy^4+Ey^6$$

§.73. Posito igitur $\frac{\partial x}{\sqrt{X}}=\partial t$ erit $\frac{\partial y}{\sqrt{Y}}=\mp\partial t$ hincque sumtis quadratis

$$\frac{\partial x^2}{\partial t^2}=X \quad \text{et} \quad \frac{\partial y^2}{\partial t^2}=Y.$$

Hinc formentur hae aequationes

$$\frac{xx\partial x^2}{\partial t^2}=xxX \quad \text{et} \quad \frac{yy\partial y^2}{\partial t^2}=yyY.$$

Iam introducantur duae novae variabiles p et q, ut sit

$$xx+yy=2p \quad \text{et} \quad xx-yy=2q,$$

ex quo fit

$$x\partial x+y\partial y=\partial p,\ x\partial x-y\partial y=\partial q \quad \text{hincque} \quad xx\partial x^2-yy\partial y^2=\partial p\partial q;$$

quamobrem habebimus

$$\frac{\partial p\partial q}{\partial t^2}=xxX-yyY,$$

quae aequatio dividatur per $xx-yy=2q$, prodibitque

$$\frac{\partial p\partial q}{2q\partial t^2}=\frac{xxX-yyY}{xx-yy},$$

quae forma valoribus pro X et Y substitutis dabit

$$\frac{\partial p\partial q}{2q\partial t^2}==B+2Cp+D\left(3pp+qq\right)+4E\left(p^3+pqq\right).$$

§.74. Nunc porro aequationes pro $\frac{\partial x^2}{\partial t^2}$ et $\frac{\partial y^2}{\partial t^2}$ differentiatae dabunt

$$\frac{2\partial x\partial x}{\partial t^2}=X' \quad \text{et} \quad \frac{2\partial\partial y}{\partial t^2}=Y'.$$

Ex priore fit $\frac{2\partial x\partial x}{\partial t^2}=xX'$, cui addatur $\frac{2\partial x^2}{\partial t^2}=2X$, ut prodeat

$$\frac{2\left(x\partial\partial x+\partial x^2\right)}{\partial t^2} = \frac{2\partial.x\partial x}{\partial t^2} = x\mathrm{X}' + 2\mathrm{X}.$$

Simili modo erit

$$\frac{2\partial.y\partial y}{\partial t^2} = y\mathrm{Y}' + 2\mathrm{Y}.$$

quae duae aequationes invicem additae dabunt

$$\frac{2\partial.\partial p}{\partial t^2} = \frac{2\partial\partial p}{\partial t^2} = x\mathrm{X}' + y\mathrm{Y}' + 2\left(\mathrm{X}+\mathrm{Y}\right).$$

Substitutis autem valoribus et facta substitutione respectu litterarum p et q reperitur

$$2\mathrm{X} + 2\mathrm{Y} = 4\mathrm{B} + 4\mathrm{C}p + 4\mathrm{D}\left(pp+qq\right) + 4\mathrm{E}p\left(pp+3qq\right).$$

Deinde ob

$$x\mathrm{X}' = 2\mathrm{C}xx + 4\mathrm{D}x^4 + 6\mathrm{E}x^6 \text{ et } y\mathrm{Y}' = 2\mathrm{C}yy + 4\mathrm{D}y^4 + 6\mathrm{E}y^6$$

erit

$$x\mathrm{X}' + y\mathrm{Y}' = 4\mathrm{C}p + 8\mathrm{D}\left(pp+qq\right) + 12\mathrm{E}p\left(pp+3qq\right),$$

ex quibus coniunctis fit

$$\frac{2\partial\partial p}{\partial t^2} = 4\mathrm{B} + 8\mathrm{C}p + 12\mathrm{D}\left(pp+qq\right) + 16\mathrm{E}p\left(pp+3qq\right).$$

§.75 . Ab hac formula subtrahatur supra inventa $\frac{\partial p\partial q}{2q\partial t^2}$ quater sumta ac remanebit

$$\frac{2\partial\partial p}{\partial t^2} - \frac{2\partial p\partial q}{q\partial t^2} = 8\mathrm{D}qq + 32\mathrm{E}pqq.$$

Nunc utrinque multiplicetur per $\frac{\partial p}{qq}$ et prodibit

$$\frac{1}{\partial t^2}\left(\frac{2\partial p\partial\partial p}{qq} - \frac{2\partial p^2\partial q}{q^3}\right) = 8\mathrm{D}\partial p + 32\mathrm{E}p\partial p,$$

cuius integrale sponte se offert ita expressum

$$\frac{\partial p^2}{qq\partial t^2} = 4\Delta + 8\mathrm{D}p + 16\mathrm{E}pp$$

ideoque extracta radice

$$\frac{\partial p}{q\partial t} = 2\sqrt{(\Delta + 2\mathrm{D}p + 4\mathrm{E}pp)}.$$

§.76. Cum nunc sit

$$\frac{\partial p}{\partial t} = x\sqrt{X} \mp y\sqrt{Y} \quad \text{et} \quad 2q = xx - yy,$$

facta substitutione orietur haec aequatio

$$\frac{x\sqrt{X} \mp y\sqrt{Y}}{xx-yy} = \sqrt{\left(\Delta + \mathrm{D}(xx+yy) + \mathrm{E}(xx+yy)^2\right)},$$

quae sumtis quadratis reducetur ad istam formam

$$\frac{xx\mathrm{X} + yy\mathrm{Y} \mp 2xy\sqrt{\mathrm{XY}}}{(xx-yy)^2} = \Delta + \mathrm{D}(xx+yy) + \mathrm{E}(xx+yy)^2.$$

Est vero

$$xx\mathrm{X} + yy\mathrm{Y} = \mathrm{B}(xx+yy) + \mathrm{C}\left(x^4+y^4\right) + \mathrm{D}\left(x^6+y^6\right) + \mathrm{E}\left(x^8+y^8\right)$$

hincque pervenietur ad hanc aequationem

$$\frac{\mathrm{B}(xx+yy) + \mathrm{C}\left(x^4+y^4\right) + \mathrm{D}xxyy(xx+yy) + 2\mathrm{E}x^4y^4 \mp 2xy\sqrt{\mathrm{XY}}}{(xx-yy)^2} = \Delta.$$

§.77. Sumamus nunc ut supra constantem Δ ita, ut posito $y = 0$ fiat

$$x = k \text{ et } \mathrm{X} = \mathrm{K} = \mathrm{B} + \mathrm{C}kk + \mathrm{D}k^4 + \mathrm{E}k^6,$$

et aequatio integralis induet hanc formam

$$\frac{\mathrm{B}(xx+yy) + \mathrm{C}\left(x^4+y^4\right) + \mathrm{D}xxyy(xx+yy) + 2\mathrm{E}x^4y^4 \mp 2xy\sqrt{\mathrm{XY}}}{(xx-yy)^2} = \frac{\mathrm{B}+\mathrm{C}kk}{kk},$$

quae aliquanto simplicior evadit, si utrinque subtrahamus C*;* erit enim

$$\frac{\mathrm{B}(xx+yy) + \mathrm{C}\left(x^4+y^4\right) + \mathrm{D}xxyy(xx+yy) + 2\mathrm{E}x^4y^4 \mp 2xy\sqrt{\mathrm{XY}}}{(xx-yy)^2} = \frac{\mathrm{B}}{kk},$$

quae egregie convenit cum integrali supra § 72 exhibito.

§.78. Hic casus notatu dignus se offert, dum $B = 0$; tum autem aequatio differentialis ita se habebit

$$\frac{\partial x}{x\sqrt{(C+Dxx+Ex^4)}} \pm \frac{\partial y}{y\sqrt{(C+Dyy+Ey^4)}} = 0,$$

cuius ergo integrale per constantem Δ expressum erit

$$\frac{C(x^4+y^4)+Dxxyy(xx+yy)+2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \Delta.$$

Hoc autem casu integratio non ita determinari potest, ut posito $y = 0$ fiat $x = k$, quia integrale posterioris membri hoc casu manifesto abit in infinitum; quamobrem alio modo integrationem determinari conveniet, veluti ut posito $y = b$ fiat $x = a$; tum autem erit ista constans

$$\Delta = \frac{C(a^4+b^4)+Daabb(aa+bb)+2Ea^4b^4 \mp 2ab\sqrt{AB}}{(aa-bb)^2}.$$

existente

$$A = C + Daa + Ea^4 \text{ et } B = C + Dbb + Eb^4$$

§.79. Qui processum Analyseos hic usitatae comparare voluerit cum methodo, qua Illustris D. De La Grange usus est in Miscellan. Taur. Tom. IV, facile perspiciet eam multo facilius ad scopum desideratum perducere atque multo commodius ad quosvis casus applicari posse. Introduxerat autem vir illustrissimus in calculum formulam $\frac{\partial t}{T}$, cuius loco hic simplici elemento ∂t sumus usi, ac deinceps quantitatem T tanquam functionem litterarum p et q spectavit, quae positio satis difficiles calculos postulavit, dum nostra methodo longe concinnius easdem integrationes investigare licuit. Quanquam autem nullum est dubium, quin ista Analyseos species insigne incrementum polliceatur, tamen nondum patet, quemadmodum ad alias integrationes ea accommodari queat praeter hos ipsos casus, quos hic tractavimus et quos olim ex aequatione canonica derivaveram.