SUPPLEMENT VI.

AT THE END OF SECTION I. BOOK I.

Concerning the Integration of Differential Formulas.

Concerning the Formulas of Double Integrals.

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§. 1. If we wish to define the volume or surface of each proposed body, or other magnitudes of this kind, that is usually done by double integration ; indeed the formula of the differential expressed by such a form $Z\partial x\partial y$ is required to be integrated twice, containing the two variables *x* and *y*, of which only the former may be considered now as variable in the first integration, truly the latter integration is now put in place regarding the latter as the variable. Hence the magnitude resulting from that twofold integration is accustomed to be indicated by a twofold integral sign in this manner $\iint Z\partial x\partial y$, evidently by which the double formula of the differential $Z\partial x\partial y$ must be understood to be integrated twice. Therefore I will call expressions of this kind, associated with the twin sign, double integrals; since the use of which appears to be very wide, here we are to enquire carefully into their nature, the properties and uses of these to be established, and to be presented with great care.

§. 2. Therefore in the first place, since *x* and *y* may be two variable quantities not depending on each other, Z truly may denote some function of these, the strength of the double integral formula $\iint Z\partial x\partial y$ thus can be put in place, so that a finite function of these two same variables *x* et *y* may be sought, which thus being differentiated twice, so that in the first differentiation only *x*, in the other truly *y* may be considered to be variable, may lead to the formula $Z\partial x\partial y$. Thus if there were $Z = a$, evidently to become $\iint a\partial x\partial y = axy$; truly more generally there will be $\iint a\partial x\partial y = axy + X + Y$, with X denoting some function of *x*, and Y of *y* itself, since these two functions by are removed from the calculation by differentiation.

§. 3. But in general if V were a function of this kind of x and y, which thus on being differentiated twice as in the manner taught shall present $Z\partial x\partial y$; indeed there certainly will be $V=\iint Z\partial x\partial y$; truly the above twofold integration introduces in addition the arbitrary functions X and Y, the one of x and the other of y, so that most generally there shall be

$$\iint Z\partial x\partial y = V + X + Y .$$

From which it is observed at once, differential formulas of this kind by necessity are associated with the product $\partial x\partial y$; nor therefore, following this designation, do such formulas as $\iint Z\partial x^2$ or $\iint Z\partial x^2$ have any meaning; since they may be excluded by the nature of the matter at hand, while in the former on integrating by x alone, truly in the latter only y is treated as the variable.

§. 4. Thus the constituted form of the twofold integral formula of this kind $\iint Z\partial x\partial y$ is not difficult to be set up, so that x and y shall be the two variable quantities which are involved, not depending in turn on each other, and Z a finite function composed from these in some manner, indeed so that, as in the first either x or y alone is considered variable, can be done in a twofold manner. Evidently with y taken for the variable the other variable x may be treated as constant and the integral $\int Zdy$ is sought, which will be a certain function of x and y; with which found the formula of the differential $\partial x\int Zdy$ may be undertaken, in which y now may be treated as constant and x alone as the variable, and of which the integral $\iint Z\partial x\partial y$ is sought, which will be the value of the formula sought of the proposed double integral $\iint Z\partial x\partial y$. If the order of the variables x and y may be interchanged in this twofold integration, thus $\int \partial y\int Z\partial x$ shall express the value sought, which will not differ from that.

§. 5. On this account it shall be agreed, that the form of such $\iint Z\partial x\partial y$ shall be able to be shown commonly, either in this manner $\int \partial x\int Z\partial y$, or this $\int \partial y\int Z\partial x$; but whatever way we may use, the common rules of integration are required to be observed, but only if it may be observed in that integration, in which either x or y may be assumed constant, the constant introduced to become some function of which. Just as if this form may be proposed

$$\iint \frac{\partial x \partial y}{xx+yy} = \int \partial x \int \frac{\partial y}{xx+yy}$$

on account of

$$\int \frac{\partial y}{xx+yy} = \frac{1}{x} \text{arctan.} \frac{y}{x} + \frac{\partial \text{X}}{\partial x}$$

with $\frac{\partial \text{X}}{\partial x}$ denoting some function of x, there will be

$$\iint \frac{\partial x \partial y}{xx+yy} = \int \frac{\partial x}{x} \text{arctan.} \frac{y}{x} + \text{X},$$

where at this stage of the integration requiring to be performed, y may be considered as constant. Truly in a similar manner there is found

$$\iint \frac{\partial x \partial y}{xx+yy} = \int \frac{\partial y}{y} \text{arctan.} \frac{y}{x} + \text{Y},$$

in which integration x is assumed as constant ; indeed, in which example the agreement of the two forms found is not evident enough.

§. 6. Yet meanwhile the agreed truth is shown easily by series; since indeed there shall be $\text{arctang.} \frac{x}{y} = \frac{\pi}{2} - \text{arctang.} \frac{y}{x}$ with $\frac{\pi}{2}$ denoting a right angle and

$$\text{arctang.} \frac{y}{x} = \frac{y}{x} - \frac{y^3}{3x^3} + \frac{y^5}{5x^5} - \frac{y^7}{7x^7} + \frac{y^9}{9x^9} - \text{etc.},$$

there will be

$$\int \frac{\partial x}{x} \text{arctang.} \frac{y}{x} = -\frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.} + f:y$$

and

$$\int \frac{\partial y}{y} \text{arctang.} \frac{y}{x} = \frac{\pi}{2} ly - \frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.} + f:x$$

from each of which there arises

$$\iint \frac{\partial x \partial y}{xx+yy} = \text{X} + \text{Y} - \frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.}$$

[Note that *f:x* denotes an early form of writing *f(x)*, and likewise *f:y*.]

Truly where both integrations arise, the agreement between them is bestowed between that at once ; so that indeed it would be superfluous to be shown by several more examples, since the account of this shall be demonstrated perfectly from the nature of differentiation and integration.

§. 7. Therefore these are required to be kept in mind concerning such double integral formulas, when the two variables x and y clearly are not connected together, thus so that it may be agreed in either integration the other truly may be taken as constant. Truly such formulas are not to be confused with these, for which, as I have said initially, the volume and surface of whatever body may be accustomed to be expressed. And even if these formulas also require a double integration and in the first place either of the two variables, for example y, may be treated alone as variable with the other x assumed as constant, yet with the first integration performed, that to be extended through all the values of y and thus finally in place of y the extreme value, which it is able to receive, must be put in place, which generally depends on x, thus so that with this value after the first integration put in place of y in the second integration y may enter as a certain function of x, nor therefore may be able to be considered as constant, from which condition it happens, so that the other integration generally may be changed, even if the former may be resolved in a similar manner as before.

§. 8. So that which distinction may be seen more clearly, it will help to have an example brought forth. Therefore the volume of a sphere is sought, the centre of which shall be C (Fig. 1) and the radius $\mathrm{CA} = a$, and indeed in the first place a part of its quadrant ACB is present, an element of which is the small column YZ*yz* of area $\mathrm{Y}y = \partial x \partial y$ with $\mathrm{CP} = x$ and $\mathrm{PY} = y$ put in place, and its height will be $\mathrm{YZ} = \sqrt{(aa - xx - yy)}$; hence the volume of the elementary column will be $= dxdy\sqrt{(aa - xx - yy)}$, which will be required to be integrated twice. In the first place the interval $\mathrm{CP} = x$ may remain constant and with the integral thus assumed $\int dy\sqrt{(aa - xx - yy)}$, so that it may vanish on putting $y = 0$, it will give a minute portion of the area P*p*Y*q* present, which therefore will be

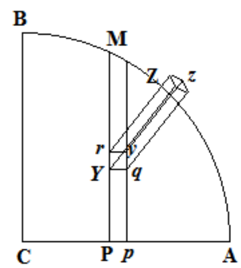

Fig. 1

$$= \tfrac{1}{2} y\sqrt{(aa - xx - yy)} + \tfrac{1}{2}(aa - xx)\arcsin.\frac{y}{\sqrt{(aa-xx)}}.$$

$$\left[Note: \int dy\sqrt{(aa - xx - yy)} = c\int dy\sqrt{\left(1 - \tfrac{y^2}{c^2}\right)}, \text{ where } c^2 = aa - xx;\right.$$

$$\text{let } y = c\sin\theta, \text{ then c}\int dy\sqrt{\left(1 - \tfrac{y^2}{c^2}\right)} = \int c^2\cos^2\theta d\theta$$

$$\left. = \tfrac{1}{2}c^2\int(1 + \cos 2\theta)\,d\theta = \tfrac{1}{2}c^2\theta + \tfrac{1}{4}\sin 2\theta.\right]$$

Now it is necessary for this value to be used in the other integration, but before that may be introduced, it must be extended through the whole distance PM, so that the element of the whole volume of the area P*p*M*m* is in place ; but with the point Y moved as far as M there becomes $y = \sqrt{(aa - xx)}$, which value therefore must be substituted in place of y, thus so that in the following integration the quantity *y* may at least be considered as constant, and this method on being treated will disagree most often with the preceding.

§. 9. Therefore on putting $y=\sqrt{(aa-xx)}$ there becomes

$$\int \partial y\sqrt{(aa-xx-yy)}=\tfrac{\pi}{4}(aa-xx),$$

since there shall be $\arcsin.1=\frac{\pi}{2}$, and thus the integral requiring to be resolved at this time shall be

$$\int \partial x\int \partial y\sqrt{(aa-xx-yy)}=\tfrac{\pi}{4}\int (aa-xx)\partial x,$$

where indeed the single variable *x* is present, but not so, since now here *y* may be considered to be constant, but surely because for *y* a certain function of *x* has been substituted. Truly this other integration thus put in place, so that it may vanish on putting $x=0$, will give the volume of the part of the sphere, which stands on the area CBMP, which therefore will be $=\frac{\pi}{4}\left(aax-\frac{1}{3}x^3\right)$; from which the octant of the sphere, or the part of the whole quadrant ACB in place will be produced by advancing the point P as far as to A, so that there shall become $x=a$. Then therefore the volume of the octant of the sphere will be $=\frac{\pi}{6}a^3$, and hence of the whole sphere $=\frac{4\pi}{3}a^3$, as it is agreed. From which example it is understood such an investigating of the volume most often differs from the twofold integration of the formula first set out.

§. 10. But if we do not wish to investigate the whole octant of the sphere, but only that part of it, which stands on the area of the rectangle CEDF (Fig. 2), the first integration as before is required to be put in place, but the value PM must be attributed to that of *y* acted on, which indeed is constant, and therefore this investigation is observed to approach to the first kind, yet truly it disagrees with that, which may give rise to a determined integral, since there indefinite functions X and Y may be carried over. Therefore as before on putting the radius of the sphere $\mathrm{CA}=a$, the sides of the rectangle CEFD shall be $\mathrm{CD}=e$ and $\mathrm{CE}=f$, and the volume with the element of area in place P*pYq* will be as before :

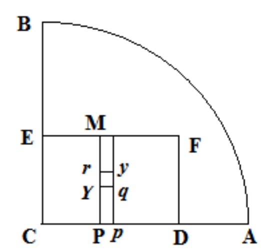

Fig. 2

$$=\tfrac{1}{2}y\sqrt{(aa-xx-yy)}+\tfrac{1}{2}(aa-xx)\arcsin.\frac{y}{\sqrt{(aa-xx)}},$$

which extended as far as to M, where there becomes $y=f$, will become

$$\tfrac{1}{2}f\sqrt{(aa-ff-xx)}+\tfrac{1}{2}(aa-xx)\arcsin.\frac{f}{\sqrt{(aa-xx)}},$$

from which the volume of the area present CPEM will be expressed by the following integral :

$$\tfrac{1}{2}f\int \partial x\sqrt{(aa-ff-xx)}+\tfrac{1}{2}\int (aa-xx)\partial x\operatorname{arcsin}.\tfrac{f}{\sqrt{(aa-xx)}},$$

if indeed it may be defined thus, so that it may vanish on putting $x=0$. Therefore we may set out these two formulas themselves.

§. 11. And the first indeed produces at once :

$$\int \partial x\sqrt{(aa-ff-xx)}=\tfrac{1}{2}x\sqrt{(aa-ff-xx)}+\tfrac{1}{2}(aa-ff)\operatorname{arcsin}.\tfrac{x}{\sqrt{(aa-xx)}},$$

but the second on account of

$$\partial.\operatorname{arcsin}.\tfrac{f}{\sqrt{(aa-xx)}}=\tfrac{fx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}}$$

is transformed thus:

$$\int (aa-xx)\partial x\operatorname{arcsin}.\tfrac{f}{\sqrt{(aa-xx)}}$$
$$=\left(aax-\tfrac{1}{3}x^3\right)\operatorname{arcsin}.\tfrac{f}{\sqrt{(aa-xx)}}-f\int \tfrac{\left(aa-\frac{1}{3}xx\right)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}},$$

to which latter part requiring to be integrated it may be observed :

$$\operatorname{arcsin}.\tfrac{fx}{\sqrt{(aa-xx)(aa-xx)}}=\int \tfrac{af\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}};$$

a certain multiple of this therefore will be given, so that on being added to that form, such a form may be produced :

$$\int \tfrac{\left(aa-\frac{1}{3}xx\right)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}}+m\operatorname{arcsin}.\tfrac{fx}{\sqrt{(aa-xx)(aa-xx)}}=\int \tfrac{\left(aaxx-\frac{1}{3}x^4+maf\right)\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}},$$

so that $aaxx-\tfrac{1}{3}x^4+maf$ may become divisible by $aa-xx$ divisible, which happens by assuming $m=-\tfrac{2a^3}{3f}$; and hence there will become :

$$\int\frac{\left(aa-\frac{1}{3}xx\right)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}}=\frac{2a^3}{3f}\operatorname{arcsin.}\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}-\frac{1}{3}\int\frac{(2aa-xx)\partial x}{\sqrt{(aa-ff-xx)}}.$$

§. 12. Therefore since there shall be

$$\int\frac{(2aa-xx)\partial x}{\sqrt{(aa-ff-xx)}}=\frac{1}{2}(3aa+ff)\arcsin\frac{x}{\sqrt{(aa-ff)}}+\frac{1}{2}x\sqrt{(aa-ff-xx)},$$

there shall be

$$\int\frac{\left(aa-\frac{1}{3}xx\right)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}}=$$

$$\frac{2a^3}{3f}\operatorname{arcsin.}\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}-\frac{1}{6}(3aa+ff)\arcsin\frac{x}{\sqrt{(aa-ff)}}-\frac{1}{6}x\sqrt{(aa-ff-xx)},$$

and hence

$$\int(aa-xx)\partial x\operatorname{arcsin.}\frac{f}{\sqrt{(aa-xx)}}$$

$$=\left(aax-\frac{1}{3}x^3\right)\operatorname{arcsin.}\frac{f}{\sqrt{(aa-xx)}}-\frac{2a^3}{3}\operatorname{arcsin.}\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}$$

$$+\frac{1}{6}f(3aa+ff)\arcsin\frac{x}{\sqrt{(aa-ff)}}+\frac{1}{6}fx\sqrt{(aa-ff-xx)}.$$

Whereby on putting $x=\mathrm{CD}=e$, the volume of the rectangular part of the sphere CDEF present will be

$$\frac{1}{4}ef\sqrt{(aa-ee-ff)}+\frac{1}{4}f(aa-ff)\operatorname{arcsin.}\frac{e}{\sqrt{(aa-ff)}}$$

$$+\frac{1}{6}e(3aa-ee)\operatorname{arcsin.}\frac{f}{\sqrt{(aa-ee)}}-\frac{1}{3}a^3\operatorname{arcsin.}\frac{ef}{\sqrt{(aa-ee)(aa-ff)}}$$

$$+\frac{1}{12}f(3aa+ff)\operatorname{arcsin.}\frac{e}{\sqrt{(aa-ee)}}+\frac{1}{12}ef\sqrt{(aa-ee-ff)},$$

which expression is reduced to this :

$$\frac{1}{3}ef\sqrt{(aa-ee-ff)}+\frac{1}{6}f(3aa-ff)\operatorname{arcsin.}\frac{e}{\sqrt{(aa-ff)}}$$

$$+\frac{1}{6}e(3aa-ee)\operatorname{arcsin.}\frac{f}{\sqrt{(aa-ee)}}-\frac{1}{3}a^3\operatorname{arcsin.}\frac{ef}{\sqrt{(aa-ee)(aa-ff)}}.$$

§. 13. Therefore if the limit F may be extended as far as to the periphery, so that there shall be $ee+ff=aa$, the first member vanishes and the three remaining affected circular arcs will change into the right angle $\frac{\pi}{2}$; and the volume becomes

$$\tfrac{\pi}{2}\left(\tfrac{1}{2}aae+\tfrac{1}{2}aaf-\tfrac{1}{6}e^3-\tfrac{1}{6}f^3-\tfrac{1}{3}a^3\right)$$

or on account of $f=\sqrt{(aa-ee)}$,

$$\tfrac{\pi}{12}\left((2aa+ee)\sqrt{(aa-ee)}-2a^3+3aae-e^3\right),$$

which volume becomes a maximum, if $f=e=\frac{a}{\sqrt{2}}$, and then that shall become

$\frac{\pi a^3}{12\sqrt{2}}\left(5-2\sqrt{2}\right)$, while the volume of the octant of the sphere is $=\frac{\pi}{6}a^3$,thus so that our volume shall be to the octant of the sphere as $5-2\sqrt{2}$ to $2\sqrt{2}$. But if the point F may not reach the periphery of the quadrant and there were $f=e$, the volume sought will be

$$=\tfrac{1}{3}ee\sqrt{(aa-2ee)}+\tfrac{1}{3}e(3aa-ee)\text{arcsin.}\frac{e}{\sqrt{(aa-ee)}}-\tfrac{1}{3}a^3\text{arcsin.}\frac{ee}{(aa-ee)}.$$

Whereby if there were

$$\text{arcsin.}\frac{e}{\sqrt{(aa-ee)}}:\text{arcsin.}\frac{ee}{aa-ee}=a^3:e(3aa-ee),$$

the volume will be expressed algebraically.

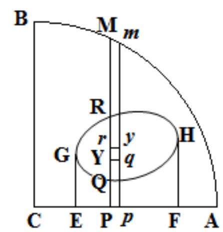

Fig. 3.

§. 14. But so that we may describe the matter more generally, we shall seek the volume of any area GQHR (Fig. 3) present ; since an element of which may stand of area $\text{Y}y=\partial x\partial y$, and that shall be $=\partial x\partial y\sqrt{(aa-xx-yy)}$, the first integration with *x* taken constant gives :

$$\tfrac{1}{2}\partial x\left(y\sqrt{(aa-xx-yy)}+(aa-xx)\text{arcsin.}\frac{y}{\sqrt{(aa-xx)}}\right).$$

Now from the nature of the curve GQHR the extreme distances shall be $\text{PQ}=q$ and $\text{PR}=r$ and the volume for the element of area QR present will be

$$\tfrac{1}{2}\partial x\left\{\begin{array}{l}+r\sqrt{(aa-xx-rr)}+(aa-xx)\text{arcsin.}\frac{r}{\sqrt{(aa-xx)}}\\-q\sqrt{(aa-xx-qq)}-(aa-xx)\text{arcsin.}\frac{q}{\sqrt{(aa-xx)}}\end{array}\right\}.$$

Whereby since *q* and *r* shall be any functions of *x*, it is evident, however much may be missing, the quantity *y* cannot be taken as constant in the following integration. But the following integration is required to be extended from the value $x=\text{CE}$ as far as to the value $x=\text{CF}$.

§. 15. If the figure of the base GCHR may be cut by the right line CA, so that the volume may be sought consisting of the base CGH (Fig. 4), the nature of which may be expressed by some equation between $\mathrm{CP} = x, \mathrm{PR} = r$,and the volume will be

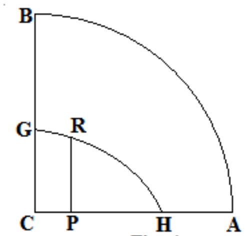

Fig. 4

$$\tfrac{1}{2}\int \partial x\left(r\sqrt{(aa-xx-rr)} + (aa-xx)\arcsin.\tfrac{r}{\sqrt{(aa-xx)}} \right),$$

where an elegant problem presents itself, where the figure of the base CGH is sought, so that the volume of that present may be expressed algebraically.

To this end, there may be put $r = u\sqrt{(aa-xx)}$, so that an indefinite volume of area CPRG may be present

$$\tfrac{1}{2}\int (aa-xx)\partial x\left(u\sqrt{(1-uu)} + arcsin.u\right),$$

which expression may be transformed into this

$$\tfrac{1}{2}\left(aax-\tfrac{1}{3}x^3\right)\cdot\left(u\sqrt{(1-uu)} + arcsin.u\right) - \int\left(aax-\tfrac{1}{3}x^3\right)\partial u\cdot\sqrt{(1-uu)}.$$

Now there becomes

$$\int\left(aax-\tfrac{1}{3}x^3\right)\partial u\cdot\sqrt{(1-uu)} = na^3\arcsin.u + a^3U$$

for some algebraic function *U* of *u* acting, and since the volume shall be

$$\tfrac{1}{2}\left(aax-\tfrac{1}{3}x^3\right)u\sqrt{(1-uu)} - a^3U + \left(\tfrac{1}{2}aax-\tfrac{1}{6}x^3 - na^3\right)\arcsin.u,$$

that will become in the algebraic case $-x^3 + 3aax = 6na^3$, provided *u* may vanish on putting $x = 0$; for then the volume will become $= na^3u\sqrt{(1-uu)} - a^3U$.

§. 16. We may put $dU = U'du$ and this equation between *x* and *u* will be produced:

$$aax - \tfrac{1}{3}x^3 = \tfrac{na^3}{1-uu} + \tfrac{a^3U'}{\sqrt{(1-uu)}}.$$

There may be devised $U = mu\sqrt{(1-uu)}$ *;* there will become $U' = \frac{m-2muu}{\sqrt{(1-uu)}}$, and so that *u* may vanish on putting $x = 0$, there must be $m = -n$, so that there becomes

$$aax - \tfrac{1}{3}x^3 = \tfrac{2na^3uu}{1-uu} \text{ or } u = \sqrt{\tfrac{3aax-x^3}{6na^3+3aax-x^3}}$$

and hence

$$r=\sqrt{\frac{(aa-xx)(3aax-x^3)}{6na^3+3aax-x^3}}.$$

Now on account of

$$u\sqrt{(1-uu)}=\frac{\sqrt{6na^3(3aax-x^3)}}{6na^3+3aax-x^3}$$

that volume becomes

$$=\frac{2na^3\sqrt{6na^3(3aax-x^3)}}{6na^3+3aax-x^3}.$$

If this volume may be found on making $x=a$, there becomes

$$n=\tfrac{1}{3},\quad r=\sqrt{\frac{(aa-xx)(3aax-x^3)}{2a^3+3aax-x^3}}=\sqrt{\frac{x(a-x)(3aa-xx)}{(a+x)(2a-x)}}$$

and on putting $x=a$, the volume will be $=\tfrac{1}{3}a^3$ and the curve found for the base is a line of the fourth order.

§. 17. [The curves] which are treated here concerned with the volume of spherical parts consistent with a given base, can be applied to any other bodies by a similar calculation, where in the formula $Z\partial x\partial y$, only the quantity Z may be determined by x and y in another way, while here it was $Z=\sqrt{(aa-xx-yy)}$. But also if the surface of any given body must be defined overhanging the base, that can be established in the same manner by the double integration of the similar differential of the formula $Z\partial x\partial y$. Thus if the body shall be a sphere, the element of the surface arising to the element of the base $\partial x\partial y$ is $\frac{a\partial x\partial y}{\sqrt{(aa-xx-yy)}}$, thus so that there shall be $Z=\frac{a}{\sqrt{(aa-xx-yy)}}$, of which the double integral is to be put in place for the reckoning of the base in a similar manner, for which the element of the surface arising is sought. And in general, any other quantities of the body as it pleases, which may correspond to a certain base, will be determined with the aid of similar operations.

[*Note*: $\frac{a\partial x\partial y}{\sqrt{(aa-xx-yy)}}$ arises as the element of the spherical surface ∂S corresponding to the element of the base $\partial x\partial y$, as the direction cosines of the associated radius are $\frac{x}{a},\frac{y}{a},\frac{z}{a}$; from which $\partial x\partial y=\partial S\cdot\frac{z}{a}$, etc.]

§. 18. Therefore whatever the function Z were of x and y, for the double integral $\iint Z\partial x\partial y$ the integral $\int Z\partial y$ is sought in the first place, with the quantity x observed as constant, and that may be extended through the whole magnitude y and thus the extreme values of y may be introduced into the calculation, which will be functions of x from the known figure of the base ; and thus for $\int Z\partial y$ a function of x will arise, which multiplied again by ∂x must be integrated in the usual manner. In a like manner it is required to be considered, if in the inverted order the first formula $\int Z\partial x$ may be integrated with y considered as constant ; while which integral may be extended through the whole interval x , the extreme values of x corresponding to the same y, which will be functions of y which will enter, and thus $\int Z\partial x$ will be changed into a function of y only, which multiplied again by ∂y thus must be integrated again, so that the integral may be extended through the whole interval y. Clearly by each method the integration is required to be extended through the whole base and the same precepts are to be observed, whatever function Z were of x and y.

§. 19. Therefore the determination of the integration likewise may be had for the given base, and if the quantity Z were constant and only the integral $\iint \partial x\partial y$ were sought, where the area of the base is expressed. Whereby according to the precepts, which it will be required to observe in the determination of these integrals, it will suffice to being established by putting $Z=1$, so that the double integral $\iint \partial x\partial y$ shall be required to be defined ; but if x or y may be taken, and the extreme values of each will be determined by the equation expressing the figure of the base. Clearly with the first integration performed, where the point Y (Fig. 5) was assumed everywhere to be within the extreme limits, then this point may be transferred onto the periphery of the base, with which agreed, x and y will become the coordinates of the base, between which the equation is given, from which henceforth either y will be determined by x or x by y.

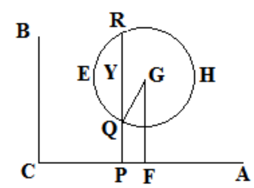

Fig. 5

§. 20. So that which may be seen more clearly, we may assume the figure of the base to be a circle, centre at G and having radius GQ and we may put
$\mathrm{CF}=f$, $\mathrm{FG}=g$ and $\mathrm{GQ}=c$; with the point Y translated onto the periphery of this circle :

$$cc=(f-x)^2+(g-y)^2.$$

Now for investigating the area of this circle initially x shall be constant and $\int \partial y = y+\mathrm{C}$, and because y has a double value on our base

$$y=g\pm\sqrt{\left(cc-(f-x)^2\right)},$$

this integration thus may be determined thus [for the lower limit], so that the integral may vanish, while the lesser of these values of y is given $g-\sqrt{\left(cc-(f-x)^2\right)}$, thus so that there shall become

$$\int \partial y = y-g+\sqrt{\left(cc-(f-x)^2\right)}.$$

Now therefore y extended as far as to the other limit $y=g+\sqrt{\left(cc-(f-x)^2\right)}$, the integral will become

$$\int \partial y = 2\sqrt{\left(cc-(f-x)^2\right)},$$

which now multiplied by ∂x and integrated gives

$$\int \partial x \int \partial y = \mathrm{C}-(f-x)\sqrt{\left(cc-(f-x)^2\right)}-cc\arcsin\tfrac{f-x}{c};$$

which so that it may vanish on putting $x=f-c$, there becomes $\mathrm{C}=cc\arcsin.1=\frac{\pi}{2}cc.$ Again there may be put
$x=f+c,$ and on account of $cc\arcsin.\frac{f-x}{c}=-cc\arcsin.1=-\frac{\pi}{2}cc$, the whole area sought will be $=\frac{\pi}{2}cc+\frac{\pi}{2}cc=\pi cc,$ as it is agreed.

§. 21. If we may consider these determinations more carefully, we see the extreme values of x to be prepared thus, so that one shall be a maximum, the other a minimum, if indeed the whole base may be defined for a certain curve returned into itself. Both these values therefore will be found, if the equation expressing the nature of the base may be differentiated and there may be put $\partial x=0$. But when the base is not defined by a certain single curved line, but may be contained by some part, such as CGH (Fig. 4), of which the base CH shall be the maximum, while the lesser limit of x evidently is $=0$, but the greater limit is equal to CH ; and in the same case the corresponding limits of the applied line PR of the abscissa $\mathrm{CP}=x$ are either $=0$, or truly $=\mathrm{CG}$. Therefore whatever the figure of that proposed base, it is required first to be examined properly and whatever its limits, to be investigated, so that the investigation of the area or of any other formula of the double integral may be able to be undertaken, but with the limits defined, by which the area shall be contained, thence the determinations of the integrations are required.

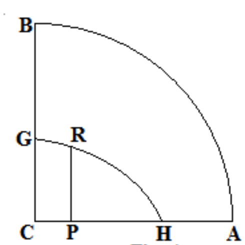

Fig. 4

§. 22. Regarding the determination of these integrations, the most conspicuous and noteworthy dispositions of double integral formulas of this kind, which occur in the transformation of these, deserve to be considered carefully. Clearly since the coordinates of the same curve are able to be taken in an infinite number of ways, thus here in place of the two variables *x* et *y* some other two variables can be introduced into the computation, either equally these shall be the coordinates, or some other quantities defined in some manner. Thus the transformation of such can be considered in general, so that in place of *x* and *y* some functions of other two variable *t* and *v* may be substituted, and with these introduced into the equation given for the base, the limits of these quantities *t* and *v* can be defined in a similar manner, from which the figure of the base is determined. But in whatever manner these substitutions may be taken, finally after the double integration it is necessary that the same quantity will always result.

§. 23. If some other pair of orthogonal coordinates may be introduced in place of *x* and *y* , for instance, *t* and *v*, which happens in general [for a rotation of the axes] by putting :

$$x = f + mt + v\sqrt{(1-mm)} \text{ and } y = g + t\sqrt{(1-mm)} - mv,$$

it is evident the element of the base, which before was $\partial x\partial y$, now must be expressed by $\partial t\partial v$. But since thence there shall be

$$\partial x = m\partial t + \partial v\sqrt{(1-mm)} \text{ et } \partial y = \partial t\sqrt{(1-mm)} - m\partial v,$$

it may be minimally apparent, how in place of $\partial x\partial y$ by these substitutions $\partial t\partial v$ may be able to arise, while rather there may appear :

$$\partial x\partial y = m\partial t^2\sqrt{(1-mm)} + (1-2mm)\partial t\partial v - m\partial v^2\sqrt{(1-mm)},$$

moreover which formula, however it may be applied to the double integration, always leads to the greatest errors. Therefore it will be allowed to deduce much less, if in place of *x* and *y* other functions of *t* and *v* may be substituted, the expression of which must be used in place of $\partial x\partial y$.

§. 24. And indeed in the first place I observe here to be no reason, why an expression on being introduced into the calculation in place of $\partial x\partial y$ must be equal to that ; which while finally it must by necessity be the case, if the two integrations may be put in place in the same manner as before, following the two variables. But now since other variables *t* and *v* shall be present, and the one integration shall be by the variability of *t*, while the other integration in terms of *v* shall be required to be administered, which operations generally differ from the preceding ones, the formula now being introduced in place of $\partial x\partial y$ not to be considered from equality, but rather adapted to the outcome which has been proposed. And since now it will be required to distinguish between the binary integrations according to the two variables *t* and *v*, it is evident the formula being used in place of

$\partial x\partial y$ to be associated by necessity with the product $\partial t\partial v$, and must have a form of this kind $\mathrm{Z}\partial t\partial v$.

§. 25. So that this may be brought about more reliably, initially *x* may remain the same and another variable *u* may be introduced in place of *y*, thus so that *y* shall be some function of *x* and *u* and $\partial y = \mathrm{P}\partial x + \mathrm{Q}\partial u$. Now if in the first integration *x* may be assumed constant, certainly there will be $\partial y = \mathrm{Q}\partial u$, hence $\iint \partial x\partial y = \int \partial x \int \mathrm{Q}du$, thus so that now in place of the formula $\partial x\partial y$ there will be considered $\mathrm{Q}\partial x\partial u$, of which the double integral therefore also will be able to be expressed in this manner $\int \partial u \int \mathrm{Q}\partial x$, where in the first integration $\int \mathrm{Q}\partial x$ the quantity *u* is taken as constant. But if now in a similar manner *u* may be retained and in place of *x* there may be introduced some function of *t* and *u*, so that there shall be $\partial x = \mathrm{R}\partial t + \mathrm{S}\partial u$, in the treatment of the formula $\int \partial u \int \mathrm{Q}\partial x$, in the first integration $\int \mathrm{Q}\partial x$, in which *u* may be placed constant, will be changed into this $\int \mathrm{QR}\partial t$, thus so that the double integral shall be $\int \partial u \int \mathrm{QR}\partial t$ or commonly $\iint \mathrm{QR}\partial t\partial u$, from which it is evident on account of both these substitutions this formula $\mathrm{QR}\partial t\partial u$ must be treated in place of the formula $\partial x\partial y$.

§. 26. Now we may introduce at once these two new variables *t* and *u* in place of *x* and *y*, by which these may be determined thus, so that there shall be

$$\partial x = \mathrm{R}\partial t + \mathrm{S}\partial u \quad \text{and} \quad \partial y = \mathrm{T}\partial t + \mathrm{V}\partial u,$$

from which with the value of ∂x substituted into the form $\partial y = \mathrm{P}\partial x + \mathrm{Q}\partial u$ there becomes

$$\partial y = \mathrm{PR}\partial t + (\mathrm{PS} + \mathrm{Q})\partial u,$$

thus so that there shall become $\mathrm{PR} = \mathrm{T}$ and $\mathrm{PS} + \mathrm{Q} = \mathrm{V}$, from which there becomes $\mathrm{P} = \frac{\mathrm{T}}{\mathrm{R}}$ and $\frac{\mathrm{ST}}{\mathrm{R}} + \mathrm{Q} = \mathrm{V}$ and thus $\mathrm{QR} = \mathrm{VR} - \mathrm{ST}$. Whereby by the strength of these substitutions in place of $\partial x\partial y$ we may be able to use the formula $(\mathrm{VR} - \mathrm{ST})\partial t\partial u$, which double integration by the determinations just summoned must give an area equal to the whole base and given by the formula $\partial x\partial y$ itself twice integrated. But because here $\iint \partial x\partial y$ has been shown for the areas of the bases, there is a place for some other $\iint \mathrm{Z}\partial x\partial y$, certainly which is transformed into this form $\iint \mathrm{Z}(\mathrm{VR} - \mathrm{ST})\partial t\partial u$ by the same substitutions, provided in Z assumed values may be substituted in place of *x* and *y*.

Indeed in a like manner, it will be required to determine the two integrations from the figure of the base.

§. 27. Therefore just as if there may be put

$$dx = Rdt + Sdu \text{ et } dy = Tdt + Vdu,$$

in place of $\partial x \partial y$ consequently we have $(\mathrm{RV}-\mathrm{ST})\partial t \partial u$, which formula generally differs from that, to which the product $\partial x \partial y$ actually is equal [recall in modern terms, that $\mathrm{RV}-\mathrm{ST}$ is the determinant of the transformation]; indeed even if the terms may be returned associated with ∂t^2 and ∂u^2, which is nonsense for the double integration, yet what remains, $(\mathrm{RV}+\mathrm{ST})\partial t \partial u$ differs on account of the sign from the true formula. Truly here no small doubt emerges, which, since the coordinates *x* and *y* travel past with equal steps, our formula may embrace the difference $\mathrm{RV}-\mathrm{ST}$ rather than the inverse $\mathrm{ST}-\mathrm{RV}$; which doubt will be increased more from that, because, if we may invert the above reasoning with respect to *x* and *y*, the same substitutions may have actually led us back to the formula $(\mathrm{ST}-\mathrm{RV})\partial t \partial u$. But because the whole distinction is associated with the sign and the one formula is the negative of the other, hence the absolute determination of the area of the base, certainly of which the absolute magnitude is sought, does not suffer any real change.

§. 28. But these may become more transparent, if we may consider more carefully the manner, by which above (§ 20) we have made use for finding the areas EQHR (Fig. 5). Clearly initially from the integration of the formula $\iint \partial x \partial y$ we have deduced this area

$$= \int \partial x (\mathrm{PR\text{-}PQ}),$$

were indeed we have subtracted PQ from PR, because clearly there was $\mathrm{PR} > \mathrm{PQ}$;but in the calculation no ratio may be present, which may come first, so that we may rather subtract PQ from PR then PR from PQ in turn, and thus we may not be able equally to turn our attention just to that area expressed by $= \int \partial x (\mathrm{PQ\text{-}PR})$, with that agreed to be negative, but was going to be produced equal to the first. From which it is understood the $+$ or $-$ sign does not affect the magnitude of the area which is sought, and the calculation may be produced just the same for each. On account of which the above doubt may be lessened, so that we may say the area sought must be expressed thus, so that it shall be $= \pm \iint \partial t \partial u (\mathrm{RV}-\mathrm{ST})$, and so that the area may emerge expressed positive, in whatever case with that sign being used, where $\pm(\mathrm{RV}-\mathrm{ST})$ may be returning a positive amount.
[Euler's work here may have stimulated Jacobi's researches.]

§. 29. Hence also the doubts, which perhaps can arise concerning the discovery of the area of curves may be resolved easily, of which the parts are disposed on each side of the axis and with which beginners often are accustomed to be somewhat disturbed. For if the total area QAR of the curve QAR (Fig. 6) must be defined relative to the axis AP corresponding to the axis $\mathrm{AP}=x$, and its parts APQ and APR themselves may be considered, it is certain, if the one APQ may be considered positive so that it shall be $=+\mathrm{Q}$, the other APR must be considered to be negative, so that it shall be $=-\mathrm{R}$. Nor yet does it follow from this that the total area QAR to become $=\mathrm{Q}-\mathrm{R}$, certainly which will vanish, if both the parts APQ and APR shall be equal ; but likewise if both the points Q and R shall be placed on the same side of the axis, the area always will be $=\pm\int\partial x(\mathrm{PR}-\mathrm{PQ})$, from which on account of

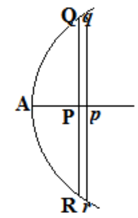

Fig. 6

$$\int \mathrm{PQ}\cdot\partial x=\mathrm{Q} \text{ and } \int \mathrm{PR}\cdot\partial x=-\mathrm{R}$$

the whole area will become $=\pm(\mathrm{Q}+\mathrm{R})$, just as the nature of the matter demands.

§. 30. Moreover with the aid of such substitutions, from which, in place of the two variables *x* and *y* some other two may be introduced *t* and *u*, most integrations repeated several times are able to be returned lightened and made easier, and in any case the most convenient substitution can be found without difficulty. Just as if the area of the circle EQHR (Fig. 5) must be defined relative to the axis GP, where on account of

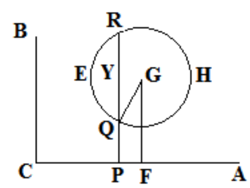

Fig. 5

$\mathrm{CF}=f, \mathrm{FG}=g,\ \mathrm{GQ}=c$ there was $cc=(f-x)^2+(g-y)^2$, it will be convenient to put

$$f-x=\frac{t}{\sqrt{(1+uu)}} \text{ and } g-y=\frac{tu}{\sqrt{(1+uu)}},$$

so that there becomes $tt=cc$ and $t=c$. Then truly on account of

$$\partial x=\frac{-\partial t}{\sqrt{(1+uu)}}+\frac{tu\partial u}{(1+uu)^{\frac{3}{2}}} \text{ and } \partial y=\frac{-u\partial t}{\sqrt{(1+uu)}}-\frac{t\partial u}{(1+uu)^{\frac{3}{2}}},$$

in place of $\partial x\partial y$, by § 27, we arrive at $\partial t\partial u\left(\frac{t}{(1+uu)^2}+\frac{tuu}{(1+uu)^2}\right)=\frac{t\partial t\partial u}{1+uu}$, of which the double integral may be expressed thus :

$$\int\frac{\partial u}{1+uu}\int t\partial t$$

Now truly there is $\int t\partial t = \frac{1}{2}tt = \frac{1}{2}cc$ and the whole area will be $\frac{1}{2}cc\int\frac{\partial u}{1+uu}$, while all the possible values of *u* are given, since *u* was no further affecting the equation of the base.

§. 31. So that we may explain this use more clearly, we may consider again the sphere having centre C and radius $\mathrm{CA} = a$, of which a part of the circular base must be acting perpendicularly. Because the radius CA can be drawn through the centre of this circle G, there shall be $\mathrm{FG} = g = 0$, so that there may become $cc = (f-x)^2 + yy$ and the volume sought $= \int \partial x\partial y(aa - xx - yy)$; now there may be put

$$x = \frac{t}{\sqrt{(1+uu)}} \text{ and } y = \frac{tu}{\sqrt{(1+uu)}},$$

so that there may become $xx + yy = tt$ and $\sqrt{(aa - xx - yy)} = \sqrt{(aa - tt)}$ and for $\partial x\partial y$ there may be given $\frac{t\partial t\partial u}{1+uu}$, thus so that the volume may be expressed thus $\iint \frac{t\partial t\partial u\sqrt{(aa-tt)}}{1+uu}$, which integrations must be determined from the equation hence arising for the figure of the base

$cc = ff - \frac{2ft}{\sqrt{(1+uu)}} + tt$, from which there becomes

either $t = \frac{f \pm \sqrt{(cc+ccuu-ffuu)}}{\sqrt{(1+uu)}}$ or $\sqrt{(1+uu)} = \frac{2ft}{ff-cc+tt}$.

§. 32. Initially we may consider *t* as constant and the integral will become

$$= \int t\partial t\sqrt{(aa-tt)}\cdot \arctan.u,$$

where a constant is not necessary to be added, because with *u* vanishing likewise *y* vanishes; initially indeed we may seek the volume present for the semicircle. But by this initial extension by the integral to the extreme limit, on account of $\arctan.u = \arccos.\frac{1}{\sqrt{(1+uu)}}$, this shall become

$$\int t\partial t\sqrt{(aa-tt)}\cdot\arccos.\frac{ff-cc+tt}{2ft},$$

of which the limits of the integration are $t = f - c$ et $t = f + c$. If we may not wish to define the volume of this part of the sphere, but the surface as if arising from the base, we will have arrived at this formula

$$\int \frac{at\partial t}{\sqrt{(aa-tt)}}\text{arccos.}\frac{ff-cc+tt}{2ft};$$

but it does not seem to be worth the effort to pursue its integration further.

§. 33. Moreover, the method of treating double integral formulas of this kind will be illustrated somewhat more, if we may apply it to the problem of that once famous Florentine [V. Viviani; there is an extensive literature on what is termed Viviani's curve; Originally proposed in the *Acta Erud*. 1692, p.274; see e.g. Mathworld for a modern view; an analysis closer to that of Euler can be found in *Examples of the progress of the Differential and Integral Calculus,* D.F. Gregory, CUP. 1866, p. 436.], were it will be required to designate geometrically a part of a spherical surface, the surface of which shall be able to be expressed algebraically.

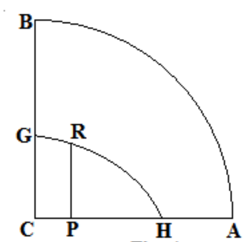

Fig. 4

The part GRH of such a spherical curve (Fig. 4) may be overhanging, the figure of which therefore is required to be determined ; in which if there may be put $\text{CP} = x$, $\text{PR} = y$, the surface of the part of the sphere overhanging [*i.e.* of the inner cylinder] is expressed by this duplicate formula of the integral $\iint \frac{a\partial x\partial y}{\sqrt{(aa-xx-yy)}}$. Now with no substitution used if initially x may be considered as constant, there will be produced

$$\int a\partial x\,\text{arcsin.}\frac{y}{\sqrt{(aa-xx)}},$$

by which an indefinite area of the sphere is expressed covering CPRG, and the question now returns to this, so that an algebraic equation of this kind may be assigned between x and y , from which a part of the spherical surface corresponding to that total area CHRG may become assignable algebraically.

§. 34. For the sake of brevity we may put $\frac{y}{\sqrt{(aa-xx)}} = v$, so that there shall be $y = v\sqrt{(aa-xx)}$ and on putting $x = 0$ there becomes $v = n$; since the upper integral must vanish on putting $x = 0$, therefore the indefinite area CPRG covering the spherical surface will be

$$= ax\,\text{arcsin}.v - a\int \frac{x\partial v}{\sqrt{(1-vv)}},$$

with this integral taken thus, so that it may vanish on putting $x = 0$. Now, we may put

$$\int \frac{x\partial v}{\sqrt{(1-vv)}} = f\,\text{arcsin}.v - a\text{V}$$

with V denoting some algebraic function of *v*, which many be changed into N on putting $x=0$, and our surface will be

$$=ax\arcsin.v-af\arcsin.v+aa\mathrm{V}+af\arcsin.n-aa\mathrm{N}$$

and *x* thus will be determined by *v*, thus so that there shall be

$$x=f-\frac{a\partial\mathrm{V}\sqrt{(1-vv)}}{\partial v};$$

now there shall be $\mathrm{CH}=h$ and there may be put $x=h$, in which case there becomes $v=m$ and $\mathrm{V}=\mathrm{M}$, and since the proposed surface shall be

$$ah\arcsin.m-af\arcsin.m+aa\mathrm{M}+af\arcsin.n-aa\mathrm{N},$$

that cannot be expressed algebraically, unless there shall be

$$h\mathrm{A}\sin.m-f\mathrm{A}\sin.m+f\mathrm{A}\sin.n=0.$$

§. 35. Therefore here in the first place the arcs, of which the sines are *m* et *n*, must be returned commensurable to each other, unless perhaps there shall be $n=0$, in which case there will suffice to become $h=f$. Because even if it can be performed easily in an infinite number of ways, yet this problem may be resolved much easier by using the substitutions set out previously. Therefore there may be put

$$x=\frac{t}{\sqrt{(1+uu)}}\text{ and }y=\frac{tu}{\sqrt{(1+uu)}},$$

so that there may become $xx+yy=tt$ and for $\partial x\partial y$, $\frac{t\partial t\partial u}{1+uu}$ may be produced , and the surface of the part of the sphere may be expressed by this double formula of the integral $\iint\frac{at\partial t\partial u}{(1+uu)\sqrt{(aa-tt)}}$. Initially *u* may be assumed constant ; from that there will become $\int\frac{a\partial u}{1+uu}\left(b-\sqrt{(aa-tt)}\right)$, which now can be easily returned completely integrable ; for it may be put equal to some algebraic function of *u*, which shall be $=\mathrm{V}$, and there shall be $b-\sqrt{(aa-tt)}=\frac{\partial\mathrm{V}(1+uu)}{a\partial u}$ and thus the indefinite part of the spherical surface will be V, where some algebraic function of *u* may be taken for V.

§. 36. The most simple solutions will be deduced from this hypothesis :

$$\mathrm{V}=\frac{a(\alpha+\beta u)}{\sqrt{(1+uu)}};$$

from which there becomes $\frac{\partial V}{a\partial u}=\frac{-\alpha u+\beta}{(1+uu)^{\frac{3}{2}}}$ and hence $b-\sqrt{(aa-tt)}=\frac{\beta-\alpha u}{\sqrt{1+uu}}$. There may be put $b=0,$ and since from the substitutions there shall be $u=\frac{y}{x}$ and $t=\sqrt{(xx+yy)},$ for which the curve sought will be

$$\sqrt{(xx+yy)}(aa-\ xx-yy)=\alpha y-\beta x$$

and for the surface:

$$V=\frac{a(\alpha x+\beta y)}{\sqrt{(xx+yy)}}.$$

Hence the most simple case arises on putting $\beta=0$ and $\alpha=a$, from which there will be produced $aaxx-(xx+yy)^2=0$ or $yy=ax-xx$, thus so that the curve GRH shall be a circle described with the diameter AC and $V=\frac{aax}{\sqrt{(xx+yy)}}$. An infinitude of other circles having the diameter $=a$ and passing through the centre of the sphere are found, if there shall be $\beta=\sqrt{(aa-\alpha\alpha)}$, from which there becomes

$$ax+y\sqrt{(aa-\alpha\alpha)}=xx+yy \text{ and } V=\frac{a\left(\alpha x+y\sqrt{(aa-\alpha\alpha)}\right)}{\sqrt{(xx+yy)}}=a\sqrt{(xx+yy)},$$

where it is required to be noted the magnitude V according to the nature of the problem, to be assumed to be some constant.

§. 37. Therefore the octant extracted from the sphere above the quadrant ACB (Fig. 7) may be considered, of which the radius is $CA=a$, which likewise shall be the diameter of the semicircle CRA ; on which if some chord CR may be drawn and the perpendicular RP, so that there shall be $CP=x$ and $PR=y$, there will be $CR=t$ and u will be the tangent of the angle ACR. Therefore since we may put $b=0$, the first integral, where u was constant, is $\sqrt{(aa-tt)}$; which since it may vanish, if $t=a$, clearly that is not by the chord $CR=t$, but by its complement extended RS. Hence that part of the spherical surface can be expressed by the repeated integral $\int\frac{adu}{1+uu}\sqrt{(aa-tt)}$, by which it overhangs by the three lines RVAS, which therefore on account of $\sqrt{(aa-tt)}=\frac{au}{\sqrt{(1+uu)}}$ is $=\frac{-aa}{\sqrt{(1+uu)}}+aa,$ evidently with the integral thus taken, so that it may vanish with the angle ACR. Whereby on account of

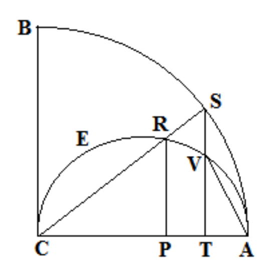

Fig. 7

$$\frac{1}{\sqrt{(1+uu)}}=\cos.ACR$$

with that perpendicular ST taken, that surface will be $a(a-\mathrm{CT})=\mathrm{CA}\cdot\mathrm{AT}=\mathrm{AV}^2$, with the chord AV removed. Consequently the part of the surface of the sphere CERASB overhanging between the quadrant and the circle intercepted is equal to the square of the radius of the sphere.

§. 38. But we may consider at this point a case of this kind, where the first integration may vanish on putting $t=0$, or there shall be $b=a$ and there may be put $\mathrm{V}=\frac{1}{2}aau$, which expression likewise will produce the surface sought. Therefore there will be

$$a-\sqrt{(aa-tt)}=\tfrac{1}{2}a(1+uu) \text{ and } \sqrt{(aa-tt)}=\tfrac{1}{2}a(1-uu),$$

thus so that there shall be

$$t=\tfrac{1}{2}a\sqrt{\left(3+2uu-u^4\right)} \text{ or } t=\tfrac{1}{2}a\sqrt{\left((1+uu)(3-uu)\right)},$$

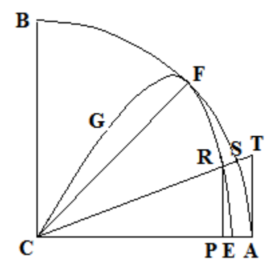

Fig. 8

where there is $\mathrm{CR}=t$ (Fig. 8) and u denotes the tangent of the angle ACR. From this equation it is apparent, if there shall be $u=0$, there becomes $t=\frac{a\sqrt{3}}{2}$; evidently the curve sought with the radius AC thus cross at E, so that there shall be $\mathrm{CE}=\mathrm{CA}\cdot\frac{\sqrt{3}}{2}$, and being perpendicular to that. Then if the angle ACR may be increased to the right angle ACF, so that there becomes $u=1$, and there will be $t=a$ and in this case the curve passes through the point F itself and there it will have a common tangent with the quadrant; and likewise the distance t shall be a maximum. Thereafter the curve is reflected inwards and t vanishes, if $u=\sqrt{3}$; that is, the curve with centre C thus dips downwards so that its tangent at C shall make an angle of $60°$ with the radius CA.

§. 39. Therefore the whole curve described in the quadrant will have the figure ERFGC and with some line CR drawn from that from C and the tangent of the angle ECR shall be $=u$; then the part of the spherical surface of the sector ECR overhanging will be able to be expressed algebraically and that will be $=\frac{1}{2}aau$. Whereby if CR may be produced to the crossing with the tangent AT, on account of $\mathrm{AT}=au$ that part will be exactly equal to the triangle CAT and the part of the sector ECF overhanging will be $=\frac{1}{2}aa$; but if the angle ECR were taken greater than a half right angle, so that there shall be $u>1$, since then $\sqrt{(aa-tt)}=\sqrt{(aa-xx-yy)}$, which is the elevation of the surface of the sphere above the quadrant, shall become negative, the surface must be taken in the lower octant. So that if we desire the equation of this curve between the coordinates $\mathrm{CP}=x$ and $\mathrm{PR}=y$, on account of $tt=xx+yy$ and $u=\frac{y}{x}$ we will have

$$4xx+4yy=aa\left(3+\frac{2yy}{xx}-\frac{y^4}{x^4}\right)=\frac{aa(xx+yy)(3xx-yy)}{x^4},$$

which divided by $xx+yy$ gives

$$4x^4=3aaxx-aayy \text{ or } yy=3xx-\frac{4x^4}{aa}.$$

§. 40. We may render this solution more general by putting $\mathrm{V}=abu$
and there will become $a-\sqrt{(aa-tt)}=b(1+uu)$, hence $\sqrt{(aa-tt)}=a-b-buu$, therefore

$$tt=2ab-bb+2(a-b)buu-bbu^4=(1+uu)(2ab-bb-bbuu).$$

With which transformed into orthogonal coordinates again it will follow divided by $xx+yy$ and it will become

$$x^4=(2ab-bb)xx-bbyy \text{ or } y=\frac{x}{b}\sqrt{(2ab-bb-xx)}$$

and the part of the sector ECR of the spherical surface of this curve overhanging will be $=\frac{aby}{x}=b\cdot\mathrm{AT}$; which expression finds a place, as long as $uu<\frac{a-b}{b}$, that is, while the tangent of the angle ECR may become $=\sqrt{\frac{a-b}{b}}$, when there becomes $t=a$. Then truly with the angle ECR increased further the perpendiculars erected above the curve to the lower hemisphere must be extended lower, in which case the surface may be increased there more. Therefore if there shall be $b=a$, because $\sqrt{(aa-tt)}$ shall become a negative quantity everywhere, the quantity $b\cdot\mathrm{AT}$ expresses a part of the spherical surface continued to the lower surface.

§. 41. At this stage let $b=a$ and there may be put $\mathrm{V}=\frac{a^2(\alpha+\beta u)}{\sqrt{(1+uu)}}-\alpha a^2$, so that the surface required to be assigned may vanish on putting $u=0$, and there will become

$$a-\sqrt{(aa-tt)}=\frac{a(\beta-\alpha u)}{\sqrt{(1+uu)}} \text{ and } \sqrt{(aa-tt)}=a-\frac{a(\beta-\alpha u)}{\sqrt{(1+uu)}},$$

where it is required to be observed, if this expression becomes negative, with this descending into the lower hemisphere. But from these there is produced

$$\frac{tt}{aa}=\frac{2(\beta-\alpha u)}{\sqrt{(1+uu)}}-\frac{(\beta-\alpha u)^2}{(1+uu)^2}.$$

Whereby with the angle ECR vanishing, of which the tangent $=u$, there will be $\frac{tt}{aa}=2\beta-\beta\beta$, but if $u=\frac{\beta}{\alpha}$, t vanishes. For the other part of the axis CA, u becomes negative and by putting $u=-v$ the negative surface is found from the expression $\mathrm{V}=\frac{a^2(\alpha-\beta v)}{\sqrt{(1+vv)}}-\alpha a^2$ and the curve will be defined by this equation

$$\frac{tt}{aa}=\frac{2(\beta+\alpha v)}{\sqrt{(1+vv)}}-\frac{(\beta+\alpha v)^2}{1+vv},$$

from which with v produced indefinitely there arises $\frac{tt}{aa}=2\alpha-\alpha\alpha$ *;* where the right line CR becomes normal to the curve, which also arises, when

$$v=\frac{\alpha}{\beta} \text{ and } \frac{tt}{aa}=2\sqrt{(\alpha\alpha+\beta\beta)}-\alpha\alpha-\beta\beta.$$

Whereby lest t becomes imaginary, it shall be required that $\sqrt{(\alpha\alpha+\beta\beta)}<2$.

§. 42 . We may consider the case, where $\alpha=-\frac{1}{\sqrt{2}}$ and $\beta=\frac{1}{\sqrt{2}}$, so that the surface shall be

$$\mathrm{V}=aa\left(\frac{1}{\sqrt{2}}-\frac{1-u}{\sqrt{2}(1+uu)}\right) \text{ and } \frac{tt}{aa}=\frac{2(1+u)}{\sqrt{2}(1+uu)}-\frac{(1+u)^2}{2(1+uu)},$$

where it is apparent, if $u=-1$, to become $t=0$; then truly, it follows thus,

if $u=0$, if $u=1$, if $u=7$, if $u=\infty$,
there will be

$$t=a\sqrt{\frac{2\sqrt{2}-1}{2}},\qquad t=a,\qquad t=a\sqrt{\frac{24}{25}},\qquad t=a\sqrt{\frac{2\sqrt{2}-1}{2}},$$

where it is required to be noted in the cases $u=1$ and $u=\infty$ the right line CR to become normal to the curve. Therefore in this quadrant our curve almost is confused with the quadrant, since everywhere there shall be approximately $t=a$, to which the overhanging part of the surface of the sphere will be $=aa\sqrt{2}$, which differs from the surface of the whole octant, which is $\frac{\pi}{2}aa$, by the small enough part $aa\left(\frac{\pi}{2}-\sqrt{2}\right)=0{,}15658aa$. For the other part of the axis CA this curve falls on the centre, where the tangent may make a half right angle with CA .

§. 43. Truly the solution given in § 35 can be enlarged much more ; for since the surface of the sphere requiring to be assigned may be expressed by this formula $\int\frac{a\partial u}{(1+uu)}\int\frac{t\partial t}{\sqrt{(aa-tt)}}$ and in the integration $\int\frac{t\partial t}{\sqrt{(aa-tt)}}$, the magnitude u may be considered as constant, thus from the integral it will be able to be shown to become $\mathrm{U}-\sqrt{(aa-tt)}$, with U denoting

some function of *u;* which formula since it vanishes, if
$\sqrt{(aa-tt)}=\mathrm{U}$ and $t=(aa-UU)$, from this limit the magnitude *t* can be considered to be extended further. Now V may denote some other function of *u*, which may pass through C on putting $u=0$, and the surface may be put in place

$$\int\frac{a\partial u}{1+uu}\cdot\left(\mathrm{U}-\sqrt{(aa-tt)}\right)=a\mathrm{V}-a\mathrm{C}$$

and hence there will be

$$\mathrm{U}-\sqrt{(aa-tt)}=\frac{\partial\mathrm{V}(1+uu)}{\partial u}$$

and thus

$$\sqrt{(aa-tt)}=\mathrm{U}-\frac{\partial\mathrm{V}(1+uu)}{\partial u},$$

from which the other limit of *t* is defined.

§. 44. Hence the solution of the Florentine problem therefore will be presented most generally. With the quadrant of the circle ACB put in place (Fig. 9), on which the octant of a sphere may stand, with the radius present CA $=a$, and with some radius drawn CS the tangent of the angle ACS may be called *u*; then initially the curve EQG thus may be constructed, so that there shall be

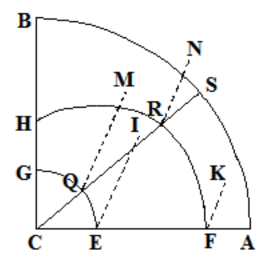

Fig. 9

$$\mathrm{CQ}=\sqrt{(aa-\mathrm{UU})}$$

and the perpendicular from Q to the sphere erected as far as to the surface $\mathrm{QM}=\mathrm{U}$ with U denoting some algebraic function of *u*. If $u=0$, CQ may be changed into CE and QM into EI. Then the other curve is described FRH, so that there shall become

$$\mathrm{CR}=\sqrt{\left(aa-\left(\mathrm{U}-\frac{\partial\mathrm{V}(1+uu)}{\partial u}\right)^2\right)}$$

and the perpendicular from R as far as reaching the sphere

$$\mathrm{RN}=\mathrm{U}-\frac{\partial\mathrm{V}(1+uu)}{\partial u}$$

with V denoting some algebraic function of *u*, which may change into C, if $u=0$; in which case likewise CR may change into CF and RN in FK. Now with these two curves constructed the part of the spherical surface of overhanging area EQRF and contained between the limits I, K, M, N may be expressed algebraically, and it will become $=a(\mathrm{V}-\mathrm{C})$.

§. 45. This concerned with the nature of double integral formulas have provided the occasion of commenting on an equally elegant and useful problem in analysis, if indeed its solution will be able to be set out. Evidently between all the bodies of the same volume that will be sought which may possess the minimum surface, because indeed for the three orthogonal coordinates *x, y* and *z* the relative position may be expressed thus analytically $dz = pdx + qdy$, so that between all the relations of these three variables, which may contain the same magnitude of this twofold integral formula $\iint z\partial x\partial y$, that may be defined, to which the minimum magnitude of this $\iint \partial x\partial y\sqrt{(1+pp+qq)}$ may correspond. Because if we may approach the problem by the theory of variation, it will require to be effected, so that there becomes

$$a\delta\iint \partial x\partial y\sqrt{(1+pp+qq)} = \delta\iint z\partial x\partial y,$$

thus so that the whole undertaking may be reduced to being investigated according to the variations of double integral formulas of this kind.
[See the final appendix to Vol. III, §. 174, of Euler's Integral Calculus, on this website.]

§. 46. So that each double integral formula may be examined, if in the first place *x* may be considered constant, our equation will be represented thus :

$$a\delta\int \partial x\int \partial y\sqrt{(1+pp+qq)} = \delta\int \partial x\int z\partial y.$$

Truly here it is to be noted properly, after the integration

$$\int \partial y\sqrt{(1+pp+qq)} \text{ and } \int z\partial y$$

will have been found, then the variable *y* no longer will be indefinite or not left depending on *x*, so that rather for *y* there will be a certain function of *x*, as the shape of the body emerges, it will be required to be substituted, so that in the following integration the magnitude *y* may be considered not as a constant or not depending on *x* . But because even now on account of the unknown figure of the body that function cannot be constructed, and by no means apparent, in whatever manner variations of this kind should be able to determine variations of the double formulas of this kind.

§.47. Truly the nature of this question besides may be seen to require other determinations, the account of which must be found in the solution. For just as if a curve is sought, which amongst all others having the same area may contain the shortest arc, not only the base AP (Fig. 10), but also the two points B and M, through which the curve may pass, are accustomed to be prescribed; thus also in our problem not only the base, to which the body may stand as a column, must be considered to be assumed known, but also the extreme limits of the surface sought. For if indeed not all

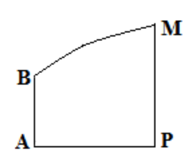

Fig. 10

these matters may be prescribed, indeed if certain parts of the question may be relinquished, for even if the base may be prescribed, truly the limits of the upper surface will be left to our discretion; it is evident, so that the higher the column, there the volume is going to be increased more with the same top surface remaining, whenever the surface of the sides does not enter into the calculation. But a problem without the prescribed base will retain less strength, since with the narrowing of the base a somewhat greater volume for the minimum surface might be able to be put together.

SUPPLEMENTUM VI.

IN FINE SECTIONIS I. TOM. I.

De

INTEGRATIONE FORMULARUM DIFFERENTIALIUM.

De Formulis integralibus duplicatis. *Novi Commentatrii Academiae Scientiar. Petropolitanae. Tom. XIV. Pars I. Pag.* 72-103.

§. 1. Si corporis cujusque propositi vel soliditatem vel superficiem vel alias hujusmodi quantitates definire velimus, id per duplicem integrationem fieri solet; formula enim differentialis bis integranda tali forma $Z\partial x\partial y$ exprimitur, binas variabiles x et y continente, quarum altera sola in priori integratione ut variabilis spectatur, posterior vera integratio ad alteram jam ut variabilem spectatam instituitur. Hinc quantitas per duplicem istam integrationem resultans duplex signum integrale praefigendo indicari solet hoc modo $\iint Z\partial x\partial y$, quippe qua duplicatione formula differentialis proposita $Z\partial x\partial y$ bis integrari debere est intelligenda. Hujusmodi igitur expressiones geminato signo summatario affectas his formulas integrales duplicatas appello, quarum usus cum latissime pateat, in earum indolem hic diligentius inquirere, earumque proprietates et affectiones accuratius evolvere constitui.

§. 2. Primum igitur cum x et y sint duae quantitates variabiles a se invicem non pendentes, Z vero denotet earum functionem quamcunque, formulae integralis duplicatae $\iint Z\partial x\partial y$ vis ita exponi potest, ut quaerenda sit functio finita binarum istarum variabilium x et y, quaesita bis differentiata, ut in altera differentiatione sola x, in altera sola y pro variabili habeatur, ad formulam $Z\partial x\partial y$ deducat. Ita si fuerit $Z = a$, evidens fore $\iint a\partial x\partial y = axy$; generalius vera erit $\iint a\partial x\partial y = axy + X + Y$, denotante X functionem x quamcunque ipsius x et Y ipsius y, quandoquidem hae duae quantitates per geminam illam differentiationem ex calculo tolluntur.

§. 3. In genere autem si V fuerit ejusmodi functio ipsarum x et y, quae bis differentiata ita ut modo est praeceptum, praebeat $Z\partial x\partial y$; erit quidem utique $V = \iint Z\partial x\partial y$; verum duplex integratio insuper functiones arbitrarias X et Y, illam ipsius x, hanc ipsius y inducit, ut sit generalissime

$$\iint Z\partial x\partial y = V + X + Y .$$

Ex quo statim perspicitur, hujusmodi formulas differentiales necessario affectas esse producto $\partial x\partial y$, neque propterea secundum hanc significationem tales formulas $\iint Z\partial x^2$ vel $\iint Z\partial x^2$ quicquam significare; siquidem per ipsam rei naturam excluduntur, dum in altera integratione sola *x*, in altera vero sola *y* ut variabilis tractatur.

§. 4. Constituta sic forma huiusmodi formularum integralium duplicatarum $\iint Z\partial x\partial y$, ita ut *x* et *y* sint duae quantitates variabiles a se invicem non pendentes et *Z* functio finita ex iis utcunque composita, haud difficile est duplicem integrationem, quam involvunt, instituere, quod quidem, prout primo vel *x* vel *y* sola variabilis consideratur, duplici modo fieri potest. Sumta scilicet primo *y* pro variabili altera *x* ut constans tractatur quaeriturque integrale $\int Zdy$, quod erit certa quaedam functio ipsarum *x* et *y;* qua inventa suscipiatur formula differentialis $\partial x\int Zdy$, in qua iam *y* ut constans solaque *x* ut variabilis tractetur, eiusque quaeratur integrale $\iint Z\partial x\partial y$, qui erit valor quaesitus formulae integralis duplicatae propositae $\iint Z\partial x\partial y$. Si in hac duplici integratione ordo variabilium *x* et *y* invertatur, valor quaesitus ita exprimetur $\int \partial y\int Z\partial x$, qui ab illo non discrepabit.

§. 5. Ob hunc consensum fit, ut talis forma $\iint Z\partial x\partial y$ promiscue sive hoc modo $\iint Z\partial x\partial y$ sive hoc $\int \partial y\int Z\partial x$ exhiberi possit; utrovis autem utamur, regulae vulgares integrationis sunt observandae, si modo notetur in ea integratione, in qua vel *x* vel *y* pro constante sumatur, constantem introductam eiusdem fore functionem quamcunque. Veluti si proponatur haec forma

$$\iint \frac{\partial x\partial y}{xx+yy} = \int \partial x\int \frac{\partial y}{xx+yy}$$

ob

$$\int \frac{\partial y}{xx+yy} = \frac{1}{x}\text{arctan.}\frac{y}{x} + \frac{\partial X}{\partial x}$$

denotante $\frac{\partial X}{\partial x}$ functionem quamcunque ipsius *x* erit

$$\iint \frac{\partial x\partial y}{xx+yy} = \int \frac{\partial x}{x}\text{arctan.}\frac{y}{x} + X,$$

ubi in integratione adhuc perficienda *y* pro constante habetur. Simili vero modo reperitur

$$\iint \frac{\partial x \partial y}{xx+yy} = \int \frac{\partial y}{y} \operatorname{arctan.} \frac{y}{x} + \mathrm{Y},$$

in qua integratione x constans assumitur; in quo quidem exemplo consensus binorum valorum inventorum non satis est perspicuus.

§. 6. Interim tamen veritas consensus per series facile ostenditur; cum enim sit $\operatorname{arctang.} \frac{x}{y} = \frac{\pi}{2} - \operatorname{arctang.} \frac{y}{x}$ denotante $\frac{\pi}{2}$ angulum rectum et

$$\operatorname{arctang.} \frac{y}{x} = \frac{y}{x} - \frac{y^3}{3x^3} + \frac{y^5}{5x^5} - \frac{y^7}{7x^7} + \frac{y^9}{9x^9} - \text{etc.},$$

erit

$$\int \frac{\partial x}{x} \operatorname{arctang.} \frac{y}{x} = -\frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.} + f:y$$

et

$$\int \frac{\partial y}{y} \operatorname{arctang.} \frac{y}{x} = \frac{\pi}{2} ly - \frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.} + f:x$$

ex quarum utraque oritur

$$\iint \frac{\partial x \partial y}{xx+yy} = \mathrm{X} + \mathrm{Y} - \frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.}$$

Verum ubi ambae integrationes succedunt, convenientia sponte se offert; quod quidem pluribus exemplis ostendisse superfluum foret, cum eius ratio ex natura differentialium et integralium perfecte sit demonstrata.

§. 7. Haec igitur tenenda sunt de istiusmodi formulis integralibus duplicatis, quando binae variabiles x et y nullo plane nexu inter se cohaerent, ita ut in altera integratione altera, in altera vero altera constans accipiatur. Verum tales formulae non confundendae sunt cum iis, quibus, ut initio dixi, soliditas et superficies corporum quorumcunque exprimi solet. Etsi enim hae formulae etiam duplicem integrationem requirunt et in priori altera binarum variabilium, puta y, sola ut variabilis tractatur altera x pro constante assumta, tamen priori integratione peracta, ea per omnes valores ipsius y extendi sicque tandem loco y extremus valor, quem recipere potest, statui debet, qui plerumque ab x pendet, ita ut hoc valore post primam integrationem loco y constituto in posteriori integratione y tanquam functio quaedam ipsius x ingrediatur neque propterea pro constanti haberi queat, qua conditione fit, ut altera integratio plurimum immutetur, etsi prior simili modo ut ante absolvatur.

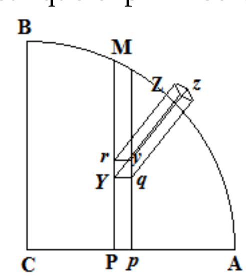

Fig. 1

§. 8. Quod discrimen quo clarius perspiciatur, exemplum attulisse iuvabit. Quaeratur ergo soliditas sphaerae, cuius centrum sit C (Fig. 1) et radius $CA = a$, ac primo quidem portio eius quadranti ACB insistens, cuius elementum est columella YZyz areolae $Yy = \partial x \partial y$ insistens positis $CP = x$ et $PY = y$, eritque eius altitudo $YZ = \sqrt{(aa - xx - yy)}$; hinc soliditas columellae elementaris $= dxdy\sqrt{(aa - xx - yy)}$, quam bis integrari oportet. Maneat primo intervallum $CP = x$ constans et integrale $\int dy\sqrt{(aa - xx - yy)}$ ita sumtum, ut evanescat posito $y = 0$, dabit portiunculam areolae PpYq insistentem, quae ergo erit

$$= \tfrac{1}{2} y\sqrt{(aa - xx - yy)} + \tfrac{1}{2}(aa - xx)\text{arcsin.}\frac{y}{\sqrt{(aa-xx)}}.$$

Iam hoc valore in altera integratione uti oportet, sed antequam is inducatur, per totam distantiam PM extendi debet, ut habeatur elementum soliditatis toti areolae PpMm insistens; puncto autem Y ad M usque promoto fit $y = \sqrt{(aa - xx)}$, qui ergo valor loco y substitui debet, ita ut in sequente integratione quantitas y minime ut constans consideretur haecque tractandi methodus plurimum a praecedente discrepet.

§. 9. Posito ergo $y = \sqrt{(aa - xx)}$ fit

$$\int \partial y\sqrt{(aa - xx - yy)} = \tfrac{\pi}{4}(aa - xx),$$

cum sit $\text{arcsin.}1 = \frac{\pi}{2}$, sicque integratio adhuc absolvenda erit

$$\int \partial x \int \partial y\sqrt{(aa - xx - yy)} = \tfrac{\pi}{4}\int (aa - xx)\partial x,$$

ubi quidem unica variabilis x inest, sed non ideo, quod iam hic y pro constanti habeatur, sed quia pro y certa quaedam functio ipsius x est substituta.
Haec altera vero integratio ita instituta, ut evanescat posito $x = 0$, dabit soliditatem portionis sphaerae, quae areae CBMP insistit, quae idcirco erit $= \frac{\pi}{4}\left(aax - \frac{1}{3}x^3\right)$; unde sphaerae octans seu portio toti quadranti ACB insistens prodibit punctum P usque in A promovendo, ut fiat $x = a$. Tum ergo soliditas octantis sphaerae erit $= \frac{\pi}{6}a^3$ hincque totius sphaerae $= \frac{4\pi}{3}a^3$, uti constat. Ex quo exemplo intelligitur talem soliditatis investigationem plurimum differre ab integratione duplicata formularum primo exposita.

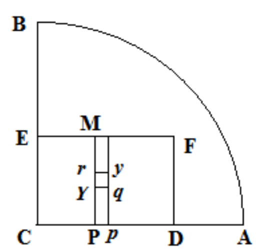

Fig. 2

§. 10. Quodsi non totum octantem sphaerae, sed eam tantum eius portionem, quae areae rectangulari CEDF (Fig.2) insistit, investigare velimus, prior integratio ut ante instituenda est, sed ea peracta ipsi y valor PM debet tribui, qui quidem est constans, et propterea haec investigatio ad primum genus videtur accedere,

verum tamen eo discrepat, quod integrale determinatum
prodeat, cum ibi functiones indefinitae X et Y inveherentur. Posito ergo ut ante sphaerae radio $CA = a$ sit rectanguli CEFD latus $CD =$ et $CE = f$ et solidum elementare areolae PpYq
insistens erit ut ante

$$= \tfrac{1}{2} y\sqrt{(aa - xx - yy)} + \tfrac{1}{2}(aa - xx)\text{arcsin.}\frac{y}{\sqrt{(aa-xx)}},$$

quod usque ad M extensum, ubi fit $y = f$, erit

$$\tfrac{1}{2} f\sqrt{(aa - ff - xx)} + \tfrac{1}{2}(aa - xx)\text{arcsin.}\frac{f}{\sqrt{(aa-xx)}},$$

unde solidum areae CPEM insistens sequenti integrali exprimetur

$$\tfrac{1}{2} f\int \partial x\sqrt{(aa - ff - xx)} + \tfrac{1}{2}\int (aa - xx)\partial x\text{arcsin.}\frac{f}{\sqrt{(aa-xx)}},$$

si quidem ita definiatur, ut evanescat posito $x = 0$. Evolvamus ergo seorsim has binas formulas.

§. 11. Ac prima quidem statim praebet

$$\int \partial x\sqrt{(aa - ff - xx)} = \tfrac{1}{2} x\sqrt{(aa - ff - xx)} + \tfrac{1}{2}(aa - ff)\text{arcsin.}\frac{x}{\sqrt{(aa-xx)}},$$

altera autem ob

$$\partial.\text{arcsin.}\frac{f}{\sqrt{(aa-xx)}} = \frac{fx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}}$$

ita transformatur

$$\int (aa - xx)\partial x\,\text{arcsin.}\frac{f}{\sqrt{(aa-xx)}}$$

$$= \left(aax - \tfrac{1}{3}x^3\right)\text{arcsin.}\frac{f}{\sqrt{(aa-xx)}} - f\int \frac{\left(aa-\frac{1}{3}xx\right)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}},$$

ad quam postremam partem integrandam notetur esse

$$\text{arcsin.}\frac{fx}{\sqrt{(aa-xx)(aa-xx)}} = \int \frac{af\,\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}};$$

huius ergo dabitur multiplum quoddam, quod illi formae adiectum praebeat talem formam

$$\int\frac{\left(aa-\frac{1}{3}xx\right)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}}+m\text{arcsin.}\frac{fx}{\sqrt{(aa-xx)(aa-xx)}}=\int\frac{\left(aaxx-\frac{1}{3}x^4+maf\right)\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}},$$

ut $aaxx-\frac{1}{3}x^4+maf$ fiat per $aa-xx$ divisibile, id quod fit sumendo $m=-\frac{2a^3}{3f}$;
hincque erit

$$\int\frac{\left(aa-\frac{1}{3}xx\right)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}}=\frac{2a^3}{3f}\text{arcsin.}\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}-\frac{1}{3}\int\frac{(2aa-xx)\partial x}{\sqrt{(aa-ff-xx)}}.$$

§. 12. Cum igitur sit

$$\int\frac{(2aa-xx)\partial x}{\sqrt{(aa-ff-xx)}}=\frac{1}{2}(3aa+ff)\text{arcsin}\frac{x}{\sqrt{(aa-ff)}}+\frac{1}{2}x\sqrt{(aa-ff-xx)},$$

erit

$$\int\frac{\left(aa-\frac{1}{3}xx\right)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}}=$$

$$\frac{2a^3}{3f}\text{arcsin.}\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}-\frac{1}{6}(3aa+ff)\text{arcsin}\frac{x}{\sqrt{(aa-ff)}}-\frac{1}{6}x\sqrt{(aa-ff-xx)},$$

hincque

$$\int(aa-xx)\partial x\,\text{arcsin.}\frac{f}{\sqrt{(aa-xx)}}$$

$$=\left(aax-\frac{1}{3}x^3\right)\text{arcsin.}\frac{f}{\sqrt{(aa-xx)}}-\frac{2a^3}{3}\text{arcsin.}\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}$$
$$+\frac{1}{6}f(3aa+ff)\text{arcsin}\frac{x}{\sqrt{(aa-ff)}}+\frac{1}{6}fx\sqrt{(aa-ff-xx)}.$$

Quare posito $x=\text{CD}=e$ erit soliditas portionis sphaerae rectangulo CDEF insistentis

$$\frac{1}{4}ef\sqrt{(aa-ee-ff)}+\frac{1}{4}f(aa-ff)\text{arcsin.}\frac{e}{\sqrt{(aa-ff)}}$$
$$+\frac{1}{6}e(3aa-ee)\text{arcsin.}\frac{f}{\sqrt{(aa-ee)}}-\frac{1}{3}a^3\text{arcsin.}\frac{ef}{\sqrt{(aa-ee)(aa-ff)}}$$
$$+\frac{1}{12}f(3aa+ff)\text{arcsin.}\frac{e}{\sqrt{(aa-ee)}}+\frac{1}{12}ef\sqrt{(aa-ee-ff)},$$

quae expressio reducitur ad hanc

$$\frac{1}{3}ef\sqrt{(aa-ee-ff)}+\frac{1}{6}f(3aa-ff)\text{arcsin.}\frac{e}{\sqrt{(aa-ff)}}$$
$$+\frac{1}{6}e(3aa-ee)\text{arcsin.}\frac{f}{\sqrt{(aa-ee)}}-\frac{1}{3}a^3\text{arcsin.}\frac{ef}{\sqrt{(aa-ee)(aa-ff)}}.$$

§. 13. Si ergo rectanguli terminus F usque ad peripheriam porrigatur, ut sit $ee+ff=aa$, primum membrum evanescit et arcus circulares tria reliqua afficientes abeunt in angulum rectum seu $\frac{\pi}{2}$; eritque soliditas

$$\tfrac{\pi}{2}\left(\tfrac{1}{2}aae+\tfrac{1}{2}aaf-\tfrac{1}{6}e^3-\tfrac{1}{6}f^3-\tfrac{1}{3}a^3\right)$$

seu ob $f=\sqrt{(aa-ee)}$

$$\tfrac{\pi}{12}\left((2aa+ee)\sqrt{(aa-ee)}-2a^3+3aae-e^3\right),$$

quod solidum fit maximum, si $f=e=\frac{a}{\sqrt{2}}$, fitque id tum $\frac{\pi a^3}{12\sqrt{2}}\left(5-2\sqrt{2}\right)$, dum soliditas octantis sphaerae est $=\frac{\pi}{6}a^3$, ita ut nostrum solidum sit ad octantem sphaerae ut $5-2\sqrt{2}$ ad $2\sqrt{2}$. Sin autem punctum F non ad peripheriam quadrantis pertingat fueritque $f=e$, erit soliditas quaesita

$$=\tfrac{1}{3}ee\sqrt{(aa-2ee)}+\tfrac{1}{3}e(3aa-ee)\,\text{arcsin.}\frac{e}{\sqrt{(aa-ee)}}-\tfrac{1}{3}a^3\text{arcsin.}\frac{ee}{(aa-ee)}.$$

Quare si fuerit

$$\text{arcsin.}\frac{e}{\sqrt{(aa-ee)}}:\text{arcsin.}\frac{ee}{aa-ee}=a^3:e(3aa-ee),$$

solidum algebraice exprimetur.

§. 14. Quo autem rem generalius complectamur, quaeramus solidum areae cuicunque GQHR (Fig. 3) insistens; cuius elementum cum areolae $\text{Y}y=\partial x\partial y$ insistat idque sit $=\partial x\partial y\sqrt{(aa-xx-yy)}$, prima integratio sumto x constante praebet

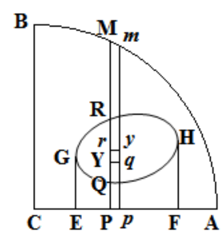

Fig. 3.

$$\tfrac{1}{2}\partial x\left(y\sqrt{(aa-xx-yy)}+(aa-xx)\,\text{arcsin.}\frac{y}{\sqrt{(aa-xx)}}\right).$$

Sint iam ex natura curvae GQHR distantiae extremae $\text{PQ}=q$ et $\text{PR}=r$ atque solidum elementare areolae QR insistens erit

$$\tfrac{1}{2}\partial x\begin{Bmatrix}+r\sqrt{(aa-xx-rr)}+(aa-xx)\,\text{arcsin.}\dfrac{r}{\sqrt{(aa-xx)}}\\ -q\sqrt{(aa-xx-rr)}-(aa-xx)\text{arcsin.}\dfrac{q}{\sqrt{(aa-xx)}}\end{Bmatrix}.$$

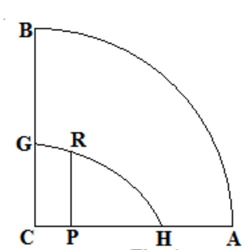

Fig. 4

Quare cum q et r possint esse functiones quaecunque ipsius x, evidens est, quantum absit, quominus quantitas y in sequente integratione pro constanti habeatur. Sequens autem integratio a valore $x = \mathrm{CE}$ usque ad valorem $x = \mathrm{CF}$ est extendenda.

§. 15. Si figura basis GCHR a recta CA traiiciatur, ut quaeratur solidum basi CGH (Fig. 4) insistens, cuius natura exprimatur aequatione quacunque inter $\mathrm{CP} = x, \mathrm{PR} = r$, erit solidum

$$\tfrac{1}{2}\int \partial x\left(r\sqrt{(aa-xx-rr)} + (aa-xx)\mathrm{arcsin.}\tfrac{r}{\sqrt{(aa-xx)}} \right),$$

ubi problema non inelegans se offert, quo figura basis CGH quaeritur, ut solidum ei insistens algebraice exprimatur.

Statuatur in hunc finem $r = u\sqrt{(aa-xx)}$, ut solidum indefinitum areae CPRG insistens sit

$$\tfrac{1}{2}\int (aa-xx)\partial x\left(u\sqrt{(1-uu)} + arcsin.u\right),$$

quae expressio transformatur in hanc

$$\tfrac{1}{2}\left(aax - \tfrac{1}{3}x^3\right)\cdot\left(u\sqrt{(1-uu)} + arcsin.u\right) - \int\left(aax - \tfrac{1}{3}x^3\right)\partial u \cdot \sqrt{(1-uu)}.$$

Fiat iam

$$\int\left(aax - \tfrac{1}{3}x^3\right)\partial u \cdot \sqrt{(1-uu)} = na^3 \arcsin.u + a^3 U$$

existente U functione algebraica ipsius u, et cum sit soliditas

$$\tfrac{1}{2}\left(aax - \tfrac{1}{3}x^3\right)u\sqrt{(1-uu)} - a^3 U + \left(\tfrac{1}{2}aax - \tfrac{1}{6}x^3 - na^3\right)\arcsin.u,$$

ea erit algebraica casu $-x^3 + 3aax = 6na^3$, dummodo u evanescat posito $x = 0$; tum enim soliditas erit $= na^3 u\sqrt{(1-uu)} - a^3 U$.

§. 16. Ponamus $dU = U'du$ ac prodibit haec inter x et u aequatio

$$aax - \tfrac{1}{3}x^3 = \tfrac{na^3}{1-uu} + \tfrac{a^3 U'}{\sqrt{(1-uu)}}.$$

Fingatur $U = mu\sqrt{(1-uu)}$; erit $U' = \frac{m-2muu}{\sqrt{(1-uu)}}$, et ut u evanescat posito $x = 0$, debet esse $m = -n$, ut fiat

$$aax - \tfrac{1}{3}x^3 = \frac{2na^3uu}{1-uu} \text{ seu } u = \sqrt{\frac{3aax-x^3}{6na^3+3aax-x^3}}$$

hincque

$$r = \sqrt{\frac{(aa-xx)(3aax-x^3)}{6na^3+3aax-x^3}}.$$

Iam ob

$$u\sqrt{(1-uu)} = \frac{\sqrt{6na^3(3aax-x^3)}}{6na^3+3aax-x^3}$$

fit soliditas illa

$$= \frac{2na^3\sqrt{6na^3(3aax-x^3)}}{6na^3+3aax-x^3}.$$

Si haec soliditas locum habere debeat facto $x = a$, fit

$$n = \tfrac{1}{3}, \quad r = \sqrt{\frac{(aa-xx)(3aax-x^3)}{2a^3+3aax-x^3}} = \sqrt{\frac{x(a-x)(3aa-xx)}{(a+x)(2a-x)}}$$

ac posito $x = a$, erit soliditas $= \tfrac{1}{3}a^3$ et curva pro basi inventa est linea quarti ordinis.

§. 17. Quae hic de soliditate portionis sphaericae datae basi insistentis sunt tradita, simili calculo ad quaevis alia corpora accommodari possunt, cum tantum in formula $Z\partial x\partial y$ quantitas Z alio modo per x et y determinetur, dum hic erat $Z = \sqrt{(aa-xx-yy)}$. Quin etiam si superficies corporis cuiuscunque datae basi imminens definiri debeat, id integratione gemina similis formulae differentialis $Z\partial x\partial y$ eodem modo expedietur. Ita si corpus sit sphaera, elementum superficiei areolae elementari basis $\partial x\partial y$ imminentis est $\frac{a\partial x\partial y}{\sqrt{(aa-xx-yy)}}$, ita ut sit $Z = \frac{a}{\sqrt{(aa-xx-yy)}}$, cuius gemina integratio pari modo pro ratione basis, cui imminens portio superficiei quaeritur, est instituenda. Atque in genere quantitates quaecunque aliae cuiusvis corporis, quae certae basi respondeant, ope similium operationum determinabuntur.

§. 18. Quaecunque ergo Z fuerit functio ipsarum x et y, pro integrali duplicato $\iint Z\partial x\partial y$ primo quaeritur integrale $\int Z\partial y$ quantitate x ut constante spectata idque extendatur per totam quantitatem y sicque extremi valores ipsius y in computum ingredientur, quae erunt functiones ipsius x ex basis figura cognitae; sicque pro $\int Z\partial y$ orietur functio ipsius x, quae in ∂x ducta denuo more solito debet integrari. Idem tenendum est, si ordine inverso primo formula $\int Z\partial x$ integretur spectato y ut constante;

quod integrale dum per totum intervallum x extenditur, extremi valores ipsius x eidem y respondentes, qui erunt functiones ipsius y, invehentur sicque $\int Z\partial x$ abibit in functionem ipsius y tantum, quae per ∂y multiplicata denuo ita integrari debet, ut integrale per totum intervallum y extendatur. Utroque scilicet modo integratio per totam basin est extendenda eademque praecepta sunt observanda, qualiscunque Z fuerit functio ipsarum x et y.

§. 19. Basi ergo data determinatio integrationum perinde se habet, ac si quantitas Z esset constans quaerereturque tantum integrale $\iint \partial x\partial y$, quo area basis exprimitur. Quare ad praecepta, quae in determinatione horum integralium observari oportet, stabilienda sufficiet posuisse $Z=1$, ut integrale duplicatum $\iint \partial x\partial y$ definiendum sit; sive autem sumatur x sive y, extremi valores utriusque determinabuntur per aequationem basis figuram exprimentem. Scilicet priori integratione peracta, ubi punctum Y (Fig. 5) ubicunque intra terminos extremos erat assumtum, tum hoc punctum in peripheriam basis transferatur, quo pacto x et y fient coordinatae basis, inter quas aequatio datur, ex qua deinceps sive y per x sive x per y determinabitur.

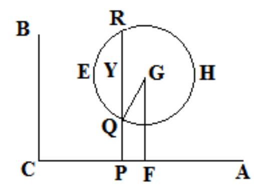

Fig. 5

§. 20. Quae quo clarius perspiciantur, sumamus basis figuram esse circulum centrum in G et radium GQ habentem ponamusque $\mathrm{CF}=f$, $\mathrm{FG}=g$ et $\mathrm{GQ}=c$; erit puncto Y in peripheriam huius circuli translato

$$cc=(f-x)^2+(g-y)^2.$$

Iam ad aream huius circuli investigandam sit primo x constans eritque $\int\partial y=y+\mathrm{C}$, et quia y habet geminum valorem in nostra basi

$$y=g\pm\sqrt{\left(cc-(f-x)^2\right)},$$

haec integratio ita determinetur, ut integrale evanescat, dum ipsi y minor horum valorum $g-\sqrt{\left(cc-(f-x)^2\right)}$ tribuitur, ita ut sit

$$\int\partial y=y-g+\sqrt{\left(cc-(f-x)^2\right)}.$$

Nunc ergo y usque ad alterum terminum $y=g+\sqrt{\left(cc-(f-x)^2\right)}$, extenso erit

$$\int\partial y=2\sqrt{\left(cc-(f-x)^2\right)},$$

quod iam per ∂x multiplicatum et integratum praebet

$$\int \partial x \int \partial y = \mathrm{C} - (f-x)\sqrt{\left(cc-(f-x)^2\right)} - cc\,\mathrm{arcsin}\,\tfrac{f-x}{c};$$

quod ut evanescat posito $x = f - c,$ fit $\mathrm{C} = cc\mathrm{arcsin}.1 = \frac{\pi}{2}cc.$ Porro statuatur $x = f + c,$ et ob $cc\,\mathrm{arcsin}.\frac{f-x}{c} = -cc\,\mathrm{arcsin}.1 = -\frac{\pi}{2}cc$ erit area quaesita tota $= \frac{\pi}{2}cc + \frac{\pi}{2}cc = \pi cc,$ uti constat.

§. 21. Si has determinationes accuratius perpendamus, videmus extremos valores ipsius *x* ita esse comparatos, ut alter sit maximus, [alter minimus,] siquidem basis tota quadam curva in se redeunte terminetur. Hi ergo ambo valores reperientur, si aequatio naturam basis exprimens differentietur et $\partial x = 0$ ponatur. Quando autem basis non una quadam linea curva terminatur, sed portione quapiam, veluti CGH (Fig. 4), continetur, cuius basis CH sit maxima, tum minor terminus ipsius *x* manifesto est $= 0$, maior autem ipsi CH aequalis; eodemque casutermini applicatae PR abscissae $\mathrm{CP} = x$ respondentia sunt alter $= 0$, alter vero $= \mathrm{CG}$. Quacunque ergo basi proposita eius figura ante probe est examinanda ipsiusque termini quaquaversus explorandi, quam investigatio areae vel cuiusvis alius formulae integralis duplicatae suscipi queat; definitis autem terminis, quibus area continetur, inde determinationes integrationum sunt petendae.

§. 22. His de integrationum determinatione expositis insignes maximeque notatu dignae affectiones huiusmodi formularum integralium duplicatarum perpendi merentur, quae in earum transformatione occurrunt. Scilicet quemadmodum coordinatae eiusdem curvae infinitis modis sumi possunt, ita hic loco binarum variabilium *x* et *y* binae quaecunque aliae variabiles in computum introduci possunt, sive eae pariter sint coordinatae sive aliae quantitates utcunque definitae. Ita talis transformatio in genere ita concipi potest, ut loco *x* et *y* functiones quaecunque aliarum duarum variabilium *t* et *v* substituantur, hisque in aequationem pro basi datam introductis simili modo limites harum quantitatum *t* et *v*, quibus figura basis terminatur, definiri poterunt. Utcunque autem hae substitutiones sumantur, tandem post duplicem integrationem semper eadem quantitas resultet necesse est.

§. 23. Si loco *x* et *y* aliae quaecunque binae coordinatae orthogonales introducantur, puta *t* et *v*, quod fit in genere ponendo

$$x = f + mt + v\sqrt{(1-mm)} \quad \text{et} \quad y = g + t\sqrt{(1-mm)} - mv,$$

manifestum est elementum areae basis, quod ante erat $\partial x \partial y$, nunc per $\partial t \partial v$ exprimi debere. Cum autem inde sit

$$\partial x = m\partial t + \partial v\sqrt{(1-mm)} \quad \text{et} \quad \partial y = \partial t\sqrt{(1-mm)} - m\partial v,$$

minime patet, quomodo loco $\partial x\partial y$ per has substitutiones oriri possit $\partial t\partial v$, dum potius prodiret

$$\partial x\partial y = m\partial t^2\sqrt{(1-mm)} + (1-2mm)\partial t\partial v - m\partial v^2\sqrt{(1-mm)},$$

quae autem formula, utcunque ad geminam integrationem adaptatur, semper in maximos errores inducet. Multo minus ergo hinc colligere licet, si loco *x* et *y* aliae functiones ipsarum *t* et *v* substituantur, cuiusmodi expressio loco $\partial x\partial y$ adhiberi debeat.

§. 24. Ac primo quidem observo nullam hic esse rationem, cur expressio loco $\partial x\partial y$ in calculum introducenda ei aequalis esse debeat; quod tum demum necesse esset, si binae integrationes eodem modo ut ante secundum binas variabiles instituerentur. Cum autem nunc aliae variabiles *t* et *v* adsint atque altera integratio per variabilitatem ipsius *t*, altera ipsius *v* sit administranda, quae operationes a praecedentibus plurimuin differunt, formula iam loco $\partial x\partial y$ inducenda non ex aequalitate aestimari, sed potius ad scopum, qui est propositus, accommodari debet. Et quoniam iam binas integrationes secundum binas variabiles *t* et *v* distingui oportet, manifestum est formulam loco $\partial x\partial y$ adhibendam necessario producto $\partial t\partial v$ affectam esse et huiusmodi formam $Z\partial t\partial v$ habere debere.

§. 25. Quo haec certius expediantur, maneat primo *x* et loco *y* introducatur alia variabilis *u*, ita ut sit *y* functio quaecunque ipsarum *x* et *u* et $\partial y = P\partial x + Q\partial u$. Si iam in priori integratione *x* constans sumatur, erit utique $\partial y = Q\partial u$, hinc $\iint \partial x\partial y = \int \partial x\int Qdu$, ita ut nunc loco formulae $\partial x\partial y$ habeatur $Q\partial x\partial u$, cuius integrale duplicatum proinde etiam hoc modo exprimi poterit $\int \partial u\int Q\partial x$, ubi in priori integratione $\int Q\partial x$ quantitas *u* sumitur pro constante. Quodsi nunc simili modo *u* retineatur et loco *x* introducatur functio quaecunque ipsarum *t* et *u*, ut sit $\partial x = R\partial t + S\partial u$, in tractatione formulae $\int \partial u\int Q\partial x$ prior integratio $\int Q\partial x$, in qua *u* constans statuitur, abibit in hanc $\int QR\partial t$, ita ut integrale duplicatum sit $\int \partial u\int QR\partial t$ seu promiscue $\iint QR\partial t\partial u$, unde manifestum est ob has ambas substitutiones loco formulae $\partial x\partial y$ hanc $QR\partial t\partial u$ tractari debere.

§. 26. Introducamus nunc statim loco *x* et *y* has duas novas variabiles *t* et *u*, per quas illae ita determinentur, ut sit

$$\partial x = R\partial t + S\partial u \quad \text{et} \quad \partial y = T\partial t + V\partial u,$$

unde valore ipsius ∂x in forma $\partial y = P\partial x + Q\partial u$ substituto fit

$$\partial y = PR\partial t + (PS + Q)\partial u,$$

ita ut sit $\mathrm{PR}=\mathrm{T}$ et $\mathrm{PS}+\mathrm{Q}=\mathrm{V}$, unde fit $\mathrm{P}=\frac{\mathrm{T}}{\mathrm{R}}$ et $\frac{\mathrm{ST}}{\mathrm{R}}+\mathrm{Q}=\mathrm{V}$ sicque $\mathrm{QR}=\mathrm{VR}-\mathrm{ST}$. Quare vi harum substitutionum loco $\partial x\partial y$ uti debemus formula $(\mathrm{VR}-\mathrm{ST})\partial t\partial u$, quae bis integrata iustis adhibitis determinationibus aeque aream totius basis praebere debet atque ipsa formula $\partial x\partial y$ bis integrata. Quod autem hic pro formula areae baseos $\iint\partial x\partial y$ est ostensum, locum habet pro quacunque alia formula $\iint \mathrm{Z}\partial x\partial y$, quippe quae per easdem substitutiones transformatur in hanc $\iint \mathrm{Z}(\mathrm{VR}-\mathrm{ST})\partial t\partial u$, dummodo in Z loco *x* et *y* assumti valores substituantur. Pari enim modo binas integrationes ex figura basis determinari oportet.

§. 27. Quodsi ergo ponatur

$$dx = Rdt + Sdu \text{ et } dy = Tdt + Vdu,$$

loco $\partial x\partial y$ consequimur $(\mathrm{RV}-\mathrm{ST})\partial t\partial u$, quae formula plurimum differt ab ea, cui productum $\partial x\partial y$ revera est aequale; etiamsi enim termini per ∂t^2 et ∂u^2 affecti, utpote ad duplicem integrationem inepti, reiiciantur, tamen, quod restat, $(RV+ST)\partial t\partial u$ ratione signi a vera formula discrepat. Verum hic non leve dubium exoritur, quod, cum coordinatae *x* et *y* pari passu ambulent, nostra formula potius differentiam $\mathrm{RV}-\mathrm{ST}$ quam inversam $\mathrm{ST}-\mathrm{RV}$ complectatur; quod dubium eo magis augetur, quod, si superius ratiocinium respectu *x* et *y* invertissemus, eaedem substitutiones nos revera ad formulam $(\mathrm{ST}-\mathrm{RV})\partial t\partial u$ perduxissent. Sed quia totum discrimen tantum in signo versatur alteraque formula alterius est negativa, hinc determinatio absoluta areae basis, quippe cuius quantitas absoluta quaeritur, nullam mutationem realem patitur.

§. 28. Haec autem magis fient perspicua, si modum, quo supra (§ 20) ad aream EQHR (Fig. 5) inveniendam usi sumus, attentius consideremus. Primum scilicet ex integratione formulae $\iint\partial x\partial y$ deduximus hanc aream

$$=\int\partial x(PR-PQ),$$

ubi quidem PQ a PR subtraximus, quia manifesto erat $\mathrm{PR}>\mathrm{PQ}$; sed in ipso calculo nulla continetur ratio, quae praecipiat, ut potius PQ a PR quam vicissim PR a PQ subtrahamus, sicque non adversante calculo potuissemus aequo iure eandem aream per $=\int\partial x(PQ-PR)$, exprimere, quo pacto ea negativa, sed priori aequalis proditura fuisset. Ex quo perspicuum est signum $+$ vel $-$ non quantitatem areae, quae quaeritur, afficere et calculum pari iure ad utrumque perducere posse. Quam ob causam superius dubium ita diluetur, ut dicamus aream quaesitam ita exprimi debere, ut sit

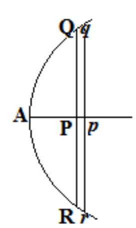

Fig. 6

$=\pm\iint\partial t\partial u\left(\mathrm{RV}-\mathrm{ST}\right)$, et ut area positive expressa prodeat, quovis casu eo signo utendum esse, quo $\pm(\mathrm{RV}-\mathrm{ST})$ reddatur quantitas positiva.

§. 29. Hinc etiam dubia, quae forte oriri possent circa inventionem areae curvarum, quarum partes utrinque ad axem sunt dispositae et quibus tirones saepe non parum turbari solent, facile resolvuntur. Si enim curvae QAR (Fig. 6) ad axem AP relatae area tota QAR abscissae $AP=x$ respondens definiri debeat eiusque partes APQ et APR seorsim considerentur, certum est, si altera APQ affirmative spectetur, ut sit $=+\mathrm{Q}$, alteram APR negative concipi debere, ut sit $=-\mathrm{R}$. Neque tamen hinc sequitur aream totam QAR fore $=\mathrm{Q}-\mathrm{R}$, quippe quae evanesceret, si ambae partes APQ et APR essent aequales; sed perinde ac si ambo puncta Q et R ad eandem axis partem sita essent, area perpetuo est $=\pm\int\partial x\left(\mathrm{PR}-\mathrm{PQ}\right)$, unde ob

$$\int\mathrm{PQ}\cdot\partial x=\mathrm{Q}\ \text{ et }\ \int\mathrm{PR}\cdot\partial x=-\mathrm{R}$$

fit tota area $=\pm(\mathrm{Q}+\mathrm{R})$, uti rei natura postulat.

§. 30. Ope autem talium substitutionum, quibus loco binarum variabilium *x* et *y* binae quaecunque aliae introducantur *t* et *u*, saepenumero integrationes plurimum sublevari facilioresque reddi possunt et quovis casu haud difficile est substitutiones maxime idoneas reperire. Veluti si area circuli EQHR (Fig. 5) ad axem GP relati definiri debeat, ubi ob $\mathrm{CF}=g,\mathrm{GQ}=c$ erat $cc=\left(f-x\right)^2+\left(\mathrm{g}-y\right)^2$, poni conveniet

$$f-x=\frac{t}{\sqrt{(1+uu)}}\ \text{et}\ \ g-y=\frac{tu}{\sqrt{(1+uu)}},$$

ut fiat $tt=cc$ et $t=c$. Tum vero ob

$$\partial x=\frac{-\partial t}{\sqrt{(1+uu)}}+\frac{tu\partial u}{(1+uu)^{\frac{3}{2}}}\ \text{ et }\ \partial y=\frac{-u\partial t}{\sqrt{(1+uu)}}-\frac{t\partial u}{(1+uu)^{\frac{3}{2}}},$$

loco $\partial x\partial y$ per § 27 adipiscimur $\partial t\partial u\left(\frac{t}{\left(1+uu\right)^2}+\frac{tuu}{\left(1+uu\right)^2}\right)=\frac{t\partial t\partial u}{1+uu}$, cuius duplex integrale ita exprimatur

$$\int\frac{\partial u}{1+uu}\int t\partial t$$

Iam vero est $\int t\partial t=\frac{1}{2}tt\ =\frac{1}{2}cc$ et area tota erit $\frac{1}{2}cc\int\frac{\partial u}{1+uu}$, dum ipsi *u* omnes valores possibiles tribuuntur, quandoquidem *u* non amplius aequationem pro basi afficiebat.

§. 31. Quo hunc usum clarius explicemus, consideremus iterum sphaeram centrum C et radium $\mathrm{CA}=a$ habentem, cuius portio basi circulari perpendiculariter insistens quaeri debeat. Quia radium CA per centrum huius circuli G ducere licet, sit $\mathrm{FG}=g=0$, ut fiat $cc=(f-x)^2+yy$ et solidum quaesitum $=\int \partial x\partial y(aa-xx-yy)$; statuatur iam

$$x=\frac{t}{\sqrt{(1+uu)}} \text{ et } y=\frac{tu}{\sqrt{(1+uu)}},$$

ut fiat $xx+yy=tt$ et $\sqrt{(aa-xx-yy)}=\sqrt{(aa-tt)}$ et pro $\partial x\partial y$ prodeat $\frac{t\partial t\partial u}{1+uu}$, ita ut soliditas quaesita ita exprimatur $\iint\frac{t\partial t\partial u\sqrt{(aa-tt)}}{1+uu}$, quae integrationes determinari debebunt ex aequatione hinc pro figura basis oriunda

$cc=ff-\frac{2ft}{\sqrt{(1+uu)}}+tt$, unde fit

$$\text{vel } t=\frac{f\pm\sqrt{(cc+ccuu-ffuu)}}{\sqrt{(1+uu)}} \text{ vel } \sqrt{(1+uu)}=\frac{2ft}{ff-cc+tt}.$$

§. 32. Consideretur primo *t* ut constans fietque integrale

$$=\int t\partial t\sqrt{(aa-tt)}\cdot\text{arctan}.u,$$

ubi constantem adiici non est necesse, quia evanescente *u* simul *y* evanescit; quaeramus enim primo solidum semicirculo insistens. At integrali hoc primo extenso ad terminum extremum ob $\text{arctan}.u=\text{arccos}.\frac{1}{\sqrt{(1+uu)}}$ fit id

$$=\int t\partial t\sqrt{(aa-tt)}\cdot\text{arccos}.\frac{ff-cc+tt}{2ft},$$

cuius integrationis limites sunt $t=f-c$ et $t=f+c$. Si non soliditatem huius portionis sphaerae, sed eius superficiem basi quasi imminentem definire voluissemus, perventuri fuissemus ad hanc formulam

$$\int\frac{a\partial t}{\sqrt{(aa-tt)}}\text{arccos}.\frac{ff-cc+tt}{2ft};$$

at operae pretium non videtur eius integrationem fusius prosequi.

§. 33. Methodus autem huiusmodi formulas integrales duplicatas tractandi haud parum illustrabitur, si eam ad problema illud quondam famosum Florentinum accommodemus, quo in superficie sphaerica portio geometrice assignabilis requirebatur, cuius superficies algebraice exprimi possit. Immineat talis sphaerae portio curvae GRH (Fig. 4), cuius propterea figura est determinanda; in qua si ponatur $\mathrm{CP}=x$, $\mathrm{PR}=y$, superficies sphaerae imminens hac formula integrali duplicata exprimitur $\iint \frac{a\partial x\partial y}{\sqrt{(aa-xx-yy)}}$. Iam nulla substitutione adhibita si primo x pro constante habeatur, prodibit

$$\int a\partial x \operatorname{arcsin.}\frac{y}{\sqrt{(aa-xx)}},$$

qua portio sphaerae aream indefinitam CPRG tegens exprimitur, et quaestio nunc huc redit, ut eiusmodi aequatio algebraica inter x et y assignetur, unde pro tota area CHRG portio superficiei sphaericae ei respondentis fiat algebraice assignabilis.

§. 34. Ponamus brevitatis gratia $\frac{y}{\sqrt{(aa-xx)}}=v$, ut sit $y=v\sqrt{(aa-xx)}$ ac posito $x=0$ fiat $v=n$; quoniam superius integrale evanescere debet posito $x=0$, erit ergo superficies sphaerica aream indefinitam CPRG tegens

$$=ax\operatorname{arcsin}.v-a\int\frac{x\partial v}{\sqrt{(1-vv)}}$$

sumto hoc integrali ita, ut evanescat posito $x=0$. Statuatur nunc

$$\int\frac{x\partial v}{\sqrt{(1-vv)}}=f\operatorname{arcsin}.v-a\mathrm{V}$$

denotante V functionem quamcunque algebraicam ipsius v, quae abeat in N posito $x=0$, eritque superficies nostra

$$=ax\operatorname{arcsin}.v-af\operatorname{arcsin}.v+aaV+af\operatorname{arcsin}.n-aaN$$

atque x per v ita determinabitur, ut sit

$$x=f-\frac{a\partial\mathrm{V}\sqrt{(1-vv)}}{\partial v};$$

sit iam $\mathrm{CH}=h$ ac ponatur $x=h$, quo casu fiat $v=m$ et $\mathrm{V}=\mathrm{M}$, et cum superficies proposita sit

$$ah\operatorname{arcsin}.m-af\operatorname{arcsin}.m+aa\mathrm{M}+af\operatorname{arcsin}.n-aa\mathrm{N},$$

ea algebraica esse nequit, nisi sit

$$h\mathrm{A}\sin.m-f\mathrm{A}\sin.m+fA\sin.n=0.$$

§. 35. Hic igitur primo arcus, quorum sinus sunt m et n, inter se commensurabiles reddi debent, nisi forte sit $n=0$, quo casu sufficit fieri $h=f$. Quod etsi facile infinitis modis praestari potest, tamen hoc problema multo facilius adhibendis substitutionibus ante expositis resolvetur. Ponatur ergo

$$x = \frac{t}{\sqrt{(1+uu)}} \text{ et } y = \frac{tu}{\sqrt{(1+uu)}},$$

ut fiat $xx + yy = tt$ et pro $\partial x \partial y$ prodeat $\frac{t\partial t\partial u}{1+uu}$, atque superficies portionis sphaericae hac formula integrali duplicata exprimetur $\iint \frac{at\partial t\partial u}{(1+uu)\sqrt{(aa-tt)}}$. Sumatur primo *u* constans; erit ea $\int \frac{a\partial u}{1+uu}\left(b - \sqrt{(aa-tt)}\right)$, quae iam facile absolute integrabilis reddi potest; ponatur enim aequalis functioni algebraicae cuicunque ipsius *u*, quae sit $= \mathrm{V}$, eritque
$b - \sqrt{(aa-tt)} = \frac{\partial \mathrm{V}(1+uu)}{a\partial u}$ et portio superficiei sphaericae adeo indefinita erit V, ubi pro V functionem algebraicam quamcunque ipsius *u* accipere licet.

§. 36. Simplicissimae solutiones deducentur ex hac hypothesi

$$\mathrm{V} = \frac{a(\alpha+\beta u)}{\sqrt{(1+uu)}}$$

unde fit $\frac{\partial \mathrm{V}}{a\partial u} = \frac{-\alpha u+\beta}{(1+uu)^{\frac{3}{2}}}$ hincque $b - \sqrt{(aa-tt)} = \frac{\beta-\alpha u}{\sqrt{1+uu}}$. Ponatur $b = 0$, et cum per substitutiones sit $u = \frac{y}{x}$ et $t = \sqrt{(xx+yy)}$, erit pro curva quaesita

$$\sqrt{(xx+yy)}\,(aa - xx - yy) = \alpha y - \beta x$$

et pro superficie

$$\mathrm{V} = \frac{a(\alpha x+\beta y)}{\sqrt{(xx+yy)}}.$$

Hinc casus simplicissimus oritur ponendo $\beta = 0$ et $\alpha = a$, unde prodit
$aaxx - (xx+yy)^2 = 0$ seu $yy = ax - xx$, ita ut curva GRH sit circulus diametro AC descriptus et $\mathrm{V} = \frac{aax}{\sqrt{(xx+yy)}}$. Infiniti alii circuli diametrum $= a$ habentes ac per centrum sphaerae transeuntes reperiuntur, si sit $\beta = \sqrt{(aa-\alpha\alpha)}$,
unde fit

$$ax + y\sqrt{(aa-\alpha\alpha)} = xx+yy \text{ et } V = \frac{a\left(\alpha x+y\sqrt{(aa-\alpha\alpha)}\right)}{\sqrt{(xx+yy)}} = a\sqrt{(xx+yy)},$$

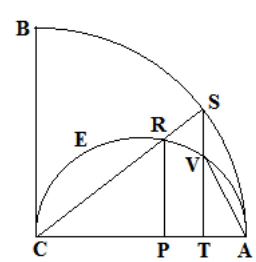

Fig. 7

ubi notandum est quantitatem *V* pro natura rei constantem quandam assumere.

§. 37. Concipiatur ergo octans sphaerae super quadrante ACB (Fig. 7) extractus, cuius radius $\mathrm{CA}=a$, qui simul sit diameter semicirculi CRA*;* in quo si ducatur corda quaecunque CR et perpendiculum RP, ut sit $\mathrm{CP}=x$ et $\mathrm{PR}=y$, erit $\mathrm{CR}=t$ et u erit tangens anguli ACR. Quoniam igitur posuimus $b=0$, prius integrale, quo u erat constans, est $\sqrt{(aa-tt)}$; quod cum evanescat, si $t=a$, evidens est id non per cordam $\mathrm{CR}=t$, sed per eius complementum RS extendi. Hinc repetita integratio $\int\frac{adu}{1+uu}\sqrt{(aa-tt)}$ eam sphaericae superficiei portionem exprimit, quae trilineo RVAS imminet, quae ergo ob $\sqrt{(aa-tt)}=\frac{au}{\sqrt{(1+uu)}}$ est $=\frac{-aa}{\sqrt{(1+uu)}}+aa,$ integrali scilicet ita sumto, ut evanescat cum angulo ACR. Quare ob

$$\frac{1}{\sqrt{(1+uu)}}=\cos.\mathrm{ACR}$$

ducto perpendiculo ST erit illa superficies $a(a-\mathrm{CT})=\mathrm{CA}\cdot\mathrm{AT}=\mathrm{AV}^2$ ducta corda *A V*. Consequenter portio superficiei sphaerae spatia CERASB inter quadrantem et semicirculum intercepto imminens aequatur quadrato radii sphaerae.

§. 38. Contemplemur autem adhuc eiusmodi casum, quo prima integratio evanescat posito $t=0$, seu sit $b=a$ ac ponatur $\mathrm{V}=\frac{1}{2}aau$, quae expressio simul superficiem quaesitam praebet. Erit ergo

$$a-\sqrt{(aa-tt)}=\tfrac{1}{2}a(1+uu) \text{ et } \sqrt{(aa-tt)}=\tfrac{1}{2}a(1-uu),$$

ita ut sit

$$t=\tfrac{1}{2}a\sqrt{\left(3+2uu-u^4\right)} \text{ seu } t=\tfrac{1}{2}a\sqrt{\left((1+uu)(3-uu)\right)},$$

ubi est $\mathrm{CR}=t$ (Fig. 8) et u denotat tangentem anguli ACR. Ex hac aequatione patet, si sit u = 0, fore $t=\frac{a\sqrt{3}}{2}$; scilicet curva quaesita radio AC ita in E occurrit, ut sit $\mathrm{CE}=\mathrm{CA}\cdot\frac{\sqrt{3}}{2}$, eique perpendiculariter insistit. Tum si angulus ACR augeatur ad semirectum ACF, ut fiat $u=1$, erit $t=a$ hocque casu curva per ipsum punctum *F* transit ibique quadrantem osculabitur; ac simul distantia t fit maxima. Dehinc curva introrsum reflectitur et t evanescit, si $u=\sqrt{3}$*;* hoc est, curva centra C ita immergitur ut eius tangens in C cum radio CA faciat angulum 60°.

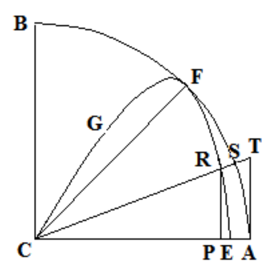

Fig. 8

§. 39. Tota ergo curva in quadrante descripta figuram habebit ERFGC et ducta in ea ex C recta utcunque CR angulique ECR tangens sit $=u$; tum portio superficiei sphaericae sectori ECR imminens algebraice poterit assignari eritque ea $=\frac{1}{2}aau$. Quare si CR ad occursum cum tangente AT producatur, ob $\mathrm{AT} = au$ ea portio praecise aequabitur triangulo CAT et portio imminens sectori ECF erit $= \frac{1}{2}aa$; si autem angulus ECR maior semirecto sumatur, ut sit $u > 1$, quia tum $\sqrt{(aa-tt)} = \sqrt{(aa-xx-yy)}$, quae est elevatio superficiei sphaericae supra quadrantem, fit negativa, superficies in inferiori octante capi debet. Quodsi huius curvae aequationem inter coordinatas $\mathrm{CP} = x$ et $\mathrm{PR} = y$ desideremus, ob $tt = xx+yy$ et $u = \frac{y}{x}$ habebimus

$$4xx+4yy = aa\left(3+\frac{2yy}{xx}-\frac{y^4}{x^4}\right) = \frac{aa(xx+yy)(3xx-yy)}{x^4},$$

quae divisa per $xx+yy$ praebet

$$4x^4 = 3aaxx - aayy \quad \text{seu} \quad yy = 3xx - \frac{4x^4}{aa}.$$

§. 40. Hanc solutionem reddere possumus generaliorem ponendo $\mathrm{V} = abu$ fietque $a-\sqrt{(aa-tt)} = b(1+uu)$, hinc $\sqrt{(aa-tt)} = a-b-buu$, ergo

$$tt = 2ab-bb+2(a-b)buu-bbu^4 = (1+uu)(2ab-bb-bbuu).$$

Qua ad coordinatas orthogonales translata divisio per $xx+yy$ iterum succedet fietque

$$x^4 = (2ab-bb)xx-bbyy \quad \text{seu} \quad y = \frac{x}{b}\sqrt{(2ab-bb-xx)}$$

ac portio superficiei sphaericae sectori ECR huius curvae imminens erit $= \frac{aby}{x} = b\cdot\mathrm{AT}$; quae expressio locum habet, quamdiu $uu < \frac{a-b}{b}$, hoc est, donec anguli ECR tangens fiat $= \sqrt{\frac{a-b}{b}}$, ubi fit $t = a$. Tum vero angulo ECR ultra aucto perpendiculares super curva erectae ad hemisphaerium inferius protendi debent, quo casu superficies eo magis augetur. Si ergo sit $b = a$, quia $\sqrt{(aa-tt)}$ ubique fit quantitas negativa, quantitas $b\cdot\mathrm{AT}$ portionem sphaericae superficiei ad inferius hemisphaerium continuatae exprimit.

§. 41. Sit adhuc $b = a$ ac ponatur $\mathrm{V} = \frac{a^2(\alpha+\beta u)}{\sqrt{(1+uu)}} - \alpha a^2$, ut superficies assignanda evanescat posito $u = 0$, eritque

$$a-\sqrt{(aa-tt)} = \frac{a(\beta-\alpha u)}{\sqrt{(1+uu)}} \quad \text{et} \quad \sqrt{(aa-tt)} = a - \frac{a(\beta-\alpha u)}{\sqrt{(1+uu)}},$$

ubi notandum est, si haec expressio fiat negativa, ibi in hemisphaerium inferius descendi. Ex his autem prodit

$$\frac{tt}{aa} = \frac{2(\beta-\alpha u)}{\sqrt{(1+uu)}} - \frac{(\beta-\alpha u)^2}{(1+uu)^2}.$$

Quare evanescente angulo ECR, cuius tangens $=u$, erit $\frac{tt}{aa}=2\beta-\beta\beta$, at si $u=\frac{\beta}{\alpha}$, evanescit *t*. Pro altera parte axis CA fit *u* negativum ac posito $u=-v$ habetur superficies negative expressa $\mathrm{V}=\frac{a^2(\alpha-\beta v)}{\sqrt{(1+vv)}}-\alpha a^2$ et curva hac definietur aequatione

$$\frac{tt}{aa}=\frac{2(\beta+\alpha v)}{\sqrt{(1+vv)}}-\frac{(\beta+\alpha v)^2}{1+vv},$$

unde posito *v* infinito prodit $\frac{tt}{aa}=2\alpha-\alpha\alpha$; ubi recta CR fit in curvam normalis, quod etiam evenit, ubi $v=\frac{\alpha}{\beta}$ et $\frac{tt}{aa}=2\sqrt{(\alpha\alpha+\beta\beta)}-\alpha\alpha-\beta\beta$.
Quare ne fiat *t* imaginarium, oportet sit $\sqrt{(\alpha\alpha+\beta\beta)}<2$.

§. 42 . Consideremus casum, quo $\alpha=-\frac{1}{\sqrt{2}}$ et $\beta=\frac{1}{\sqrt{2}}$ ut sit superficies

$$\mathrm{V}=aa\left(\frac{1}{\sqrt{2}}-\frac{1-u}{\sqrt{2}(1+uu)}\right) \text{ et } \frac{tt}{aa}=\frac{2(1+u)}{\sqrt{2}(1+uu)}-\frac{(1+u)^2}{2(1+uu)},$$

ubi patet, si $u=-1$, fore $t=0$; tum vero, ut sequitur,
si $u=0$, si $u=1$, si $u=7$, si $u=\infty$,
erit

$$t=a\sqrt{\frac{2\sqrt{2}-1}{2}},\qquad t=a,\qquad t=a\sqrt{\frac{24}{25}},\qquad t=a\sqrt{\frac{2\sqrt{2}-1}{2}}$$

ubi notandum casibus $u=1$ et $u=\infty$ rectam CR fore in curvam normalem. In hoc ergo quadrante curva nostra fere cum quadrante confunditur, cum ubique sit proxime $t=a$, cui portio superficiei sphaericae imminens erit $=aa\sqrt{2}$, quae deficit a superficie totius octantis, quae est $\frac{\pi}{2}aa$, parte satis parva $aa\left(\frac{\pi}{2}-\sqrt{2}\right)=0{,}15658aa$. Ad alteram axis CA partem haec curva in centrum incidit, ubi tangens cum CA faciet angulum semirectum.

§. 43. Verum solutio § 35 data multo magis amplificari potest; cum enim superficies sphaerae assignanda hac formula exprimatur $\int\frac{a\partial u}{(1+uu)}\int\frac{t\partial t}{\sqrt{(aa-tt)}}$ et in integratione $\int\frac{t\partial t}{\sqrt{(aa-tt)}}$ quantitas *u* ut constans consideretur, integrale ita exhiberi poterit $\mathrm{U}-\sqrt{(aa-tt)}$ denotante U functionem quamcunque ipsius *u;* quae formula quoniam evanescit, si $\sqrt{(aa-tt)}=\mathrm{U}$ et $t=(aa-UU)$, ab hoc termino quantitas *t* ulterius protendi est concipienda. Denotet iam V aliam quamcunque functionem ipsius *u*, quae abeat in C posito $u=0$, ac ponatur superficies

$$\int \frac{a\partial u}{1+uu}\cdot\left(\mathrm{U}-\sqrt{(aa-tt)}\right)=a\mathrm{V}-a\mathrm{C}$$

eritque hinc

$$\mathrm{U}-\sqrt{(aa-tt)}=\frac{\partial \mathrm{V}(1+uu)}{\partial u}$$

ideoque

$$\sqrt{(aa-tt)}=\mathrm{U}-\frac{\partial \mathrm{V}(1+uu)}{\partial u},$$

unde alter terminus ipsius *t* definitur.

§. 44. Hinc igitur solutio problematis Florentini ita generalissime adornabitur. Constituto quadrante circuli ACB (Fig. 9), cui octans sphaerae insistat, radio *CA* existente $=a$, ductoque radio quocunque CS vocetur anguli ACS tangens *u*; tum primo curva EQG ita construatur, ut sit

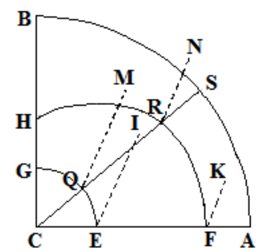

Fig. 9

$$\mathrm{CQ}=\sqrt{(aa-\mathrm{UU})}$$

et perpendiculum ex Q ad sphaericam usque superficiem erectum $\mathrm{QM}=\mathrm{U}$ denotante U functionem quamcunque algebraicam ipsius *u*. Si $u=0$, abeat CQ in CE et QM in EI. Deinde alia describatur curva FRH, ut sit

$$\mathrm{CR}=\sqrt{\left(aa-\left(\mathrm{U}-\frac{\partial \mathrm{V}(1+uu)}{\partial u}\right)^2\right)}$$

et perpendiculum ex R ad sphaeram usque pertingens

$$\mathrm{RN}=\mathrm{U}-\frac{\partial \mathrm{V}(1+uu)}{\partial u}$$

denotante V aliam quamcunque functionem algebraicam ipsius *u*, quae abeat in C, si $u=0$; quo casu simul CR in CF et RN in FK abeat. Iam his duabus curvis constructis portio superficiei sphaericae areae EQRF imminens et intra terminos I, K, M, N contenta algebraice exprimetur eritque $=a(\mathrm{V}-\mathrm{C})$.

§. 45. Haec de natura formularum integralium duplicatarum commentandi occasionem praebuit problema aeque elegans atque utile in Analysi, si quidem eius solutionem evolvere liceret. Quaerebatur scilicet inter omnia corpora eiusdem soliditatis id, quod minima superficie contineretur, quod quidem ad ternas coordinatas orthogonales *x*, *y* et *z* relatum posito $dz=pdx+qdy$ ita analytice exprimitur, ut inter omnes relationes harum trium variabilium, quae eandem quantitatem huius formulae integralis duplicatae $\iint z\partial x\partial y$ contineant, ea definiatur, cui minima quantitas huius $\iint \partial x\partial y\sqrt{(1+pp+qq)}$ respondeat. Quod problema si per theoriain variationum aggrediamur, effici oportebit, ut fiat

$$a\delta\iint \partial x\partial y\sqrt{(1+pp+qq)}=\delta\iint z\partial x\partial y\,,$$

ita ut totum negotium ad variationes huiusmodi formularum integralium duplicatarum indagandas reducatur.

§. 46. Quoniam utraque formula duplicem integrationem exigit, si in priori *x* pro constante habeatur, nostra aequatio ita repraesentabitur

$$a\delta\int \partial x\int \partial y\sqrt{(1+pp+qq)}=\delta\int \partial x\int z\partial y.$$

Verum hic probe animadvertendum est, postquam integralia

$$\int \partial y\sqrt{(1+pp+qq)} \text{ et } \int z\partial y$$

fuerint inventa, tum variabilem *y* non amplius indefinitam seu ab *x* non pendentem relinqui, quin potius pro *y* certam functionem ipsius *x*, quam figura corporis exigit, substitui oportere, ita ut in secunda integratione quantitas *y* non ut constans seu ab *x* non pendens spectari queat. Quia autem ob figuram corporis etiamnunc incognitam ista functio non constat, neutiquam apparet, quomodo variationes istiusmodi formularum duplicatarum determinari debeant.

§.47. Ipsa vero huius quaestionis natura alias praeterea determinationes requirere videtur, quarum ratio in solutione haberi debeat. Nam quemadmodum, si curva quaeritur, quae inter omnes alias eandem aream includentes brevissimo arcu contineatur, non solum basis AP (Fig. 10), sed etiam duo puncta B et M, per quae curva transeat, praescribi solent, ita etiam in nostro problemate non modo basis, cui corpus tanquam columna insistat, pro cognita assumi debere videtur, sed etiam ipsi extremi termini superficiei quaesitae. Quodsi enim hae res non praescribantur omnes, ne quaestioni quidem certae locus relinquitur; nam, etiamsi basis praescriberetur, termini vero supremi superficiei arbitrio nostro relinquerentur, manifestum est, quo altior fuerit columna, eo magis soliditatem auctum iri eadem manente superficie suprema, quandoquidem superficies laterum non in computum ducitur. Multo minus autem problema sine basis praescriptione ullam vim retineret, quoniam basi coarctanda quantumvis magna soliditas cum minima superficie posset esse coniuncta.

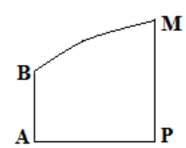

Fig. 10