7). A Method of finding Integral Formulas which in certain cases maintain a given Ratio amongst themselves; where likewise a Method is treated of summing Continued Fractions.

Opuscula analytica II, 1785, p. 178-216; [E594].

§.196. Just as in recurring series where some term may be determined from one or more preceding terms according to a certain constant law, thus here I am going to consider a series of this kind, in which some term may be determine from one or more preceding terms following some variable law. But since in such series the general formula expressing the individual terms is not algebraic but transcending, it will be appropriate to show the individual terms by integral formulas ; which, so that the values may be determined, I assume the determined value to be attributed after the integration of the variable quantity, thus so that the individual terms may produce determined values ; and now the equation of the principles here is returned, just as these same integral formulas must be prepared, so that any term may be determined according to a given rule from one or several preceding terms.

§.197. So that these ideas may be examined more clearly, we will consider the most noteworthy series of these integral formulas :

$$\int \frac{\partial x}{\sqrt{(1-xx)}},\ \int \frac{xx\partial x}{\sqrt{(1-xx)}},\ \int \frac{x^4\partial x}{\sqrt{(1-xx)}},\ \int \frac{x^6\partial x}{\sqrt{(1-xx)}},\ \text{etc.;}$$

which if the individual members thus may be integrated, so that they vanish on putting $x=0$, then truly the value 1 may be attributed to the variable *x*, any term thus will depend on the preceding term, to that there shall become :

$$\int \frac{xx\partial x}{\sqrt{(1-xx)}} = \frac{1}{2}\int \frac{\partial x}{\sqrt{(1-xx)}},$$

$$\int \frac{x^4\partial x}{\sqrt{(1-xx)}} = \frac{3}{4}\int \frac{xx\partial x}{\sqrt{(1-xx)}},$$

$$\int \frac{x^6\partial x}{\sqrt{(1-xx)}} = \frac{5}{6}\int \frac{x^4\partial x}{\sqrt{(1-xx)}},$$

and in general

$$\int \frac{x^n\partial x}{\sqrt{(1-xx)}} = \frac{n-1}{n}\int \frac{x^{n-2}\partial x}{\sqrt{(1-xx)}}.$$

From which it appears this general formula can be seen as the general term of that series, and any term to arise from the preceding one, if that may be multiplied by $\frac{n-1}{n}$.

§.198. According to the likeness of this case, we may therefore put in place a series of integral formulas in general :

$$\int \partial v,\ \int x\partial v,\ \int xx\partial v,\ \int x^3\partial v,\ \int x^4\partial v,\ \text{etc.},$$

thus so that the term corresponding to the index *n* shall be :

$$\int x^{n-1}\partial v,$$

which individual integrals thus we may assume to be accepted, so that they may vanish on putting $x = 0$; but after the integration we may attribute some constant value to the variable *x*, such as $x = 1$ or some other number. With which in place the question here is returned, a function of *x* must be assumed for *v* of such a kind shall be determined , so that any term may be determined by one or two or more preceding terms according to some given law, depending in some manner on the variable or on the index *n* ; where indeed with that expressed, so that the index *n* will be expressed with respect to that, the index may ascent in the scale of the proposed relation by some number of dimensions ; but generally there will not be a need to rise beyond the first dimension. Therefore we may hence treat the following problems.

PROBLEM 1

§.199. *To find a function v, so that a relation may itself be found between these same two terms succeeding each other*

$$\int x^n\partial v = \tfrac{\alpha n+a}{\beta n+b}\int x^{n-1}\partial v.$$

Therefore it is required here, that there shall be

$$(\alpha n+a)\int x^{n-1}\partial v = (\beta n+b)\int x^n\partial v,$$

clearly if a certain value may be given to the variable *x* after the integration. Therefore since at last that same condition must be found, after that same constant value were given to the variable *x*, we may put in general, while *x* is variable, this equation can be found:

$$(\alpha n+a)\int x^{n-1}\partial v = (\beta n+b)\int x^n\partial v + \mathrm{V},$$

but the quantity V is required to be prepared thus, so that it may vanish, after the value for that variable determined were assigned. Truly besides, because we assume both integrals to be taken thus, so that they vanish on putting $x=0$, it is necessary, that also this same quantity V shall vanish in the same case also.

§.200. Since this equality must remain for all indices *n*, which we may regard as positive always, it is easily understood that same quantity V must have a factor x^n ; with which agreed upon now for that same condition to be satisfied, so that on putting $x=0$, there shall become also $\mathrm{V}=0$. On account of which we may put $\mathrm{V}=x^n\mathrm{Q}$, where Q may denote a function of *x* adapted for the proposition, and so that we may wish to prepare likewise thus, so that it may vanish if a certain value may be given to *x* itself.

§.201. Therefore since there must become

$$(\alpha n+a)\int x^{n-1}\partial v=(\beta n+b)\int x^n\partial v+x^n\mathrm{Q},$$

this same equation may be differentiated and with the differential divided by x^{n-1}, the differential equation will be come upon :

$$(\alpha n+a)\partial v=(\beta n+b)x\partial v+n\mathrm{Q}\partial x+x\partial\mathrm{Q},$$

which since the equation must remain true for all the values of *n*, the terms associated with that letter must be taken separately, from which we come upon these two equations :

$$\text{I. } (\alpha-\beta x)\partial v=\mathrm{Q}\partial x \text{ and II. } (a-bx)\partial v=x\partial\mathrm{Q}.$$

From the first there shall become $\partial v=\frac{\mathrm{Q}\partial x}{\alpha-\beta x}$, from the second truly $\partial v=\frac{x\partial\mathrm{Q}}{a-bx}$, which two values equated to each other provide this equation $\frac{\partial\mathrm{Q}}{\mathrm{Q}}=\frac{\partial x}{x}\cdot\frac{a-bx}{\alpha-\beta x}$, which equation is resolved into these parts :

$$\frac{\partial\mathrm{Q}}{\mathrm{Q}}=\frac{a}{\alpha}\cdot\frac{\partial x}{x}+\frac{a\beta-b\alpha}{\alpha}\cdot\frac{\partial x}{\alpha-\beta x},$$

the integral of which therefore will be

$$l\mathrm{Q}=\frac{a}{\alpha}\cdot lx-\frac{a\beta-b\alpha}{\alpha\beta}l(\alpha-\beta x),$$

from which it is deduced:

$$\mathrm{Q}=\mathrm{C}x^{\frac{a}{\alpha}}(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}}.$$

§.202. From this value found for Q it is at once apparent to vanish in the case $x=\frac{\alpha}{\beta}$, but only if there were $\frac{b\alpha-a\beta}{\alpha\beta}>0$; but if it may eventuate otherwise, it is not apparent, how this quantity may be able to vanish in any case. However with this value found for Q, thence there will be found :

$$\partial v = \mathrm{C}x^{\frac{a}{\alpha}}\partial x\left(\alpha-\beta x\right)^{\frac{b\alpha-a\beta}{\alpha\beta}-1}$$

and hence the term of our series corresponding to the index *n* will become :

$$\int x^{n-1}\partial v = \mathrm{C}\int x^{n+\frac{a}{\alpha}-1}\partial x\left(\alpha-\beta x\right)^{\frac{b\alpha-a\beta}{\alpha\beta}-1},$$

then truly there will become

$$\mathrm{V} = \mathrm{C}x^{n+\frac{a}{\alpha}}\left(\alpha-\beta x\right)^{\frac{b\alpha-a\beta}{\alpha\beta}}.$$

Where the matter therefore is reduced in the first place, so that the quantity itself besides the case $x=0$ will vanish in addition for another case.

COROLLARY 1

§.203. Here two cases occur, which demand special expansions ; the first is, where $\alpha=0$; but then it will be required to begin from the equation $\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{(a-bx)\partial x}{\beta xx}$, from which by integrating there will be elicited $l\mathrm{Q}=\frac{a}{\beta x}+\frac{b}{\beta}lx,$ and hence by taking *e* for the number, of which the hyperbolic logarithm $=1$, there will be deduced :

$$\mathrm{Q}=e^{\frac{a}{\beta x}}x^{\frac{b}{\beta}},$$

which formula cannot be reduced to nothing, unless there may become $\frac{a}{\beta x}=-\infty$ and thus $x=0$, and thus two cases will not be obtained, in which there may become $\mathrm{V}=0$, when still two will be desired. But meanwhile hence there will become :

$$\partial v = \frac{e^{\frac{a}{\beta x}}x^{\frac{b}{\beta}}\partial x}{-\beta x}$$

COROLLARY 2

§.204. But the other case demanding a special integration will be $\beta=0$; but then there will be $\frac{\partial \mathrm{Q}}{\mathrm{Q}}=\frac{\partial x(a-bx)}{\alpha x},$ from which here becomes $l\mathrm{Q}=\frac{a}{\alpha}lx-\frac{bx}{\alpha}$ and thus $\mathrm{Q}=e^{\frac{a}{\alpha}}x^{\frac{-bx}{\alpha}}$, which formula vanishes in the case $x=\infty$, but only if $\frac{b}{\alpha}$ were a positive number ; but if $\frac{b}{\alpha}$ were a negative number, then Q vanishes in the case $x=-\infty$. Again truly in this case there will become

$$\partial v=\frac{x^{\frac{a}{\alpha}}e^{\frac{-bx}{\alpha}}\partial x}{\alpha}.$$

SCHOLIUM

§.205. Generally, from these observations we may set out some special cases, for which we will attribute certain values to the letters $\alpha,\ \beta$ and a, b, which shall lead now to well-known cases.

EXAMPLE 1

§.206. *The integral formulas may be sought, so that there may become :*

$$\int x^n\partial v=\frac{2n-1}{2n}\int x^{n-1}\partial v.$$

Therefore since here there must be $(2n-1)\int x^{n-1}\partial v=2n\int x^n\partial v$, in this case there will be $\alpha=2$ and $a=-1$, then truly $\beta=2$ and $b=0$; hence there becomes

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{a\partial x}{2x(1-x)}=-\frac{\partial x}{2x}-\frac{\partial x}{2(1-x)},$$

thence on integrating,

$$lQ=-\tfrac{1}{2}lx+\tfrac{1}{2}l(1-x)$$

and thus

$$\mathrm{Q}=\mathrm{C}\sqrt{\tfrac{1-x}{x}},\ \text{therefore}\ \mathrm{V}=\mathrm{C}x^n\sqrt{\tfrac{1-x}{x}}$$

Again since here there shall be $\partial v=\frac{\mathrm{Q}\partial x}{2(1-x)}$, there will be

$$\partial v=\frac{\mathrm{C}\sqrt{\frac{1-x}{x}}\partial x}{2(1-x)}=\frac{\mathrm{C}\partial x}{2\sqrt{(x-xx)}};$$

therefore on taking $\mathrm{C}=2$ there will be $\partial v=\frac{\partial x}{\sqrt{(x-xx)}}$ and our general formula

$$\int x^{n-1}\partial v=\int\frac{x^{n-1}\partial x}{\sqrt{(x-xx)}};$$

from which since there shall be $\mathrm{V}=x^{n}\sqrt{\frac{1-x}{x}}$, this quantity clearly will vanish on taking $x=1$, thus so that our formula, if after the integration we may put $x=1$, shall satisfy the question.
But if now we may put $x=yy$, that same formula will adopt this form

$$2\int\frac{y^{2n-2}\partial y}{\sqrt{(1-yy)}},$$

which by putting $y=1$ after the integration, gives this relation

$$\int\frac{y^{2n}\partial y}{\sqrt{(1-yy)}}=\frac{2n-1}{2n}\int\frac{y^{2n-2}\partial y}{\sqrt{(1-yy)}},$$

which contains the above-mentioned relations (§.197) ; hence indeed there becomes

$$\int\frac{yy\partial y}{\sqrt{(1-yy)}}=\frac{1}{2}\int\frac{\partial y}{\sqrt{(1-yy)}},$$

$$\int\frac{y^{4}\partial y}{\sqrt{(1-yy)}}=\frac{3}{4}\int\frac{yy\partial y}{\sqrt{(1-yy)}},$$

and

$$\int\frac{y^{6}\partial y}{\sqrt{(1-yy)}}=\frac{5}{6}\int\frac{y^{4}\partial y}{\sqrt{(1-yy)}}.$$

EXAMPLE 2

§.207. *The integral formulas may be sought, so that there may become :*

$$\int x^{n}\partial v=\frac{\alpha n-1}{\alpha n}\int x^{n-1}\partial v.$$

Therefore since here there must become $(\alpha n-1)\int x^{n-1}\partial v=\alpha n\int x^{n}\partial v$, in this case there will be $a=-1,\ \beta=\alpha$ and $b=0$, from which by the formulas given above [$\mathrm{Q}=\mathrm{C}x^{\frac{a}{\alpha}}(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}}$;$\partial v=\mathrm{C}x^{\frac{a}{\alpha}}\partial x(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}-1}$.] there is deduced :

$$\mathrm{Q}=\mathrm{C}x^{\frac{-1}{\alpha}}(\alpha-\alpha x)^{\frac{\alpha}{\alpha^{2}}}=\mathrm{C}x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}},$$

which quantity evidently vanishes on putting $x=1$. But then there will become :

$$\partial v=\left[Cx^{\frac{a}{\alpha}}\partial x(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}-1}\right]=\frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{a}}\partial x}{1-x}$$

[We have to assume that the extra constant factor cancels with the value of C in order to become 1.]
from which our general formula will be

$$\partial v=\frac{x^{\frac{-1}{a}}(1-x)^{\frac{+1}{a}}\partial x}{1-x}$$

$$\int x^{n-1}\partial v=\int x^{n-\frac{1}{a}-1}(1-x)^{+\frac{1}{a}-1}\partial x=\int\frac{x^{n-\frac{1}{a}-1}}{(1-x)^{1-\frac{1}{a}}}\partial x,$$

which is reduced more concisely by making $x=y^{\alpha}$; then indeed that will adopt this form :

$$\int\frac{y^{\alpha n-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}},$$

where again after the integration there must be put in place $y=1$. Hence there will become :

$$\int\frac{y^{\alpha n+\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}=\frac{\alpha n-1}{\alpha n}\int\frac{y^{\alpha n-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}},$$

and hence these special cases will arise:

$$\int\frac{y^{2\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}=\frac{\alpha-1}{\alpha}\int\frac{y^{\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}$$

and

$$\int\frac{y^{3\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}=\frac{2\alpha-1}{\alpha}\int\frac{y^{2\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}.$$

§.208. Hence therefore if there may be taken $\alpha=1$, so that there must become

$$\int x^{n}\partial v=\frac{n-1}{n}\int x^{n-1}\partial v,$$

our general formula now expressed in terms of *y* will become $\int y^{n-2}\partial y$,the value of which is $\frac{1}{n-1}y^{n-1}=\frac{1}{n-1}$, from which the whole series of our integral formulas will be changed into this :

$$\tfrac{1}{0},\ \tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \tfrac{1}{7}\ \text{ etc.}$$

§.209. We may take also $\alpha=\frac{1}{2}$ [*i.e.* $a=-1,\ \beta=\alpha=\frac{1}{2}$ and $b=0$] and now there will no further need for *y* to proceed.
Therefore in this case there will be

$$\mathrm{Q}=\left[Cx^{\frac{-1}{\alpha}}\left(1-x\right)^{\frac{+1}{\alpha}}\right]=\frac{(1-x)^2}{xx}\ \text{ and }\ \partial v=\frac{(1-x)\partial x}{xx},$$

from which our general formula shall become

$$\int x^{n-1}\partial v=\int x^{n-3}\left(1-x\right)\partial x,$$

of which the value expressed algebraically will be

$$\tfrac{1}{n-2}x^{n-2}-\tfrac{1}{n-1}x^{n-1}=\frac{1}{(n-1)(n-2)},$$

from which our series of formulas emerges :

$$\tfrac{1}{0\cdot-1},\ \tfrac{1}{0\cdot1},\ \tfrac{1}{1\cdot2},\ \tfrac{1}{2\cdot3},\ \tfrac{1}{3\cdot4},\ \tfrac{1}{4\cdot5},\ \tfrac{1}{5\cdot6},\ \tfrac{1}{6\cdot7}\ \text{ etc.}$$

EXAMPLE 3

§.210. *The integral formulas are sought, so that there shall be*

$$\int x^n\partial v=n\int x^{n-1}\partial v.$$

Therefore since there must become $n\int x^{n-1}\partial v=\int x^n\partial v$, there will be $\alpha=1,\ a=0,\ b=1,\ \beta=0$. Therefore since there shall be $\beta=0$, the case of Corollary 2 will be considered here and thence there will be $\mathrm{Q}=e^{-x}$ and thus $\mathrm{V}=e^{-x}x^n$, which quantity vanishes in these two cases $x=0$ and $x=\infty$. Again truly there will be $\partial v=e^{-x}\partial x$ and hence our general formula will become $\int x^{n-1}e^{-x}\partial x$, from which the terms of the series themselves will be had in the following manner :

$$\int e^{-x}\partial x, \quad \int e^{-x}x\partial x, \quad \int e^{-x}xx\partial x, \quad \int e^{-x}x^3\partial x, \quad \text{etc.}$$

thus with which integrated, so that they may vanish on putting $x=0$, then truly on putting $x=\infty$ the following simple enough series will arise :

$$1,\ 1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4,\ 1\cdot 2\cdot 3\cdot 4\cdot 5 \text{ etc.},$$

which is the hypergeometric series of Wallis, of which the general term is therefore :

$$\int x^{n-1}e^{-x}\partial x = 1\cdot 2\cdot 3\cdot 4\cdots(n-1).$$

[See Wallis, *Arithmeticam infinitorum*; Scholium adjoining Prop. 190.]

§.211. Therefore with the help of this general term, it will be possible to interpolate this series. Thus if the mean is sought between the two first terms, there may be put $n=\frac{3}{2}$ and the value of this term will be $\int e^{-x}\sqrt{x}\partial x$, but the value of which cannot be expressed algebraically in any way. Moreover this term itself is found in a particular way to be itself equal to $\frac{1}{2}\sqrt{\pi}$ with π denoting the periphery of the circle, of which the diameter $=1$, from which in turn we know $\int e^{-x}\sqrt{x}\partial x = \frac{\sqrt{\pi}}{2}$, clearly on putting $x=\infty$ after the integration. But the term preceding this corresponding to the index $\frac{1}{2}$ will be $=\sqrt{\pi}$, to which therefore the formula $\int \frac{e^{-x}\partial x}{\sqrt{x}}$ may be equated. But if here we may put $e^x = y$, thus so that on putting $x=0$, there shall be $y=1$, but on putting $x=\infty$ there shall become $y=\infty$, then therefore this same formula $\int \frac{e^{-x}\partial x}{\sqrt{x}}$ will be changed into this $\int \frac{\partial y}{yy\sqrt{ly}}$, which formula, if thus it may be integrated, so that it may vanish on putting $y=1$, then truly there it may become fiat $y=\infty$, produces the value of $\sqrt{\pi}$. Again, if there may be put $y=\frac{1}{z}$, the terms of the integration will become $z=1$ and $z=0$ and the formula of the integral will be :

$$-\int \frac{\partial z}{\sqrt{-lz}} \begin{bmatrix} \text{from} & z=1 \\ \text{to} & z=0 \end{bmatrix} = \sqrt{\pi},$$

or if with the terms of the integration interchanged, it will become :

$$\int \frac{\partial z}{\sqrt{-lz}} \begin{bmatrix} \text{from} & z=0 \\ \text{to} & z=1 \end{bmatrix} = \sqrt{\pi},$$

just as I have noted previously. [See E19 and E421]

EXAMPLE 4

§.212. *The integral formulas may be sought, so that there shall be*

$$\int x^n \partial v = \tfrac{1}{n}\int x^{n-1}\partial v \;\; or \;\; \int x^{n-1}\partial v = n\int x^n \partial v.$$

Here there is $\alpha = 0$ and $a = 1$, $\beta = 1$ and $b = 0$; which therefore is the case treated in Corollary 1, from which there is deduced to be $\mathrm{Q} = e^{\frac{1}{x}}$ and thus $\mathrm{V} = x^n e^{\frac{1}{x}}$, which formula does not even vanish on taking $x = 0$, since the formula $e^{\frac{1}{0}}$ will be the equivalent to the infinitude of an infinite power. But here I am amazed how it arises, so that in the case $x = -0$ the formula $e^{-\frac{1}{0}}$ may be returned suddenly to be vanishing. Clearly if ω will denote an infinitely small quantity, there will be $e^{\frac{1}{\omega}} = \infty$, then truly at once there will become $e^{-\frac{1}{\omega}} = \frac{1}{\infty} = 0$, which formula for this reason is hence shown not to lie within our scope. Indeed it will be found that $\partial v = -e^{\frac{1}{x}}\frac{\partial x}{x}$, thus so that our general formula is going to become $-\int x^{n-2}\partial x e^{\frac{1}{x}}$, but which cannot be put to any outstanding use by us.

§.213. But if here we may put $\frac{1}{x} = y$, this same general formula will be changed into this $+\int \frac{e^y \partial y}{y^n}$. But truly now there will become $\mathrm{V} = \frac{e^y}{y^n}$, which formula vanishes on putting $y = -\infty$. But in whatever manner we may transform this expression, the same inconvenience will occur always occurs. Yet meanwhile this case also will be able to be resolved in the following manner. Indeed let the first term of the series which we seek be $= \omega$, from which thus so that the terms will proceed according to the following prescribed rule

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & n \\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1\cdot 2}, & \frac{\omega}{1\cdot 2\cdot 3}, & \frac{\omega}{1\cdot 2\cdot 3\cdot 4}, & \cdots\frac{\omega}{1\cdot 2\cdot 3\cdots(n-1)}. \end{array}$$

But above we have seen the value of this series $1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(n-1)$ to be expressed by this integral $\int x^{n-1}e^{-x}\partial x$ with the integration extended from $x = 0$ to $x = \infty$; therefore there is only a need, that we may change this integral formula into a denominator, and the general term of the series which we seek, will be

$$\frac{1}{\int x^{n-1}e^{-x}\partial x},$$

from which it is understood well enough that the matter cannot be set out in a simpler form, which likewise is required to be extended to other cases, in which the quantity V cannot vanish in two cases ; for then there is only a need to invert the fraction $\frac{\alpha n+a}{\beta n+b}$ and to transfer the integral formula into the denominator.

SCHOLIUM

§.214. Unless either $\alpha = 0$ or $\beta = 0$, which cases we have now investigated, the resolution of our problem can always be reduced to the case, in which both the letters α and β are equal to unity. Indeed since there must be

$$\int x^n\partial v = \frac{\alpha n+a}{\beta n+b}\int x^{n-1}\partial v,$$

there may be put $x = \frac{\alpha y}{\beta}$ and there will become

$$\frac{\alpha}{\beta}\int y^n\partial v = \frac{\alpha n+a}{\beta n+b}\int y^{n-1}\partial v,$$

which equation is reduced to this form

$$\int y^n\partial v = \frac{n+a:\alpha}{n+b:\beta}\int y^{n-1}\partial v,$$

But if now in place of $\frac{a}{\alpha}$ we may write a and b in place of $\frac{b}{\beta}$, this formula will be required to be resolved

$$\int y^n\partial v = \frac{n+a}{n+b}\int y^{n-1}\partial v,$$

the resolution of which, if in place of x we may write y and unity in place of the letters α and β, from the above solution in the first place there will be given :

$$\mathrm{Q} = \mathrm{C}y^a(1-y)^{b-a}$$

which therefore vanishes on putting $y = 1$, but only if there were $b > a$; but then there will be this formula

$$\int y^{n-1}\partial v = \mathrm{C}\int y^{n+a-1}\partial y(1-y)^{b-a-1};$$

but if there were $b < a$, this solution, as we have seen, cannot have a place ; truly in this case this form $\frac{1}{\int y^{n-1}\partial v}$ for the limit of our series, thus so that then there must become

$$\frac{1}{\int y^{n}\partial v} = \frac{n+a}{n+b}\cdot\frac{1}{\int y^{n-1}\partial v}$$

or

$$\int y^{n}\partial v = \frac{n+b}{n+a}\cdot\int y^{n-1}\partial v$$

the resolution of which by the interchange of the letters *a* and *b* gives

$$\mathrm{Q} = \mathrm{C}y^{b}\left(1-y\right)^{a-b}$$

which now vanishes in the case $y = 1$, if there were $a > b$; and then the general formula will become :

$$\int y^{n-1}\partial v = \mathrm{C}\int y^{n+b-1}\partial y\left(1-y\right)^{a-b-1}.$$

Therefore whether there shall be $b > a$ or $a > b$, the solution will not be troubled with further difficulty.

§. 215. But if there were either $\alpha = 0$ or $\beta = 0$, also in place of the other one can be written; from which if there should be :

$$\int x^{n}\partial v = \frac{n+a}{b}\cdot\int x^{n-1}\partial v,$$

on account of $\alpha = 1$ and $\beta = 0$, our general solution gives

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}} = \frac{\partial x}{x}\left(a-bx\right),$$

from which there is gathered, $\mathrm{Q} = \mathrm{C}x^{a}e^{-bx}$, which formula vanishes on putting $x = \infty$, but only if *b* were a positive number; moreover then the general term will become

$$\int x^{n-1}\partial v = \mathrm{C}\int x^{n+a-1}\partial x e^{-bx}.$$

But truly the number *b* cannot be negative, because otherwise the prescribed condition shall become incongruous.

§.216. We will consider also the other case, in which $\alpha = 0$ and $\beta = 1$ and thus the prescribed condition :

$$\int x^n \partial v = \frac{a}{n+b} \cdot \int x^{n-1} \partial v$$

from which there becomes :

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}} = \frac{-\partial x}{xx}\left(a - bx\right).$$

But hence the value will arise for Q, which shall not be able to vanish besides the case $x = 0$; for which reason the general formula must be established to become $\frac{1}{\int x^{n-1}\partial v}$, thus so that there must become

$$\int x^n \partial v = \frac{n+b}{a} \int x^{n-1} \partial v\,,$$

from which there arises

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}} = \frac{\partial x}{x}\left(b - ax\right) \quad \text{and thus } \mathrm{Q} = \mathrm{C}e^{-ax}x^b,$$

which expression vanishes on putting $x = \infty$, since *a* by necessity must be a positive number; then also there will become

$$dv = \mathrm{C}e^{-ax}x^b \partial x\,,$$

from which the general formula of the series will be

$$\frac{1}{\mathrm{C}\int x^{n+b-1}\partial x e^{-ax}}\,.$$

PROBLEM 2

§.217. T *will denote a term in the series corresponding to the index n, as we have been required to take into consideration, but truly* T' *shall be the following term and this condition shall be proposed requiring to be fulfilled:*

$$\mathrm{T}' = \frac{(\alpha n+a)(\alpha' n+a')}{(\beta n+b)(\beta' n+b')}\mathrm{T}.$$

SOLUTION

Since here twin values occur, this condition will be satisfied most conveniently, if the general term T may be seen as the product of two factors. Therefore there may be put $\mathrm{T} = \mathrm{RS}$ and the following term shall be $= \mathrm{R'S'}$ and the formulas R and S may be sought, so that there shall become

$$\mathrm{R}' = \frac{\alpha n+a}{\beta n+b}\mathrm{R} \quad \text{et} \quad \mathrm{S}' = \frac{\alpha' n+a'}{\beta' n+b'}\mathrm{S}\,;$$

for then by taking $\mathrm{T} = \mathrm{RS}$ the prescribed condition clearly will be satisfied. Therefore in this manner the formulas may be found for R and S either of this kind $\int x^{n-1}\partial v$, or of the inverse $\frac{1}{\int x^{n-1}\partial v}$, which suffices for the general solution, concerning which we will illustrate by an example.

EXAMPLE

§.218. *The general formula* T *may be sought*, *so that there may become*

$$\mathrm{T}' = \frac{nn-cc}{nn}\mathrm{T}.$$

Therefore we may resolve T into the two factors R and S and put

$$\mathrm{R}' = \frac{n-c}{n}\mathrm{R} \quad \text{and} \quad \mathrm{S}' = \frac{n+c}{n}\mathrm{S}.$$

For the first form, if we may put $\mathrm{R} = \int x^{n-1}\partial v$, from the general solution, where there will become $\alpha = 1,\ a = -c,\ \beta = 1$ and $b = 0$, there will become

$$\mathrm{Q} = \mathrm{C}x^{-c}(1-x)^{c},$$

which form clearly vanishes on putting $x = 1$; and hence since there becomes

$$\mathrm{V} = \mathrm{C}x^{n-c}(1-x)^{c}$$

this form also vanished in the case $x = 0$, but only if *n* were $> c$, that which can be assumed with care, because we assume the exponent *n* to increase successively to infinity and generally only fractions are accustomed to be accepted for *c*. Hence there will be therefore

$$\mathrm{R} = \mathrm{C}\int x^{n-c-1}(1-x)^{c-1}\partial x.$$

§.219. Hence now the other value of the letter *S* may be deduced only by writing $-c$ in place of *c,* but then no longer will there become $Q = 0$ on putting $x = 1$, on account of which it will be required to assume the inverse formula $\frac{1}{\int x^{n-1}dv}$ for S, so that there may become

$$\int x^{n}\partial v = \frac{n}{n+c}\int x^{n-1}\partial v;$$

where since there shall become $\alpha = 1, a = 0, \beta = 1$ and $b = c,$ there will be found

$$\mathrm{Q} = \mathrm{C}(1-x)^{c}$$

which form evidently will become fit $=0$ on putting $x=1$; but hence there will be produced

$$\partial v = C(1-x)^{c-1}\partial x,$$

therefore we will have

$$S = \frac{1}{C\int x^{n-1}(1-x)^{c-1}\partial x};$$

consequently our general formula sought will be

$$T = \frac{\int x^{n-c-1}(1-x)^{c-1}\partial x}{\int x^{n-1}(1-x)^{c-1}\partial x}$$

§.220. But therefore if we may put the first term of our series formed by the preceding factors $=A$, that same series will become:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ A, & \frac{1-cc}{1}A, & \frac{1-cc}{1}\cdot\frac{4-cc}{4}A, & \frac{1-cc}{1}\cdot\frac{4-cc}{4}\cdot\frac{9-cc}{9}A \quad \text{etc.;} \end{array}$$

from which if we may assume $c=\frac{1}{2}$, there will be this series :

$$A, \quad \frac{1\cdot 3}{2\cdot 2}A, \quad \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}A, \quad \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}A \quad \text{etc.;}$$

of which the term corresponding to the index *n* is

$$\frac{\int x^{n-\frac{3}{2}}(1-x)^{-\frac{1}{2}}\partial x}{\int x^{n-1}(1-x)^{-\frac{1}{2}}\partial x},$$

which on putting $x=yy$ will change into this form :

$$\frac{\int y^{2n-2}(1-y)^{-\frac{1}{2}}\partial y}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}\partial y},$$

from which it is apparent the first term becomes

$$A = \int\frac{\partial y}{\sqrt{(1-yy)}} : \int\frac{y\partial y}{\sqrt{(1-yy)}} = \frac{\pi}{2},$$

clearly, on putting $y=1$, after the integration .

PROBLEM 3

§.221. T *shall denote the term of the series corresponding to the index n , and* T' *and* T" *the following terms for the indices* $n+1$ *and* $n+2$; *if such a relation may be proposed between the three terms following each other, so that there shall be*

$$(\alpha n+a)\mathrm{T}=(\beta n+b)\mathrm{T}'+(\gamma n+c)\mathrm{T}'',$$

to find the formula for T, *by which the general term of this series may be expressed.*

SOLUTION

The integral formula $\int x^{n-1}\partial v$ may be assumed for T, and the integral of this may be taken thus, so that it may vanish on putting $x=0$, and the following terms will be $\mathrm{T}'=\int x^{n}\partial v$ and $\mathrm{T}''=\int x^{n+1}\partial v$, if indeed after the integration a certain value may be attributed to the variable *x*. But as long as this quantity *x* may be considered as variable, we may put

$$(\alpha n+a)\mathrm{T}=(\beta n+b)\mathrm{T}'+(\gamma n+c)\mathrm{T}''+x^{n}\mathrm{Q}$$

and it is evident Q must be a function of *x* of this kind, which may vanish, if in place of *x* that value determined may be substituted, but which is required to be different from zero, since we assume now all these formulas must become zero on putting $x=0$. So that if indeed a complete solution cannot be found satisfying this condition, that will be an indication our problem cannot be resolved by this method, so that clearly its general term T may be shown by such a simple differential formula $\int x^{n-1}\partial v$.

§.222. Now we will differentiate the equation in the manner established, and with a division made by x^{n-1}, the following equation will be produced

$$(\alpha n+a)\partial v=(\beta n+b)x\partial v+(\gamma n+c)xx\partial v+n\mathrm{Q}\partial x+x\partial\mathrm{Q},$$

which, because the terms associated with the letter *n* must cancel each other, may be separated into the two following equations

$$1.\ \alpha\partial v=\beta x\partial v+\gamma xx\partial v+\mathrm{Q}\partial x,$$
$$2.\ a\partial v=bx\partial v+cxx\partial v+x\partial\mathrm{Q},$$

from the first of which there becomes

$$\partial v=\frac{\mathrm{Q}\partial x}{\alpha-\beta x-\gamma xx},$$

truly from the other

$$\partial v = \frac{x\partial \mathrm{Q}}{a-bx-cxx},$$

of which the latter value divided by the first gives

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}} = \frac{\partial x(a-bx-cxx)}{x(\alpha-\beta x-\gamma xx)},$$

therefore from which integration the value of Q itself must be elicited, with which done it will be readily apparent, that in whatever case besides $x=0$ each may be able to vanish. But here initially it will be convenient to observe, if this integral involves a factor of this kind $e^{\frac{1}{x}}$, then this integral solution also in the succeeding is going to be removed, since on putting $x=0$ this same factor will be involve such a great power to the infinite so that, even if it may be multiplied by x^n, the product even now will remain infinite.

§.223. So that if therefore it may be allowed to satisfy these prescribed conditions, then with the value of the letter Q found, which we may be able to put $=0$ on putting $x=f$, there will be obtained :

$$\partial v = \frac{\mathrm{Q}\partial x}{\alpha-\beta x-\gamma xx},$$

and the nature of the general formula generating the series will be

$$\mathrm{T} = \int x^{n-1}\partial v = \int \frac{x^{n-1}\mathrm{Q}\partial x}{\alpha-\beta x-\gamma xx}$$

certainly the integral of which from the limit $x=0$ extended as far as to the limit $x=f$ will provide the value of the term T corresponding to any index *n*.

SCHOLIUM

§.224. But with the relation found between the three terms of some such series themselves exceeding succeeding each other in turn thence in the customary manner will be able to form a continued fraction, of which the value will be able to be assigned. For if the characters

T', T'', T''', T'''' etc.

may denote all the terms in order following after T to infinity, from the relations, which they hold between themselves, the following formulas will be deduced. From the relation

$$(\alpha n+a)\mathrm{T} = (\beta n+b)\mathrm{T}' + (\gamma n+c)\mathrm{T}''$$

there is deduced

$$(\alpha n+a)\tfrac{\mathrm{T}}{\mathrm{T}'}=\beta n+b+\tfrac{(\gamma n+c)(\alpha n+\alpha+a)}{(\alpha n+\alpha+a)\mathrm{T}':\mathrm{T}''}$$

From the following relation :

$$(\alpha n+\alpha+a)\mathrm{T}'=(\beta n+\beta+b)\mathrm{T}''+(\gamma n+\gamma+c)\mathrm{T}'''$$

there is deduced :

$$(\alpha n+\alpha+a)\tfrac{\mathrm{T}'}{\mathrm{T}''}=\beta n+\beta+b+\tfrac{(\gamma n+\gamma+c)(\alpha n+2\alpha+a)}{(\alpha n+2\alpha+a)\mathrm{T}'':\mathrm{T}'''}.$$

In a similar manner the following relations will be supplied :

$$(\alpha n+2\alpha+a)\tfrac{\mathrm{T}''}{\mathrm{T}'''}=\beta n+2\beta+b+\tfrac{(\gamma n+2\gamma+c)(\alpha n+3\alpha+a)}{(\alpha n+3\alpha+a)\mathrm{T}''':\mathrm{T}''''},$$

$$(\alpha n+3\alpha+a)\tfrac{\mathrm{T}'''}{\mathrm{T}''''}=\beta n+3\beta+b+\tfrac{(\gamma n+3\gamma+c)(\alpha n+4\alpha+a)}{(\alpha n+4\alpha+a)\mathrm{T}'''':\mathrm{T}'''''};$$

from which it is evident, if in the first formula the following values may be substituted in order, a continued fraction is going to be produced, the value of which will be equal to the formula $(\alpha n+a)\tfrac{\mathrm{T}}{\mathrm{T}'}$.

§.225. So that, if in place of *n* we may therefore write successively the numbers 1, 2, 3, 4 etc., we will be able to resolve the following problems with continued fractions.

PROBLEM 4

For the continued fraction of this form :

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{etc.}}}}}}$$

: to investigate its value.

SOLUTION

§.226. We may consider in general this same relation between the three quantities succeeding quantities each other T, T', T'', which shall become

$$(\alpha n+a)\mathrm{T}=(\beta n+b)\mathrm{T}'+(\gamma n+c)\mathrm{T}'',$$

and from the preceding problem the value of T may be sought, if indeed it is able to be done, expressed in this manner :

$$\mathrm{T}=\int x^{n-1}\partial v=\int\frac{x^{n-1}\mathrm{Q}\partial v}{a-\beta x-\gamma xx},$$

the integral of which may extend from $x=0$ as far as to $x=f$, where the formula found may be put :

$$\int\frac{\mathrm{Q}\partial x}{a-\beta x-\gamma xx}=\mathrm{A}\quad\text{and}\quad\int\frac{x\mathrm{Q}\partial x}{a-\beta x-\gamma xx}=\mathrm{B},$$

thus so that A and B shall be the values of T for the cases $n=1$ and $n=2$; from which defined, the value of the continued fraction proposed by the preceding will be $=\frac{(\alpha+a)\mathrm{A}}{\mathrm{B}}$. Therefore we may apply this investigation to the following examples.

EXAMPLE 1

§.227. *To investigate the value of the continued fraction, that Brouncker proposed at one time for the quadrature of the circle, which is*

$$2+\cfrac{1\cdot 1}{2+\cfrac{3\cdot 3}{2+\cfrac{5\cdot 5}{2+\text{etc.}}}}$$

[See, *e.g.*, Wikipedia article on Brouncker.]

Because all the integer parts looking to the left are constants $=2$, for our general form there will become

$$\beta+b=2,\ \ 2\beta+b=2,\ \ 3\beta+b=2\ \ \text{etc.};$$

therefore there will be $\beta=0$ and $b=2$; but for the numerators of the following fractions, since they depend on two factors, for the first factors there will become :

$$\gamma+c=1,\ 2\gamma+c=3,\ 3\gamma+c=5,\ 4\gamma+c=7\ \text{etc.},$$

from which it is concluded $\gamma=2$ and $c=-1$, truly for the other there will be:

$$2\alpha+a=1,\ 3\alpha+a=3,\ 4\alpha+a=5\ \text{etc.},$$

from which $\alpha = 2$ and $a = -3$. But from these values we deduce this equation :

$$\frac{\partial Q}{Q}\left[=\frac{\partial x(a-bx-cxx)}{x(\alpha-\beta x-\gamma xx)}\right]=-\frac{\partial x(3+2x-xx)}{2x(1-xx)}$$

which by suppressing $1+x$ gives

$$\frac{\partial Q}{Q}=-\frac{\partial x(3-x)}{2x(1-x)},$$

from which on integrating there becomes :

$$lQ=-\tfrac{3}{2}lx+l(1-x) \text{ and hence } Q=\frac{1-x}{x^{\frac{3}{2}}},$$

from which value again there is found:

$$A=\int\frac{(1-x)\partial x}{2x^{\frac{3}{2}}(1-xx)}=\int\frac{\partial x}{2x(1+x)\sqrt{x}}$$

$$B=\int\frac{(1-x)\partial x}{2x^{\frac{1}{2}}(1-xx)}=\int\frac{\partial x}{2(1+x)\sqrt{x}}.$$

§.228. But the same inconvenience arises from these values, because the first integral is unable to be returned vanishing on putting $x = 0$. But this inconvenience can be easily removed, if we may truncate the continued fraction from the uppermost part, and we seek the value of this fraction :

$$2+\cfrac{3\cdot 3}{2+\cfrac{5\cdot 5}{2+\text{etc.}}};$$

which if it were found to be $=s$, the value of the proposed itself will be $=b+\frac{1}{s}$. Now truly with the comparison put in place as before $\beta = 0$ and $b = 2$, then truly $\gamma = 2$ and $c = +1$, $\alpha = 2$ and $a = -1$, from which it follows

$$\frac{\partial Q}{Q}=-\frac{\partial x(1+2x+xx)}{2x(1-xx)}=-\frac{\partial x(1+x)}{2x(1-x)},$$

from which by integrating there becomes

$$lQ=-\tfrac{1}{2}lx+l(1-x) \text{ and thus } Q=\frac{1-x}{\sqrt{x}},$$

from which value we will now have

$$\mathrm{A} = \int \frac{(1-x)\partial x}{2(1-xx)\sqrt{x}} = \frac{1}{2}\int \frac{\partial x}{(1+x)\sqrt{x}}$$

and

$$\mathrm{B} = \frac{1}{2}\int \frac{\partial x\sqrt{x}}{1+x};$$

where since there shall be $\mathrm{Q} = \frac{1-x}{\sqrt{x}}$, its value clearly will vanish on putting $x=1$, on account of which that these integrations are to be extended from $x=0$ as far as to $x=1$.

§.229. Now so that we may elicit these integrals more easily, we may put $x = zz$, thus so that now the terms of the integration shall be $z=0$ and $z=1$, and there will become

$$\mathrm{A} = \int \frac{dz}{1+zz} = \mathrm{Atang}.z = \frac{\pi}{4}$$

and

$$B = \int \frac{zzdz}{1+zz} = 1 - \frac{\pi}{4}$$

and thus we will have $s = \frac{\pi}{4-\pi}$, on account of which the value of Brouncker's fraction is $1+\frac{4}{\pi}$, completely as Brouncker had now found some time ago.

EXAMPLE 2

§.230. *To investigate the value of Brouncker's continued fraction established more generally*

$$b+\cfrac{1\cdot 1}{b+\cfrac{3\cdot 3}{b+\cfrac{5\cdot 5}{b+\text{etc.}}}}$$

Here so that we may avoid the above inconvenience, we may omit the uppermost member and we may seek

$$s = b+\cfrac{3\cdot 3}{b+\cfrac{5\cdot 5}{b+\text{etc.},}}$$

since then the value sought will be $= b+\frac{1}{s}$. Therefore now there will be $\beta = 0$ and $b = b, \gamma = 2,\ c = 1,\ \alpha = 2$ and $a = -1$, from which there becomes

$$\frac{\partial Q}{Q}=-\frac{\partial x(1+bx+xx)}{2x(1-xx)}$$

and hence,

$$lQ=-\tfrac{1}{2}lx-\tfrac{b-2}{4}l(1+x)+\tfrac{b+2}{4}l(1-x)$$

and hence

$$Q=\frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}}\sqrt{x}},$$

which formula evidently becomes $=0$ on putting $x=1$, if indeed $b+2$ were a positive number, from which there becomes

$$\partial v=\frac{(1-x)^{\frac{b-2}{4}}\partial x}{2(1+x)^{\frac{b+2}{4}}\sqrt{x}}.$$

But hence it will be deduced

$$A=\tfrac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}\partial x}{(1+x)^{\frac{b+2}{4}}\sqrt{x}}\quad\text{and}\quad B=\tfrac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}\partial x\sqrt{x}}{(1+x)^{\frac{b+2}{4}}}$$

or on putting $x=zz$ we will have

$$A=\int\frac{(1-zz)^{\frac{b-2}{4}}\partial z}{(1+zz)^{\frac{b+2}{4}}}\quad\text{and}\quad B=\int\frac{(1-zz)^{\frac{b-2}{4}}zz\partial z}{(1+zz)^{\frac{b+2}{4}}},$$

both which integrals are to be extended from $z=0$ as far as to $z=1$. Moreover from these values A and B there will be $s=\frac{A}{B}$; therefore the value of the proposed fraction will be $=b+\frac{1}{s}=b+\frac{B}{A}$.

§.231. But if here we may put $b=2$, the case will be produced set out before depending on the squaring of the circle, certainly for which case the formula becomes rational. But when the exponents $\frac{b-2}{4}$ and $\frac{b+2}{4}$ are not whole numbers, then the letters A and B cannot be expressed either by circular arcs or by logarithms. Just as if there were $b=4$, there will be

$$A=\int\frac{\partial z\sqrt{(1-zz)}}{(1+zz)^{\frac{3}{2}}},$$

the value of which may be able to be expressed by an elliptical arc. But if *b* were an odd number, these values emerge much more transcending, thus so that we must be content with these letters A and B themselves. But on the other hand if these exponents were made whole number, the whole business will be able to be set out by circular arcs .

§.232. But these exponents $\frac{b-2}{4}$ and $\frac{b+2}{4}$ will be whole numbers, whenever b were a number of this form

$$b = 4i + 2;$$

then indeed there will be

$$\mathrm{A} = \int \frac{(1-zz)^{i}\partial z}{(1+zz)^{i+1}} \quad \text{and} \quad \mathrm{B} = \int \frac{(1-zz)^{i}zz\partial z}{(1+zz)^{i+1}};$$

therefore there will be a need to show how it may be necessary to set out those cases, just as Wallis has now considered these.

§.233. Since this whole business is returned to the reduction of integral formulas of this kind to simpler forms, we may consider in general the form $\mathrm{P} = \frac{z^{m}}{(1+zz)^{n}}$, the differential of which can be shown in the following forms:

$$1.\quad \partial\mathrm{P} = \frac{mz^{m-1}\partial z}{(1+zz)^{n}} - \frac{2nz^{m+1}\partial z}{(1+zz)^{n+1}},$$

$$2.\quad \partial\mathrm{P} = \frac{mz^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{(2n-m)z^{m+1}\partial z}{(1+zz)^{n+1}},$$

$$3.\quad \partial\mathrm{P} = -\frac{(2n-m)z^{m-1}\partial z}{(1+zz)^{n}} + \frac{2nz^{m-1}\partial z}{(1+zz)^{n+1}},$$

from this threefold reduction of the integral we deduce

$$\text{I.}\quad \int \frac{z^{m+1}\partial z}{(1+zz)^{n+1}} = \frac{m}{2n}\int \frac{z^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{1}{2n}\frac{z^{m}}{(1+zz)^{n}},$$

$$\text{II.}\quad \int \frac{z^{m+1}\partial z}{(1+zz)^{n+1}} = \frac{m}{2n-m}\int \frac{z^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{1}{2n-m}\frac{z^{m}}{(1+zz)^{n}},$$

$$\text{III.}\quad \int \frac{z^{m-1}\partial z}{(1+zz)^{n+1}} = \frac{2n-m}{2n}\int \frac{z^{m-1}\partial z}{(1+zz)^{n}} + \frac{1}{2n}\frac{z^{m}}{(1+zz)^{n}},$$

of which reduction with the aid in cases $b = 4i + 2$ the whole business to be resolved and will be able to be reduced to the form $\frac{\pi}{4}$, if indeed after the integration there may be taken $z = 1$.

§.234. Let $i = 1$ and thus $b = 6$ and there will be

$$\mathrm{A} = \int \frac{(1-zz)\partial z}{(1+zz)^{2}} \quad \text{and} \quad \mathrm{B} = \int \frac{(1-zz)zz\partial z}{(1+zz)^{2}}.$$

Therefore now by the third reduction we find

$$\int \frac{\partial z}{(1+zz)^2} = \frac{1}{2}\int \frac{\partial z}{1+zz} + \frac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} + \frac{1}{4}$$

and by the first reduction

$$\int \frac{zz\partial z}{(1+zz)^2} = \frac{1}{2}\int \frac{\partial z}{1+zz} - \frac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} - \frac{1}{4},$$

again,

$$\int \frac{z^4\partial z}{(1+zz)^2} = \frac{3}{2}\int \frac{zz\partial z}{1+zz} - \frac{1}{2}\cdot\frac{z^3}{1+zz} = \frac{5}{4} - \frac{3\pi}{8}.$$

Now, it is gathered from these values, $\mathrm{A} = \frac{1}{2}$ and $\mathrm{B} = \frac{\pi}{2} - \frac{3}{2}$ and thus $\frac{\mathrm{B}}{\mathrm{A}} = \pi - 3$, whereby this summation will arise

$$3 + \pi = 6 + \cfrac{1\cdot 1}{6 + \cfrac{3\cdot 3}{6 + \cfrac{5\cdot 5}{6 + \cfrac{7\cdot 7}{6 + \text{etc.}}}}}$$

§.235. Now let $i = 2$ and $b = 10$ and there becomes

$$\mathrm{A} = \int \frac{(1-zz)^2\partial z}{(1+zz)^3} \quad \text{and} \quad \mathrm{B} = \int \frac{zz(1-zz)^2\partial z}{(1+zz)^3}.$$

So that we may investigate the values of these, we may establish the following formulas

$$\int \frac{\partial z}{(1+zz)^3} = \frac{3}{4}\int \frac{\partial z}{(1+zz)^2} + \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{3\pi}{32} + \frac{1}{4},$$

$$\int \frac{zz\partial z}{(1+zz)^3} = \frac{1}{4}\int \frac{\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{\pi}{32},$$

$$\int \frac{z^4\partial z}{(1+zz)^3} = \frac{3}{4}\int \frac{zz\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^3}{(1+zz)^2} = \frac{3\pi}{32} - \frac{1}{4},$$

$$\int \frac{z^6\partial z}{(1+zz)^3} = \frac{5}{4}\int \frac{z^4\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^5}{(1+zz)^2} = \frac{3}{2} - \frac{15\pi}{32}.$$

Now from which values there is deduced $\mathrm{A} = \frac{\pi}{8}$ and $\mathrm{B} = 2 - \frac{5\pi}{8}$ and thus $\frac{\mathrm{B}}{\mathrm{A}} = \frac{16-5\pi}{\pi}$, from which the following summation arises :

$$\frac{5\pi+16}{\pi}=10+\cfrac{1\cdot 1}{10+\cfrac{3\cdot 3}{10+\cfrac{5\cdot 5}{10+\text{etc.}}}}$$

§.236. If b were a negative number, the investigation evidently should proceed without difficulty. If indeed in general there were

$$s=-a+\cfrac{\alpha}{-b+\cfrac{\beta}{-c+\cfrac{\gamma}{-d+\cfrac{\delta}{-e+\text{etc.},}}}}$$

there will be always :

$$-s=a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\text{etc.},}}}}$$

from which, if the value of this expression may be had, likewise the value of that taken negative will be given.

EXAMPLE 3

§.237. *This same continued fraction shall be proposed, the value of which may be investigated, be found,*

$$1+\cfrac{1\cdot 1}{3+\cfrac{3\cdot 3}{5+\cfrac{5\cdot 5}{7+\cfrac{7\cdot 7}{9+\text{etc.}}}}}$$

So that the above fractions may be removed [§ 225], with the above part omitted there shall become :

$$s=3+\cfrac{3\cdot 3}{5+\cfrac{5\cdot 5}{7+\cfrac{7\cdot 7}{9+\text{etc.}}}}$$

and there will be $\beta+b=3,\ 2\beta+b=5$ and thus $\beta=2$ and $b=1$, then truly as before $\alpha=2,\ a=-1,\ \gamma=2$ and $c=+1$; but with s found, the value sought will be $=1+\frac{1}{s}$. Therefore now we will have

$$\frac{\partial Q}{Q}=-\frac{\partial x(1+x+xx)}{2x(1-x-xx)}.$$

Truly there is :

$$\frac{1+x+xx}{x(1-x-xx)}=\frac{1}{x}+\frac{2+2x}{1-x-xx},$$

from which there becomes :

$$lQ=-\tfrac{1}{2}lx-\int\frac{\partial x(1+x)}{1-x-xx}.$$

Truly again for the formula $\int\frac{\partial x(1+x)}{1-x-xx}$ requiring to be found, we may put the denominator

$$1-x-xx=(1-fx)(1-gx)$$

and there will be $f+g=-1$ and $fg=-1$, from which there becomes :

$$f=\tfrac{1+\sqrt{5}}{2}\ \text{ and }\ g=\tfrac{1-\sqrt{5}}{2}.$$

Now there may be put in place

$$\frac{1+x}{1-x-xx}=\frac{\mathfrak{A}}{1-fx}+\frac{\mathfrak{B}}{1-gx},$$

from which there will be found:

$$\mathfrak{A}=\frac{1+f}{f-g}\ \text{ and }\ \mathfrak{B}=-\frac{1+g}{f-g},$$

or with the values substituted for f and g given above , there will become :

$$\mathfrak{A}=\frac{\sqrt{5}+3}{2\sqrt{5}}\ \text{ and }\ \mathfrak{B}=\frac{\sqrt{5}-3}{2\sqrt{5}},$$

with which found there will be :

$$\int\frac{\partial x(1+x)}{1-x-xx}=-\frac{\mathfrak{A}}{f}l(1-fx)-\frac{\mathfrak{B}}{g}l(1-gx),$$
$$=-\frac{1+\sqrt{5}}{2\sqrt{5}}l(1-fx)-\frac{\sqrt{5}-1}{2\sqrt{5}}l(1-gx),$$

on account of which there becomes :

$$lQ=-\tfrac{1}{2}lx+\frac{\sqrt{5}+1}{2\sqrt{5}}l(1-fx)+\frac{\sqrt{5}-1}{2\sqrt{5}}l(1-gx),$$

consequently

$$Q=\frac{(1-fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}}(1-gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

which value will vanish in the two cases, the one, where

$$x=\frac{1}{f}=\frac{2}{1+\sqrt{5}}=\frac{\sqrt{5}-1}{2},$$

truly the other, where

$$x=\frac{1}{g}=-\frac{1+\sqrt{5}}{2};$$

but as we may use one or the other, the argument returned will be the same.

§.238. Moreover, from this value we will obtain :

$$\mathrm{A}=\int\frac{\mathrm{Q}\partial x}{1-x-xx}\ \text{ and }\ \mathrm{B}=\int\frac{\mathrm{Q}x\partial x}{1-x-xx},$$

and here again there is deduced :

$$s=(\alpha+a)\frac{\mathrm{A}}{\mathrm{B}}=\frac{\mathrm{A}}{\mathrm{B}};$$

hence the sum of the proposed fractions will be $1+\frac{\mathrm{B}}{\mathrm{A}}$. But hence nothing further can be concluded on account not only because of the irrational differential formulas, but also because of the transcendental exponent surds.

EXAMPLE 4

§.239. *This continued fraction shall be proposed*

$$b+\cfrac{1\cdot 1}{b+\cfrac{2\cdot 2}{b+\cfrac{3\cdot 3}{b+\cfrac{4\cdot 4}{b+\text{etc.}}}}}$$

where there is $\beta=0,\ b=b$.

Now we will consider this form:

$$s=b+\cfrac{2\cdot 2}{b+\cfrac{3\cdot 3}{b+\text{etc.}}}$$

certainly so that for the value found that sought will be $=b+\frac{1}{s}$. Therefore we will have $\gamma+c=2,\ 2\gamma+c=3$ and therefore $\gamma=1$ and $c=1,$ then there will be $\alpha=\gamma=1,\ a=0$ and $c=1.$ Hence we therefore deduce

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{\partial x(bx+xx)}{x(1-xx)}=-\frac{\partial x(b+x)}{1-xx}$$

and therefore

$$lQ = -\tfrac{b}{2} l \tfrac{1+x}{1-x} + \tfrac{1}{2} l\left(1-xx\right)$$

and hence

$$Q = \frac{(1-x)^{\frac{b}{2}}\sqrt{(1-xx)}}{(1+x)^{\frac{b}{2}}} = \frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

which quantity clearly vanishes on putting $x=1$. Hence therefore there will become

$$A = \int \frac{Q\partial x}{1-xx} = \int \frac{(1-x)^{\frac{b+1}{2}}\partial x}{(1+x)^{\frac{b-1}{2}}(1-xx)} = \int \frac{(1-x)^{\frac{b-1}{2}}\partial x}{(1+x)^{\frac{b+1}{2}}}$$

and

$$B = \int \frac{x(1-x)^{\frac{b-1}{2}}\partial x}{(1+x)^{\frac{b+1}{2}}},$$

so then there will become $s = \left(\alpha + a\right)\frac{A}{B} = \frac{A}{B}$, and thus the sum sought $b + \frac{B}{A}$.

§.240. Now we may run through particular cases and the first shall be $b=1$, and there will be

$$A = \int \tfrac{\partial x}{1+x} = l\left(1+x\right) = l2 \text{ and } B = \int \tfrac{x\partial x}{1+x} = x - \int \tfrac{\partial x}{1+x} = 1 - l2$$

and thus $b + \frac{B}{A} = \frac{1}{l2}$; therefore this same sum will be produced

$$\frac{1}{l2} = 1 + \cfrac{1\cdot 1}{1 + \cfrac{2\cdot 2}{1 + \cfrac{3\cdot 3}{1 + \text{etc.}}}}$$

§.241. Now let $b=2$ and there will be

$$A = \int \frac{\partial x\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}} \quad \text{and} \quad B = \int \frac{x\partial x\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

For these formulas being required to be rendered rational, we may put

$$\frac{\sqrt{(1-x)}}{\sqrt{(1+x)}} = z$$

and there will be $x = \frac{1-zz}{1+zz}$, from which for the limits of the integration $x = 0$ and $x = 1$ there will correspond $z = 1$ et $z = 0$; then truly there will be

$$1 + x = \frac{2}{1+zz} \text{ and } \partial x = -\frac{4z\partial z}{(1+zz)^2}$$

and hence there will be deduced

$$\mathrm{A} = -2\int \frac{zz\partial z}{1+zz} = -2z + 2\mathrm{Atang.}z = 2 - \frac{\pi}{2},$$

again there becomes

$$\mathrm{B} = -2\int \frac{zz\partial z}{(1+zz)^2} + 2\int \frac{z^4\partial z}{(1+zz)^2}$$

Therefore by the reductions shown above (§.232), clearly if here we interchange the limits of the integration $z = 1$ and $z = 0$, so that we may have

$$\mathrm{B} = +2\int \frac{zz\partial z}{(1+zz)^2} - 2\int \frac{z^4\partial z}{(1+zz)^2}$$

there will be

$$\mathrm{B} = 2(\tfrac{\pi}{8} - \tfrac{1}{4}) - 2\left(\tfrac{5}{4} - \tfrac{3\pi}{8}\right) = \pi - 3,$$

from which it follows this same summation :

$$\frac{2}{4-\pi} = 2 + \cfrac{1\cdot 1}{2 + \cfrac{2\cdot 2}{2 + \cfrac{3\cdot 3}{2 + \cfrac{4\cdot 4}{2 + \text{etc.},}}}}$$

which adds nothing to the BROUNCKER simplicity.

§.242. If we may put $b = 0$, the continued fraction will change into the following continued product

$$\frac{1\cdot1}{2\cdot2} \cdot \frac{3\cdot3}{4\cdot4} \cdot \frac{5\cdot5}{6\cdot6} \cdot \frac{7\cdot7}{88} \cdot \text{etc. };$$

but in this case there becomes

$$\mathrm{A} = \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ and } B = \int \frac{x\partial x}{\sqrt{(1-xx)}} = 1,$$

from which the value of that same product is gathered to be $\frac{2}{\pi}$, that which agrees exceedingly well with that known already, since this product is the Wallis progression itself. [The Wallis product is $\frac{4}{\pi}=\frac{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot\text{etc.}}{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot\text{etc.}}$.]

EXAMPLE 5

§.243. *This continued fraction shall be proposed, where* $\beta=0,\ b=b$ *and the numerators shall be the triangular numbers,*

$$b+\cfrac{1}{b+\cfrac{3}{b+\cfrac{6}{b+\cfrac{10}{b+\text{etc.}}}}}$$

With the upper member omitted, we may put

$$s=b+\cfrac{3}{b+\cfrac{6}{b+\cfrac{10}{b+\text{etc.}}}}$$

and in the first place we may represent the numerators by products in this manner

$$3=2\cdot\tfrac{3}{2},\quad 6=3\cdot\tfrac{4}{2},\quad 10=4\cdot\tfrac{5}{2},$$

of which the former may be compared with the formulas $\gamma+c,\ 2\gamma+c,\ 3\gamma+c,$ truly the latter with the formulas $2\alpha+a,\ 3\alpha+a,\ 4\alpha+a$, and there will become $\gamma=1,\ c=1,\ \alpha=\frac{1}{2},\ a=\frac{1}{2}$, from which there will be :

$$\frac{\partial Q}{Q}=\frac{\partial x\left(\frac{1}{2}-bx-xx\right)}{x\left(\frac{1}{2}-xx\right)}=\frac{\partial x(1-2bx-2xx)}{x(1-2xx)}$$

or

$$\frac{\partial Q}{Q}=\frac{\partial x}{x}-\frac{2b\partial x}{1-2xx},$$

of which the integral is

$$lQ=lx-\frac{b}{\sqrt{2}}\,l\,\frac{1+x\sqrt{2}}{1-x\sqrt{2}},$$

therefore

$$Q=\frac{x\left(1-x\sqrt{2}\right)^{\frac{b}{\sqrt{2}}}}{\left(1+x\sqrt{2}\right)^{\frac{b}{\sqrt{2}}}},$$

which formula vanishes in the case $x=\frac{1}{\sqrt{2}}$. Hence there will be therefore

$$\partial v=\frac{2x\left(1-x\sqrt{2}\right)^{\frac{b}{\sqrt{2}}}\partial x}{\left(1-2xx\right)\left(1+x\sqrt{2}\right)^{\frac{b}{\sqrt{2}}}}.$$

Let $\frac{b}{\sqrt{2}}=\lambda$, and there will be

$$\mathrm{A}=2\int\frac{x\left(1-x\sqrt{2}\right)^{\lambda}\partial x}{\left(1-2xx\right)\left(1+x\sqrt{2}\right)^{\lambda}}=2\int\frac{x\left(1-x\sqrt{2}\right)^{\lambda-1}\partial x}{\left(1+x\sqrt{2}\right)^{\lambda+1}}$$

and

$$\mathrm{B}=2\int\frac{xx\left(1-x\sqrt{2}\right)^{\lambda-1}\partial x}{\left(1+x\sqrt{2}\right)^{\lambda+1}},$$

where after the integration there may be put $x=\frac{1}{\sqrt{2}}$; but then there becomes $s=\frac{\mathrm{A}}{\mathrm{B}}$ and hence the value of the proposed fraction will be $=b+\frac{\mathrm{B}}{\mathrm{A}}$.

§.244. Therefore unless $\lambda=\frac{b}{\sqrt{2}}$ were a rational number, it is not possible to assign these convenient values. Therefore let $b=\sqrt{2}$ or $\lambda=1$, and there will become

$$\mathrm{A}=2\int\frac{x\partial x}{\left(1+x\sqrt{2}\right)^{2}}\quad\text{and}\quad\mathrm{B}=2\int\frac{xx\partial x}{\left(1+x\sqrt{2}\right)^{2}}.$$

Hence on integration there will be deduced :

$$\mathrm{A}=l\left(1+x\sqrt{2}\right)-\frac{x\sqrt{2}}{1+x\sqrt{2}}$$

and thus on putting $x\sqrt{2}=1$ there will become $\mathrm{A}=l2-\frac{1}{2}$; then truly there will be found :

$$\mathrm{B}=\frac{3}{2\sqrt{2}}-\sqrt{2}\cdot l2,$$

whereby on account of $b=\sqrt{2}$ there will become $b+\frac{\mathrm{B}}{\mathrm{A}}=\frac{1}{\sqrt{2}(2l2-1)}$, from which this summation follows :

$$\frac{1}{\sqrt{2}(2l2-1)}=\sqrt{2}+\cfrac{1}{\sqrt{2}+\cfrac{3}{\sqrt{2}+\cfrac{6}{\sqrt{2}+\text{etc.}}}}$$

SCHOLIUM

§.245. But continued fractions, which we are able to deduce generally from a numerical calculation, are accustomed to have this form

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

where all the numerators are ones, and truly the denominators *a, b, c,* d, *e* etc. are whole numbers. With the aid of our method the more difficult of these values will be enabled to be elicited, even if the numbers *a, b, c, d, e* constitute an arithmetical progression, which we will show in the following example.

EXAMPLE

§.246. *This continued fraction shall be proposed :*

$$\beta+b+\cfrac{1}{2\beta+b+\cfrac{1}{3\beta+b+\cfrac{1}{4\beta+b++\cfrac{1}{5\beta+b+\text{etc.},}}}}$$

where $\alpha=0,\ \gamma=0,\ a=1,\ c=1.$
Hence there becomes

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{\partial x(1-bx-xx)}{\beta xx},$$

from which

$$l\mathrm{Q}=\tfrac{1}{\beta x}+\tfrac{b}{\beta}lx+\tfrac{x}{\beta}\ \text{ and }\ \mathrm{Q}=e^{\frac{1+xx}{\beta x}}x^{\frac{b}{\beta}},$$

but which expression under no circumstances can vanish, even if it may be multiplied by x^n , if indeed β were a positive number. Truly if a negative number were taken for β , consider for example $\beta=-m$, then the value $\mathrm{Q}=x^{\frac{-b}{m}}e^{\frac{-(1+xx)}{mx}}$ evidently will vanish, both if $x=0$ as well as if $x=\infty$. Hence moreover there will be

$$\partial v=\frac{x^{\frac{-b}{m}}e^{\frac{-1-xx}{mx}}\partial x}{mxx},$$

on account of which we will have

$$\mathrm{A}=\frac{1}{m}\int\frac{\partial x}{x^{2+\frac{b}{m}}e^{\frac{1+xx}{mx}}}\ \text{ and }\ \mathrm{B}=\frac{1}{m}\int\frac{\partial x}{x^{1+\frac{b}{m}}e^{\frac{1+xx}{mx}}}\ .$$

From these two value found the formula $\frac{\mathrm{A}}{\mathrm{B}}$ will express the sum of these continued fractions

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

on account of which that formula taken negative $-\frac{\mathrm{A}}{\mathrm{B}}$ will express the value of this continued fraction :

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\cfrac{1}{5m-b+\text{etc.}}}}}$$

which therefore it will be allowed to assign, but only if the integral formulas A and B can be worked out and they may be able to be extended from the limit $x=0$ to $x=\infty$. Truly these formulas have been prepared thus, so that the integration of those plainly in no case may be able to be expressed by known quantities, so that yet it will not be impeded, whereby the fraction $\frac{\mathrm{A}}{\mathrm{B}}$ may not be able to involve known values, even if at this point there is no way we may be able to assign these.

§.247. But of such continued fractions indeed the two following are known to me, the values of which it will be permitted to show conveniently :

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}}=\frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}$$

and

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}}=\cot.\tfrac{1}{n}.$$

[Note that the latter formula follows from the former by writing $n\sqrt{-1}$ in place of n.]

Of these fractions the first taken together with the formulas of the latter example produces
$m-b=n,\ 2m-b=3n$ and thus $m=2n$ and $b=n,$ from which there becomes

$$\text{A}=\tfrac{1}{2n}\int\frac{\partial x}{x^{\frac{5}{2}}e^{\frac{1+xx}{2nx}}} \text{ and } \text{B}=\tfrac{1}{2n}\int\frac{\partial x}{x^{\frac{3}{2}}e^{\frac{1+xx}{2nx}}},$$

from which we learn now, if these two formulas may be integrated from the limit $x=0$ as far as to the limit $x=\infty$, then there becomes

$$\frac{\text{A}}{\text{B}}=\frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

although at this point no analytical way is apparent how this may be demonstrated.

7). Methodus inveniendi formulas integrales quae certis casibus datam inter se teneant rationem, ubi simul methodus traditur fractiones continuas summandi.

Opuscula analytica II, 1785, p. 178-216

§.196. Quemadmodum in seriebus recurrentibus quilibet terminus ex uno pluribusve praecedentibus secundum legem quandam constantem determinatur, ita hic eiusmodi series sum consideraturus, in quibus quilibet terminus ex uno pluribusve praecedentibus secundum quampiam legem variabilem determinatur. Quoniam autem in talibus seriebus formula generalis singulos terminos exprimens plerumque non est algebraica sed transcendens, singulos terminos per formulas integrales exhiberi conveniet; quae ut valores determinatos praebeant, post integrationem quantitati variabili valorem determinatum tribui assumo, ita ut singuli termini prodeant quantitates determinatae; atque nunc quaestio principalis huc redit, quemadmodum istae formulae integrales debeant esse comparatae, ut quilibet terminus secundum datam legem ex uno pluribusve praecedentibus determinetur.

§.197. Quod quo clarius perspiciatur, contemplemur seriem notissimam harum formularum integralium

$$\int\frac{\partial x}{\sqrt{(1-xx)}},\ \int\frac{xx\partial x}{\sqrt{(1-xx)}},\ \int\frac{x^4\partial x}{\sqrt{(1-xx)}},\ \int\frac{x^6\partial x}{\sqrt{(1-xx)}},\ \text{etc.};$$

quae si singulae ita integrentur, ut evanescant posito $x=0$, tum vero variabili x tribuatur valor 1, quilibet terminus a praecedente ita pendet, ut sit

$$\int\frac{xx\partial x}{\sqrt{(1-xx)}}=\frac{1}{2}\int\frac{\partial x}{\sqrt{(1-xx)}},$$
$$\int\frac{x^4\partial x}{\sqrt{(1-xx)}}=\frac{3}{4}\int\frac{xx\partial x}{\sqrt{(1-xx)}},$$
$$\int\frac{x^6\partial x}{\sqrt{(1-xx)}}=\frac{5}{6}\int\frac{x^4\partial x}{\sqrt{(1-xx)}},$$

atque in genere

$$\int\frac{x^n\partial x}{\sqrt{(1-xx)}}=\frac{n-1}{n}\int\frac{x^{n-2}\partial x}{\sqrt{(1-xx)}}.$$

Unde patet hanc formuIam generalem spectari posse tanquam terminum generalem illius seriei atque quemlibet terminum ex praecedente oriri, si iste multiplicetur per $\frac{n-1}{n}$.

§.198. Ad similitudinem igitur huius casus seriem formularum integralium ita in genere constituamus

$$\int \partial v,\ \int x\partial v,\ \int xx\partial v,\ \int x^3\partial v,\ \int x^4\partial v,\ \text{etc.},$$

ita ut terminus indici *n* respondens sit

$$\int x^{n-1}\partial v,$$

quae singula integralia ita accipi sumamus, ut evanescant posito $x = 0$; post integrationem autem quantitati variabili *x* tribuamus quempiam valorem constantem, veluti $x = 1$ vel alii cuipiam numero. Quibus positis quaestio huc redit, qualis pro *v* assumi debeat functio ipsius *x*, ut quilibet terminus per unum vel duos pluresve praecedentes secundum legem quandam datam, utcunque variabilem sive ab indice *n* pendentem, determinetur; ubi quidem imprimis eo erit respiciendum, ad quot dimensiones index *n* in scala relationis proposita ascendat; plerumque autem non ultra primam dimensionem assurgere erit opus. Hinc igitur sequentia problemata pertractemus.

PROBLEMA 1

§.199. *Invenire functionem v, ut ista relatio inter bios terminos sibi succedentes locum habeat*

$$\int x^n\partial v = \tfrac{\alpha n+a}{\beta n+b}\int x^{n-1}\partial v.$$

Requiritur igitur hic, ut sit

$$(\alpha n+a)\int x^{n-1}\partial v = (\beta n+b)\int x^n\partial v,$$

si scilicet post integrationem variabili *x* certus valor tribuatur. Quoniam igitur ista conditio tum demum locum habere debet, postquam variabili *x* iste valor constans fuerit datus, ponamus in genere, dum *x* est variabilis, hanc aequationem locum habere

$$(\alpha n+a)\int x^{n-1}\partial v = (\beta n+b)\int x^n\partial v + \mathrm{V},$$

quantitatem autem V ita esse comparatam, ut evanescat, postquam variabili ille valor determinatus fuerit assignatus. Praeterea vero, quia ambo integralia ita capi assumimus, ut evanescant posito $x = 0$, necesse est, ut etiam ista quantitas V eodem quoque casu evanescat.

§.200. Quoniam haec aequalitas subsistere debet pro omnibus indicibus *n,* quos quidem semper ut positivos spectamus, facile intelligitur quantitatem istam V factorem habere debere x^n; quo pacto iam isti conditioni satisfit, ut posito $x = 0$, etiam fiat $\mathrm{V} = 0$. Quamobrem statuamus $\mathrm{V} = x^n\mathrm{Q}$, ubi Q denotet functionem ipsius *x* proposito accommodatam, et quam simul ita comparatam esse desideramus, ut evanescat, si ipsi x certus quidam valor tribuatur.

§.201. Cum igitur esse debeat

$$(\alpha n+a)\int x^{n-1}\partial v=(\beta n+b)\int x^{n}\partial v+x^{n}\mathrm{Q},$$

differentietur ista aequatio ac differentiali per x^{n-1} diviso pervenietur ad hanc aequationem differentialem

$$(\alpha n+a)\partial v=(\beta n+b)x\partial v+n\mathrm{Q}\partial x+x\partial\mathrm{Q},$$

quae cum subsistere debeat pro omnibus valoribus ipsius *n*, termini ista littera affecti seorsim se tollere debent, unde nanciscimur has duas aequalitates

$$\text{I. } (\alpha-\beta x)\partial v=\mathrm{Q}\partial x \text{ et}$$
$$\text{II. } (a-bx)\partial v=x\partial\mathrm{Q}.$$

Ex priore fit $\partial v=\frac{\mathrm{Q}\partial x}{\alpha-\beta x}$, ex altera vero $\partial v=\frac{x\partial\mathrm{Q}}{a-bx}$, qui duo valores inter se aequati suppeditant hanc aequationem $\frac{\partial\mathrm{Q}}{\mathrm{Q}}=\frac{\partial x}{x}\cdot\frac{a-bx}{\alpha-\beta x}$ quae aequatio resolvitur in has partes

$$\frac{\partial\mathrm{Q}}{\mathrm{Q}}=\frac{a}{\alpha}\cdot\frac{\partial x}{x}+\frac{a\beta-b\alpha}{\alpha}\cdot\frac{\partial x}{\alpha-\beta x},$$

cuius ergo integrale erit

$$l\mathrm{Q}=\frac{a}{\alpha}\cdot lx-\frac{a\beta-b\alpha}{\alpha\beta}l(\alpha-\beta x),$$

unde deducitur

$$\mathrm{Q}=\mathrm{C}x^{\frac{a}{\alpha}}(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}}.$$

§.202. Ex hoc valore pro Q invento statim patet, eum evanescere casu $x=\frac{\alpha}{\beta}$, si modo fuerit $\frac{b\alpha-a\beta}{\alpha\beta}>0$; sin autem secus eveniat, non patet, quomodo haec quantitas ullo casu evanescere queat. Invento autem hoc valore Q inde reperietur

$$\partial v=\mathrm{C}x^{\frac{a}{\alpha}}\partial x(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}-1}$$

hincque nostrae seriei terminus indici *n* respondens erit

$$\int x^{n-1}\partial v = \mathrm{C}\int x^{n+\frac{a}{\alpha}-1}\partial x(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}-1},$$

tum vero erit

$$\mathrm{V} = \mathrm{C}x^{n+\frac{a}{\alpha}}(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}}.$$

Ubi res imprimis eo redit, ut ista quantitas praeter casum $x=0$ insuper alio casu evanescat.

COROLLARIUM 1

§.203. Hic duo casus occurrunt, qui peculiarem evolutionem postulant; prior est, quo $\alpha=0$; tum autem inchoandum erit ab aequatione $\frac{\partial \mathrm{Q}}{\mathrm{Q}} = -\frac{(a-bx)\partial x}{\beta xx}$, unde integrando elicitur $l\mathrm{Q} = \frac{a}{\beta x}+\frac{b}{\beta}lx$, hincque sumendo *e* pro numero, cuius logarithmus hyperbolicus $=1$, colligitur

$$\mathrm{Q} = e^{\frac{a}{\beta x}}x^{\frac{b}{\beta}},$$

quae formula in nihilum abire nequit, nisi fiat $\frac{a}{\beta x}=-\infty$ ideoque $x=0$, sicque non duo haberentur casus, quibus fieret $\mathrm{V}=0$, cum tamen duo desiderentur. Interim autem hinc fiet

$$\partial v = \frac{e^{\frac{a}{\beta x}}x^{\frac{b}{\beta}}\partial x}{-\beta x}$$

COROLLARIUM 2

§.204. Alter casus peculiarem integrationem postulans erit $\beta=0$; tum autem erit $\frac{\partial \mathrm{Q}}{\mathrm{Q}} = \frac{\partial x(a-bx)}{\alpha x}$, unde fit $l\mathrm{Q} = \frac{a}{\alpha}lx-\frac{bx}{\alpha}$ ideoque $\mathrm{Q} = e^{\frac{a}{\alpha}}x^{\frac{-bx}{\alpha}}$, quae formula casu $x=\infty$ evanescit, si modo fuerit $\frac{b}{\alpha}$ numerus positivus; sin autem $\frac{b}{\alpha}$ fuerit numerus negativus, tum Q evanescit casu $x=-\infty$. Porro vero hoc casu fiet

$$\partial v = \frac{x^{\frac{a}{\alpha}}e^{\frac{-bx}{\alpha}}\partial x}{\alpha}.$$

SCHOLION

§.205. His in genere observatis aliquot casus speciales evolvamus, quibus litteris α, β et *a, b* certos valores tribuemus, qui ad casus iam ·satis cognitos perducant.

EXEMPLUM 1

§.206. *Quaerantur formulae integrales, ut fiat*

$$\int x^n \partial v = \tfrac{2n-1}{2n}\int x^{n-1}\partial v.$$

Cum igitur hic esse debeat $(2n-1)\int x^{n-1}\partial v = 2n\int x^n \partial v$, erit hoc casu $\alpha = 2$ et $a = -1$, tum vero $\beta = 2$ et $b = 0$; hinc fit

$$\tfrac{\partial \mathrm{Q}}{\mathrm{Q}} = -\tfrac{a\partial x}{2x(1-x)} = -\tfrac{\partial x}{2x} - \tfrac{\partial x}{2(1-x)},$$

inde integrando

$$lQ = -\tfrac{1}{2}lx + \tfrac{1}{2}l(1-x)$$

ideoque

$$\mathrm{Q} = \mathrm{C}\sqrt{\tfrac{1-x}{x}},\ \text{ergo}\ \mathrm{V} = \mathrm{C}x^n\sqrt{\tfrac{1-x}{x}}$$

Porro cum hic sit $\partial v = \frac{\mathrm{Q}\partial x}{2(1-x)}$, erit

$$\partial v = \frac{\mathrm{C}\sqrt{\frac{1-x}{x}}\partial x}{2(1-x)} = \frac{\mathrm{C}\partial x}{2\sqrt{(x-xx)}};$$

sumto ergo $\mathrm{C} = 2$ erit $\partial v = \frac{\partial x}{\sqrt{(x-xx)}}$ et formula nostra generalis

$$\int x^{n-1}\partial v = \int \tfrac{x^{n-1}\partial x}{\sqrt{(x-xx)}};$$

unde cum sit $\mathrm{V} = x^n\sqrt{\frac{1-x}{x}}$, haec quantitas manifesto evanescit sumto $x = 1$, ita ut nostra formula, si post integrationem statuatur $x = 1$, quaesito satisfaciat.
Quodsi iam ponamus $x = yy$, ista formula induet hanc formam

$$2\int \tfrac{y^{2n-2}\partial y}{\sqrt{(1-yy)}},$$

quae posito post integrationem $y = 1$ praebet hanc relationem

$$\int \tfrac{y^{2n}\partial y}{\sqrt{(1-yy)}} = \tfrac{2n-1}{2n}\int \tfrac{y^{2n-2}\partial y}{\sqrt{(1-yy)}},$$

quae continet relationes supra (§.197) commemoratas; hinc enim fiet

$$\int \frac{yy\partial y}{\sqrt{(1-yy)}} = \frac{1}{2}\int \frac{\partial y}{\sqrt{(1-yy)}},$$

$$\int \frac{y^4\partial y}{\sqrt{(1-yy)}} = \frac{3}{4}\int \frac{yy\partial y}{\sqrt{(1-yy)}},$$

et

$$\int \frac{y^6\partial y}{\sqrt{(1-yy)}} = \frac{5}{6}\int \frac{y^4\partial y}{\sqrt{(1-yy)}}.$$

EXEMPLUM 2

§.207. *Quaerantur formulae integrales, ut fiat*

$$\int x^n \partial v = \frac{\alpha n-1}{\alpha n}\int x^{n-1}\partial v.$$

Cum igitur hic esse debeat $(\alpha n-1)\int x^{n-1}\partial v = \alpha n\int x^n \partial v$, erit hoc casu $a=-1,\ \beta=\alpha$ et $b=0$, unde per formulas supra datas colligitur

$$\mathrm{Q} = \mathrm{C}x^{\frac{-1}{a}}(\alpha - ax)^{\frac{-\alpha}{a}} = \mathrm{C}x^{\frac{-1}{a}}(1-x)^{\frac{+1}{a}},$$

quae quantitas manifesto evanescit posito $x=1$. Tum autem erit

$$\partial v = \frac{x^{\frac{-1}{a}}(1-x)^{\frac{+1}{a}}\partial x}{1-x}$$

unde formula nostra generalis erit

$$\partial v = \frac{x^{\frac{-1}{a}}(1-x)^{\frac{+1}{a}}\partial x}{1-x}$$

$$\int x^{n-1}\partial v = \int x^{n-\frac{1}{a}-1}(1-x)^{+\frac{1}{a}-1}\partial x = \int \frac{x^{n-\frac{1}{a}-1}}{(1-x)^{1-\frac{1}{a}}}\partial x,$$

quae concinnior redditur faciendo $x=y^{\alpha}$; tum enim ea induet hanc formam

$$\int \frac{y^{\alpha n-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}},$$

ubi iterum post integrationem statui debet $y=1$. Erit hinc

$$\int\frac{y^{\alpha n+\alpha-2}\partial y}{\left(1-y^{\alpha}\right)^{\frac{\alpha-1}{\alpha}}}=\frac{\alpha n-1}{\alpha n}\int\frac{y^{\alpha n-2}\partial y}{\left(1-y^{\alpha}\right)^{\frac{\alpha-1}{\alpha}}},$$

atque hinc orientur sequentes casus speciales

$$\int\frac{y^{2\alpha-2}\partial y}{\left(1-y^{\alpha}\right)^{\frac{\alpha-1}{\alpha}}}=\frac{\alpha-1}{\alpha}\int\frac{y^{\alpha-2}\partial y}{\left(1-y^{\alpha}\right)^{\frac{\alpha-1}{\alpha}}}$$

et

$$\int\frac{y^{3\alpha-2}\partial y}{\left(1-y^{\alpha}\right)^{\frac{\alpha-1}{\alpha}}}=\frac{2\alpha-1}{\alpha}\int\frac{y^{2\alpha-2}\partial y}{\left(1-y^{\alpha}\right)^{\frac{\alpha-1}{\alpha}}}.$$

§.208. Hinc igitur si sumatur $\alpha=1$, ut fieri debeat

$$\int x^{n}\partial v=\frac{n-1}{n}\int x^{n-1}\partial v,$$

formula nostra generalis iam in y expressa erit $\int y^{n-2}\partial y$, cuius ergo valor est $\frac{1}{n-1}y^{n-1}=\frac{1}{n-1}$, unde tota series nostrarum formularum integralium abibit in hanc

$$\frac{1}{0},\ \frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7}\ \text{ etc.}$$

§.209. Sumamus etiam $\alpha=\frac{1}{2}$ et iam non amplius opus erit ad y procedere. Hoc igitur casu erit

$$\mathrm{Q}=\frac{(1-x)^{2}}{xx}\quad\text{et}\quad\partial v=\frac{(1-x)\partial x}{xx},$$

unde formula nostra generalis fit

$$\int x^{n-1}\partial v=\int x^{n-3}\left(1-x\right)\partial x,$$

cuius ergo valor algebraice expressus erit

$$\frac{1}{n-2}x^{n-2}-\frac{1}{n-1}x^{n-1}=\frac{1}{(n-1)(n-2)},$$

unde series nostrarum formularum evadet

$$\frac{1}{0\cdot -1},\ \frac{1}{0\cdot 1},\ \frac{1}{1\cdot 2},\ \frac{1}{2\cdot 3},\ \frac{1}{3\cdot 4},\ \frac{1}{4\cdot 5},\ \frac{1}{5\cdot 6},\ \frac{1}{6\cdot 7}\ \text{ etc.}$$

EXEMPLUM 3

§.210. *Quaerantur formulae integrales, ut sit*

$$\int x^n \partial v = n\int x^{n-1}\partial v.$$

Cum igitur esse debeat $n\int x^{n-1}\partial v = \int x^n \partial v$, erit $\alpha = 1,\ a = 0,\ b = 1,\ \beta = 0$. Cum igitur sit $\beta = 0$, casus Corollarii 2 hic locum habet indeque erit $\mathrm{Q} = e^{-x}$ ideoque $\mathrm{V} = e^{-x}x^n$, quae quantitas his duobus casibus evanescit $x = 0$ et $x = \infty$. Porro vero erit $\partial v = e^{-x}\partial x$ hincque formula nostra generalis fiet $\int x^{n-1}e^{-x}\partial x$, unde ipsi seriei termini ab initio sequenti modo se habebunt:

$$\int e^{-x}\partial x,\quad \int e^{-x}x\partial x,\quad \int e^{-x}xx\partial x,\quad \int e^{-x}x^3\partial x,\quad \text{etc.}$$

quibus integratis ita, ut evanescant posito $x = 0$, tum vero posito $x = \infty$ orietur sequens series satis simplex

$$1,\ 1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4,\ 1\cdot 2\cdot 3\cdot 4\cdot 5\ \text{etc.},$$

quae est series hypergeometrica WALLISII, cuius ergo terminus generalis est

$$\int x^{n-1}e^{-x}\partial x = 1\cdot 2\cdot 3\cdot 4\cdots(n-1).$$

§. 211. Ope ergo huius termini generalis hanc seriem interpolare licebit. Ita si quaeratur terminus medius inter duos primos, poni debet $n = \frac{3}{2}$ ac valor huius termini erit $\int e^{-x}\sqrt{x}\partial x$, cuius autem valor nullo modo algebraice exprimi potest. Inveni autem singulari modo hunc ipsum terminum aequari $\frac{1}{2}\sqrt{\pi}$ denotante *n* peripheriam circuli, cuius diameter $= 1$, unde hic vicissim cognoscimus esse $\int e^{-x}\sqrt{x}\partial x = \frac{\sqrt{\pi}}{2}$, posito scilicet post integrationem $x = \infty$. Terminus autem hunc praecedens indici $\frac{1}{2}$ respondens erit $= \sqrt{\pi}$, cui ergo aequatur formula $\int \frac{e^{-x}\partial x}{\sqrt{x}}$. Quodsi hic ponamus $e^x = y$, ita ut posito $x = 0$, sit $y = 1$, at posito $x = \infty$ fiat $y = \infty$, tum ergo ista formula $\int \frac{e^{-x}\partial x}{\sqrt{x}}$ abit in hanc $\int \frac{\partial y}{yy\sqrt{ly}}$, quae formula, si ita integretur, ut evanescat posito $y = 1$, tum vero fiat $y = \infty$, praebet valorem ipsius $\sqrt{\pi}$. Si porro fiat $y=\frac{1}{z}$, erunt termini integrationis $z=1$ et $z=0$ et formula integralis erit

$$-\int\frac{\partial z}{\sqrt{-lz}}\begin{bmatrix}\text{a} & z=1\\ \text{ad} & z=0\end{bmatrix}=\sqrt{\pi}\ ,$$

sive permutatis terminis integrationis erit

$$\int\frac{\partial z}{\sqrt{-lz}}\begin{bmatrix}\text{a} & z=1\\ \text{ad} & z=0\end{bmatrix}=\sqrt{\pi}\ ,$$

quemadmodum iam olim observari.

EXEMPLUM 4

§.212. *Quaerantur formulae integrales, ut sit*

$$\int x^n\partial v=\frac{1}{n}\int x^{n-1}\partial v\ \ sive\ \int x^{n-1}\partial v=n\int x^n\partial v.$$

Hic est $\alpha=0$ et $a=1,\beta=1$ et $b=0$; qui ergo est casus in Corollario 1 tractatus, unde colligitur fore $\mathrm{Q}=e^{\frac{1}{x}}$ ideoque $\mathrm{V}=x^ne^{\frac{1}{x}}$, quae formula nequidem evanescit sumto $x=0$, quandoquidem formula $e^{\frac{1}{0}}$ aequivalet infinito infinitesimae potestatis. Hic autem miro modo evenit, ut casus $x=-0$ reddat formulam $e^{-\frac{1}{0}}$ subito evanescentem. Scilicet si ω denotet quantitatem infinite parvam, erit $e^{\frac{1}{\omega}}=\infty$, tum vero repente fiet $e^{-\frac{1}{\omega}}=\frac{1}{\infty}=0$, quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem $\partial v=-e^{\frac{1}{x}}\frac{\partial x}{x}$, ita ut formula nostra generalis futura $-\int x^{n-2}\partial xe^{\frac{1}{x}}$, quae autem nobis nullum usum praestare potest.

§.213. Quodsi hic ponamus $\frac{1}{x}=y$, formula ista generalis transit in hanc $+\int\frac{e^y\partial y}{y^n}$ At vero nunc erit $\mathrm{V}=\frac{e^y}{y^n}$, quae formula evanescit posito $y=-\infty$. Quomodocunque autem hanc expressionem transformemus, semper idem incommodum occurret. Interim tamen etiam hunc casum sequenti modo resolvere licebit. Sit enim seriei, quam quaerimus, primus terminus $=\omega$, ex quo per regulam praescriptam sequentes ordine ita procedent

$$\begin{matrix}1 & 2 & 3 & 4 & 5 & & n\\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1\cdot 2}, & \frac{\omega}{1\cdot 2\cdot 3}, & \frac{\omega}{1\cdot 2\cdot 3\cdot 4}, & \cdots & \frac{\omega}{1\cdot 2\cdot 3\cdots(n-1)}.\end{matrix}$$

Supra autem vidimus huius formulae $1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(n-1)$ valorem exprimi per hoc integrale $\int x^{n-1}e^{-x}\partial x$ integratione ab $x=0$ ad $x=\infty$ extensa; tantum igitur opus est, ut hanc formulam integralem in denominatorem transferamus, et seriei, quam quaerimus, terminus generalis erit

$$\frac{1}{\int x^{n-1}e^{-x}\partial x},$$

unde satis intelligitur negotium non per simplicem formulam integralem expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantitas V non duobus casibus evanescere potest; tum enim tantum opus est fractionem $\frac{\alpha n+a}{\beta n+b}$ invertere atque formulam integralem in denominatorem transferre.

SCHOLION

§.214. Nisi sit vel $\alpha=0$ vel $\beta=0$, quos casus iam expedivimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae α et β sunt aequales unitati. Cum enim esse debeat

$$\int x^n\partial v=\frac{\alpha n+a}{\beta n+b}\int x^{n-1}\partial v,$$

ponatur $x=\frac{\alpha y}{\beta}$ fietque

$$\frac{\alpha}{\beta}\int y^n\partial v=\frac{\alpha n+a}{\beta n+b}\int y^{n-1}\partial v,$$

quae aequatio reducitur ad hanc formam

$$\int y^n\partial v=\frac{n+a:\alpha}{n+b:\beta}\int y^{n-1}\partial v,$$

Quodsi iam nunc loco $\frac{a}{\alpha}$ scribamus a et b loco $\frac{b}{\beta}$, resolvenda erit haec formula

$$\int y^n\partial v=\frac{n+a}{n+b}\int y^{n-1}\partial v,$$

cuius resolutio, si loco x scribamus y et loco litterarum α et β unitatem, ex superiori solutione praebet primo

$$\mathrm{Q}=\mathrm{C}y^a(1-y)^{b-a}$$

quod ergo evanescit posito $y=1$, si modo fuerit $b>a$; tum autem erit ipsa formula

$$\int y^{n-1}\partial v=\mathrm{C}\int y^{n+a-1}\partial y(1-y)^{b-a-1};$$

sin autem fuerit $b<a$, haec solutio, uti vidimus, locum habere nequit; verum hoc casu pro termino nostrae seriei assumi debet haec forma $\frac{1}{\int y^{n-1}\partial v}$, ita ut tum esse debeat

$$\frac{1}{\int y^{n}\partial v}=\frac{n+a}{n+b}\cdot\frac{1}{\int y^{n-1}\partial v}$$

sive

$$\int y^{n}\partial v=\frac{n+b}{n+a}\cdot\int y^{n-1}\partial v$$

cuius resolutio permutatis litteris *a* et *b* praebet

$$\mathrm{Q}=\mathrm{C}y^{b}\left(1-y\right)^{a-b}$$

quae iam casu $y=1$ evanescit, si fuerit $a>b$; atque tum erit formula generalis

$$\int y^{n-1}\partial v=\mathrm{C}\int y^{n+b-1}\partial y\left(1-y\right)^{a-b-1}.$$

Sive igitur sit $b>a$ sive $a>b$, solutio nulla amplius laborat difficultate.

§. 215. Sin autem fuerit vel $\alpha=0$ vel $\beta=0$, loco alterius etiam scribi poterit unitas; unde si esse debeat

$$\int x^{n}\partial v=\frac{n+a}{b}\cdot\int x^{n-1}\partial v,$$

ob $\alpha=1$ et $\beta=0$ solutio nostra generalis dat

$$\frac{\partial\mathrm{Q}}{\mathrm{Q}}=\frac{\partial x}{x}\left(a-bx\right),$$

unde colligitur $\mathrm{Q}=\mathrm{C}x^{a}e^{-bx}$, quae formula evanescit posito $x=\infty$, si modo *b* fuerit numerus positivus; tum autem fit terminus generalis

$$\int x^{n-1}\partial v=\mathrm{C}\int x^{n+a-1}\partial xe^{-bx}.$$

At vero numerus *b* negativus esse nequit, quia alioquin conditio praescripta esset incongrua.

§.216. Consideremus etiam alterum casum, quo $\alpha=0$ et $\beta=1$ ideoque conditio praescripta

$$\int x^{n}\partial v=\frac{a}{n+b}\cdot\int x^{n-1}\partial v$$

unde fit

$$\frac{\partial Q}{Q}=\frac{-\partial x}{xx}(a-bx).$$

Hinc autem pro *Q* orietur valor, qui praeter casum $x=0$ evanescere non posset; quam ob causam formula generalis statui debet $\frac{1}{\int x^{n-1}\partial v}$, ita ut esse debeat

$$\int x^n\partial v=\frac{n+b}{a}\int x^{n-1}\partial v,$$

unde prodit

$$\frac{\partial Q}{Q}=\frac{\partial x}{x}(b-ax)\quad \text{ideoque } Q=Ce^{-ax}x^b,$$

quae expressio evanescit posito $x=\infty$, quoniam *a* necessaria debet esse numerus positivus; tum autem erit

$$dv=Ce^{-ax}x^b\partial x,$$

unde formula generalis seriei erit

$$\frac{1}{C\int x^{n+b-1}\partial xe^{-ax}}.$$

PROBLEMA 2

§.217. *Denotet* T *terminum indici n respondentem in serie, quam considerandam suscepimus, at vero* T' *terminum sequentem atque proponatur haec conditio adimplenda*

$$T'=\frac{(\alpha n+a)(\alpha' n+a')}{(\beta n+b)(\beta' n+b')}T.$$

SOLUTIO

Quoniam hic valores geminati occurrunt, huic conditioni commodissime satisfiet, si terminus generalis T tanquam productum ex duobus factoribus spectetur. Statuatur igitur $T=RS$ sitque terminus sequens $=R'S'$ et quaerantur formulae R et S, ut fiat

$$R'=\frac{\alpha n+a}{\beta n+b}R\quad \text{et}\quad S'=\frac{\alpha' n+a'}{\beta' n+b'}S;$$

tum enim sumendo $T=RS$ conditioni praescriptae manifesto satisfiet. Hoc igitur modo pro R et S vel huiusmodi formulae $\int x^{n-1}\partial v$ vel inversae $\frac{1}{\int x^{n-1}\partial v}$ reperientur, id quod pro solutione generali sufficit, unde rem exemplo illustremus.

EXEMPLUM

§.218. *Quaeratur formula generalis* T, *ut fiat*

$$\mathrm{T}' = \frac{nn-cc}{nn}\mathrm{T}.$$

Resolvamus igitur T in duos factores R et S ac statuamus

$$\mathrm{R}' = \frac{n\text{-}c}{n}\mathrm{R} \ \text{ et } \ \mathrm{S}' = \frac{n+c}{n}\mathrm{S}.$$

Pro priore forma si statuamus $\mathrm{R} = \int x^{n-1}\partial v$, ex solutione generali, ubi erit $\alpha = 1, a = -c, \ \beta = 1$ et $b = 0$, fiet

$$\mathrm{Q} = \mathrm{C}x^{-c}\left(1-x\right)^{c},$$

quae forma manifesto evanescit posito $x = 1$; hincque quia fit

$$\mathrm{V} = \mathrm{C}x^{n-c}\left(1-x\right)^{c}$$

haec forma etiam casu $x = 0$ evanescit, si modo *n* fuerit > *c*, id quod tuto assumi potest, quia exponentem *n* successive in infinitum crescere assumimus ac plerumque pro *c* fractiones tantum accipi solent. Hinc ergo erit

$$\mathrm{R} = \mathrm{C}\int x^{n-c-1}\left(1-x\right)^{c-1}\partial x\,.$$

§.219 Hinc iam alter valor litterae *S* deduci posset scribendo tantum $-c$ loco *c*, tum autem non amplius fieret $Q = 0$ posito $x = 1$, quamobrem pro S formulam inversam $\frac{1}{\int x^{n-1}dv}$ assumi oportet, ut esse debeat

$$\int x^{n}\partial v = \frac{n}{n+c}\int x^{n-1}\partial v;$$

ubi cum sit $\alpha = 1, a = 0, \beta = 1$ et $b = c$, reperitur

$$\mathrm{Q} = \mathrm{C}\left(1-x\right)^{c}$$

quae forma manifesto fit $= 0$ posito $x = 1$; hinc autem prodit

$$\partial v = \mathrm{C}(1-x)^{c-1}\partial x\,,$$

ergo habebimus

$$S = \frac{1}{\mathrm{C}\int x^{n-1}(1-x)^{c-1}\partial x};$$

consequenter formula nostra generalis quaesita erit

$$\mathrm{T}=\frac{\int x^{n-c-1}(1-x)^{c-1}\partial x}{\int x^{n-1}(1-x)^{c-1}\partial x}$$

§.220. Quodsi ergo nostrae seriei per factores procedentis primum terminum ponamus $=\mathrm{A}$, ipsa series erit

1	2	3	4	
A,	$\frac{1-cc}{1}\mathrm{A}$,	$\frac{1-cc}{1}\cdot\frac{4-cc}{4}\mathrm{A}$,	$\frac{1-cc}{1}\cdot\frac{4-cc}{4}\cdot\frac{9-cc}{9}\mathrm{A}$	etc.;

unde si sumamus $c=\frac{1}{2}$, erit haec series

$$\mathrm{A},\quad \frac{1\cdot3}{2\cdot2}\mathrm{A},\quad \frac{1\cdot3}{2\cdot2}\cdot\frac{3\cdot5}{4\cdot4}\mathrm{A},\quad \frac{1\cdot3}{2\cdot2}\cdot\frac{3\cdot5}{4\cdot4}\cdot\frac{5\cdot7}{6\cdot6}\mathrm{A}\quad \text{etc.;}$$

cuius ergo terminus indici *n* respondens est

$$\frac{\int x^{n-\frac{3}{2}}(1-x)^{-\frac{1}{2}}\partial x}{\int x^{n-1}(1-x)^{-\frac{1}{2}}\partial x},$$

qui posito $x=yy$ transit in hanc formam

$$\frac{\int y^{2n-2}(1-y)^{-\frac{1}{2}}\partial y}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}\partial y},$$

unde patet terminum primum fore

$$\mathrm{A}=\int\frac{\partial y}{\sqrt{(1-yy)}}:\int\frac{y\partial y}{\sqrt{(1-yy)}}=\frac{\pi}{2},$$

posito scilicet post integrationem $y=1$.

PROBLEMA 3

§.221. *Denotet* T *terminum seriei indici n respondentem sintque* T' *et* T'' *termini sequentes pro indicibus* $n+1$ *et* $n+2$; *si proponatur inter ternos terminos se insequentes talis relatio, ut sit*

$$(\alpha n+a)\mathrm{T}=(\beta n+b)\mathrm{T}'+(\gamma n+c)\mathrm{T}'',$$

investigare formulam pro T, *qua terminus generalis huius seriei exprimatur.*

SOLUTIO

Assumatur pro T formula integralis $\int x^{n-1}\partial v$ huiusque integrale ita capiatur, ut evanescat posito $x=0$, eruntque termini sequentes $\mathrm{T}'=\int x^{n}\partial v$ et $\mathrm{T}''=\int x^{n+1}\partial v$, siquidem post integrationem variabili *x* certus valor determinatus tribuatur. Quamdiu autem haec quantitas *x* ut variabilis spectatur, ponamus esse

$$(\alpha n+a)\mathrm{T}=(\beta n+b)\mathrm{T}'+(\gamma n+c)\mathrm{T}''+x^{n}\mathrm{Q}$$

ac perspicuum est Q eiusmodi functionem esse debere ipsius *x*, quae evanescat, si loco *x* valor ille determinatus substituatur, quem autem a cyphra diversum esse oportet, quoniam iam assumsimus omnes istas formulas in nihilum abire posito $x=0$. Quodsi vero absoluto calculo huic conditioni nullo modo satisfieri poterit, id erit indicatio problema nostrum hac ratione resolvi non posse, ut scilicet eius terminus generalis T per talem formulam differentialem simplicem $\int x^{n-1}\partial v$ exhibeatur.

§.222. Differentiemus nunc aequationem modo stabilitam ac divisione fecta per x^{n-1} sequens prodibit aequatio

$$(\alpha n+a)\partial v=(\beta n+b)x\partial v+(\gamma n+c)xx\partial v+n\mathrm{Q}\partial x+x\partial\mathrm{Q},$$

quae, quia termini littera *n* affecti seorsim se destruere debent, discerpetur in binas sequentes aequationes

$$1.\ \alpha\partial v=\beta x\partial v+\gamma xx\partial v+\mathrm{Q}\partial x,$$
$$2.\ a\partial v=bx\partial v+cxx\partial v+x\partial\mathrm{Q},$$

ex quarum priore fit

$$\partial v=\frac{\mathrm{Q}\partial x}{\alpha-\beta x-\gamma xx},$$

ex altera vero fit

$$\partial v=\frac{x\partial\mathrm{Q}}{a-bx-cxx},$$

quorum valorum posterior per priorem divisus praebet

$$\frac{\partial\mathrm{Q}}{\mathrm{Q}}=\frac{\partial x(a-bx-cxx)}{x(\alpha-\beta x-\gamma xx)},$$

ex cuius ergo integratione valor ipsius Q elici debet, quo facto facile patebit, utrum is certo quodam casu praeter $x=0$ evanescere possit. Imprimis autem hic notari convenit, si hoc integrale involvat huiusmodi factorem $e^{\frac{1}{x}}$, tum solutionem quoque successu esse cariturам, quandoquidem posito $x=0$ iste factor tantam involvet infiniti potestatem, ut, etiamsi per x^n multiplicetur, productum etiamnum infinitum maneat.

§.223. Quodsi igitur his conditionibus praescriptis satisfacere licuerit, tum invento valore litterae Q, quem ponamus fieri $=0$ posito $x=f$, habebitur

$$\partial v=\frac{Q\partial x}{\alpha-\beta x-\gamma xx}$$

et formula generalis naturam seriei complectens erit

$$T=\int x^{n-1}\partial v=\int\frac{x^{n-1}Q\partial x}{\alpha-\beta x-\gamma xx}$$

quippe cuius integrale a termino $x=0$ usque ad terminum $x=f$ extensum praebebit valorem termini T indici cuicunque *n* respondentis.

SCHOLION

§.224. Inventa autem tali relatione inter ternos terminos cuiuspiam seriei sibi invicem succedentes inde more solito formari poterit fractio continua, cuius valorem assignare licebit. Si enim characteres.

$$T',\ T'',\ T''',\ T''''\ \text{etc.}$$

denotent ordine omnes terminos post *T* sequentes in infinitum, ex relationibus, quas inter se tenent, sequentes formulae deducentur. Ex relatione

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T''$$

deducitur

$$(\alpha n+a)\frac{T}{T'}=\beta n+b+\frac{(\gamma n+c)(\alpha n+\alpha+a)}{(\alpha n+\alpha+a)T':T''}$$

Ex relatione sequente

$$(\alpha n+\alpha+a)T'=(\beta n+\beta+b)T''+(\gamma n+\gamma+c)T'''$$

deducitur

$$(\alpha n+\alpha+a)\frac{T'}{T''}=\beta n+\beta+b+\frac{(\gamma n+\gamma+c)(\alpha n+2\alpha+a)}{(\alpha n+2\alpha+a)T'':T'''}.$$

Simili modo sequentes relationes suppeditabunt

$$(\alpha n+2\alpha+a)\frac{\mathrm{T''}}{\mathrm{T'''}}=\beta n+2\beta+b+\frac{(\gamma n+2\gamma+c)(\alpha n+3\alpha+a)}{(\alpha n+3\alpha+a)\mathrm{T'''}:\mathrm{T''''}},$$

$$(\alpha n+3\alpha+a)\frac{\mathrm{T'''}}{\mathrm{T''''}}=\beta n+3\beta+b+\frac{(\gamma n+3\gamma+c)(\alpha n+4\alpha+a)}{(\alpha n+4\alpha+a)\mathrm{T''''}:\mathrm{T'''''}};$$

unde manifestum est, si in prima formula continua sequentes valores ordine substituantur, prodituram esse fractionem continuam, cuius valor aequalis erit formulae $(\alpha n+a)\frac{\mathrm{T}}{\mathrm{T'}}$.

§.225. Quodsi ergo loco *n* successive scribamus numeros 1, 2, 3, 4 etc., sequens problema circa fractiones continuas resolvere poterimus.

PROBLEMA 4

Proposita fractione continua huius formae

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{etc.}}}}}}$$

eius valorem investigare.

SOLUTIO

§.226.Consideretur in genere ista relatio inter ternas quantitates sibi succedentes T, T', T'', quae sit

$$(\alpha n+a)\mathrm{T}=(\beta n+b)\mathrm{T}'+(\gamma n+c)\mathrm{T}'',$$

atque ex praecedente problemate quaeratur valor ipsius *T,* siquidem fieri potest, hoc modo expressus .

$$\mathrm{T}=\int x^{n-1}\partial v=\int\frac{x^{n-1}\mathrm{Q}\partial v}{a-\beta x-\gamma xx},$$

cuius integrale ab $x=0$ usque ad $x=f$ extendatur, qua formula inventa ponatur

$$\int\frac{\mathrm{Q}\partial x}{a-\beta x-\gamma xx}=\mathrm{A}\quad\text{et}\quad\int\frac{x\mathrm{Q}\partial x}{a-\beta x-\gamma xx}=\mathrm{B},$$

ita ut A et B sint valores ipsius T pro casibus $n=1$ et $n=2$; quibus definitis fractionis continuae propositae valor per praecedentia erit $=\frac{(\alpha+a)\mathrm{A}}{\mathrm{B}}$.
Hanc igitur investigationem ad sequentia exempla accommodemus.

EXEMPLUM 1

§.227. *Investigare valorem fractionis continuae notissimae, quam olim BROUNCKERUS pro quadratura circuli protulit, quae est*

$$2+\cfrac{1\cdot1}{2+\cfrac{3\cdot3}{2+\cfrac{5\cdot5}{2+\text{etc.}}}}$$

Quia omnes partes integrae laevam respicientes sunt constantes $=2$, pro nostra forma generali fiet

$$\beta+b=2,\ \ 2\beta+b=2,\ \ 3\beta+b=2\ \ \text{etc.};$$

erit ergo $\beta=0$ et $b=2$; at pro numeratoribus sequentium fractionum, quandoquidem constant binis factoribus, erit pro factoribus prioribus

$$\gamma+c=1,\ 2\gamma+c=3,\ 3\gamma+c=5,\ 4\gamma+c=7\ \text{etc.},$$

unde concluditur $\gamma=2$ et $c=-1$, pro alteris vero erit

$$2\alpha+a=1,\ 3\alpha+a=3,\ 4\alpha+a=5\ \text{etc.},$$

unde $\alpha=2$ et $a=-3$. Ex his autem valoribus colligimus hanc aequationem

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}\left[=\frac{\partial x(a-bx-cxx)}{x(\alpha-\beta x-\gamma xx)}\right]=-\frac{\partial x(3+2x-xx)}{2x(1-xx)}$$

quae per $1+x$ depressa praebet

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{\partial x(3-x)}{2x(1-x)},$$

unde integrando fit

$$l\mathrm{Q}=-\tfrac{3}{2}lx+l(1-x)\ \text{et hinc}\ \mathrm{Q}=\frac{1-x}{x^{\frac{3}{2}}},$$

ex quo valore porro sequitur

$$A = \int \frac{(1-x)\partial x}{2x^{\frac{3}{2}}(1-xx)} = \int \frac{\partial x}{2x(1+x)\sqrt{x}}$$

$$B = \int \frac{(1-x)\partial x}{2x^{\frac{1}{2}}(1-xx)} = \int \frac{\partial x}{2(1+x)\sqrt{x}}.$$

§.228. In his autem valoribus istud incommodum deprehenditur, quod prius integrale evanescens reddi nequit posito $x = 0$. Hoc autem incommodum facile removeri potest, si fractionem continuam supremo membro truncemus et quaeramus valorem istius fractionis

$$2 + \frac{3\cdot 3}{2 + \frac{5\cdot 5}{2 + \text{etc.};}}$$

qui si repertus fuerit $= s$, erit ipsius propositae valor $= b + \frac{1}{s}$. Nunc vero comparatione instituta fit quidem ut ante $\beta = 0$ et $b = 2$, tum vero $\gamma = 2$ et $c = +1$, $\alpha = 2$ et $a = -1$, unde sequitur

$$\frac{\partial Q}{Q} = -\frac{\partial x(1+2x+xx)}{2x(1-xx)} = -\frac{\partial x(1+x)}{2x(1-x)},$$

unde integrando fit

$$lQ = -\tfrac{1}{2}lx + l(1-x) \text{ ideoque } Q = \frac{1-x}{\sqrt{x}},$$

ex quo valore iam habebimus

$$A = \int \frac{(1-x)\partial x}{2(1-xx)\sqrt{x}} = \frac{1}{2}\int \frac{\partial x}{(1+x)\sqrt{x}}$$

et

$$B = \frac{1}{2}\int \frac{\partial x\sqrt{x}}{1+x};$$

ubi cum sit $Q = \frac{1-x}{\sqrt{x}}$, eius valor manifesto evanescit posito $x = 1$, quamobrem illa integralia a termino $x = 0$ usque ad $x = 1$ sunt extendenda.

§.229. Quo nunc haec integralia facilius eruamus, statuamus $x = zz$, ita ut termini integrationis etiamnunc sint $z = 0$ et $z = 1$, eritque

$$A = \int \frac{dz}{1+zz} = \text{Atang.}z = \frac{\pi}{4}$$

et

$$B = \int \frac{zzdz}{1+zz} = 1 - \frac{\pi}{4}$$

sicque habebimus $s=\frac{\pi}{4-\pi}$, quocirca ipsius fractionis BROUNCKERIANAE valor est $1+\frac{4}{\pi}$, omnino uti olim BROUNCKERUS iam invenerat.

EXEMPLUM 2

§.230. *Investigare valorem huius fractionis continuae BROUNOKERIANAE latius patentis*

$$b+\cfrac{1\cdot 1}{b+\cfrac{3\cdot 3}{b+\cfrac{5\cdot 5}{b+\text{etc.}}}}$$

Ut hic incommodum superius evitemus, omittamus membrum supremum et quaeramus

$$s=b+\cfrac{3\cdot 3}{b+\cfrac{5\cdot 5}{b+\text{etc.},}}$$

quandoquidem tum erit valor quaesitus $=b+\frac{1}{s}$. Nunc igitur erit $\beta=0$ et $b=b, \gamma=2,\ c=1,\ \alpha=2$ et $a=-1$, unde fit

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{\partial x(1+bx+xx)}{2x(1-xx)}$$

ac proinde

$$lQ=-\tfrac{1}{2}lx-\tfrac{b-2}{4}l(1+x)+\tfrac{b+2}{4}l(1-x)$$

hincque

$$Q=\frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}}\sqrt{x}},$$

quae formula manifesto fit $=0$ ponendo $x=1$, siquidem $b+2$ fuerit numerus positivus, unde fit

$$\partial v=\frac{(1-x)^{\frac{b-2}{4}}\partial x}{2(1+x)^{\frac{b+2}{4}}\sqrt{x}}.$$

Hinc autem colligetur

$$\mathrm{A}=\frac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}\partial x}{(1+x)^{\frac{b+2}{4}}\sqrt{x}}\quad\text{et}\quad\mathrm{B}=\frac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}\partial x\sqrt{x}}{(1+x)^{\frac{b+2}{4}}}$$

sive ponendo $x=zz$ habebimus

$$A=\int\frac{(1-zz)^{\frac{b-2}{4}}\partial z}{(1+zz)^{\frac{b+2}{4}}}\quad\text{et}\quad B=\int\frac{(1-zz)^{\frac{b-2}{4}}zz\partial z}{(1+zz)^{\frac{b+2}{4}}},$$

quae ambo integralia a $z=0$ usque ad $z=1$ sunt extendenda. Ex his autem valoribus A et B erit $s=\frac{A}{B}$; ipsius igitur fractionis propositae valor erit $=b+\frac{1}{s}=b+\frac{B}{A}$.

§.231. Quodsi hic ponatur $b=2$, prodit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes $\frac{b-2}{4}$ et $\frac{b+2}{4}$ non sunt numeri integri, tum litteras A et B neque per arcus circulares neque per logarithmos exprimere licet. Veluti si fuerit $b=4$, erit

$$A=\int\frac{\partial z\sqrt{(1-zz)}}{(1+zz)^{\frac{3}{2}}},$$

cuius valor per arcus ellipticos exhiberi posset. At si b fuerit numerus impar, hi valores multo magis evadunt transcendentes, ita ut his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi fiant numeri integri, totum negotium per arcus circulares expedire licebit.

§.232. Exponentes autem illi $\frac{b-2}{4}$ et $\frac{b+2}{4}$ erunt numeri integri, quoties fuerit b numerus huius formae

$$b=4i+2;$$

tum enim erit

$$A=\int\frac{(1-zz)^{i}\partial z}{(1+zz)^{i+1}}\quad\text{et}\quad B=\int\frac{(1-zz)^{i}zz\partial z}{(1+zz)^{i+1}};$$

quos ergo casus quomodo evolvi oporteat, operae pretium erit docere, quoniam Wallisius eos iam est contemplatus.

§.233. Quoniam hoc negotium totum redit ad reductionem huiusmodi formularum integralium ad formas simpliciores, consideremus in genere formam $P=\frac{z^{m}}{(1+zz)^{n}}$, cuius differentiale sub sequentibus formis exhiberi potest:

$$1.\quad\partial P=\frac{mz^{m-1}\partial z}{(1+zz)^{n}}-\frac{2nz^{m+1}\partial z}{(1+zz)^{n+1}},$$

$$2.\quad\partial P=\frac{mz^{m-1}\partial z}{(1+zz)^{n+1}}-\frac{(2n-m)z^{m+1}\partial z}{(1+zz)^{n+1}},$$

$$3.\quad\partial P=-\frac{(2n-m)z^{m-1}\partial z}{(1+zz)^{n}}+\frac{2nz^{m-1}\partial z}{(1+zz)^{n+1}},$$

unde hanc triplicem reductionem integralium deducimus

$$\text{I.}\quad \int \frac{z^{m+1}\partial z}{(1+zz)^{n+1}} = \frac{m}{2n}\int \frac{z^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{1}{2n}\frac{z^m}{(1+zz)^n},$$

$$\text{II.}\quad \int \frac{z^{m+1}\partial z}{(1+zz)^{n+1}} = \frac{m}{2n-m}\int \frac{z^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{1}{2n-m}\frac{z^m}{(1+zz)^n},$$

$$\text{III.}\quad \int \frac{z^{m-1}\partial z}{(1+zz)^{n+1}} = \frac{2n-m}{2n}\int \frac{z^{m-1}\partial z}{(1+zz)^{n}} + \frac{1}{2n}\frac{z^m}{(1+zz)^n},$$

quarum reductionum ope casibus $b = 4i+2$ totum negotium absolvi et ad formulam $\frac{\pi}{4}$: reduci poterit, siquidem post integrationem sumatur $z = 1$.

§.234. Sit $i = 1$ ideoque $b = 6$ eritque

$$\text{A} = \int \frac{(1-zz)\partial z}{(1+zz)^2} \quad \text{et} \quad \text{B} = \int \frac{(1-zz)zz\partial z}{(1+zz)^2}.$$

Nunc igitur reperiemus per reductionem tertiam

$$\int \frac{\partial z}{(1+zz)^2} = \frac{1}{2}\int \frac{\partial z}{1+zz} + \frac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} + \frac{1}{4}$$

et per reductionem primam

$$\int \frac{zz\partial z}{(1+zz)^2} = \frac{1}{2}\int \frac{\partial z}{1+zz} - \frac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} - \frac{1}{4},$$

porro

$$\int \frac{z^4\partial z}{(1+zz)^2} = \frac{3}{2}\int \frac{zz\partial z}{1+zz} - \frac{1}{2}\cdot\frac{z^3}{1+zz} = \frac{5}{4} - \frac{3\pi}{8}.$$

Ex his iam valoribus colligitur $\text{A} = \frac{1}{2}$ et $\text{B} = \frac{\pi}{2} - \frac{3}{2}$ ideoque $\frac{\text{B}}{\text{A}} = \pi - 3$,
quocirca orietur ista summatio

$$3 + \pi = 6 + \cfrac{1\cdot1}{6+\cfrac{3\cdot3}{6+\cfrac{5\cdot5}{6+\cfrac{7\cdot7}{6+\text{etc.}}}}}$$

§.235. Sit nunc $i = 2$ et $b = 10$ eritque

$$\text{A} = \int \frac{(1-zz)^2\partial z}{(1+zz)^3} \quad \text{et} \quad \text{B} = \int \frac{zz(1-zz)^2\partial z}{(1+zz)^3}.$$

Quo harum integralium valores investigemus, sequentes evolvamus formulas

$$\int \frac{\partial z}{(1+zz)^3} = \frac{3}{4}\int \frac{\partial z}{(1+zz)^2} + \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{3\pi}{32} + \frac{1}{4},$$

$$\int \frac{zz\partial z}{(1+zz)^3} = \frac{1}{4}\int \frac{\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{\pi}{32},$$

$$\int \frac{z^4\partial z}{(1+zz)^3} = \frac{3}{4}\int \frac{zz\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^3}{(1+zz)^2} = \frac{3\pi}{32} - \frac{1}{4},$$

$$\int \frac{z^6\partial z}{(1+zz)^3} = \frac{5}{4}\int \frac{z^4\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^5}{(1+zz)^2} = \frac{3}{2} - \frac{15\pi}{32}.$$

Ex quibus iam valoribus deducitur $\mathrm{A} = \frac{\pi}{8}$ et $\mathrm{B} = 2 - \frac{5\pi}{8}$ ideoque $\frac{\mathrm{B}}{\mathrm{A}} = \frac{16-5\pi}{\pi}$, unde emergit sequens summatio

$$\frac{5\pi+16}{\pi} = 10 + \cfrac{1\cdot1}{10+\cfrac{3\cdot3}{10+\cfrac{5\cdot5}{10+\text{etc.}}}}$$

§.236. Si b esset numerus negativus, investigatio nulla prorsus laboraret difficultate. Si enim in genere fuerit

$$s = -a + \cfrac{\alpha}{-b+\cfrac{\beta}{-c+\cfrac{\gamma}{-d+\cfrac{\delta}{-e+\text{etc.},}}}}$$

semper erit

$$-s = a + \cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\text{etc.},}}}}$$

unde, si habeatur valor istius expressionis, idem negative sumtus dabit valorem illius.

EXEMPLUM 3

§.237. *Proposita sit fractio continua, cuius valorem investigari oporteat, ista*

$$1 + \cfrac{1\cdot1}{3+\cfrac{3\cdot3}{5+\cfrac{5\cdot5}{7+\cfrac{7\cdot7}{9+\text{etc.}}}}}$$

Quo fractiones supra [§ 225] allegatas [adhibeamus,] omisso membro supremo sit

$$s = 3 + \cfrac{3\cdot3}{5+\cfrac{5\cdot5}{7+\cfrac{7\cdot7}{9+\text{etc.}}}}$$

eritque $\beta+b=3,\ 2\beta+b=5$ ideoque $\beta=2$ et $b=1$, tum vero ut ante $\alpha=2,\ a=-1,\ \gamma=2$ et $c=+1$; invento autem s erit valor quaesitus $=1+\frac{1}{s}$. Nunc igitur habebimus

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{\partial x(1+x+xx)}{2x(1-x-xx)}.$$

Est vero

$$\frac{1+x+xx}{x(1-x-xx)}=\frac{1}{x}+\frac{2+2x}{1-x-xx},$$

unde fit

$$l\mathrm{Q}=-\tfrac{1}{2}lx-\int\frac{\partial x(1+x)}{1-x-xx}.$$

Porro vero pro formula $\int\frac{\partial x(1+x)}{1-x-xx}$ invenienda statuamus denominatorem

$$1-x-xx=(1-fx)(1-gx)$$

eritque $f+g=-1$ et $fg=-1$, unde fit

$$f=\frac{1+\sqrt{5}}{2} \text{ et } g=\frac{1-\sqrt{5}}{2}.$$

Nunc statuatur

$$\frac{1+x}{1-x-xx}=\frac{\mathfrak{A}}{1-fx}+\frac{\mathfrak{B}}{1-gx},$$

unde reperietur

$$\mathfrak{A}=\frac{1+f}{f-g} \text{ et } \mathfrak{B}=-\frac{1+g}{f-g},$$

sive substitutis pro f et g valoribus supra datis erit

$$\mathfrak{A}=\frac{\sqrt{5}+3}{2\sqrt{5}} \text{ et } \mathfrak{B}=\frac{\sqrt{5}-3}{2\sqrt{5}},$$

quibus inventis erit

$$\int\frac{\partial x(1+x)}{1-x-xx}=-\frac{\mathfrak{A}}{f}l(1-fx)-\frac{\mathfrak{B}}{g}l(1-gx),$$
$$=-\frac{1+\sqrt{5}}{2\sqrt{5}}l(1-fx)-\frac{\sqrt{5}-1}{2\sqrt{5}}l(1-gx),$$

quocirca fiet

$$l\mathrm{Q}=-\tfrac{1}{2}lx+\frac{\sqrt{5}+1}{2\sqrt{5}}l(1-fx)+\frac{\sqrt{5}-1}{2\sqrt{5}}l(1-gx),$$

consequenter

$$\mathrm{Q}=\frac{(1-fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}}(1-gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

qui valor duobus casibus evanescit, altero, quo

$$x=\tfrac{1}{f}=\tfrac{2}{1+\sqrt{5}}=\tfrac{\sqrt{5}-1}{2},$$

altero vero, quo

$$x=\tfrac{1}{g}=-\tfrac{1+\sqrt{5}}{2};$$

utrovis autem utamur, res eodem redibit.

§.238. Ex hoc autem valore habebimus

$$\mathrm{A}=\int\frac{\mathrm{Q}\partial x}{1-x-xx}\ \text{ et }\ \mathrm{B}=\int\frac{\mathrm{Q}x\partial x}{1-x-xx},$$

unde porro deducitur

$$s=(\alpha+a)\tfrac{\mathrm{A}}{\mathrm{B}}=\tfrac{\mathrm{A}}{\mathrm{B}};$$

hinc propositae fractionis summa erit $1+\frac{\mathrm{B}}{\mathrm{A}}$. Hinc autem nihil ulterius concludere licet ob formulas differentiales non solum irrationales, sed etiam vere transcendentes ob exponentes surdos.

EXEMPLUM 4

§.239. *Proposita sit haec fractio continua*

$$b+\cfrac{1\cdot 1}{b+\cfrac{2\cdot 2}{b+\cfrac{3\cdot 3}{b+\cfrac{4\cdot 4}{b+\text{etc.}}}}}$$

ubi est $\beta=0,\ b=b$.

Nunc consideremus hanc formam

$$s=b+\cfrac{2\cdot 2}{b+\cfrac{3\cdot 3}{b+\text{etc.}}}$$

quippe quo valore invento quaesitus erit $=b+\frac{1}{s}$. Habebimus igitur
$\gamma+c=2,\ 2\gamma+c=3$ ideoque $\gamma=1$ et $c=1,$ deinde erit $\alpha=\gamma=1, a=0$ et $c=1$. Hinc igitur colligimus

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{\partial x(bx+xx)}{x(1-xx)}=-\frac{\partial x(b+x)}{1-xx}$$

ideoque

$$lQ=-\tfrac{b}{2}l\tfrac{1+x}{1-x}+\tfrac{1}{2}l\left(1-xx\right)$$

hincque

$$Q=\frac{(1-x)^{\frac{b}{2}}\sqrt{(1-xx)}}{(1+x)^{\frac{b}{2}}}=\frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

quae quantitas manifesto evanescit posito $x=1$. Hinc igitur fiet

$$A=\int\frac{Q\partial x}{1-xx}=\int\frac{(1-x)^{\frac{b+1}{2}}\partial x}{(1+x)^{\frac{b-1}{2}}(1-xx)}=\int\frac{(1-x)^{\frac{b-1}{2}}\partial x}{(1+x)^{\frac{b+1}{2}}}$$

et

$$B=\int\frac{x(1-x)^{\frac{b-1}{2}}\partial x}{(1+x)^{\frac{b+1}{2}}},$$

tum antem erit $s=\left(\alpha+a\right)\frac{A}{B}=\frac{A}{B}$ ideoque summa quaesita $b+\frac{B}{A}$.

§.240. Percurramus nunc casus praecipuos ac primo sit $b=1$ eritque

$$A=\int\frac{\partial x}{1+x}=l\left(1+x\right)=l2 \text{ et } B=\int\frac{x\partial x}{1+x}=x-\int\frac{\partial x}{1+x}=1-l2$$

ideoque $b+\frac{B}{A}=\frac{1}{l2}$; ergo hinc prodibit ista summatio

$$\frac{1}{l2}=1+\cfrac{1\cdot 1}{1+\cfrac{2\cdot 2}{1+\cfrac{3\cdot 3}{1+\text{etc.}}}}$$

§.241. Sit nunc $b=2$ eritque

$$A=\int\frac{\partial x\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}} \quad \text{et} \quad B=\int\frac{x\partial x\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

Ad has formulas rationales reddendas statuamus

$$\frac{\sqrt{(1-x)}}{\sqrt{(1+x)}}=z$$

eritque $x=\frac{1-zz}{1+zz}$, unde terminis integrationis $x=0$ et $x=1$ respondebunt $z=1$ et $z=0$; tum vero erit

$$1+x=\frac{2}{1+zz} \text{ et } \partial x=-\frac{4z\partial z}{(1+zz)^2}$$

hincque colligitur

$$\mathrm{A}=-2\int\frac{zz\partial z}{1+zz}=-2z+2\mathrm{Atang}.z=2-\frac{\pi}{2},$$

porro fit

$$\mathrm{B}=-2\int\frac{zz\partial z}{(1+zz)^2}+2\int\frac{z^4\partial z}{(1+zz)^2}$$

Per reductiones igitur supra (§.232) monstratas, si hic scilicet terminos integrationis $z=1$ et $z=0$ permutemus, ut habeamus

$$\mathrm{B}=+2\int\frac{zz\partial z}{(1+zz)^2}-2\int\frac{z^4\partial z}{(1+zz)^2}$$

erit

$$\mathrm{B}=2(\tfrac{\pi}{8}-\tfrac{1}{4})-2\left(\tfrac{5}{4}-\tfrac{3\pi}{8}\right)=\pi-3,$$

unde sequitur ista summatio

$$\frac{2}{4-\pi}=2+\cfrac{1\cdot 1}{2+\cfrac{2\cdot 2}{2+\cfrac{3\cdot 3}{2+\cfrac{4\cdot 4}{2+\text{etc.},}}}}$$

quae BROUNCKERIANAE simplicitate nihil cedit.

§.242. Si ponamus $b=0$, fractio continua abit in sequens continuum productum

$$\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 3}{4\cdot 4}\cdot\frac{5\cdot 5}{6\cdot 6}\cdot\frac{7\cdot 7}{88}\cdot\text{etc. };$$

hoc autem casu fit

$$\mathrm{A}=\int\frac{\partial x}{\sqrt{(1-xx)}}=\frac{\pi}{2} \text{ et } B=\int\frac{x\partial x}{\sqrt{(1-xx)}}=1,$$

unde istius producti valor colligitur $\frac{2}{\pi}$, id quod egregie convenit cum iam dudum cognitis, quandoquidem hoc productum est ipsa progressio WALLISIANA.

EXEMPLUM 5

§.243. *Proposita sit haec fractio continua, ubi* $\beta=0,\ b=b$ *et numeratores numeri trigonales,*

$$b+\cfrac{1}{b+\cfrac{3}{b+\cfrac{6}{b+\cfrac{10}{b+\text{etc.}}}}}$$

Omisso supremo membro statuamus

$$s=b+\cfrac{3}{b+\cfrac{6}{b+\cfrac{10}{b+\text{etc.}}}}$$

et primo numeratores per producta repraesentemus hoc modo

$$3=2\cdot\tfrac{3}{2},\quad 6=3\cdot\tfrac{4}{2},\quad 10=4\cdot\tfrac{5}{2},$$

quorum priores comparentur cum formulis $\gamma+c,\ 2\gamma+c,\ 3\gamma+c,$ posteriores vero cum formulis $2\alpha+a,\ 3\alpha+a,\ 4\alpha+a$, eritque $\gamma=1,\ c=1,\ \alpha=\frac{1}{2},\ a=\frac{1}{2}$, unde erit

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=\frac{\partial x\left(\frac{1}{2}-bx-xx\right)}{x\left(\frac{1}{2}-xx\right)}=\frac{\partial x(1-2bx-2xx)}{x(1-2xx)}$$

sive

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=\frac{\partial x}{x}-\frac{2b\partial x}{1-2xx},$$

cuius integrale est

$$l\mathrm{Q}=lx-\frac{b}{\sqrt{2}}\,l\,\frac{1+x\sqrt{2}}{1-x\sqrt{2}},$$

ergo

$$\mathrm{Q}=\frac{x\left(1-x\sqrt{2}\right)^{\frac{b}{\sqrt{2}}}}{\left(1+x\sqrt{2}\right)^{\frac{b}{\sqrt{2}}}},$$

quae formula evanescit casu $x=\frac{1}{\sqrt{2}}$. Hinc igitur erit

$$\partial v=\frac{2x\left(1-x\sqrt{2}\right)^{\frac{b}{\sqrt{2}}}\partial x}{\left(1-2xx\right)\left(1+x\sqrt{2}\right)^{\frac{b}{\sqrt{2}}}}.$$

Sit $\frac{b}{\sqrt{2}}=\lambda$, eritque

$$\mathrm{A}=2\int\frac{x\left(1-x\sqrt{2}\right)^{\lambda}\partial x}{(1-2xx)\left(1+x\sqrt{2}\right)^{\lambda}}=2\int\frac{x\left(1-x\sqrt{2}\right)^{\lambda-1}\partial x}{\left(1+x\sqrt{2}\right)^{\lambda+1}}$$

et

$$\mathrm{B}=2\int\frac{xx\left(1-x\sqrt{2}\right)^{\lambda-1}\partial x}{\left(1+x\sqrt{2}\right)^{\lambda+1}},$$

ubi post integrationem statuitur $x=\frac{1}{\sqrt{2}}$; tum autem fit $s=\frac{\mathrm{A}}{\mathrm{B}}$ hincque valor fractionis propositae $=b+\frac{\mathrm{B}}{\mathrm{A}}$.

§.244. Nisi igitur fuerit $\lambda=\frac{b}{\sqrt{2}}$, numerus rationalis, hos valores commode assignare non licet. Sit igitur $b=\sqrt{2}$ sive $\lambda=1$ eritque

$$\mathrm{A}=2\int\frac{x\partial x}{\left(1+x\sqrt{2}\right)^{2}}\quad\text{et}\quad\mathrm{B}=2\int\frac{xx\partial x}{\left(1+x\sqrt{2}\right)^{2}}.$$

Hinc integrando colligitur

$$\mathrm{A}=l\left(1+x\sqrt{2}\right)-\frac{x\sqrt{2}}{1+x\sqrt{2}}$$

ideoque posito $x\sqrt{2}=1$ fiet $\mathrm{A}=l2-\frac{1}{2}$; tum vero reperitur

$$\mathrm{B}=\frac{3}{2\sqrt{2}}-\sqrt{2}\cdot l2$$

quare ob $b=\sqrt{2}$ erit $b+\frac{\mathrm{B}}{\mathrm{A}}=\frac{1}{\sqrt{2}(2l2-1)}$, unde sequitur haec summatio

$$\frac{1}{\sqrt{2}(2l2-1)}=\sqrt{2}+\cfrac{1}{\sqrt{2}+\cfrac{3}{\sqrt{2}+\cfrac{6}{\sqrt{2}+\text{etc.}}}}$$

SCHOLION

§.245. Fractiones autem continuae, ad quas plerumque calculo numerico deducimur, huiusmodi formam habere solent

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

ubi omnes numeratores sunt unitates, denominatores vero *a, b, c,* d, *e* etc. numeri integri. Verum ope nostrae methodi difficulter talium formarum valores eruere licet, etiamsi numeri *a, b, c, d, e* progressionem arithmeticam constituant, id quod sequenti exemplo ostendamus.

EXEMPLUM

§.246. *Proposita sit ista fractio continua*

$$\beta+b+\cfrac{1}{2\beta+b+\cfrac{1}{3\beta+b+\cfrac{1}{4\beta+b++\cfrac{1}{5\beta+b+\text{etc.},}}}}$$

ubi $\alpha=0,\ \gamma=0,\ a=1,\ c=1.$
Hinc fit

$$\frac{\partial \mathrm{Q}}{\mathrm{Q}}=-\frac{\partial x(1-bx-xx)}{\beta xx},$$

unde

$$l\mathrm{Q}=\frac{1}{\beta x}+\frac{b}{\beta}lx+\frac{x}{\beta}\ \text{ et }\ \mathrm{Q}=e^{\frac{1+xx}{\beta x}}x^{\frac{b}{\beta}},$$

quae autem expressio nullo casu evanescere potest, etiamsi per x^n multiplicetur, siquidem β fuerit numerus positivus. Verum si pro β sumamus numeros negativos, puta $\beta=-m$, tum valor $\mathrm{Q}=x^{\frac{-b}{m}}e^{\frac{-(1+xx)}{mx}}$ manifesto evanescit, tam si $x=0$ quam si $x=\infty$. Hinc autem erit

$$\partial v=\frac{x^{\frac{-b}{m}}e^{\frac{-1-xx}{mx}}\partial x}{mxx},$$

quamobrem habebimus

$$\mathrm{A}=\frac{1}{m}\int\frac{\partial x}{x^{2+\frac{b}{m}}e^{\frac{1+xx}{mx}}}\ \text{ et }\ \mathrm{B}=\frac{1}{m}\int\frac{\partial x}{x^{1+\frac{b}{m}}e^{\frac{1+xx}{mx}}}$$

His valoribus inventis formula $\frac{\mathrm{A}}{\mathrm{B}}$ exprimet summam huius fractionis continuae

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

quamobrem formula illa negative sumta $-\frac{\mathrm{A}}{\mathrm{B}}$ exprimet valorem huius fractionis continuae

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\cfrac{1}{5m-b+\text{etc.}}}}}$$

quem igitur assignare liceret, si modo formulae integrales A et B expediri et a termino $x=0$ ad $x=\infty$ extendi possent. Verum istae formulae ita sunt comparatae, ut earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impedit, quominus fractio $\frac{\mathrm{A}}{\mathrm{B}}$ valores satis cognitos involvere queat, etiamsi eos nullo adhuc modo assignare valeamus.

§.247. Talium autem fractionum continuarum mihi quidem binae sequentes innotuere, quarum valores commode exhibere licet:

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}}=\frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}$$

et

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}}=\cot.\frac{1}{n}.$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m-b=n,\ 2m-b=3n$ ideoque $m=2n$ et $b=n,$ unde fit

$$\mathrm{A}=\frac{1}{2n}\int\frac{\partial x}{x^{\frac{5}{2}}e^{\frac{1+xx}{2nx}}}\ \text{ et }\ \mathrm{B}=\frac{1}{2n}\int\frac{\partial x}{x^{\frac{3}{2}}e^{\frac{1+xx}{2nx}}},$$

unde iam discimus, si hae duae formulae integrentur a termino $x=0$ usque ad terminum $x=\infty$, tum fore

$$\frac{\mathrm{A}}{\mathrm{B}}=\frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet hanc convenientiam demonstrandi.