4) Concerning the values of integrals extended from the limit of the variable $x=0$ as far as $x=\infty$.

M. S. Academiae exhib. d. 30 *Aprilis* 1781.

[E 675; Institutiones Calculi Integralis 4, 1794, p. 337-345]

§. 124. Of such formulas, the values of which are chosen from the limit $x=0$ as far as to the limit $x=\infty$, the most simple is of the circle $\int\frac{\partial x}{1+xx}$, of which the value is $\frac{\pi}{2}$; with π denoting the periphery of the circle for the diameter 1. Then also, in a straight forwards manner, the individual integrals to be found will be

$$\int\frac{x^{m-1}\partial x}{1+x^{n}}\begin{bmatrix}\text{a} & x=0\\ \text{ad} & x=\infty\end{bmatrix}=\frac{\pi}{n\sin.\frac{m\pi}{n}}.$$

Truly in addition in this manner I have extricated several other formulas of this kind, in the differentials of which not only algebraic functions of *x*, but also of *lx* may enter.

§. 125. Moreover certain other formulas of this kind have occurred to me involving even transcendental functions, the desired values of which may be seen to be rejected at this time by all known methods. As it was , I was looking for that curved line, in which the radius of osculation should be inversely proportional to the arc of the curve, thus so that on putting the arc $=s$ and the radius of osculation $=r$ there would be

$$rs=aa.$$

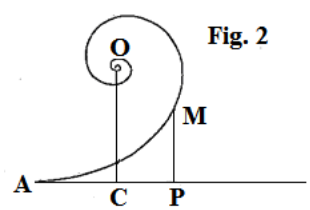

Fig. 2

Hence indeed the figure of the curve sought can be described freely by hand without difficulty, since that must have such a shape (Fig. 2). Without doubt with the initial curvature if the curve established at A thence the curve will continually become more curved, and finally after an infinite spiral it will be gathered together at a certain point O, which will be called the *pole* of this curve. Therefore it was proposed for me to investigate more accurately the position of this pole and the magnitudes of the coordinates AC and CO to be sought for that.

§. 126. To this end in the calculation of any part introduced as it pleases $\text{AM}=s$, with the amplitude $=\varphi$, so that there shall be $r=\frac{ds}{d\varphi}$, and hence there becomes

$$s\partial s=aa\partial\varphi \text{ and hence } ss=2aa\varphi, \text{ and } s=a\sqrt{2\varphi}=2c\sqrt{\varphi}.$$

Hence now there will be produced $\partial s=\frac{c\partial\varphi}{\sqrt{\varphi}}$, from which with the abscissa put for this arc $\text{AP}=x$ and with the applied line $\text{PM}=y$ there is deduced to be

$$x = c\int\frac{\partial\varphi\cos.\varphi}{\sqrt{\varphi}} \text{ and } y = c\int\frac{\partial\varphi\sin.\varphi}{\sqrt{\varphi}}.$$

§. 127. Hence therefore for the pole O being determined, the values of these two integrals are required, after they were extended from the limit $\varphi = 0$ as far as to $\varphi = \infty$. Indeed initially I have observed these values cannot be obtained otherwise than by being approximated, while each formula is evaluated successively by parts ; evidently at first from $\varphi = 0$ as far as to $\varphi = \pi$, then from $\varphi = \pi$ as far as to $\varphi = 2\pi$, again from $\varphi = 2\pi$ as far as to $\varphi = 3\pi$ etc., certainly with which agreed on, the series readily converge well enough. Truly it is clear this operation requires a long enough tedious calculation, which indeed I have not ventured to set out. But recently by good fortune by a singular method, I have seen at once, both

$$\int\frac{\partial\varphi\cos.\varphi}{\sqrt{\varphi}}\begin{bmatrix}\text{from} & \varphi=0\\ \text{to} & \varphi=\infty\end{bmatrix} = \sqrt{\frac{\pi}{2}},$$

as well as

$$\int\frac{\partial\varphi\sin.\varphi}{\sqrt{\varphi}}\begin{bmatrix}\text{from} & \varphi=0\\ \text{to} & \varphi=\infty\end{bmatrix} = \sqrt{\frac{\pi}{2}},$$

thus so that in place of the pole sought O there shall be $\mathrm{AC} = c\sqrt{\frac{\pi}{2}}$ and $\mathrm{CO} = c\sqrt{\frac{\pi}{2}}$.

§. 128. Therefore because the method, which I have been led to here, may be seen to be bountiful, I have considered it is going to be gratefully received by the geometers, if I can set that out here carefully. And because it may be appear to be much wider in application than to these formulas, I am going to propose that also to any extension, all of which I have deduced in a satisfactory manner from the consideration of this simple formula

$$\int x^{n-1}\partial x e^{-x},$$

whose integral therefore it will be agreed to investigate for various values of the exponent n.

§. 129. And indeed at first for the case $n = 1$ the integral of which formula $\int\partial x e^{-x}$ clearly is $1 - e^{-x}$, which vanishes in the case $x = 0$, but on making $x = \infty$ it will become unity. Besides, since the differential of this formula $x^{\lambda}e^{-x}$ shall be

$$\lambda x^{\lambda-1}\partial x e^{-x} - x^{\lambda}\partial x e^{-x},$$

in turn there will be :

$$\int x^{\lambda}\partial x e^{-x} = \lambda\int x^{\lambda-1}\partial x e^{-x} - x^{\lambda}e^{-x},$$

which latter term vanishes both for the case $x=0$ as for the case $x=\infty$, but only if there were $\lambda>0$. Then truly for the limits of our integration there will become :

$$\int x^{\lambda}\partial xe^{-x}=\lambda\int x^{\lambda-1}\partial xe^{-x},$$

of which formula with the aid of $\int\partial xe^{-x}=1$ the following values of the integration may be deduced :

$$\int x\partial xe^{-x}=1,$$
$$\int x^{2}\partial xe^{-x}=1\cdot 2,$$
$$\int x^{3}\partial xe^{-x}=1\cdot 2\cdot 3,$$
$$\int x^{4}\partial xe^{-x}=1\cdot 2\cdot 3\cdot 4,$$

and thus in general:

$$\int x^{n-1}\partial xe^{-x}=1\cdot 2\cdot 3\cdot 4\cdots\cdot(n-1),$$

the values of which product, as often as *n* were a whole positive number, at once are produced; but when *n* is a fraction, as I have shown elsewhere [E19 & E122], how the values are able to be shown for the quadratures of algebraic curves. Thus for the case $n=\frac{1}{2}$, it is agreed the value to be $=\sqrt{\pi}$.

§. 130. Therefore since all the values of this infinite product $1\cdot 2\cdot 3\cdot 4\cdots(n-1)$ may be regarded as known, I will designate these by the letter $\varDelta$, thus so that there shall be

$$\varDelta=1\cdot 2\cdot 3\cdot 4\cdots(n-1),$$

and thus now we have arrived at this integral formula sign

$$\int x^{n-1}\partial xe^{-x}=\varDelta,$$

evidently of the integral extended from $x=0$ to $x=\infty$; and I have deduced everything from this same formula, which pertain to the case mentioned before, where indeed the particular reasoning must be followed closely, which therefore here I am going to set out more carefully.

§. 131. Initially I have put $x=ky$, and because both the limits of the integration remain the same, there will be also :

$$k^n \int y^{n-1} \partial y e^{-ky} = \Delta,$$

since this formula also is extended from $y = 0$ as far as to $y = \infty$;
on account of which by dividing by kn we will have

$$\int y^{n-1} \partial y e^{-ky} = \frac{\Delta}{k^n},$$

but where it must be observed no negative numbers can be accepted for k, because otherwise the formula e^{-ky} no longer will vanish in the case $y = \infty$; and here such values are themselves required to be excluded, that so that also imaginary values may be able to be used in place of k , and hence I have attended to these hard integrations.

§. 132. Therefore we may put

$$k = p + q\sqrt{-1}\,,$$

and since there shall be

$$e^{-qy\sqrt{-1}} = \cos.qy - \sqrt{-1} \cdot \sin.qy$$

and

$$e^{+qy\sqrt{-1}} = \cos.qy + \sqrt{-1} \cdot \sin.qy,$$

our formula will now adopt this form

$$\int y^{n-1} \partial y e^{-py} \left(\cos.qy - \sqrt{-1} \cdot \sin.qy \right) = \frac{\Delta}{\left(p + q\sqrt{-1} \right)^n}.$$

On account of which if we may change the sign of the imaginary formula, in a similar manner there will be

$$\int y^{n-1} \partial y e^{-py} \left(\cos.qy + \sqrt{-1} \cdot \sin.qy \right) = \frac{\Delta}{\left(p - q\sqrt{-1} \right)^n}.$$

§. 133. In order that the values found may be allowed to be expressed more conveniently, we may put

$$p = f\cos.\theta \text{ and } q = f\sin.\theta$$

and there will become:

$$\left(p + q\sqrt{-1} \right)^n = f^n \left(\cos.n\theta + \sqrt{-1} \cdot \sin.n\theta \right)$$

and

$$\left(p - q\sqrt{-1} \right)^n = f^n \left(\cos.n\theta - \sqrt{-1} \cdot \sin.n\theta \right);$$

where it will help to be noted that $\tan g.\theta = \frac{q}{p}$, from which from the assumed values p and q, there will become also $f = \sqrt{(pp+qq)}$. Therefore in this manner for the first case there becomes :

$$\frac{\varDelta}{\left(p+q\sqrt{-1}\right)^n} = \frac{\varDelta}{f^n\left(\cos.n\theta+\sqrt{-1}\cdot\sin.n\theta\right)},$$

for the other

$$\frac{\varDelta}{\left(p-q\sqrt{-1}\right)^n} = \frac{\varDelta}{f^n\left(\cos.n\theta-\sqrt{-1}\cdot\sin.n\theta\right)}.$$

On account of which if these two formulas may be added, there will be produced :

$$\frac{2\varDelta\cos.n\theta}{f^n}$$

But the difference of these two formulas gives :

$$\frac{2\varDelta\sqrt{-1}\cdot\sin.n\theta}{f^n} \quad .$$

§. 134. Therefore we may also add these integral formulas themselves and we will have :

$$\int y^{n-1}\partial y e^{-py}\cos.qy = \frac{\varDelta\cos.n\theta}{f^n}.$$

But if we may subtract and divide by $2\sqrt{-1}$, there arises :

$$\int y^{n-1}\partial y e^{-py}\sin.qy = \frac{\varDelta\sin.n\theta}{f^n},$$

which two formulas now appear to be the most general, since the numbers p and q in short are left arbitrary by us, with only that requiring to be observed, negative numbers cannot be taken for p. Therefore it will be worth the effort to set out these two integral formulas in the following two theorems.

THEOREM 1

On putting

$$\varDelta = 1\cdot 2\cdot 3\cdot\cdot\cdot(n-1)$$

and by taking any two positive numbers for the letters p and q, thence there shall become

$$\sqrt{(pp+qq)} = f$$

and the angle θ may be sought, so that there shall be

$$\text{tang}.\theta = \frac{q}{p},$$

and this memorable integral will be obtained

$$\int x^{n-1}\partial x e^{-px}\cos.qx \begin{bmatrix}\text{from} & x=0\\ \text{to} & x=\infty\end{bmatrix} = \frac{\Delta\cos.n\theta}{f^n}.$$

THEOREM 2

On putting

$$\Delta = 1\cdot 2\cdot 3 \cdot\cdot\cdot (n-1)$$

and by taking any two positive numbers for the letters p and q, thence there shall become

$$\sqrt{(pp+qq)} = f$$

and the angle θ may be sought, so that there shall be

$$\text{tang}.\theta = \frac{q}{p},$$

and this memorable integral will be obtained :

$$\int x^{n-1}\partial x e^{-px}\sin.qx \begin{bmatrix}\text{from} & x=0\\ \text{to} & x=\infty\end{bmatrix} = \frac{\Delta\sin.n\theta}{f^n}.$$

§. 135. Therefore since for the case considered above we will arrive at these integral formulas

$$\int\frac{\partial\varphi\cos.\varphi}{\sqrt{\varphi}} \qquad \text{and} \qquad \int\frac{\partial\varphi\sin.\varphi}{\sqrt{\varphi}},$$

from the applied line made there will be $n = \frac{1}{2}$ and thus $\Delta = \sqrt{\pi}$, then truly there will be $p = 0$ and $q = 1$, from which there becomes $f = 1$ and $\text{tang}.\theta = \frac{q}{p} = \infty$ and thus $\theta = \frac{\pi}{2}$, therefore $\cos.n\theta = \frac{1}{\sqrt{2}} = \sin.n\theta.$

Hence therefore there becomes

$$\int\frac{\partial\varphi\cos.\varphi}{\sqrt{\varphi}} \begin{bmatrix}\text{from} & \varphi=0\\ \text{to} & \varphi=\infty\end{bmatrix} = \sqrt{\frac{\pi}{2}}$$

and likewise

$$\int\frac{\partial\varphi\sin.\varphi}{\sqrt{\varphi}} \begin{bmatrix}\text{from} & \varphi=0\\ \text{to} & \varphi=\infty\end{bmatrix} = \sqrt{\frac{\pi}{2}}.$$

§. 136. But it will be worth the effort to set out this case in general, where $n=\frac{1}{2}$ and $\Delta=\sqrt{\pi}$; and since we shall have put

$$\sqrt{(pp+qq)} = f \text{ and } \operatorname{tang}.\theta=\frac{q}{p},$$

there will be

$$\sin.\theta=\frac{q}{f} \text{ and } \cos.\theta=\frac{p}{f}.$$

Hence therefore initially,

$$\sin.\tfrac{1}{2}\theta=\sqrt{\frac{1-\cos.\theta}{2}}=\sqrt{\frac{f-p}{2f}} \text{ and } \cos.\tfrac{1}{2}\theta=\sqrt{\frac{1+\cos.\theta}{2}}=\sqrt{\frac{f+p}{2f}};$$

from which there becomes for the integral values :

$$\frac{\Delta\sin.\frac{1}{2}\theta}{\sqrt{f}}=\frac{\sqrt{\pi}}{f}\cdot\sqrt{\frac{f-p}{2}} \text{ and } \frac{\Delta\cos.\frac{1}{2}\theta}{\sqrt{f}}=\frac{\sqrt{\pi}}{f}\cdot\sqrt{\frac{f+p}{2}}$$

On account of which we will have the two following integral formulas :

$$\int\frac{\partial x}{\sqrt{x}}e^{-px}\sin.qx=\frac{\sqrt{\pi}}{f}\cdot\sqrt{\frac{f-p}{2}}$$

$$\int\frac{\partial x}{\sqrt{x}}e^{-px}\cos.qx=\frac{\sqrt{\pi}}{f}\cdot\sqrt{\frac{f+p}{2}}.$$

§. 137. But the case, in which a positive whole number is taken for n and thus Δ can be shown completely by whole numbers, thus have been prepared, so that also by known methods, clearly with the aid of reduced integral formulas known well enough, may be able to be extricated and thus to be shown for integrals in general. But this operation demands quite an involved calculation, on account of which our formulas nevertheless are worthy of attention, clearly simple enough for the case $x=\infty$. But when we wish to attribute negative values to the exponent n, these cases immediately at the start of the integration demand the addition of infinite constants, as clearly the integrals vanish in the case $x=0$, and thus indeed the values of the integral, which we seek here, will remain infinite and thus are not to be referred to in our situation.

§. 138. But the most memorable case occurs here, when $n=0$ and which in short demands a singular skill; which therefore we will set out more carefully.
Since we shall have put

$$\Delta=1\cdot2\cdot3\cdot4\cdots(n-1),$$

in a similar manner we may put

$$\Delta' = 1\cdot 2\cdot 3\cdot 4\cdots n \text{ and } \Delta'' = 1\cdot 2\cdot 3\cdot 4\cdots(n+1)$$

and clearly there will be

$$\Delta = \frac{\Delta'}{n} \text{ et } \Delta' = \frac{\Delta''}{n+1}$$

and thus:

$$\Delta = \frac{\Delta''}{n(n+1)}.$$

Now we may take $n = \omega$ with ω being infinitely small, and since there shall be $\Delta'' = 1$, thence there becomes $\Delta = \frac{1}{\omega}$ and thus its value will be infinite. But since for the first integral formula there shall be $\sin.n\theta = \omega\theta$, clearly there shall become $\Delta\sin.n\theta = \theta$; on account of which this same first integral formula will become

$$\int \frac{\partial x}{x} e^{-px} \sin.qx = \theta,$$

while truly the integral is extended from the limit $x = 0$ as far as to the limit $x = \infty$. Moreover the other value of the other formula of our integral,

$$\int \frac{\partial x}{x} e^{-px} \cos.qx$$

will be infinitely great. But that case generally deserves to be noted, so that we may include that with the singular theorem.

THEOREM 3

§. 139. *If the letters p and q may denote any positive numbers and hence the angle θ is sought, so that there shall be*

$$\text{tang.}\theta = \frac{q}{p},$$

the following most memorable integration will be considered :

$$\int \frac{\partial x}{x} e^{-px} \sin.qx \begin{bmatrix} \text{from} & x=0 \\ \text{to} & x=\infty \end{bmatrix} = \theta,$$

the demonstration of which theorem without doubt can be investigated otherwise only by approximations.

§.140. But the simplest case, where $p = 0$ and $q = 1$, now may be seen to be overcome at this point by all the known artifices of the calculation; as in this case there shall be

$$\text{tang.}\theta = \frac{1}{0} = \infty,$$

there will become

$$\theta = \frac{\pi}{2},$$

from which this integration arises :

$$\int \frac{\partial x}{x} \sin.x = \frac{\pi}{2}.$$

Yet meanwhile concerning the truth of that, there less may be doubted, because approximations used will lead to just about the same value. But if we may compare this case with the former mentioned

$$\int \frac{\partial x}{\sqrt{x}} \sin.x = \sqrt{\frac{\pi}{2}},$$

the remarkably similar sum merits attention, since the integral of this shall be precisely the square root of that.

4) De valoribus integralium a termino variabilis $x = 0$ usque ad $x = \infty$ extensorum.

M. S. Academiae exhib. d. 30 *Aprilis* 1781.

[Commentatio 675 indicis ENESTROEMIANI
Institutiones calculi integralis 4, 1794, p. 337-345]

§. 124. Talium formularum, quae a termino $x = 0$ usque ad terminum $x = \infty$ extensae finitum sortiuntur valorem, simplicissima est circularis $\int \frac{\partial x}{1+xx}$, cujus valor est $\frac{\pi}{2}$; denotante π peripheriam pro diametro 1. Deinde etiam methodo prorsus singulari inveni esse

$$\int \frac{x^{m-1}\partial x}{1+x^n} \begin{bmatrix} \text{a} & x=0 \\ \text{ad} & x=\infty \end{bmatrix} = \frac{\pi}{n\sin.\frac{m\pi}{n}}.$$

Praeterea vero hoc modo plures alias formulas huius generis expedivi, in quarum differentialia non solum functiones algebraicae ipsius *x*, sed etiam *lx* ingrediatur.

§. 125. Obtulerunt se mihi autem quondam aliae huiusmodi formulae etiam functiones transcendentes involventes, quarum valores desiderati omnes methodos adhuc cognitas respuere videantur. Quaesiveram scilicet eam lineam curvam, in qua radius osculi ubique reciproce esset proportionalis arcui curvae, ita ut posito arcu $= s$ et radio osculi $= r$ esset

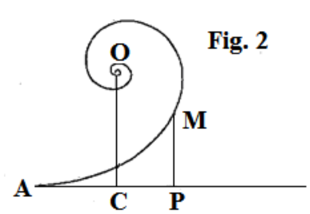

Fig. 2

$$rs = aa.$$

Hinc enim haud difficile est figuram curvae libero quasi manus ductu describere, quandoquidem ea talem habere debet figuram (Fig. 2). Initio nimirum curvae in A constituto inde curva continuo magis incurvabitur et tandem post infinitas spiras in certum punctum O glomerabitur, quod *polum* huius curvae appellare licebit. Propositum igitur mihi fuerat locum huius poli accuratius investigare pro eoque quantitatem coordinatarum AC et CO perscrutari.

§. 126. Hunc in finem introducta in calculum portionis cuiusvis $\mathrm{AM} = s$ amplitudine $= \varphi$, ut sit $r = \frac{ds}{d\varphi}$, fit
hincque

$$s\partial s = aa\partial\varphi \text{ hincque } ss = 2aa\varphi \text{ et } s = a\sqrt{2}\varphi = 2c\sqrt{\varphi}.$$

Hinc iam prodit $\partial s = \frac{c\partial\varphi}{\sqrt{\varphi}}$, unde posita abscissa pro hoc arcu $\mathrm{AP} = x$ et applicata $\mathrm{PM} = y$ colligitur fore

$$x = c\int \frac{\partial\varphi \cos.\varphi}{\sqrt{\varphi}} \text{ et } y = c\int \frac{\partial\varphi \sin.\varphi}{\sqrt{\varphi}}.$$

§. 127. Hinc ergo pro polo O determinando requiruntur valores harum duarum formularum integralium, postquam a termino $\varphi = 0$ usque ad $\varphi = \infty$ fuerint extensae. Initio quidem sum arbitratus hos valores aliter obtineri non posse nisi approximando, dum utraque formula successive per partes evolvatur; primo scilicet a $\varphi = 0$ usque ad $\varphi = \pi$, deinde a $\varphi = \pi$ usque ad $\varphi = 2\pi$, porro a $\varphi = 2\pi$ usque ad $\varphi = 3\pi$ etc., quippe quo pacto series prodibunt satis prompte convergentes. Verum evidens est hanc operationem longos calculos satis taediosos requirere, quos quidem evolvere non sum ausus. Nuper autem forte fortuna per methodum prorsus singularem perspexi esse tam

$$\int \frac{\partial\varphi \cos.\varphi}{\sqrt{\varphi}} \begin{bmatrix} \text{a} & \varphi=0 \\ \text{ad} & \varphi=\infty \end{bmatrix} = \sqrt{\frac{\pi}{2}},$$

quam

$$\int \frac{\partial\varphi \sin.\varphi}{\sqrt{\varphi}} \begin{bmatrix} \text{a} & \varphi=0 \\ \text{ad} & \varphi=\infty \end{bmatrix} = \sqrt{\frac{\pi}{2}},$$

ita ut pro loco poli quaesito O sit $\mathrm{AC} = c\sqrt{\frac{\pi}{2}}$ et $\mathrm{CO} = c\sqrt{\frac{\pi}{2}}$.

§. 128. Quoniam igitur methodus, qua huc sum perductus, non parum polliceri videtur, Geometris haud ingratum fore arbitror, si eam omni cura hic exposuero. Et quia multo latius quam ad istas formulas patet, eam etiam omni extensione sum propositurus, quae omnia ex consideratione huius formulae satis simplicis

$$\int x^{n-1}\partial x e^{-x}$$

deduxi, cuius ergo integrale pro variis valoribus exponentis *n* investigate convenit.

§. 129. Ac primo quidem pro casu $n=1$ huius formulae $\int \partial x e^{-x}$ integrale manifestum est $1-e^{-x}$, quod casu $x=0$ evanescit, facto autem $x=\infty$ abit in unitatem. Praeterea, cum huius formulae $x^{\lambda}e^{-x}$ differentiale sit

$$\lambda x^{\lambda-1}\partial x e^{-x} - x^{\lambda}\partial x e^{-x},$$

erit vicissim

$$\int x^{\lambda}\partial x e^{-x} = \lambda\int x^{\lambda-1}\partial x e^{-x} - x^{\lambda}e^{-x},$$

quod postremum membrum tam pro casu $x=0$ quam $x=\infty$ evanescit, si modo fuerit $\lambda>0$. Tum igitur pro nostris terminis integrationis erit

$$\int x^{\lambda}\partial x e^{-x} = \lambda\int x^{\lambda-1}\partial x e^{-x},$$

cuius formulae ope ob $\int \partial x e^{-x} = 1$ sequentes integralium valores deducuntur

$$\int x\partial x e^{-x} = 1,$$
$$\int x^2\partial x e^{-x} = 1\cdot 2,$$
$$\int x^3\partial x e^{-x} = 1\cdot 2\cdot 3,$$
$$\int x^4\partial x e^{-x} = 1\cdot 2\cdot 3\cdot 4,$$

sicque in genere

$$\int x^{n-1}\partial x e^{-x} = 1\cdot 2\cdot 3\cdot 4\cdots(n-1),$$

cuius producti valores, quoties *n* fuerit numerus integer positivus, sponte se produnt; quando autem nest numerus fractus, olim ostendi, quomodo valores per quadraturas curvarum algebraicarum exhiberi queant. Sic pro casu $n=\frac{1}{2}$ constat istum valorem esse $=\sqrt{\pi}$.

§. 130. Cum igitur omnes valores huius producti infiniti $1\cdot 2\cdot 3\cdot 4\cdots(n-1)$ tanquam cogniti spectari queant, eos littera Δ designabo, ita ut sit

$$\Delta = 1\cdot 2\cdot 3\cdot 4\cdots(n-1),$$

sicque iam adepti sumus hanc insignem formulam integralem

$$\int x^{n-1}\partial x e^{-x} = \Delta,$$

integrali scilicet ab $x=0$ ad $x=\infty$ extenso; atque ex hac ipsa formula omnia deduxi, quae ad casum ante memoratum pertinent, ubi quidem ratiocinia penitus singularia adhiberi debent, quae igitur hic diligentius sum expositurus.

§. 131. Posui autem primo $x=ky$, et quoniam ambo termini integralis iidem manent, erit etiam

$$k^n \int y^{n-1}\partial y e^{-ky} = \Delta,$$

quandoquidem haec formula etiam ab $y=0$ ad $y=\infty$ usque extenditur; quamobrem per *kn* dividendo habebimus

$$\int y^{n-1}\partial y e^{-ky} = \frac{\Delta}{k^n},$$

ubi autem notari oportet pro *k* nullos numeros negativos accipi posse, quia alioquin formula e^{-ky} non amplius evanesceret casu $y=\infty$; atque hic isti soli valores sunt excludendi, ita ut etiam valores imaginarii loco *k* adhiberi queant, atque hinc illas arduas integrationes sum assecutus.

§. 132. Ponamus ergo

$$k = p + q\sqrt{-1},$$

et cum sit

$$e^{-qy\sqrt{-1}} = \cos.qy - \sqrt{-1}\cdot\sin.qy$$

et

$$e^{+qy\sqrt{-1}} = \cos.qy + \sqrt{-1}\cdot\sin.qy,$$

nostra formula nunc induet hanc formam

$$\int y^{n-1}\partial y e^{-py}\left(\cos.qy - \sqrt{-1}\cdot\sin.qy\right) = \frac{\Delta}{\left(p+q\sqrt{-1}\right)^n}.$$

Quamobrem si formulae imaginariae signum mutemus, erit simili modo

$$\int y^{n-1}\partial y e^{-py}\left(\cos.qy + \sqrt{-1}\cdot\sin.qy\right) = \frac{\Delta}{\left(p-q\sqrt{-1}\right)^n}.$$

§. 133. Quo valores inventos commodius exprimere liceat, ponamus

$$p = f\cos.\theta \;\text{ et }\; q = f\sin.\theta$$

eritque

$$\left(p+q\sqrt{-1}\right)^n = f^n\left(\cos.n\theta+\sqrt{-1}\cdot\sin.n\theta\right)$$

et

$$\left(p-q\sqrt{-1}\right)^n = f^n\left(\cos.n\theta-\sqrt{-1}\cdot\sin.n\theta\right);$$

ubi notasse iuvabit fore $\tang.\theta = \frac{q}{p}$ unde ex valoribus *p* et *q* assumtis erit etiam $f = \sqrt{(pp+qq)}$. Hoc ergo modo fit priore casu

$$\frac{\Delta}{\left(p+q\sqrt{-1}\right)^n} = \frac{\Delta}{f^n\left(\cos.n\theta+\sqrt{-1}\cdot\sin.n\theta\right)},$$

pro altero

$$\frac{\Delta}{\left(p-q\sqrt{-1}\right)^n} = \frac{\Delta}{f^n\left(\cos.n\theta-\sqrt{-1}\cdot\sin.n\theta\right)}.$$

Quamobrem si hae duae formulae addantur, prodibit

$$\frac{2\Delta\cos.n\theta}{f^n}$$

Differentia autem harum formularum dat

$$\frac{2\Delta\sqrt{-1}\cdot\sin.n\theta}{f^n}\;.$$

§. 134. Addamus igitur quoque ipsas formulas integrales et habebimus

$$\int y^{n-1}\partial y e^{-py}\cos.qy = \frac{\Delta\cos.n\theta}{f^n}.$$

Sin autem subtrahamus et per $2\sqrt{-1}$ dividamus, oritur

$$\int y^{n-1}\partial y e^{-py}\sin.qy = \frac{\Delta\sin.n\theta}{f^n},$$

quae iam duae formulae integrales latissime patent, cum numeri per *q* prorsus arbitrio nostro relinquuntur, id tantum observando, ne pro *p* et *q* numeri negativi accipiantur. Operae igitur pretium erit has duas formulas integrales sequentibus binis Theorematibus complecti.

THEOREMA 1

Posito

$$\Delta = 1\cdot 2\cdot 3 \cdots (n-1)$$

et pro litteris p et q numeros quoscunque positivos accipiendo fiat inde

$$\sqrt{(pp+qq)} = f$$

et quaeratur angulus θ, ut sit

$$\text{tang.}\theta = \frac{q}{p},$$

et habebitur ista integratio memorabilis

$$\int x^{n-1}\partial x e^{-px}\cos.qx \begin{bmatrix}\text{a} & x=0\\ \text{ad} & x=\infty\end{bmatrix} = \frac{\Delta\cos.n\theta}{f^n}.$$

THEOREMA 2

Posito

$$\Delta = 1\cdot 2\cdot 3 \cdots (n-1)$$

et pro litteris p et q numeros quoscunque positivos accipiendo fiat inde

$$\sqrt{(pp+qq)} = f$$

et quaeratur angulus θ, ut sit

$$\text{tang.}\theta = \frac{q}{p},$$

et habebitur ista integratio memorabilis

$$\int x^{n-1}\partial x e^{-px}\sin.qx \begin{bmatrix}\text{a} & x=0\\ \text{ad} & x=\infty\end{bmatrix} = \frac{\Delta\sin.n\theta}{f^n}.$$

§. 135. Cum igitur pro casu curvae supra consideratae pervenerimus ad has formulas integrales

$$\int \frac{\partial\varphi\cos.\varphi}{\sqrt{\varphi}} \qquad \text{et} \qquad \int \frac{\partial\varphi\sin.\varphi}{\sqrt{\varphi}},$$

facta applicatione erit $n = \frac{1}{2}$ ideoque $\Delta = \sqrt{\pi}$, tum vero erit $p = 0$ et $q = 1$,
unde fit $f = 1$ et $\text{tang.}\theta = \frac{q}{p} = \infty$ ideoque $\theta = \frac{\pi}{2}$, ergo $\cos.n\theta = \frac{1}{\sqrt{2}} = \sin.n\theta.$
Hinc igitur fiet

$$\int \frac{\partial\varphi\cos.\varphi}{\sqrt{\varphi}} \begin{bmatrix}\text{a} & \varphi=0\\ \text{ad} & \varphi=\infty\end{bmatrix} = \sqrt{\frac{\pi}{2}}$$

simulque

$$\int \frac{\partial\varphi \sin.\varphi}{\sqrt{\varphi}} \left[\begin{array}{l}\text{a } \varphi=0\\ \text{ad } \varphi=\infty\end{array}\right] = \sqrt{\frac{\pi}{2}}.$$

§. 136. Operae autem pretium erit hunc casum, quo $n=\frac{1}{2}$ et $\Delta=\sqrt{\pi}$, in genere evolvere; et cum posuerimus

$$\sqrt{(pp+qq)} = f \text{ et } \tan g.\theta = \frac{q}{p},$$

erit

$$\sin.\theta = \frac{q}{f} \quad \text{et} \quad \cos.\theta = \frac{p}{f}.$$

Hinc ergo primo

$$\sin.\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos.\theta}{2}} = \sqrt{\frac{f-p}{2f}} \quad \text{et} \quad \cos.\tfrac{1}{2}\theta = \sqrt{\frac{1+\cos.\theta}{2}} = \sqrt{\frac{f+p}{2f}};$$

unde fit pro valoribus integralibus

$$\frac{\Delta \sin.\frac{1}{2}\theta}{\sqrt{f}} = \frac{\sqrt{\pi}}{f}\cdot\sqrt{\frac{f-p}{2}} \quad \text{et} \quad \frac{\Delta \cos.\frac{1}{2}\theta}{\sqrt{f}} = \frac{\sqrt{\pi}}{f}\cdot\sqrt{\frac{f+p}{2}}$$

Quamobrem habebimus binas sequentes formulas integrales

$$\int \frac{\partial x}{\sqrt{x}} e^{-px} \sin.qx = \frac{\sqrt{\pi}}{f}\cdot\sqrt{\frac{f-p}{2}}$$

$$\int \frac{\partial x}{\sqrt{x}} e^{-px} \cos.qx = \frac{\sqrt{\pi}}{f}\cdot\sqrt{\frac{f+p}{2}}.$$

§. 137. Casus autem, quibus pro *n* sumitur numerus integer positivus ideoque Δ absolute per numeros integros exhiberi potest, ita sunt comparati, ut etiam per methodos cognitas, ope scilicet formularum integralium reductionis satis notae, expediri queant atque adeo integralia in genere exhiberi. Haec autem operatio postulat calculos non parum prolixos, quamobrem formulae nostrae satis simplices pro casu scilicet $x=\infty$ nihilominus omni attentione sunt dignae. Quando autem exponenti *n* valores negativos tribuere voluerimus, hi casus statim in initio integrationis additionem constantis infinitae postulant, ut scilicet integralia evanescant casu $x=0$, sicque adeo valores integralium, quos hic quaerimus, manebunt infiniti ideoque ad institutum nostrum non sunt referendi.

§. 138. Casus autem maxime memorabilis hic occurrit, quo $n=0$ et qui prorsus singularem sollertiam postulat; quem igitur accuratius evolvamus.
Quoniam posuimus

$$\Delta = 1\cdot 2\cdot 3\cdot 4\cdots(n-1),$$

statuamus simili modo

$$\Delta' = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \text{ et } \Delta'' = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n+1)$$

eritque manifesto

$$\Delta = \frac{\Delta'}{n} \text{ et } \Delta' = \frac{\Delta''}{n+1}$$

ideoque

$$\Delta = \frac{\Delta''}{n(n+1)}$$

Sumamus nunc $n = \omega$ existente ω infinite parvo, et cum sit $\Delta'' = 1$, inde fit $\Delta = \frac{1}{\omega}$ ideoque eius valor erit infinitus. Cum autem pro formula integrali priore sit $\sin.n\theta = \omega\theta$, evidens est fore $\Delta \sin.n\theta = \theta$; quamobrem ista prior formula integralis erit

$$\int \frac{\partial x}{x} e^{-px} \sin.qx = \theta,$$

dum nempe integrale a termino $x = 0$ usque ad terminum $x = \infty$ extenditur. Alterius autem formulae nostrae integralis

$$\int \frac{\partial x}{x} e^{-px} \cos.qx$$

valor erit infinite magnus. Ille autem casus omnino meretur, ut eum singulari Theoremate complectamur.

THEOREMA 3

§. 139. *Si litterae p et q denotent numeros positivos quoscunque atque hinc quaeratur angulus θ, ut sit*

$$\text{tang.}\theta = \frac{q}{p},$$

habebitur sequens integratio maxime memorabilis

$$\int \frac{\partial x}{x} e^{-px} \sin.qx \begin{bmatrix} \text{a} & x=0 \\ \text{ad} & x=\infty \end{bmatrix} = \theta,$$

cuius theorematis demonstratio dubito quin alio modo quam per approximationes investigari queat.

§.140. Casus autem simplicissimus, quo $p = 0$ et $q = 1$, iam omnia calculi artificia adhuc cognita superare videtur; quia autem hoc casu fit

$$\text{tang.}\theta = \frac{1}{0} = \infty,$$

erit

$$\theta = \frac{\pi}{2},$$

unde oritur haec integratio

$$\int \frac{\partial x}{x} \sin . x = \frac{\pi}{2}.$$

Interim tamen de eius veritate eo minus dubitare licet, quod approximationes adhibitae ad eundem valorem propemodum perducant. Quodsi hunc casum cum initio memorato

$$\int \frac{\partial x}{\sqrt{x}} \sin . x = \sqrt{\frac{\pi}{2}}$$

comparemus, ingens similitudo summam attentionem meretur, cum huius integrale sit praecise radix quadrata illius.