2). Comparison of the values of the integral formula

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}},$$

from the limit $x=0$ extended as far as to $x=1$.

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§. 57. In this formula, the letters *n*, *p* and *q* always designate positive integers, and for any number *n* all the values may be attributed to *p* and *q*, thus so that hence for any number *n* innumerable integral formulas of this kind may arise, the values of which provide many outstanding relations between themselves; from which if some of these were known, all the rest may be able to be defined from these. Now indeed some time ago I had demonstrated several relations of this kind [see, e.g. Ch. 8, Vol. I, *Integral Calculus*, and E321 in this series of translations] ; but as of that time I had by no means exhausted the argumentum, now I have decided to inquire more accurately into these relations, and I will use a method of this kind, which shall be showing clearly all the relations of this kind ; for indeed from these found innumerable theorems will be able to be stored away, from which the whole of analysis may be agreed to be more than a little enriched.

§. 58. Therefore since in this manner for any number *n* both the letters *p* and *q* are able to receive infinite values, initially here it is agreed to be observed, all these innumerable cases can be recalled always to a finite number. Indeed however great the numbers may be taken for the letters *p* and *q*, it is allowed always to reduce these cases to other cases, in which the numbers *p* and *q* shall be diminished by the magnitude *n*. Therefore in this way all the cases of this kind finally will be able to be redirected there, so that both the numbers *p* and *q* may be reduced to be less than the exponent *n* ; from which for any number *n* it will suffice to be examining only these cases, for which the letters *p* and *q* may receive values less than *n*, or at least hence will not surpass this limit. Therefore in this manner, for any number *n*, a multitude of cases which arise in the computation, and which it is necessary to compare amongst themselves, will be determined at once.

§.59. But just as by this same reduction of the letters *p* and *q* to numbers continually smaller must be put in place, as that is noted well enough in common, yet it will help to be adapted to the formula present. Evidently if this algebraic formula may be put in place: $x^{p}\left(1-x^{n}\right)^{\frac{q}{n}}=\mathrm{V}$; and there will be

$$l\mathrm{V}=plx+\tfrac{q}{n}l\left(1-x^{n}\right),$$

hence by differentiating :

$$\frac{\partial\mathrm{V}}{\mathrm{V}}=\frac{p\partial x}{x}-\frac{qx^{n-1}\partial x}{1-x^{n}}=\frac{p\partial x-(p+q)x^{n-1}\partial x}{x\left(1-x^{n}\right)},$$

where if we may multiply by V, and we integrate by parts, this same equation will arise :

$$\mathrm{V}=p\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}}-(p+q)\int x^{p+n-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}}.$$

Therefore since the quantity V vanishes for each term of the integration, we arrive hence at this reduction

$$\int x^{p+n-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}}=\frac{p}{p+q}\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}},$$

therefore with the aid of which reduction the exponent of *x* will be able to be reduced continually by the amount *n*, while at last it may be lowered below *n* .

§. 60. Thence the formula found for

$$\frac{\partial\mathrm{V}}{\mathrm{V}}=\frac{p\partial x-(p+q)x^{n-1}\partial x}{x\left(1-x^{n}\right)}$$

will be referred in this manner

$$\frac{\partial\mathrm{V}}{\mathrm{V}}=\frac{(p+q)\partial x\left(1-x^{n}\right)-q\partial x}{x\left(1-x^{n}\right)},$$

which formula multiplied by V and integrated by parts will give again :

$$\mathrm{V}=(p+q)\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q}{n}}-q\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}},$$

from which since by putting $x=1$ there becomes $\mathrm{V}=0$, this reduction arises :

$$\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q}{n}}=\frac{q}{p+q}\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}},$$

with the aid of which reduction the exponent of the binomial $1-x^{n}$ is diminished by one, or what returns the same, the number *q* is diminished by the number *n* . Therefore with the aid of such a reduction, repeated as often as there were a need, the exponent *q* finally will be able to be depressed below *n* .

§. 61. Therefore, since for any number *n*, it is allowed to consider both the exponents *p* and *q* as smaller than *n* , we will represent the proposed formula expressed in this manner

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}.$$

Here evidently for any number *n* it will suffice for all the values of the letters *p* and *q* to be granted values smaller than *n* itself, with which agreed upon, the multitude of all the cases pertaining to some exponent *n* may be reduced to a small enough number, which yet emerges greater there, where the exponent *n* were made greater.

§. 62. But the number of diverse cases will be diminished much more, if we consider both the letters *p* and *q* to be able to be permuted between each other, thus so that the value of this formula

$$\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$$

in short will not disagree with that. Towards showing which, we may put

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} = S,$$

clearly if this same formula of the integral may be extended from $x=0$ as far as $x=1$. Now we may put $1-x^n = y^n$, so that the formula shall become

$$S = \int \frac{x^{p-1}\partial x}{y^{n-q}};$$

then truly because $x^n = 1-y^n$, there will be $x = \left(1-y^n\right)^{\frac{1}{n}}$, and hence $x^p = \left(1-y^n\right)^{\frac{p}{n}}$, from which on being differentiated there becomes :

$$px^{p-1}\partial x = -py^{n-1}\partial y\left(1-y^n\right)^{\frac{p-n}{n}},$$

with which value substituted there will become:

$$S = -\int y^{q-1}\partial y\left(1-y^n\right)^{\frac{p-n}{n}},$$

as it will be required to extend the formula from $x=0$ as far as $x=1$, that is, from $y=1$ as far as to $y=0$; therefore with these terms permuted there will become :

$$S=\int \frac{y^{q-1}\partial y}{\sqrt[n]{\left(1-y^n\right)^{n-p}}} \quad \begin{bmatrix}\text{from } y=0\\ \text{to} \quad\; y=1\end{bmatrix}.$$

And thus it has been shown both the letters *p* and *q* always to be permutable between each other.

§. 63. From these premises, by which the following calculations may be more considered more together

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-p}}},$$

we may write this symbol (*p, q*), where in the same way, either *p* may precede *q*, or vice versa ; but here always a certain exponent *n* must be understood. But here two cases occur which are most memorable before the rest. The first case is where either of the numbers *p* and *q* is equal to the exponent *n* itself ; if indeed there were $q=n$, there will become from the first formula $(p,n,)=\int x^{p-1}\partial x=\frac{1}{p}$, and thus we will have always $(p,n,)=\frac{1}{p}$, and hence always $(q,n,)=\frac{1}{q}$. The other most noteworthy case to be considered, is when $p+q=n$, in which case there is always

$$(p,q)=\frac{\pi}{n\sin.\frac{p\pi}{n}}=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

In order to show this, let $q=n-p$, and hence in the proposed formula $\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$,

there may then be put $\frac{x}{\sqrt[n]{\left(1-x^n\right)}}=z$, and because $\frac{x^p}{\sqrt[n]{\left(1-x^n\right)^p}}=z^p$, there will become

$S=\int \frac{z^p\partial x}{x}$. But from the substitution made, it follows that $x^n=\frac{z^n}{1+z^n}$ and hence

$$nlx=nlz-l\left(1+z^n\right),$$

therefore on differentiating,

$$\frac{\partial x}{x}=\frac{\partial z}{z}-\frac{z^{n-1}\partial z}{1+z^n}=\frac{\partial z}{z\left(1+z^n\right)},$$

thus so that now there shall be

$$S=\int \frac{z^{p-1}\partial z}{1+z^n}.$$

But because on taking $x=0$ there will become also $z=0$, but truly by taking $x=1$ produces $z=\infty$, this integral must be extended from the limit $z=0$ as far as $z=\infty$. Moreover it is to be observed that in this manner the result is $\frac{\pi}{n\sin.\frac{p\pi}{n}}$.

[See E60, especially §32.]

§. 64. Now we may proceed to the foundation itself, from which it is agreed all the relations which we seek are to be derived, and which depends on the above reduction ; from which there shall be :

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} = \frac{p+q}{p} \cdot \int \frac{x^{n+p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$$

where in place of $\sqrt[n]{\left(1-x^n\right)^{n-q}}$ we may write X, so that there shall become :

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} \cdot \int \frac{x^{n+p-1}\partial x}{X} :$$

now in a similar manner, if in place of p we may write $n+p$, there will become

$$\int \frac{x^{n+p-1}\partial x}{X} = \frac{n+p+q}{n+p} \cdot \int \frac{x^{2n+p-1}\partial x}{X},$$

and hence there follows to become:

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \int \frac{x^{2n+p-1}\partial x}{X}.$$

But if we may progress further in a similar manner, we will arrive at this equation :

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \cdot \frac{2n+p+q}{2n+p} \int \frac{x^{3n+p-1}\partial x}{X}.$$

Whereby if we may progress indefinitely in this way, we will have

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \cdot \frac{2n+p+q}{2n+p} \cdots\cdots \frac{in+p+q}{in+p} \int \frac{x^{(i+1)n+p-1}\partial x}{X},$$

where *i* denotes an infinitely great number.

§. 65. But if now in place of *p*, we may assume some other number *r*, with itself equally less than *n* ; in a similar manner there will become :

$$\int \frac{x^{r-1}\partial x}{X} = \frac{r+q}{r}\cdot\frac{n+r+q}{n+r}\cdot\frac{2n+r+q}{2n+r}\cdots\cdots\frac{in+r+q}{in+r}\int\frac{x^{(i+1)n+r-1}\partial x}{X},$$

where the letter *i* may designate the same infinite number, thus so that both sides may have the same number of factors present. Now we may divide the first expression by this, and because of the extreme integral formulas, on account of the letters *p* and *r* vanishing in comparison with $(i+1)n$, are required to be considered as equal to each other, with the division made by the individual factors we will find this equation:

$$\frac{\int x^{p-1}\partial x:X}{\int x^{r-1}\partial x:X} = \frac{r(p+q)}{p(r+q)}\cdot\frac{(n+r)(n+p+q)}{(n+p)(n+r+q)}\cdot\frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)}\cdot\frac{(3n+r)(3n+p+q)}{(3n+p)(3n+r+q)}\cdot\text{etc.}$$

Now we may restore the symbols established before in place of the integral formulas, and we will arrive at this same noteworthy relation :

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)}\cdot\frac{(n+r)(n+p+q)}{(n+p)(n+r+q)}\cdot\frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)}\cdot\frac{(3n+r)(3n+p+q)}{(3n+p)(3n+r+q)}\cdot\text{etc.}$$

which product is composed from an infinite number of members, of which the singular ones are fractions, of which both the numerators as well as the denominators consist of two factors. It requires that these individual factors be increased by the same number *n*, while we progress from some member to the following, from which it will suffice for only the first product to be known, which therefore we will represent thus :

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)}\cdot\text{etc.}$$

§. 66. Since the letters *p* et *q* as it were signify for us indefinite numbers, we shall make use of letters from the start of the alphabet for designating determinate numbers, and in this way there will become :

$$\frac{(a,b)}{(\alpha,b)} = \frac{\alpha(a+b)}{a(\alpha+b)}\cdot\frac{(n+\alpha)(n+a+b)}{(n+a)(n+\alpha+b)}\cdot\text{etc.}$$

Now here in place of α we may write $a+c$; and the infinite product adopts this form:

$$\frac{(a,b)}{(a+c,b)} = \frac{(a+c)(a+b)}{a(a+c+b)}\cdot\frac{(n+a+c)(n+a+b)}{(n+a)(n+a+c+b)}\cdot\text{etc.},$$

in which product clearly both the letters *b* and *c* can be permuted, from which the same infinite product also expresses the value of this form $\frac{(a,c)}{(a+b,c)}$, from which this most memorable equality follows to be :

$$\frac{(a,b)}{(a+c,b)} = \frac{(a,c)}{(a+b,c)};$$

therefore with the fractions removed we will have this conspicuous theorem :

$$(a,b)(a+b,c) = (a,c)(a+c,b)$$

and the whole analysis which we will use will be built on this theorem.

§. 67. Since on account of the reasons advanced above, the numbers *p* and *q* must not exceed the exponent *n*, also in the form of a theorem only individual terms brought forwards occur there , which are, in any case must not exceed the exponent *n*, and thus neither $a+b$ nor $a+c$ will be able to be taken greater than *n*. But I observe here initially the letters *b* and *c*, must be put in place unequal to each other: indeed if there were $c=b$, the equality expressed in the theorem would be identical, hence on that account we will assume $b>c$ always, thus so that the maximum term in the theorem shall be $a+b$, which therefore in any case ought not to exceed the exponent n , on account of which we may distribute the evolution of the general form into theorems thus contained in classes, which may be distinguished amongst themselves by the maximum value of the term $a+b$. Therefore since non of the letters *a, b, c* may be able to be taken equal to zero, and there must be $b>c$, the smallest value, that the term $a+b$ can receive, will be 3, in which therefore we may constitute the first class; truly the following classes may be put in place, while the term $a+b$ may be granted the values 4, 5, 6, 7, etc.

I. Establishing the class in which $a+b=3$.

§ . 68. Here therefore by necessity there will be $a=1,\ b=2$ and $c=1$, thus so that here no variations will be found, and our theorem gives us this single relation $(1,2)(3,1)=(1,1)(2,2)$. Therefore in the same manner the exponent *n* will not have been less than 3, this conspicuous relation always has a place ,

$$\int \frac{\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-2}}} \cdot \int \frac{xx\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-1}}} = \int \frac{\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-1}}} \cdot \int \frac{x\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-2}}},$$

of which the form, because in any symbol the terms can be permuted between themselves, can also be represented in this form :

$$\int \frac{x\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-1}}} \cdot \int \frac{\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-3}}} = \int \frac{\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-1}}} \cdot \int \frac{x\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-2}}}.$$

II. Establishing the class in which $a+b=4$.

§. 69. Because *b* cannot be less than two, here there will be either $b=2,$ or $b=3$. Therefore in the first place, let $b=2,$ and there will be $a=2$ and $c=1;$ hence this relation follows from our theorem $(2,2)(4,1)=(2,1)(3,2)$, which form clearly arises from the first class, if there the first terms of each symbol may be increased by one; that which thence also may be understood, because all the first terms contain the letter *a*, which increased by unity always shall proceed to the following class.

§.70. Then truly here $b=3$ can be put in place also, so that there shall be $a=1$; but truly the letter *c* now has the choice of two values, 1 or 2 ; in the first case, where $c=1$, this equation will be produced $(1,3)(4,1)=(1,1)(2,3)$; truly the other case, where $c=2$, provides this equation $(1,3)(4,2)=(1,2)(3,3)$. And thus this class will contain generally the three following relations :

$$1^{\text{o}}.(2,2)(4,1)=(2,1)(3,2),$$
$$2^{\text{o}}.(1,3)(4,1)=(1,1)(2,3),$$
$$3^{\text{o}}.(1,3)(4,2)=(1,2)(3,3).$$

III. Establishing the class in which $a+b=5$.

§.71. Therefore the three preceding relations occur in this class, but only if the first terms of the symbols of which may be increased by one : hence indeed the cases arise, in which there is either $b=2,$ or $b=3$. Therefore here new cases arise, in which $b=4$ and $a=1$, where therefore there will be either $c=1,$ $c=2,$ or $c=3$, therefore from which three cases set out in this class, the class will contain in all six relations, which will be :

$$1^{\text{o}}.(3,2)(5,1)=(3,1)(4,2),$$
$$2^{\text{o}}.(2,3)(5,1)=(2,1)(3,3),$$
$$3^{\text{o}}.(2,3)(5,2)=(2,2)(4,3),$$
$$4^{\text{o}}.(1,4)(5,1)=(1,1)(2,4),$$
$$5^{\text{o}}.(1,4)(5,2)=(1,2)(3,4),$$
$$6^{\text{o}}.(1,4)(5,3)=(1,3)(4,4).$$

IV. Establishing the class in which $a+b=6$.

§. 72. Therefore here all the preceding nearby relations occur first, but only if the first terms of each symbol may be increased by one: clearly these arise , if there were either $b=2,\ b=3,$ or $b=4$. Truly besides in addition the cases $b=5$ and $a=1$ occur, where the letter *c* will receive the values 1, 2, 3, 4, and thus, in all the ten following relations occur in this class

$$1^{\text{o}}.(4,2)(6,1)=(4,1)(5,2),$$
$$2^{\text{o}}.(3,3)(6,1)=(3,1)(4,3),$$
$$3^{\text{o}}.(3,3)(6,2)=(3,2)(5,3),$$
$$4^{\text{o}}.(2,4)(6,1)=(2,1)(3,4),$$
$$5^{\text{o}}.(2,4)(6,2)=(2,2)(4,4),$$
$$6^{\text{o}}.(2,4)(6,3)=(2,3)(5,4),$$
$$7^{\text{o}}.(1,5)(6,1)=(1,1)(2,5),$$
$$8^{\text{o}}.(1,5)(6,2)=(1,2)(3,5),$$
$$9^{\text{o}}.(1,5)(6,3)=(1,3)(4,5),$$
$$10^{\text{o}}.(1,5)(6,4)=(1,4)(5,5).$$

V. Establishing the class in which $a+b=7$.

§. 73. Here therefore in the first place all the relations of the class IV occur, clearly after we will have increased all the first terms of the individual symbols by one, which therefore it will not be necessary to be established here, and it will suffice to set out here only these relations, which are to be added new, and which arise from the value $b=6$, with $a=1$ being present ; where the numbers 1, 2, 3, 4, 5, thus will be able to be taken for c , thus so that the number of these shall be five. Therefore these relations are :

$$(1,6)(7,1)=(1,1)(2,6)$$
$$(1,6)(7,2)=(1,2)(3,6)$$
$$(1,6)(7,3)=(1,3)(4,6)$$
$$(1,6)(7,4)=(1,4)(5,6)$$
$$(1,6)(7,5)=(1,5)(6,6).$$

VI. Establishing the class in which $a+b=8$.

§. 74. Now in this class in the first place all the ten relations occur of class IV, while clearly all the first terms will be increased by two; besides also the five relations brought from class V are to be added, while the first parts will be augmented by one; truly besides these the following 6 relations are to be added arising from the values $a=1$ and $b=7$, while the values of the letters c are attributed with the orders 1, 2, 3, 4, 5, & 6 , which therefore will be

$$(1,7)(8,1)=(1,1)(2,7)$$
$$(1,7)(8,2)=(1,2)(3,7)$$
$$(1,7)(8,3)=(1,3)(4,7)$$
$$(1,7)(8,4)=(1,4)(5,7)$$
$$(1,7)(8,5)=(1,5)(6,7)$$
$$(1,7)(8,6)=(1,6)(7,7).$$

VII. Establishing the class in which $a+b=9$.

§. 75. So that we may obtain all the relations pertaining to this class, in the first place it is to be observed, here ten relations of class IV occur, while the first parts will be increased by three. In the second place it is required to add five relations shown in the V class, where the first parts must be increased by two. Thus in the third place six relations must be referred to of the VI th class, the first parts being increased by one. Truly in addition seven new relations are added arising from the values $a=1$ and $b=8$,
while the values 1, 2, 3, 4, 5, 6, 7 are contributed by the order of the letter c. These relations are :

$$(1,8)(9,1)=(1,1)(2,8)$$
$$(1,8)(9,2)=(1,2)(3,8)$$
$$(1,8)(9,3)=(1,3)(4,8)$$
$$(1,8)(9,4)=(1,4)(5,8)$$
$$(1,8)(9,5)=(1,5)(6,8)$$
$$(1,8)(9,6)=(1,6)(7,8)$$
$$(1,8)(9,7)=(1,7)(8,8).$$

§. 76. Hence now the order of the progression is clearly seen, so that it would be superfluous to pursue these derivations further; since on account of the huge number of relations, it would be exceedingly troublesome to run through all which occur in the following classes. Moreover our situation scarcely allows to be seen, if indeed we would have wished to enumerate all the relations pertaining to that, so that we may augment the exponent *n* in our general formula beyond six or seven. But if only a little thought may be expended, the classes introduced suffice abundantly, while the first terms of each class will be increased by some number.

§. 77. Now with these classes set out, we will treat the proposed integral formula $\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$ following the diverse values of the exponent *n*, while evidently we may assume successively $n=3,\ n=4,\ n=5,$ etc. , and for any order we may consider all the relations which can occur in that. But it is evident, whatever number may be attributed to the exponent n ; the formulas of all the lesser classes, in which clearly the term $a+b$ will not exceed *n*, can be called into use. From which it is understood, if $n=3$ a single relation will be found; but immediately *n* may be increased more, the number of all the relations thus soon increases, so that all to be examined shall become exceedingly troublesome. Therefore we will involve diverse orders constituted from the exponent *n*, starting from the first.

Order I, where $n=3$ and the formula

$$(p,q)=\int\frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-p}}}.$$

§.78. Since here there shall be $n=3$, there will be $(3,1)=1$; but there will be three integral formulas of this order ; clearly 1°. $(1,1)$, 2°. $(1,\ 2\ )$, 3°. $(\ 2,2\ )$, of which the middle one, on account of $1+2=3$, will depend on the circle, which therefore, because it is known, there may be put

$$(1,2)=\frac{\pi}{3\sin.\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}=\text{A}.$$

Therefore here only the first class can be found, which this single equation gives us:

$$\text{A}=(1,1)(2,2).$$

§. 79. Hence it is therefore apparent, the product from the two transcending formulas (1, 1), and (2, 2) to be equal to the circular magnitude $\text{A}=\frac{2\pi}{3\sqrt{3}}$, thus so that for the integral formulas themselves we will have this relation :

$$\int\frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}\cdot\int\frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)}}=\frac{2\pi}{3\sqrt{3}};$$

so that if either of these two formulas were known, the value of the other could be assigned also. Therefore we will consider the first as if known to us, even if it shall be transcending, and we may put that

$$(1,1)=\int\frac{\partial x}{\sqrt[3]{\left(1-x^{3}\right)^{2}}}=P,$$

and there will become $(2,2)=\frac{\mathrm{A}}{\mathrm{P}}$. And thus nothing besides is left to be noted in this order.

Order II, where $n=4$ and the formula

$$(p,q)=\int\frac{x^{p-1}\partial x}{\sqrt[4]{\left(1-x^{4}\right)^{4-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[4]{\left(1-x^{4}\right)^{4-p}}}.$$

§. 80. Therefore since here there shall be $n=4$, there will be $(4,1)=1$ and $(4,\ 2)=\frac{1}{2}$; moreover there shall be the following six formulas pertaining to this order :

$$1^{\mathrm{o}}.(1,1),2°.(1,2),3°.(1,3),4°.\ (2,2),5°.(2,3),6°.(3,3),$$

among which therefore these two circular formulas (1,3) and (2,2) are found, which we will designate by the letters A and B , by putting

$$(1,3)=\frac{\pi}{4\sin.\frac{\pi}{4}}=\frac{2\pi}{2\sqrt{2}}=\mathrm{A},$$

and

$$(2,2)=\frac{\pi}{4\sin.\frac{2\pi}{4}}=\frac{\pi}{4}=\mathrm{B},$$

thus so that there shall be $\frac{\mathrm{A}}{\mathrm{B}}=\sqrt{2}.$

§. 81. Therefore in this order both equations of the first as well as second order can be found; moreover the second class provides us with these three equations :

$$1°.\ \mathrm{B}=(2,1)(3,2),\ 2°.\ \mathrm{A}=(1,1)(2,3),\ 3°.\ \mathrm{A}=2(1,2)(3,3),$$

indeed the first class above gives this equation $\mathrm{A}(1,2)=(1,1)\mathrm{B}$,

or $\frac{\mathrm{A}}{\mathrm{B}}=\frac{(1,1)}{(1,2)}$, but which equation is deduced now from the two previous equations ; for indeed on account of $(3,2)=(2,3)$, the second divided by the first will give

$\frac{\text{A}}{\text{B}} = \frac{(1,1)}{(1,2)} = \sqrt{2}$, thus so that the ratio between these two formulas shall be algebraic, which deserves especially to be noted :

$$\int \frac{\partial x}{\sqrt[4]{\left(1-x^{4}\right)^{3}}} : \int \frac{\partial x}{\sqrt{\left(1-x^{4}\right)}} = \sqrt{2}.$$

§. 82. Now in this order, besides the two circular formulas $(1,3) = \text{A}$ and $(2,\ 2) = \text{B}$; we may introduce the formula (1, 2) as known, which was circular in the preceding order, but now is transcending, and we may put $(1,2) = \int \frac{\partial x}{\sqrt{\left(1-x^{4}\right)}} = \text{P};$ were it may be warned, neither the letters A and P should be confused with these, which we have used in the preceding formulas, which is also the case in the following orders. With these letters introduced our equations will become the three following :

$$1.\ \text{B} = \text{P}(3,2),\ 2.\ \text{A} = (1,1)(2,3),\ 3.\ \text{A} = 2\text{P}(3,3),$$

as we have seen, four are now contained in the preceding.

§. 83. Therefore with the aid of these three equations, the three integral formulas even now unknown, which we consider given by the three A, B and P will be allowed to be determined also. Indeed from the first there shall be $(3,2) = \frac{\text{B}}{\text{P}}$; moreover from the third there shall become $(3,3) = \frac{\text{A}}{2\text{P}}$; then truly from the second there is deduced $(1,1) = \frac{\text{A}}{(3,2)} = \frac{\text{AP}}{\text{B}}$. Therefore since in this order there shall be six integral formulas in total, of these three can be defined in terms of the remaining three, therefore it will help to have a look at these determinations :

$$1.\ (1,3) = \text{A} = \frac{\pi\sqrt{2}}{2},$$
$$2.\ (2,2) = \text{B} = \frac{\pi}{4},$$
$$3.\ (1,2) = \text{P} = \int \frac{\partial x}{\sqrt{\left(1-x^{4}\right)}},$$
$$4. (1,1) = \frac{\text{AP}}{\text{B}},$$
$$5.\ (2,3) = \frac{\text{B}}{\text{P}},$$
$$6.\ (3,3) = \frac{\text{A}}{2\text{P}}.$$

Therefore from the latter there will be

$$(2,3):(3,3) = 2\text{B}:\text{A} = \sqrt{2}:1,$$

thus so that also these two formulae have an algebraic ratio between each other, which is :

$$\int \frac{xx\partial x}{\sqrt{\left(1-x^4\right)}} = \sqrt{2}\int \frac{xx\partial x}{\sqrt[4]{\left(1-x^4\right)}}.$$

Here we will not tarry with other significant relationships, as being known well enough.

Order III, where $n=5$ and the formula

$$(p,q) = \int \frac{x^{p-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-p}}}.$$

§. 84. Therefore here on account of $n=5$ there will be initially :

$$(5,1)=1,\ (5,2)=\tfrac{1}{2},\ (5,3)=\tfrac{1}{3};$$

moreover there will be ten integral formulas of this kind :

$$1.(1,1),\ \ 2.(1,2),\ 3.(1,3),\ 4.(1,4),\ \ 5.(2,2),$$
$$6.(2,3), 7.(2,4),\ 8.(3,3),\ 9.(3,4),\ 10.(4,4),$$

among which, 4. and 6. are circular, which therefore we may designate thus :

$$(1,4) = \frac{\pi}{5\sin.\frac{1}{5}\pi} = \mathrm{A}$$

and

$$(2,3) = \frac{\pi}{5\sin.\frac{2\pi}{5}} = B.$$

Truly besides two formulas, which were circular in the preceding orders, but now are transcending, we may denote also by special letters, clearly

$$(1,3) = \mathrm{P} \text{ and } (2,2) = \mathrm{Q}.$$

Indeed it will be apparent soon, while these formulas also may be considered as known, all the remaining six can be determined from these four.

§. 85. Because here three former classes can be found, initially we will consider equations, which the three classes support, and which will be, with these values introduced :

$$1.\ \mathrm{B}=\mathrm{P}(4,2),$$
$$2.\ \mathrm{B}=(2,1)(3,3),$$
$$3.\ \mathrm{B}=2\mathrm{Q}(4,3),$$
$$4.\ \mathrm{A}=(1,1)(2,4),$$
$$5.\ \mathrm{A}=2(1,2)(3,4),$$
$$6.\ \mathrm{A}=3\mathrm{P}(4,4).$$

Which can be represented more succinctly in this manner :

$$\mathrm{A}=(1,1)(2,4)=2(1,2)(3,4)=3\mathrm{P}(4,4),$$
$$\mathrm{B}=\mathrm{P}(4,2)=(2,1)(3,3)=2\mathrm{Q}(4,3),$$

where six products arise from the two integrable formulas, which each are equal to the individual circular quantity, from which the same outstanding theorems may be formed, unless they may be seen clearly now.

§. 86. Now we may consider, how many integral formulas we may be able to define from the four known formulas A, B, P and Q, but truly the first gives $(4,2)=\frac{\mathrm{B}}{\mathrm{P}}$, the third presents $(4,3)=\frac{\mathrm{B}}{2\mathrm{Q}}$, the sixth gives $(4,4)=\frac{\mathrm{A}}{3\mathrm{P}}$; hence moreover again we deduce from the fourth : $(1,1)=\frac{\mathrm{A}}{(2,4)}=\frac{\mathrm{AP}}{\mathrm{B}}$, truly from the fifth we deduce $(1,2)=\frac{\mathrm{A}}{2(3,4)}=\frac{\mathrm{AQ}}{\mathrm{B}}$. Finally from the second we elicit $(3,3)=\frac{\mathrm{B}}{(2,1)}=\frac{\mathrm{BB}}{\mathrm{AQ}}$ and thus from these six equations we have arrived at six determinations; and therefore according to that we have assigned the values of all the remaining letters by the letters A, B, P and Q.

§. 87. Therefore since up to this point we have made use of the third class only, we will consider also the equations of the second class, which are :

$$1.\ \mathrm{AQ}=\mathrm{B}(2,1),$$
$$2.\ \mathrm{AP}=\mathrm{B}(1,1)$$

and

$$3.\ \mathrm{P}(4,2)=(1,2)(3,3);$$

in truth if we may substitute only these values found, purely identical equations will result, so that hence no new determination may follow. The same arises from the use of the first class of equations, which was $(2,1)(3,1)=(1,1)(2,2)$, with which substitution make it will be identical also, thus so that the two first classes involve nothing new. Yet neither hence can it be concluded, in the following orders also the preceding classes may be put aside, since in the following order a contrary result will show itself at once.

§.88 Therefore since this order may contain ten integral formulas, we have set out thus for the order the values of these are to be seen from the four letters A, B, P and Q :

$$\begin{array}{ll}
1.\ (1,1)=\frac{\mathrm{AP}}{\mathrm{B}}, & 6.\ (2,3)=\mathrm{B}, \\
2.\ (1,2)=\frac{\mathrm{AQ}}{\mathrm{B}}, & 7.\ (2,4)=\frac{\mathrm{B}}{\mathrm{P}}, \\
3.\ (1,3)=\mathrm{P}, & 8.\ (3,3)=\frac{\mathrm{BB}}{\mathrm{AQ}}, \\
4.\ (1,4)=\mathrm{A}, & 9.\ (3,4)=\frac{\mathrm{B}}{2\mathrm{Q}}, \\
5.\ (2,2)=\mathrm{Q}, & 10.\ (4,4)=\frac{\mathrm{A}}{3\mathrm{P}}.
\end{array}$$

§. 89. Since there shall be

$$\frac{\mathrm{A}}{\mathrm{B}}=\frac{\sin.\frac{2}{5}\pi}{\sin.\frac{1}{5}\pi}=\cos.\frac{1}{5}\pi,$$

then truly

$$\cos.\frac{1}{5}\pi=\frac{1+\sqrt{5}}{2},$$

there will be

$$\frac{\mathrm{A}}{\mathrm{B}}=\frac{1+\sqrt{5}}{2}$$

and thus are algebraic quantities. Hence therefore some equal number of integral formulas will be able to be shown, which maintain an algebraic relation among themselves; indeed there will be

$$\frac{(1,1)}{(1,3)}=\frac{1+\sqrt{5}}{2},\quad \frac{(1,2)}{(2,2)}=\frac{1+\sqrt{5}}{2},\quad \frac{(3,4)}{(3,3)}=\frac{1+\sqrt{5}}{4},\quad \frac{(4,4)}{(2,4)}=\frac{1+\sqrt{5}}{6},$$

from which the same number of outstanding theorems may be hidden away, unless they may be elicited from these formulas.

Order IV where $n=6$ and the formula

$$\left(p,q\right)=\int \frac{x^{p-1}\partial x}{\sqrt[6]{\left(1-x^6\right)^{6-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[6]{\left(1-x^6\right)^{6-p}}}.$$

§. 90. Because here there is $n=6$, we will have initially

$$\left(6,1\right)=1,\ \left(6,2\right)=\tfrac{1}{2},\ \left(6,3\right)=\tfrac{1}{3},\ \left(6,4\right)=\tfrac{1}{4};$$

moreover the number of integral formulas occurring in this order is fifteen, which are :

$$\begin{array}{l} 1.\ (1,1),\quad 2.(1,2),\quad 3.(1,3),\ 4.(1,4),\quad 5.(1,5),\\ 6.\ (2,2),\quad 7.\ (2,3),\quad 8.(2,4),\ 9.(2,5),\ 10.(3,3),\\ 11.\ (3,4),\ 12.\ (3,5),\ 13.(4,4),\ 14.(4,5),\ 15.(5,5), \end{array}$$

between which there will be found three circular ones, which we will designate in a separate manner, evidently there shall be

$$\left(1,5\right)=\frac{\pi}{6\sin.\frac{1}{6}\pi}=\frac{\pi}{3}=\mathrm{A},$$
$$\left(2,4\right)=\frac{\pi}{6\sin.\frac{2\pi}{6}}=\frac{\pi}{3\sqrt{3}}=\mathrm{B},$$

and

$$\left(3,3\right)=\frac{\pi}{6\sin.\frac{3\pi}{6}}=\frac{\pi}{6}=\mathrm{C},$$

thus so that there shall be

$$\mathrm{A}=2\mathrm{C}.$$

Truly besides both the formulas, which were circles in the preceding order, truly now are transcending, and we may put

$$\left(1,4\right)=\mathrm{P}\text{ and }\left(2,3\right)=\mathrm{Q}.$$

With these denominations made we may now set out ten equations of the fourth class, which are :

$$\begin{array}{ll} 1.\ \mathrm{B}=\mathrm{P}\left(5,\,2\right), & 6.\mathrm{B}=3\mathrm{Q}(5,4),\\ 2.\ \mathrm{C}=\left(3,1\right)\left(4,3\right), & 7.\mathrm{A}=\left(1,1\right)\left(5,2\right),\\ 3.\ \mathrm{C}=2\mathrm{Q}(5,3), & 8.\mathrm{A}=2\left(1,2\right)\left(3,5\right),\\ 4.\ \mathrm{B}=\left(2,1\right)\left(3,4\right), & 9.\mathrm{A}=3\left(1,3\right)\left(4,5\right),\\ 5.\ \mathrm{B}=2\left(2,2\right)\left(4,4\right), & 10.\mathrm{A}=4\mathrm{P}\left(5,5\right), \end{array}$$

which thus may be referred to more succinctly

$$\begin{aligned}
&\mathrm{A} = (1,1)(5,2) = 2(1,2)(3,5) = 3(1,3)(4,5) = 4\mathrm{P}(5,5),\\
&\mathrm{B} = \mathrm{P}(5,2) = (2,1)(3,4) = 2(2,2)(4,4) = 3\mathrm{Q}(4,5),\\
&\mathrm{C} = (3,1)(4,3) = 2\mathrm{Q}(5,3).
\end{aligned}$$

Behold therefore ten products from the two integrable formulas, of which the two individual quantities are circular.

§. 91. Then since there shall be $\frac{\mathrm{A}}{\mathrm{B}} = \sqrt{3}$ and $\frac{\mathrm{A}}{\mathrm{C}} = 2$, then also $\frac{\mathrm{B}}{\mathrm{C}} = \frac{2}{\sqrt{3}}$, several pairs of the two integral formulas can be shown, which maintain an algebraic ratio between themselves ; for there will be

$$\begin{aligned}
&\frac{\mathrm{A}}{\mathrm{B}} = \sqrt{3} = \frac{(1,1)}{(1,4)} = \frac{2(3,5)}{(3,4)} = \frac{(1,3)}{(2,3)} = \frac{4(5,5)}{(5,2)},\\
&\frac{\mathrm{A}}{\mathrm{C}} = 2 = \frac{(1,2)}{(2,3)} = \frac{3(4,5)}{(4,3)},\\
&\frac{\mathrm{B}}{\mathrm{C}} = \frac{2}{\sqrt{3}} = \frac{(1,2)}{(1,3)} = \frac{3(4,5)}{2(3,5)}.
\end{aligned}$$

§. 92. But if now the five letters A, B, C, P and Q designate the formulas we may consider known, we shall see, how the remaining formulas may be able to be defined by these. And indeed we run across the ten equations of the fourth class brought forwards above, of which the first will give $(5,2) = \frac{\mathrm{B}}{\mathrm{P}}$, the third gives $(5,3) = \frac{\mathrm{C}}{2\mathrm{Q}}$, the sixth provides $(5,4) = \frac{\mathrm{B}}{3\mathrm{Q}}$, and the tenth gives $(5,5) = \frac{\mathrm{A}}{4\mathrm{P}}$. But if now we may replace these values in the remaining, the seventh will give $(1,1) = \frac{\mathrm{A}}{(5,2)} = \frac{\mathrm{AP}}{\mathrm{B}}$, the eighth $(1,\ 2) = \frac{\mathrm{A}}{2(3,5)} = \frac{\mathrm{AQ}}{\mathrm{C}}$, and the ninth $(3,1) = \frac{\mathrm{A}}{3(4,5)} = \frac{\mathrm{AQ}}{\mathrm{B}}$. Truly the fourth gives again $(3,4) = \frac{\mathrm{B}}{(2,1)} = \frac{\mathrm{BC}}{\mathrm{AQ}}$, which value the second gave also. But truly we can elicit no value from the fifth, because even now neither the formula (2, 2) nor (4, 4) agree. The reason being, because the two of the remaining equations will produce the same outcome.

§. 93. Therefore we are to recourse to equations of the previous classes collected and thus from the first class :

$$(1,2)(3,1) = (1,1)(2,2)$$

we gather at once

$$(2,2) = \frac{(1,2)(3,1)}{(1,1)} = \frac{\mathrm{AQQ}}{\mathrm{CP}},$$

which value substituted into the fifth equation supplies the latter equation, namely

$$(4,4)=\frac{\mathrm{B}}{2(2,2)}=\frac{\mathrm{BCP}}{2\mathrm{AQQ}}.$$

Therefore we will refer all these values here with the order:

1. $(1,1)=\frac{\mathrm{AP}}{\mathrm{B}}$,	4. $(1,4)=\mathrm{P}$,	7. $(2,3)=Q$,
2. $(1,2)=\frac{\mathrm{AQ}}{\mathrm{C}}$,	5. $(1,5)=\mathrm{A}$,	8. $(2,4)=\mathrm{B}$,
3. $(1,3)=\frac{\mathrm{AQ}}{\mathrm{B}}$,	6. $(2,2)=\frac{\mathrm{AQQ}}{\mathrm{CP}}$,	9. $(2,5)=\frac{\mathrm{B}}{\mathrm{P}}$,
10. $(3,3)=\mathrm{C}$,	12. $(3,5)=\frac{\mathrm{C}}{2\mathrm{Q}}$,	14. $(4,5)=\frac{\mathrm{B}}{3\mathrm{Q}}$,
11. $(3,4)=\frac{\mathrm{BC}}{\mathrm{AQ}}$,	13. $(4,4)=\frac{\mathrm{BCP}}{2\mathrm{AQQ}}$	15. $(5,5)=\frac{\mathrm{A}}{4\mathrm{P}}$.

§. 94. But since in this order also equations both from the second as well as from the third class must prevail, we may see, whether values found from these classes may be appropriate or truly perhaps will supply a new determination? But with a substitution made into the three equations of the second class it becomes an identity, which also must pertain in equations of the third class, because that will soon becomes apparent on expanding . From which it is to be observed that all the equations contained in the four first classes, of which the number is 20, only ten determinations are themselves to be included.

Order V, where $n=7$ and the formula

$$(p,q)=\int \frac{x^{p-1}\partial x}{\sqrt[7]{\left(1-x^7\right)^{7-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[7]{\left(1-x^7\right)^{7-p}}}.$$

§. 95. Because here $n=7$, before everything we will have the absolute values :

$$(7,1)=1,\ (7,2)=\tfrac{1}{2},\ (7,3)=\tfrac{1}{3},\ (7,4)=\tfrac{1}{4}\ \text{et}\ (7,5)=\tfrac{1}{5},$$

then between the integral formulas of this order the circular ones must be noted especially, which we will designate in this manner:

$$\left(1,6\right)=\frac{\pi}{7\sin.\frac{\pi}{7}}=\text{A},$$
$$\left(2,5\right)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=\text{B},$$
$$\left(3,4\right)=\frac{\pi}{7\sin.\frac{3\pi}{7}}=\text{C}.$$

Truly besides these formulas will be specified by their own letters, which were circular in the preceding order, but here the values are allocated transcending values, which shall be

$$\left(1,5\right)=\text{P},\quad \left(2,4\right)=\text{Q}\ \text{ and }\ \left(3,3\right)=\text{R};$$

indeed by these six letters we will see that all the remaining formulas of this order are able to be determined.

§. 96. Since above we have not expressed all the equations of the fifth class, we may show these here jointly, and we may adapt to our case

I.	(1,6)(7,1) = (1,1)(2,6)	A = (1,1)(2,6),
II.	(1,6)(7,2) = (1,2)(3,6)	A = 2 (1,2)(3,6),
III.	(1,6)(7,3) = (1,3)(4,6)	A = 3(1,3)(4,6),
IV.	(1,6)(7,4) = (1,4)(5,6)	A = 4(1,4)(5,6),
V.	(1,6)(7,5) = (1,5)(6,6)	A = 5P (6,6),
VI.	(2,5)(7,1) = (2,1)(3,5)	B = (2,1)(3,5),
VII.	(2,5)(7,2) = (2, 2)(4,5)	B = 2(2,2)(4,5),
VIII.	(2,5)(7,3) = (2,3)(5,5)	B = 3 (2,3)(5,5),
IX.	(2,5)(7,4) = (2,4)(6,5)	B = 4 Q (6,5),
X.	(3,4)(7,1) = (3,1)(4,4)	C = (3,1)(4,4),
XI.	(3,4)(7,2) = (3,2)(5,4)	C = 2 (3,2)(5,4),
XII.	(3,4)(7,3) = (3,3)(6,4)	C = 3 R (6,4),
XIII.	(4,3)(7,1) = (4,1)(5,3)	C = (4,1)(5,3),
XIV.	(4,3)(7,2) = (4,2)(6,3)	C = 2 Q (6,3),
XV.	(5,2)(7,1) = (5,1)(6,2)	B = P(6, 2).

Therefore here we have five products of the formula A equal to just as many formulas of B and C.

§. 97. Moreover in this order 21 integral formulas occur altogether, which we will designate by the six letters A, B, C, P, Q and R, by which therefore it will be necessary to define the remaining fifteen integral formulas, which are :

$$1^{\text{o}}.\ (1,1),\ 2^{\text{o}}.\ (1,2),\ 3^{\text{o}}.\ (1,3),\ 4^{\text{o}}.\ (2,2),\ 5^{\text{o}}.\ (1,4),$$
$$6^{\text{o}}.\ (2,3),\ 7^{\text{o}}.\ (2,6),\ 8^{\text{o}}.\ (3,5),\ 9^{\text{o}}.\ (4,4),\ 10^{\text{o}}.\ (3,6),$$
$$11^{\text{o}}.\ (4,5),\ 12^{\text{o}}.\ (4,6),\ 13^{\text{o}}.\ (5,5),\ 14^{\text{o}}.\ (5,6),\ 15^{\text{o}}.\ (6,6),$$

§. 98. Therefore we shall se, how many of these formulas may be allowed to be determined from the above fifteen equations, and indeed initially from the equations V, IX, XII, XIV and XV, the following formulas are deduced immediately:

$$(6,6)=\frac{\text{A}}{5\text{P}},\ (6,5)=\frac{\text{B}}{4\text{Q}},\ (6,4)=\frac{\text{C}}{3\text{R}},\ (6,3)=\frac{\text{C}}{2\text{Q}},\ (6,2)=\frac{\text{B}}{\text{P}}.$$

Now with these found from the equations I, II, III and IV we may derive these formulas

$$(1,1)=\frac{\text{AP}}{\text{B}},\ (1,2)=\frac{\text{AQ}}{\text{C}},\ (1,3)=\frac{\text{AR}}{\text{C}},\ (1,4)=\frac{\text{AQ}}{\text{B}}.$$

Truly from these values by equations VI, X and XIII, we deduce :

$$(3,5)=\frac{\text{BC}}{\text{AQ}},\ (4,4)=\frac{\text{CC}}{\text{AR}}\ \text{and}\ (5,3)=\frac{\text{BC}}{\text{AQ}},$$

where it will be pleasing to note the same value (5, 3) to be produced from equations VI and XIII. But nothing can be concluded from the remaining equations VII, VIII and XI , from which these four formulas (2, 2), (2, 3), (5, 4) et (5, 5) even now remain unknown to us.

§.99. Therefore we are to return to the collected equations of the preceding class, certainly which pertain equally to our order and to the equations of the fifth class ; on this account in a similar manner we may place here the equations of the fourth class, and we may apply to our case:

I. (1,5)(6,1) = (1,1)(2,5)	PA = (1,1)B,
II. (1,5)(6,2) = (1,2)(3,5)	P(6,2) = (1,2)(3,5),
III. (1,5)(6,3) = (1,3)(4,5)	P(6,3) = (1,3)(4,5),
IV. (1,5)(6,4) = (1,4)(5,5)	P(6,4) = (1,4)(5,5),
V. (2,4)(6,1) = (2,1)(3,4)	QA = (2,1)C,
VI. (2,4)(6,2) = (2,2)(4,4)	Q(6,2) = (2,2)(4,4),
VII. (2,4)(6,3) = (2,3)(5,4)	Q(6,3) = (2,3)(5,4)
VIII. (3,3)(6,1) = (3,1)(4,3)	BA = (3,1)C,
IX. (3,3)(6,2) = (3,2)(5,3)	R(6,2) = (3,2)(5,3),
X. (4,2) (6,1) = (4,1)(5,2)	QA = (4,1)B.

§.100. From the equations I, V, VIII and X we can infer these formulas at once :

$$(1,1)=\frac{\mathrm{PA}}{\mathrm{B}},\ (2,1)=\frac{\mathrm{QA}}{\mathrm{C}},\ (3,1)=\frac{\mathrm{AR}}{\mathrm{C}},\ (4,1)=\frac{\mathrm{AQ}}{\mathrm{B}},$$

which values moreover we have applied now before. The second equation, if the formulas now found may be substituted, produces an identical equation. But from the third we will be able to define the formula (4,5), the value of which hence we deduce :

$$(4,5)=\frac{\mathrm{CCP}}{2\mathrm{AQR}}.$$

In a similar manner we elicit from the fourth:

$$(5,5)=\frac{\mathrm{BCP}}{3\mathrm{AQR}}.$$

Again from the sixth equation we may infer :

$$(2,2)=\frac{\mathrm{ABQR}}{\mathrm{CCP}}.$$

Then the seventh equation gives :

$$(2,3)=\frac{\mathrm{AQR}}{\mathrm{CP}}.$$

Truly the ninth equation also produces $(3,2)=\frac{\mathrm{AQR}}{\mathrm{CP}}$. And thus we have determined all fifteen unknown formulas by the six known letters A, B, C, P, Q and R.

§.101. Therefore we may set out the values of all our formulas of this order to be viewed together :

$(1,6)=\text{A}$	$(6,2)=\frac{\text{B}}{\text{P}}$	$(1,1)=\frac{\text{AP}}{\text{B}}$	$(3,5)=\frac{\text{BC}}{\text{AQ}}$	$(2,3)=\frac{\text{AQR}}{\text{CP}}$
$(2,5)=\text{B}$	$(6,3)=\frac{\text{C}}{2\text{Q}}$	$(1,2)=\frac{\text{AQ}}{\text{C}}$	$(4,4)=\frac{\text{CC}}{\text{AR}}$	$(4,5)=\frac{\text{CCP}}{2\text{AQR}}$
$(3,4)=\text{C}$	$(6,4)=\frac{\text{C}}{3\text{R}}$	$(1,3)=\frac{\text{AR}}{\text{C}}$		$(5,5)=\frac{\text{BCP}}{3\text{AQR}}$
$(1,5)=\text{P}$	$(6,5)=\frac{4}{4\text{Q}}$	$(1,4)=\frac{\text{AQ}}{\text{B}}$		$(2,2)=\frac{\text{ABQR}}{\text{CCP}}$.
$(2,4)=\text{Q}$	$(6,6)=\frac{\text{A}}{5\text{P}}$			
$(3,3)=\text{R}$				

§. 102. But because equations of the first, second, and third class also prevail in this order, if we substitute into these the values found here, we will always arrive at identical equations. Thus, since the equation of the first class shall be

$$(1,2)(3,1)=(1,1)(2,2),$$

with the substitution made there will be found

$$(1,2)(3,1)=\frac{\text{AAQR}}{\text{CC}};$$

but truly $(1,1)(2,2)$ becomes $=\frac{\text{AAQR}}{\text{CC}}$, and this identity also will be taken in the three equations of the second class and also in the six equations of the third class, just as the calculation soon to be put in place will make apparent.

§.103. In a similar manner this investigation will be extended without difficulty to higher orders, without even the law being observed, according to which the determinations of the individual formulas of any order are advanced. Yet meanwhile it will help to observe in the following sixth order, where $n=8$ and 28 formulas occur, all these initially can be determined by the four circle formulas

$$(1,7)=\text{A},(2,6)=\text{B},(3,5)=\text{C},(4,4)=\text{D},$$

as well as truly by these three transcending formulas

$$(1,6)=\text{P},\ (2,5)=\text{Q} \text{ and } (3,4)=\text{R}.$$

Therefore for any order, the determination of the individual formulas, besides the circular formulas, which certainly can be considered as known, also demand some transcending formulas, if perhaps we may wish to know the values of these formulas approximately, a method may be desired at this point, being required to define these values approximately, or as in decimal fractions. Therefore we will add such a method here in place of corollaries.

Problem.

For the proposed integral formula of any order,

$$\mathrm{S}=\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}},$$

requiring to be extended from the limit $x=0$ *as far as to* $x=1$, *to investigate the converging series, which that same value* S *may express*.

Solution.

§.104. Since there shall be

$$\frac{1}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}=\left(1-x^{n}\right)^{-\frac{(n-q)}{n}},$$

with the expansion of this binomial power made in the usual manner, there will be found:

$$\frac{1}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}=1+\frac{n-q}{n}x^{n}+\frac{n-q}{n}\cdot\frac{2n-q}{2n}x^{2n}+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}x^{3n}+\text{etc.}$$

If this series may be multiplied by $x^{p-1}\partial x$ and integrated, there will be produced

$$\mathrm{S}=\frac{x^{p}}{p}+\frac{n-q}{n}\frac{x^{n+p}}{n+p}+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{x^{2n+p}}{2n+p}$$
$$+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{x^{3n+p}}{3n+p}+\text{etc.}$$

which series now will vanish on putting $x=0$; from which if we may put $x=1$, the value sought of our formula will become

$$\mathrm{S}=\frac{1}{p}+\frac{n-q}{n}\frac{1}{n+p}+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\frac{1}{2n+p}$$
$$+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{1}{3n+p}+\text{etc.}$$

§. 105. Truly this same series, whatever numbers may be taken for the letters *n*, *p* and *q*, converges exceedingly slowly, so that from these values the value of S itself may be able to be defined well enough to perhaps three or four decimal figures exactly; on account of which another expansion may be appropriate to be put in place, while evidently we will resolve the value sought into two parts. Therefore we may put :

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}\begin{bmatrix}\text{from} & x=0\\ \text{to} & x^n=\frac{1}{2}\end{bmatrix}=\mathrm{P}$$

and

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}\begin{bmatrix}\text{from} & x^n=\frac{1}{2}\\ \text{to} & x=1\end{bmatrix}=\mathrm{Q},$$

and it will become evident that $\mathrm{S}=\mathrm{P}+\mathrm{Q}$. But now the series both for P as well as Q will be able to be shown to be converging well enough without difficulty.

§. 106. Because first for the value P attains, that we will easily derive by putting $x^n=\frac{1}{2}$ from the general value, which we have found above for S, thus so that there shall be $x=\sqrt[n]{\frac{1}{2}}$ and $x^p=\frac{1}{\sqrt[n]{2^p}}$, with which done we will find this series for P:

$$\mathrm{P}=\frac{1}{\sqrt[n]{2^p}}\left\{\begin{aligned}&\frac{1}{p}+\frac{n-q}{2n}\frac{1}{n+p}+\frac{n-q}{2n}\cdot\frac{2n-q}{4n}\frac{1}{2n+p}\\ &\qquad+\frac{n-q}{2n}\cdot\frac{2n-q}{4n}\cdot\frac{3n-q}{6n}\cdot\frac{1}{3n+p}+\text{etc.}\end{aligned}\right\}.$$

In which series the individual terms decrease more than in the ratio of two, that so that for example the tenth term now is going to be much smaller than from $\frac{1}{1024}$, certainly if we wish to the millionth part, it will suffice to extend the calculation indeed not as far as to the twentieth term.

§.107. Then since we will have put in place :

$$\mathrm{Q}=\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}\begin{bmatrix}\text{from} & x^n=\frac{1}{2}\\ \text{to} & x=1\end{bmatrix},$$

we may put $1-x^n=y^n$, so that there shall be:

$$\mathrm{Q}=\int \frac{x^{p-1}\partial x}{y^{n-q}};$$

then truly there shall be $x^{n}=1-y^{n}$ and thus $x^{p}=\sqrt[n]{\left(1-y^{n}\right)^{p}}$, from which on differentiating there is deduced :

$$x^{p-1}\partial x=-y^{n-1}\partial y\left(1-y^{n}\right)^{\frac{p-n}{n}},$$

with which value substituted there will be

$$\mathrm{Q}=-\int y^{n-1}\partial y\left(1-y^{n}\right)^{\frac{p-n}{n}}\begin{bmatrix}\text{from} & y^{n}=\frac{1}{2}\\ \text{to} & y=0\end{bmatrix}.$$

For when there becomes $x^{n}=\frac{1}{2}$, then also there will be $y^{n}=\frac{1}{2}$, but on making $x=1$ clearly there becomes $y=0$; whereby if we may interchange the terms of the integral, also the sign of the formula must be unchanged, and thus there will become :

$$\mathrm{Q}=\int y^{q-1}\partial y\left(1-y^{n}\right)^{\frac{p-n}{n}}\begin{bmatrix}\text{from } y=0\\ \text{to } \; y^{n}=\frac{1}{2}\end{bmatrix}.$$

§.108. But this formula found for Q is entirely similar to that, which we have found for P, yet with this distinction, that the letters *p* and *q* are themselves interchanged ; on which account, if the integration may be represented by a series, the following will emerge :

$$\mathrm{Q}=\frac{1}{\sqrt[n]{2^{q}}}\left\{\begin{aligned}&\frac{1}{q}+\frac{n-p}{2n}\frac{1}{n+q}+\frac{n-p}{2n}\cdot\frac{2n-p}{4n}\frac{1}{2n+q}\\&\quad+\frac{n-p}{n}\cdot\frac{2n-p}{2n}\cdot\frac{3n-p}{6n}\cdot\frac{1}{3n+q}+\text{etc.}\end{aligned}\right\},$$

which series equally will converges as well as the preceding found for P. But the value will always be sought by recalling from these two series :

$$\mathrm{S=P+Q}.$$

COROLLARY 1

§.109. This same calculation will be much contracted in these cases, in which there is put $p=q$; then indeed there will become $P=Q$ in these cases , in which

$$\mathrm{S}=\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-p}}},$$

the value of which formula extended from $x=0$ to $x=1$ will be

$$\mathrm{S} = \frac{2}{\sqrt[n]{2^p}}\left\{\begin{array}{l} \frac{1}{p} + \frac{n-p}{2n}\frac{1}{n+p} + \frac{n-p}{2n}\cdot\frac{2n-p}{4n}\frac{1}{2n+p} \\ \qquad + \frac{n-p}{2n}\cdot\frac{2n-p}{4n}\cdot\frac{3n-p}{6n}\cdot\frac{1}{3n+p} + \text{etc.} \end{array}\right\}.$$

COROLLARY 2.

§. 110. Therefore since in individual orders some of these formulas (*p*, *p*) occur, and immediately a number of values of formulas of this kind may be recalled as decimal calculations, because circular formulas by themselves are known, from these it would be allowed to assign the values of all the remaining formulas of the same order.

EXAMPLE

§. 111. This formula of the first order shall be proposed, where $p = q = 2$ and

$$\mathrm{S} = \int \frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)}} .$$

Therefore a series for S will be found :

$$\mathrm{S} = \sqrt[3]{2}\left(\frac{1}{2} + \frac{1}{6}\cdot\frac{1}{5} + \frac{1}{6}\cdot\frac{4}{12}\cdot\frac{1}{8} + \frac{1}{6}\cdot\frac{4}{12}\cdot\frac{7}{18}\cdot\frac{1}{11} + \frac{1}{6}\cdot\frac{4}{12}\cdot\frac{7}{18}\cdot\frac{10}{24}\cdot\frac{1}{14} + \text{etc.}\right).$$

But with the calculation performed there is found :

$$\mathrm{S} = 0,54326\sqrt[3]{2} = 0,68446,$$

which therefore is the value of the formula (2,2) in the first order (§. 22), where we found $\left(2,2\right) = \frac{\mathrm{A}}{\mathrm{P}}$, thus so that now there shall be $\mathrm{P} = \frac{\mathrm{A}}{(2,2)}$. Truly there is

$$A = \frac{2\pi}{3\sqrt{3}} = 1,20920,$$

hence there will become $\mathrm{P} = 1,76664 = \left(1,1\right)$, from which in decimal fractions the three formulas of the first order will be :

$$\left(1,1\right) = 1,76664,\ \left(1,\ 2\right) = 1,20918,\ \left(2,2\right) = 0,68445.$$

And in this manner also all the formulas of the following orders will be allowed to be expanded out.

3.

ADDITION TO THE DISSERTATION ON THE VALUES OF THE INTEGRAL FORMULAS $\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}$ EXTENDED FROM $x=0$ TO $x=1$.

Shown to the Convened Meeting on the 17th of October, 1776.
New Acts of the Petersburg Academy of Sciences 5 (1787), 1789, p. 118-129

§. 112. If we may wish to transfer the method treated in the preceding dissertation to orders higher than $n=7$, on account of the nature of the equations requiring to be considered, the labour will become exceedingly troublesome. But since we have seen, not all these same equations concur on the values of the individual formulas being determined, the effort will be raised considerably, if in any case we take only these equations into a calculation, which lead immediately to the determinations of the formulas, just as I am about to show here for the case $n=10$.

The determination of these formulas for the case $n=10$, where the formula

$$\left(p,q\right)=\int \frac{x^{p-1}\partial x}{\sqrt[10]{\left(1-x^{10}\right)^{10-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[10]{\left(1-x^{10}\right)^{10-p}}}.$$

§.113. Therefore in this case the formulas receiving an absolute value are :

$$\left(10,1\right)=1,\ \left(10,2\right)=\tfrac{1}{2},\ \left(10,3\right)=\tfrac{1}{3},\text{ and in general }(10,\alpha)=\tfrac{1}{\alpha}.$$

Then all the formulas, in which there is $p+q=10$, will depend on the circle, and thus are taken as known, which therefore we may designate by their own :

$$\left(1,9\right)=\frac{\pi}{10\sin.\frac{1}{10}\pi}=\text{A},\qquad \left(6,4\right)=\frac{\pi}{10\sin.\frac{6}{10}\pi}=\text{D},$$

$$\left(2,8\right)=\frac{\pi}{10\sin.\frac{2}{10}\pi}=\text{B},\qquad \left(7,3\right)=\frac{\pi}{10\sin.\frac{7}{10}\pi}=\text{C},$$

$$\left(3,7\right)=\frac{\pi}{10\sin.\frac{3}{10}\pi}=\text{C},\qquad \left(8,2\right)=\frac{\pi}{10\sin.\frac{8}{10}\pi}=\text{B},$$

$$\left(4,6\right)=\frac{\pi}{10\sin.\frac{4}{10}\pi}=\text{D},\qquad \left(9,1\right)=\frac{\pi}{10\sin.\frac{9}{10}\pi}=\text{A}.$$

$$\left(5,5\right)=\frac{\pi}{10\sin.\frac{5}{10}\pi}=\text{E},$$

§. 114. But by no means is it allowed to determine the remaining formulas contained in the general form by these circular formulas, but in addition it is necessary to call in some transcending formulas to help, from which with these taken jointly with the circular formulas it will be possible to assign the values of all the remaining formulas. Moreover in our case, where $n=10$, the following formulas may be agreed to be considered as if known, which in the preceding order, where $n=9$, were circular formulas, but now pass over into the order of transcending functions. Therefore we may designate these in the following manner :

$$(1,8)=\mathrm{P},\ (2,7)=\mathrm{Q},\ (3,6)=\mathrm{B},\ (4,5)=\mathrm{S},$$
$$(5,4)=\mathrm{S},\ (6,3)=\mathrm{R},\ (7,2)\ =\mathrm{Q},\ (8,1)=\mathrm{P}.$$

Evidently if we may regard the values of these letters as known also, through these taken with the circular ones we will be able to determine all the remaining formulas contained in this order. Therefore since the number of all the formulas contained in this order $n=10$ shall be 45, but from these nine may be considered as known, the remaining 36 must be determined from these capital letters.

§. 115. But these determinations ought to be sought from the general equation demonstrated above [§.66], which is contained in this form

$$(a,b)(a+b,c)=(a,c)(a+c,b),$$

where it may always be assumed that $b>c$, because, if there were $c=b$, the equation would become. Therefore in the first place, so that hence the equations may be arrived at, which the determinations produce at once, we may assume $a+b=10$, so that there shall become $(10,c)=\frac{1}{c}$; then truly there may be taken $c=b-1$, with which done for *a* by writing the numbers 1, 2, 3, etc. for the orders, the following determinations will be produced :

$$(1,9)(10,8)=(1,8)(9,9) \text{ or } \tfrac{1}{8}\mathrm{A}=\mathrm{P}(9,9), \quad \text{therefore } (9,9)=\tfrac{\mathrm{A}}{8\mathrm{P}},$$
$$(2,8)(10,7)=(2,7)(9,8) \text{ " } \tfrac{1}{7}\mathrm{B}=\mathrm{Q}(9,8), \quad \text{" } (9,8)=\tfrac{B}{7Q},$$
$$(3,7)(10,6)=(3,6)(9,7) \text{ " } \tfrac{1}{6}\mathrm{C}=\mathrm{R}(9,7), \quad \text{" } (9,7)=\tfrac{\mathrm{C}}{6\mathrm{R}},$$
$$(4,6)(10,5)=(4,5)(9,6) \text{ " } \tfrac{1}{5}\mathrm{D}=\mathrm{S}(9,6), \quad \text{" } (9,6)=\tfrac{\mathrm{D}}{5\mathrm{S}},$$
$$(5,5)(10,4)=(5,4)(9,5) \text{ " } \tfrac{1}{4}\mathrm{E}=\mathrm{S}(9,5), \quad \text{" } (9,5)=\tfrac{\mathrm{E}}{4S},$$
$$(6,4)(10,3)=(6,3)(9,4) \text{ " } \tfrac{1}{3}\mathrm{D}=\mathrm{R}(9,4), \quad \text{" } (9,4)=\tfrac{\mathrm{D}}{3R},$$
$$(7,3)(10,2)=(7,2)(9,3) \text{ " } \tfrac{1}{2}\mathrm{D}=\mathrm{Q}(9,3), \quad \text{" } (9,3)=\tfrac{\mathrm{C}}{2\mathrm{Q}},$$
$$(8,2)(10,1)=(8,1)(9,2) \text{ " } \mathrm{B}=\mathrm{P}(9,2), \quad \text{" } (9,2)=\tfrac{\mathrm{B}}{\mathrm{P}}.$$

§. 116. Therefore from these unknown formulas, from the number 36 we have now determined eight which sets the path for new determinations, which initially we will determine from the general equation by taking $a=1,\ b=9$ and by writing for c the numbers 1, 2, 3, ... 8 in the orders, from which calculation there will be obtained thus :

$(1,9)(10,1)=(1,1)(2,9)$	$\mathrm{A}=(1,1)\frac{\mathrm{B}}{\mathrm{P}}$, t'fore $(1,1)=\frac{\mathrm{AP}}{\mathrm{B}}$,
$(1,9)(10,2)=(1,2)(3,9)$	$\frac{1}{2}\mathrm{A}=(1,2)\frac{\mathrm{C}}{2\mathrm{Q}}$, " $(1,2)=\frac{\mathrm{AQ}}{\mathrm{C}}$,
$(1,9)(10,3)=(1,3)(4,9)$	$\frac{1}{3}\mathrm{A}=(1,3)\frac{\mathrm{D}}{3\mathrm{R}}$, " $(1,3)=\frac{\mathrm{AR}}{\mathrm{D}}$,
$(1,9)(10,4)=(1,4)(5,9)$	$\frac{1}{4}\mathrm{A}=(1,4)\frac{\mathrm{E}}{4\mathrm{S}}$, " $(1,4)=\frac{\mathrm{AS}}{\mathrm{E}}$,
$(1,9)(10,5)=(1,5)(6,9)$	$\frac{1}{5}\mathrm{A}=(1,5)\frac{\mathrm{D}}{5S}$, " $(1,5)=\frac{\mathrm{AS}}{\mathrm{D}}$,
$(1,9)(10,6)=(1,6)(7,9)$	$\frac{1}{6}\mathrm{A}=(1,6)\frac{\mathrm{C}}{6\mathrm{R}}$, " $(1,6)=\frac{\mathrm{AR}}{\mathrm{C}}$,
$(1,9)(10,7)=(1,7)(8,9)$	$\frac{1}{7}\mathrm{A}=(1,7)\frac{\mathrm{B}}{7\mathrm{Q}}$, " $(1,7)=\frac{\mathrm{AQ}}{\mathrm{B}}$,
$(1,9)(10,8)=(1,8)(9,9)$	$\frac{1}{8}\mathrm{A}=(1,8)\frac{\mathrm{A}}{8\mathrm{P}}$, " $(1,8)=\frac{\mathrm{AP}}{\mathrm{A}}$;

and in this way we have gained seven new determinations.

§.117. Now from these values found we may consider the equations arising from the values $a=1,\ b=8,\ c=1,\ 2,\ 3,\ ...\ 7$ and there will be :

$(1,8)(9,1)=(1,1)(2,8)$	$\mathrm{AP}=(1,1)\mathrm{B}$	identical,
$(1,8)(9,2)=(1,2)(3,8)$	$\mathrm{B}=(3,8)\frac{\mathrm{AQ}}{\mathrm{C}}$	$(3,8)=\frac{\mathrm{BC}}{\mathrm{AQ}}$,
$(1,8)(9,3)=(1,3)(4,8)$	$\frac{\mathrm{CP}}{2\mathrm{Q}}=(4,8)\frac{\mathrm{AR}}{\mathrm{D}}$	$(4,8)=\frac{\mathrm{CDP}}{2\mathrm{AQR}}$,
$(1,8)(9,4)=(1,4)(5,8)$	$\frac{\mathrm{DP}}{3\mathrm{R}}=(5,8)\frac{\mathrm{AS}}{\mathrm{E}}$	$(5,8)=\frac{\mathrm{DEP}}{3\mathrm{ARS}}$,
$(1,8)(9,5)=(1,5)(6,8)$	$\frac{\mathrm{EP}}{4S}=(6,8)\frac{\mathrm{AS}}{\mathrm{D}}$	$(6,8)=\frac{\mathrm{DEP}}{4ASS}$,
$(1,8)(9,6)=(1,6)(7,8)$	$\frac{\mathrm{DP}}{5\mathrm{S}}=(7,8)\frac{\mathrm{AR}}{\mathrm{C}}$	$\frac{\mathrm{AR}}{\mathrm{C}}(7,8)=\frac{\mathrm{CDP}}{5\mathrm{ARS}}$,
$(1,8)(9,7)=(1,7)(8,8)$	$\frac{\mathrm{CP}}{6\mathrm{R}}=(8,8)\frac{\mathrm{AQ}}{\mathrm{B}}$	$(8,8)=\frac{\mathrm{BCP}}{6\mathrm{AQR}}$.

§.118. New determinations may be found by putting $a=1,\ b=7,\ c=3,\ 4,\ 5,\ 6;$ hence indeed we obtain the following determinations :

$(1,7)(8,3)=(1,3)(4,7)$	$\mathrm{D}=(4,7)\frac{\mathrm{AR}}{\mathrm{D}}$	$(4,7)=\frac{\mathrm{CD}}{\mathrm{AR}}$,
$(1,7)(8,4)=(1,4)(5,7)$	$\frac{\mathrm{CDP}}{2\mathrm{BR}}=(5,7)\frac{\mathrm{AS}}{\mathrm{E}}$	$(5,7)=\frac{\mathrm{CDEP}}{2\mathrm{ABRS}}$,
$(1,7)(8,5)=(1,5)(6,7)$	$\frac{\mathrm{DEPQ}}{3\mathrm{BRS}}=(6,7)\frac{\mathrm{AS}}{\mathrm{D}}$	$(6,7)=\frac{\mathrm{DDEPQ}}{3\mathrm{ABRSS}}$,
$(1,7)(8,6)=(1,6)\ (7,7)$	$\frac{\mathrm{DEPQ}}{4\mathrm{BSS}}=(7,7)\frac{\mathrm{AR}}{\mathrm{C}}$	$(7,7)=\frac{\mathrm{CDEPQ}}{4\mathrm{ABRSS}}$.

§.119. Now we may take: $a=1,\ b=6,\ c=4,\ 5$ and there will be:

$(1,6)(7,4)=(1,4)(5,6)$	$\mathrm{D}=(5,6)\frac{\mathrm{AS}}{\mathrm{E}}$	$(5,6)=\frac{\mathrm{DE}}{\mathrm{AS}}$,
$(1,6)(7,5)=(1,5)(6,6)$	$\frac{\mathrm{DEP}}{2\mathrm{BS}}=(6,6)\frac{\mathrm{AS}}{\mathrm{D}}$	$(6,6)=\frac{\mathrm{DDEP}}{2\mathrm{ABSS}}$,

Therefore at this point we have determined all the formulas (p, q), in which $p+q>10$. But from the rest, where $p+q<9$, we have obtained these :

$$(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (1,7),$$

thus so that finally these remain to be determined :

$$(2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6),$$
$$(3,3),\ (3,4),\ (3,5),$$
$$(4,4).$$

§. 120. For finding these we may take $a=1$ and $c=1$, but for b we may take the numbers 2, 3 etc. with the order and we follow with these equations :

$(1,2)(3,1)=(1,1)(2,2)$	$\frac{\text{AAQR}}{\text{CD}}=(2,2)\frac{\text{AP}}{\text{B}}$	$(2,2)=\frac{\text{ABQR}}{\text{CDP}}$,
$(1,3)(4,1)=(1,1)(2,3)$	$\frac{\text{AARS}}{\text{DE}}=(2,3)\frac{\text{AP}}{\text{B}}$	$(2,3)=\frac{\text{ABRS}}{\text{DEP}}$
$(1,4)(5,1)=(1,1)(2,4)$	$\frac{\text{AASS}}{\text{DE}}=(2,4)\frac{\text{AP}}{\text{B}}$	$(2,4)=\frac{\text{ABSS}}{\text{DEP}}$,
$(1,5)(6,1)=(1,1)(2,5)$	$\frac{\text{AARS}}{\text{CD}}=(2,5)\frac{\text{AP}}{\text{B}}$	$(2,5)=\frac{\text{ABRS}}{\text{CDP}}$,
$(1,6)(7,1)=(1,1)(2,6)$	$\frac{\text{AAQR}}{\text{BC}}=(2,6)\frac{\text{AP}}{\text{B}}$	$(2,6)=\frac{\text{ABQR}}{\text{BCP}}$.

and thus even now the formulas etiamnunc determinandae restant formulae $(3,3)$, $(3,4)$, $(3,5)$ and $(4,4)$ remain to be determined.

§. 121. For these there may be taken $a=1,\ c=2$ et $b=3,\ 4,\ 5$ etc.; then indeed these equations wil be produced :

$(1,3)(4,2)=(1,2)(3,3)$	$\frac{\text{AABRSS}}{\text{DDEP}}=(3,3)\frac{\text{AQ}}{\text{C}}$	$(3,3)=\frac{\text{ABCRSS}}{\text{DDEPQ}}$,
$(1,4)(5,2)=(1,2)(3,4)$	$\frac{\text{AABRSS}}{\text{CDEP}}=(3,4)\frac{\text{AQ}}{\text{C}}$	$(3,4)=\frac{\text{ABRSS}}{\text{DEPQ}}$
$(1,5)(6,2)=(1,2)(3,5)$	$\frac{\text{AAQRS}}{\text{CDP}}=(3,5)\frac{\text{AQ}}{\text{C}}$	$(3,5)=\frac{\text{ARS}}{\text{DP}}$.

Therefore a single formula remains requiring to be determined, evidently (4,4), which may be defined from this equation

$$(1,4)(5,3)=(1,3)(4,4);$$

indeed there will become $\frac{\text{AARSS}}{\text{DEP}}=(4,4)\frac{\text{AR}}{\text{D}}$, and thus $(4,4)=\frac{\text{ASS}}{\text{EPP}}$.

§.122. So that now we may put all these determinations to be seen at once, since in this order $n=10$, in all 45 integral formulas occur, if from those we may consider the following nine as known

$$(1,9)=\text{A},\ (2,8)=\text{B},\ (3,7)=\text{C},\ (4,6)=\text{D},\ (5,5)=\text{E},$$
$$(1,8)=\text{P},\ (2,7)=\text{Q},\ (3,6)=\text{R},\ (4,6)=\text{S},$$

the remaining thirty six will be determined from these in the following manner :

1.	$(9,9)=\frac{A}{8P}$	8.	$(9,2)=\frac{B}{P}$
2.	$(9,8)=\frac{B}{7Q}$	9.	$(1,1)=\frac{AP}{B}$
3.	$(9,7)=\frac{C}{6R}$	10.	$(1,2)=\frac{AQ}{C}$
4.	$(9,6)=\frac{D}{5S}$	11.	$(1,3)=\frac{AR}{D}$
5.	$(9,5)=\frac{E}{4S}$	12.	$(1,4)=\frac{AS}{E}$
6.	$(9,4)=\frac{D}{3R}$	13.	$(1,5)=\frac{AS}{D}$
7.	$(9,3)=\frac{C}{2Q}$	14.	$(1,6)=\frac{AR}{C}$
15.	$(1,7)=\frac{AQ}{B}$	26.	$(8,8)=\frac{BCP}{6AQR}$
16.	$(3,8)=\frac{BC}{AQ}$	27.	$(2,2)=\frac{ABQR}{CDP}$
17.	$(4,7)=\frac{CE}{AR}$	28.	$(2,3)=\frac{ABRS}{DEP}$
18.	$(5,6)=\frac{DE}{AS}$	29.	$(2,4)=\frac{ABSS}{DEP}$
19.	$(2,6)=\frac{AQR}{CP}$	30.	$(2,5)=\frac{ABRS}{CDP}$
20.	$(3,5)=\frac{ARS}{DP}$	31.	$(5,7)=\frac{CDEP}{2ABRS}$
21.	$(4,4)=\frac{ASS}{EP}$	32.	$(6,6)=\frac{DDEP}{2ABSS}$
22.	$(4,8)=\frac{CDP}{2AQR}$	33.	$(3,4)=\frac{ABRSS}{DEPQ}$
23.	$(5,8)=\frac{DEP}{3ARS}$	34.	$(6,7)=\frac{DDEPQ}{3ABRSS}$
24.	$(6,8)=\frac{DEP}{4ASS}$	35.	$(7,7)=\frac{CDEPQ}{4ABRSS}$
25.	$(7,8)=\frac{CDP}{5ARS}$	36.	$(3,3)=\frac{ABCRSS}{DDEPQ}$.

§.123. By the same method, which we have used here for the case $n=10$, higher orders will be established without difficulty ; yet neither hence at this stage may it be shown, in what manner all the determinations may be advanced by the law, since the values of certain formulas continually emerge more complicated. In other respects values, which we have found here, are taken from all the equations in the general form

$$(a,b)(a+b,c)=(a,c)(a+c,b)$$

to be in agreement with the contents, thus so that an identical equation always will result nor on that account thence may any new relation between our capital letters be deduced. Finally this need to be noted here, because in all the orders besides the formulas depending most conveniently on these circular formulas these formulas, which in the nearby preceding order were circular, here also may be able to be accepted as known, certainly by which all the determinations are able to be perfected most successfully.

3. (cont'd)

The general method of determining the values of the formula

$$\left(p,q\right)=\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-p}}},$$

extended from the limit $x=0$ as far as to $x=1$: where besides the formulas involving the circle, in which there is $p+q=n$, also these may be accepted as known, in which $p+q=n-1$.

I. Since the general equation, from which all these determinations have been sought, shall be

$$\left(a,b\right)\left(a+b,c\right)=\left(a,c\right)\left(a+c,b\right),$$

initially there may be taken $a=n-\alpha,$ $b=\alpha,$ and $c=\alpha-1$, and the equation will become:

$$\left(n-\alpha,\alpha\right)\left(n,\alpha-1\right)=\left(n-\alpha,\alpha-1\right)\left(n-1,\alpha\right),$$

where

$$\left(n,\alpha-1\right)=\frac{1}{\alpha-1}.$$

In the first factor there is given $p+q=n$ on account of $p=n-\alpha$ and $q=\alpha$. Moreover, in the third factor, thus equally there is given $p+q=n-1$, where $p=n-\alpha$ and $q=\alpha-1$. Hence we deduce therefore

$$\left(n-1,\alpha\right)=\frac{1}{\alpha-1}\cdot\frac{\left(n-\alpha,\alpha\right)}{\left(n-\alpha,\alpha-1\right)},$$

where there must be $\alpha>1$, thus so that all the numbers from 2 as far as to $n-1$ may be able to be accepted for α; but truly in the case $\alpha=1$ the value of the formula is known by itself.

II. Now in the general equation there may be taken $a=\beta,$ $b=n-\beta-1$ and $c=1$ and our equation will become : [from $\left(a,b\right)\left(a+b,c\right)=\left(a,c\right)\left(a+c,b\right)$;]

$$\left(\beta,\ n-\beta-1\right)\left(n-1,1\right)=\left(\beta,1\right)\left(\beta+1,n-\beta-1\right),$$

and there is deduced from the equation :

$$\left(\beta,1\right)=\frac{\left(\beta,\ n-\beta-1\right)\left(n-1,1\right)}{\left(\beta+1,n-\beta-1\right)},$$

where here must be $\beta < n-1$, thus so that hence all the formulas $(\beta,1)$ may be defined from the value $\beta = 1$ as far as to $\beta = n-1$, where in the latter case the formula $(n-1,1)$ is itself known .

III. Hence so that we may elicit the other formulas also, we may take
$a = 1,\ b = n-2,\ c = \gamma$ so that this equation arises :

$$(1,n-2)(n-1,\gamma) = (1,\gamma)(1+\gamma,n-2),$$

where the first and the third factors are given by II, the second truly by I; from which the fourth is derived, namely

$$(1+\gamma,n-2) = \frac{(1,n-2)(n-1,\gamma)}{(1,\gamma)},$$

where the values of $1+\gamma$ can be increased from 2 as far as to $n-2$. Then since by I there shall be

$$(n-1,\gamma) = \frac{1}{\gamma-1}\cdot\frac{(n-\gamma,\gamma)}{(n-\gamma,\gamma-1)},$$

and indeed by II there shall be

$$(\gamma,1) = \frac{(\gamma,n-\gamma-1)(n-1,1)}{(\gamma+1,n-\gamma-1)},$$

with these values substituted there will become

$$(n-2,1+\gamma) = \frac{1}{\gamma-1}\cdot\frac{(1,n-2)(n-\gamma,\gamma)(\gamma+1,n-\gamma-1)}{(n-\gamma,\gamma-1)(\gamma,n-\gamma-1)(n-1,1)}.$$

IV. Now we may take $a = 1,\ b = n-3,\ c = \delta$ and this equation will be produced :

$$(1,n-3)(n-2,\delta) = (1,\delta)(1+\delta,n-3),$$

from which it is gathered,

$$(n-3,1+\delta) = \frac{(n-3,1)(n-2,\delta)}{(\delta,1)},$$

where therefore $1+\delta$ contains the numbers $2,\ 3,\ 4,\ \ldots\ n-3$, thus so that hence the formula $(n-3,1)$ may be excluded, but which is given by II. But these values found before may be substituted, and there will become

$$\left(n-3,1+\delta\right)=\frac{1}{\delta-2}\cdot\frac{(n-3,2)(n-2,1)(n-\delta+1,\delta-1)(\delta,n-\delta)(\delta+1,n-\delta-1)}{(n-2,2)(n-\delta+1,\delta-2)(\delta-1,n-\delta)(n-1,1)(\delta,n-\delta-1)},$$

from which it is apparent there must be $\delta>2$ and in the same manner for the preceding formula $\gamma>1$, thus so that here the cases $(n-3,1)$ and $(n-3,2)$ may be excluded, of which indeed the former is given by II , the other truly by itself.

V. Now we may put $a=1,\ b=n-4$ and $c=\varepsilon$ and this equation will be produced :

$$\left(1,n-4\right)(n-3,\varepsilon)=\left(1,\varepsilon\right)\left(1+\varepsilon,n-4\right),$$

from which it is concluded

$$\left(n-4,1+\varepsilon\right)=\frac{(n-4,1)(n-3,\varepsilon)}{(1,\varepsilon)};$$

where if in place of $(n-3,\varepsilon)$ the value found before may be substituted, the absolute factor may be introduced $\frac{1}{\varepsilon-3}$, thus so that there must be $\varepsilon>3$ and thus $1+\varepsilon>4$, from which here the cases $(n-4,1),\ (n-4,2),\ (n-4,3)$ are excluded, of which indeed the first is given by II, but the third by itself, truly the middle one actually remains unknown.

VI. Again we may put $a=1,\ b=n-5,\ c=\zeta$ and the equation will become

$$\left(1,n-5\right)(n-4,\zeta)=(1,\zeta)(1+\zeta,n-5),$$

so that there becomes

$$(n-5,1+\zeta)=\frac{(n-5,1)(n-4,\zeta)}{(1,\zeta)},$$

where on account of the formula $(n-4,\zeta)$ there must become $\zeta>4$ and thus $1+\zeta>5$, from which hence the cases are excluded $(n-5,1),(n-5,2),(n-5,3),(n-5,4)$, of which the first is agreed from II , truly the fourth is given by itself, thus so that here two cases even now remain unknown, $(n-5,2)$ and $(n-5,3)$.

VII. In a similar manner if we may assume further $a=1, b=n-6$ and $c=\eta$, there will be produced :

$$(n-6,1+\eta)=\frac{(n-6,1)(n-5,\eta)}{(1,\eta)},$$

where actually the three cases occur $(n-6,2),(n-6,3),(n-6,4)$, which at this point remain unknown, and it will be allowed to proceed in this manner, as far as necessary ; from which it is apparent the number of unknown cases continues to grow, thus so that of the terms p and q one or the other shall become either 2, 3 or 4 etc., which cases therefore at this stage remain to be defined.

VIII. Now initially we may take $a=1, b=\theta,\ c=1$, so that our equation may become :

$$(1,\theta)(1+\theta,1)=(1,1)(2,\theta),$$

from which we conclude :

$$(2,\theta)=\frac{(1,\theta)(1+\theta,1)}{(1,1)},$$

which formula now supplies all the excluded cases, in which the other term was 2.

IX. Then we may take $a=2, b=\chi$ and $c=1$, so that the equation may be produced :

$$(2,\chi)(2+\chi,1)=(2,1)(3,\chi),$$

from which there becomes

$$(3,\chi)=\frac{(2,\chi)(2+\chi,1)}{(2,1)};$$

where since $(2,\chi)$ is given by the preceding number, now these cases will be known also, when the other term was 3.

X. Again we may take $a=3, b=\chi,\ c=1$ and there will become

$$(3,\chi)(3+\chi,1)=(3,1)(4,\chi),$$

from which there becomes

$$(4,\chi)=\frac{(3,\chi)(3+\chi,1)}{(3,1)},$$

from which therefore these cases are elicited, where the other term was 4.

It is proceeding in the same manner for the remaining cases and thus plainly all the cases contained in the proposed formula have been determined.

2). Comparatio valorum formulae integralis

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}},$$

a termino $x=0$ usque ad $x=1$ extensae.

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§. 57. In hac formula, litterae *n, p* et *q* perpetuo designant numeros integros positives, et pro quolibet numero *n* binis litteris *p* et *q* omnes valores tribui concipiuntur, ita ut hinc pro quovis numero *n* innumerae nascantur hujusmodi formulae integrales, quarum valores plurimas egregias relationes inter se servant; unde si eorum aliquot fuerint cogniti, reliquae omnes ex iis definiri queant. Jam dudum equidem plures hujusmodi relationes demonstravi ; cum autem hoc argumentum tum temporis neutiquam exhausissem, nunc accuratius in istas relationes inquirere constitui, et ejusmodi methodum adhibebo, quae omnes plane hujus generis relationes sit exhibitura; his enim inventis innumerabilia theoremata condi poterunt, quibus universa analysis non mediocriter locupletari erit censenda.

§. 58. Quoniam igitur hoc modo pro quolibet numero *n* ambae litterae *p* et *q* infinitos valores recipere possunt, ante omnia hic observari convenit, omnes hos innumerabiles casus semper ad numerum finitum revocari posse. Quantumvis enim magni numeri pro litteris *p* et *q* accipiantur, eos casus semper ad alios reducere licet, in quibus numeri *p* et *q* quantitate *n* futuri sint diminuti. Hoc igitur modo omnes hujusmodi casus tandem eo redigi poterunt, ut ambo numeri *p* et *q* infra exponentem *n* deprimantur; unde pro quolibet numero *n* eos tantum casus considerasse sufficiet, quibus litterae *p* et *q* minores valores recipiant quam *n*, vel saltem hunc limitem non superent. Hoc igitur modo pro quovis numero *n* multitudo casuum, qui in computum veniunt, et quos inter se comparari oportet, prorsus erit determinata.

§.59. Quemadmodum autem ista reductio litterarum *p* et *q* ad numeros continua minores institui debeat, quamquam id satis in vulgus est notum, tamen ad formulam praesentem accommodasse juvabit. Statuatur scilicet haec formula algebraica

$x^p\left(1-x^n\right)^{\frac{q}{n}}=\mathrm{V}$; eritque

$$l\mathrm{V}=plx+\tfrac{q}{n}l\left(1-x^n\right),$$

hinc differentiando

$$\frac{\partial \mathrm{V}}{\mathrm{V}}=\frac{p\partial x}{x}-\frac{qx^{n-1}\partial x}{1-x^{n}}=\frac{p\partial x-(p+q)x^{n-1}\partial x}{x\left(1-x^{n}\right)},$$

ubi si per V multiplicemus, ac per partes integremus, orietur ista aequatio

$$\mathrm{V}=p\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}}-(p+q)\int x^{p+n-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}}.$$

Quoniam igitur quantitas V pro utroque integrationis termino evanescit, hinc adipiscimur istam reductionem

$$\int x^{p+n-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}}=\frac{p}{p+q}\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}},$$

cujus ergo reductionis ope exponens ipsius x continuo quantitate n diminui poterit, donec tandem infra n deprimatur.

§. 60. Deinde formula pro

$$\frac{\partial \mathrm{V}}{\mathrm{V}}=\frac{p\partial x-(p+q)x^{n-1}\partial x}{x\left(1-x^{n}\right)}$$

inventa hoc modo referri poterit

$$\frac{\partial \mathrm{V}}{\mathrm{V}}=\frac{(p+q)\partial x\left(1-x^{n}\right)-q\partial x}{x\left(1-x^{n}\right)},$$

quae forma per V multiplicata ac denuo per partes integrata dabit

$$\mathrm{V}=(p+q)\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q}{n}}-q\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}},$$

unde quia posito $x=1$ fit $\mathrm{V}=0$, oritur haec reductio

$$\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q}{n}}=\frac{q}{p+q}\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q-n}{n}},$$

cujus reductionis ope exponens Binomii $1-x^{n}$ unitate minuitur, sive quod eodem redit, numerus q numero n imminuitur. Tali igitur reductione, quoties opus fuerit, repetita, exponens q tandem infra n deprimi poterit.

§. 61. Quoniam igitur pro quovis numero n ambos exponentes p et q tanquam minores quam n spectare licet, formulam propositam hoc modo expresssam repraesentemus

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}.$$

Hic scilicet pro quovis numero *n* sufficiet litteris *p* et *q* omnes valores ipso *n* minores tribuisse, quo pacto multitudo omnium casuum ad quemlibet exponentem *n* pertinentium ad numerum satis modicum reducetur, qui tamen eo major evadit, quo major fuerit exponens *n*.

§. 62. Multo magis autem numerus casuum diversorum diminuetur, si perpendamus, ambas litteras *p* et *q* inter se permutari posse, ita ut hujus formulae

$$\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$$

valor ab illo prorsus non discrepet. Ad quod ostendendum ponamus

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} = S,$$

si scilicet ista formula integralis ab $x=0$ usque ad $x=1$ extendatur. Jam faciamus $1-x^n = y^n$, ut formula sit

$$S = \int \frac{x^{p-1}\partial x}{y^{n-q}};$$

tum vera quia $x^n = 1-y^n$, erit $x = \left(1-y^n\right)^{\frac{1}{n}}$, hincque $x^p = \left(1-y^n\right)^{\frac{p}{n}}$, unde differentiando fit

$$px^{p-1}\partial x = -py^{n-1}\partial y\left(1-y^n\right)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$S = -\int y^{q-1}\partial y\left(1-y^n\right)^{\frac{p-n}{n}},$$

quam formulam ab $x=0$ usque ad $x=1$, hoc est ab $y=1$ usque ad $y=0$, extendi oportet; permutatis igitur his terminis erit

$$S = \int \frac{y^{q-1}\partial y}{\sqrt[n]{\left(1-y^n\right)^{n-p}}} \quad \begin{bmatrix} ab\ y=0 \\ ad\ y=1 \end{bmatrix}.$$

Sicque demonstratum est ambas litteras *p* et *q* semper inter se esse permutabiles.

§. 63. His praemissis, quo calculos sequentes magis in compendium redigere liceat, loco formulae hujus integralis

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-p}}}$$

scribamus hunc characterem (*p, q*), ubi perinde est, sive *p* ante *q*, sive *q* ante *p* collocetur; semper autem hic certus exponens *n* subintelligi debet. Hic autem duo casus prae reliquis maxime memorabiles occurrunt. Prior casus est quo numerorum *p* et *q* alteruter ipsi exponenti *n* est aequalis ; si enim fuerit $q=n$, erit ex priore formula $(p,n,)=\int x^{p-1}\partial x=\frac{1}{p}$, sicque perpetuo habebimus $(p,n,)=\frac{1}{p}$, hincque etiam $(q,n,)=\frac{1}{q}$. Alter casus notatu dignissimus locum habet, quando $p+q=n$ quo casu semper est

$$(p,q)=\frac{\pi}{n\sin.\frac{p\pi}{n}}=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

Ad hoc ostendendum sit $q=n-p$, hincque formula proposita $\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$, tum ponatur $\frac{x}{\sqrt[n]{\left(1-x^n\right)}}=z$, et quia $\frac{x^p}{\sqrt[n]{\left(1-x^n\right)^p}}=z^p$, erit $S=\int\frac{z^p\partial x}{x}$. Ex facta autem positione sequitur $x^n=\frac{z^n}{1+z^n}$ hincque

$$nlx=nlz-l\left(1+z^n\right),$$

ergo differentiando

$$\frac{\partial x}{x}=\frac{\partial z}{z}-\frac{z^{n-1}\partial z}{1+z^n}=\frac{\partial z}{z\left(1+z^n\right)},$$

ita ut jam sit

$$S=\int\frac{z^{p-1}\partial z}{1+z^n}.$$

Quia autem sumto $x=0$ fit etiam $z=0$, at vero sumto $x=1$ prodit $z=\infty$, hoc integrale a termino $z=0$ usque ad $z=\infty$ extendi debet. Notum autem est valorem hoc modo resultantem esse $\frac{\pi}{n\sin.\frac{p\pi}{n}}$.

§. 64. Progrediamur nunc ad ipsum fundamentum, unde omnes relationes, quas quaerimus, derivari convenit, et quod reductioni priori innititur; unde fit

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}=\frac{p+q}{p}\cdot\int\frac{x^{n+p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$$

ubi loco $\sqrt[n]{\left(1-x^n\right)^{n-q}}$ scribamus X, ut sit

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} \cdot \int \frac{x^{n+p-1}\partial x}{X} :$$

jam simili modo, si loco *p* scribamus $n+p$, erit

$$\int \frac{x^{n+p-1}\partial x}{X} = \frac{n+p+q}{n+p} \cdot \int \frac{x^{2n+p-1}\partial x}{X},$$

hincque sequitur fore

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \int \frac{x^{2n+p-1}\partial x}{X}.$$

Quodsi simili modo ulterius progrediamur, perveniemus ad hanc aequationem

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \cdot \frac{2n+p+q}{2n+p} \int \frac{x^{3n+p-1}\partial x}{X}.$$

Quare si hoc modo in infinitum progrediamur, habebimus

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \cdot \frac{2n+p+q}{2n+p} \cdot\cdots\cdot \frac{in+p+q}{in+p} \int \frac{x^{(i+1)n+p-1}\partial x}{X},$$

ubi *i* denotat numerum infinite magnum.

§. 65. Quodsi jam loco *p* alium quemcunque numerum *r*, pariter ipso *n* minorem, assumamus; erit simili modo

$$\int \frac{x^{r-1}\partial x}{X} = \frac{r+q}{r} \cdot \frac{n+r+q}{n+r} \cdot \frac{2n+r+q}{2n+r} \cdot\cdots\cdot \frac{in+r+q}{in+r} \int \frac{x^{(i+1)n+r-1}\partial x}{X},$$

ubi littera *i* eundem numerum infinitum designat, ita ut utrinque idem factorum numerus adsit. Dividamus jam priorem expressionem per istam, et quoniam extremae formulae integrales, ob litteras *p* et *r* prae $(i+1)n$ evanescentes, pro aequalibus inter se sunt habendae, facta divisione per singulos factores reperiemus hanc aequationem :

$$\frac{\int x^{p-1}\partial x : X}{\int x^{r-1}\partial x : X} = \frac{r(p+q)}{p(r+q)} \cdot \frac{(n+r)(n+p+q)}{(n+p)(n+r+q)} \cdot \frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)} \cdot \frac{(3n+r)(3n+p+q)}{(3n+p)(3n+r+q)} \cdot \text{etc.}$$

Restituamus jam loco harum formularum integralium characteres ante stabilitos, atque adipiscemur istam relationem notatu dignissimam

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)} \cdot \frac{(n+r)(n+p+q)}{(n+p)(n+r+q)} \cdot \frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)} \cdot \frac{(3n+r)(3n+p+q)}{(3n+p)(3n+r+q)} \cdot \text{etc.}$$

quod productum ex infinitis membris componitur, quorum singula sunt fractiones, quarum tam numeratores quam denominatores ex binis factoribus constant. Hos factores singulos eodem numero *n* augeri oportet, dum a quovis membro ad sequens progredimur, unde sufficiet solum primum productum nosse, quod ergo ita repraesentabimus

$$\frac{(p,q)}{(r,q)}=\frac{r(p+q)}{p(r+q)}\cdot \text{etc.}$$

§. 66. Quoniam litterae *p* et *q* nobis numeros quasi indefinitos significant, utamur litteris alphabeti initialibus ad numeros determinates designandos, eritque eodem modo

$$\frac{(a,b)}{(\alpha,b)}=\frac{\alpha(a+b)}{a(\alpha+b)}\cdot\frac{(n+\alpha)(n+a+b)}{(n+a)(n+\alpha+b)}\cdot \text{etc.}$$

Hic jam loco α scribamus $a+c$; et productum infinitum hanc induct formam

$$\frac{(a,b)}{(a+c,b)}=\frac{(a+c)(a+b)}{a(a+c+b)}\cdot\frac{(n+a+c)(n+a+b)}{(n+a)(n+a+c+b)}\cdot \text{etc.},$$

in quo producto ambae litterae *b* et *c* manifesto permutari possunt, unde idem productum infinitum etiam exprimet valorem hujus formae $\frac{(a,c)}{(a+b,c)}$, unde sequitur ista aequalitas maxime memorabilis

$$\frac{(a,b)}{(a+c,b)}=\frac{(a,c)}{(a+b,b)};$$

fractionibus igitur sublatis habebimus istud insigne theorema

$$(a,b)(a+b,c)=(a,c)(a+c,b)$$

huicque theoremati universa analysis, qua utemur, erit superstructa.

§. 67. Cum ob rationes supra allegatas numeri *p* et *q* exponentem *n* superare non debeant, etiam in forma theorematis modo allati singuli termini ibi occurrentes, qui sunt, quovis casu exponentem *n* superare non debent, sicque nec $a+b$, neque $a+c$ maior capi poterit quam *n*. Hic autem prima observo litteras *b* et *c*, inter se inaequales statui debere : si enim esset $c=b$, aequalitas in theoremate expressa foret identica hanc ob rem perpetuo assumemus $b>c$, ita ut maximus terminus in theoremate sit $a+b$, quem ergo exponentem *n* quovis casu excedere non oportet, quamobrem evolutionem formae generalis in theoremate contentae ita in classes distribuamus, quae inter se per maximum. valorem termini $a+b$ distinguantur. Cum igitur nulla litterarum *a, b, c* nihilo aequalis sumi queat, ac esse debeat $b>c$, minimus valor, quem terminus $a+b$ recipere potest, erit 3, in quo ergo primam classem constituemus; sequentes vero classes constituentur , dum termino $a+b$ valores 4, 5, 6, 7, etc. tribuantur.

I. Evolutio classis qua $a+b=3$.

§ . 68. Hic ergo necessaria erit $a=1$, $b=2$ et $c=1$, ita ut hic nulla varietas locum inveniat, und theorema nostrum suppeditat hanc unicam relationem $(1,2)(3,1)=(1,1)(2,2)$. Dummodo igitur exponens *n* non fuerit minor quam 3, semper haec insignis relatio iocum habet quae forma,

$$\int \frac{\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-2}}} \cdot \int \frac{xx\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-1}}} = \int \frac{\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-1}}} \cdot \int \frac{x\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-2}}}$$

quia in quolibet charactere terminos inter se permutare licet, etiam hoc modo repraesentari poterit

$$\int \frac{x\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-1}}} \cdot \int \frac{\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-3}}} = \int \frac{\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-1}}} \cdot \int \frac{x\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-2}}}$$

II. Evolutio classis qua $a+b=4$.

§. 69. Quoniam *b* binario minor esse nequit, hic erit vel $b=2$, vel $b=3$. Sit igitur primo $b=2$, eritque $a=2$ et $c=1$; unde ex nostro theoremate sequitur haec relatio $(2,2)(4,1)=(2,1)(3,2)$, quae forma manifesto oritur ex classe prima, si ibi termini priores cujusque characteris unitate augeantur; id quod etiam inde intelligere licet, quod omnes termini priores litteram *a* continent, qua unitate aucta processus semper fit, ad classem sequentem.

§.70. Deinde vero hic quoque statui potest $b=3$, unde sit $a=1$; at vero littera *c* jam duos valores, vel 1, vel 2 sortiri poterit; priore casu; quo $c=1$, prodibit ista aequatio $(1,3)(4,1)=(1,1)(2,3)$; alter vero casus, quo $c=2$, praebet hanc aequationem $(1,3)(4,2)=(1,2)(3,3)$. Sicque haec classis omnino sequentes tres relationes continebit

$$1^{\text{o}}.(2,2)(4,1)=(2,1)(3,2),$$
$$2^{\text{o}}.(1,3)(4,1)=(1,1)(2,3),$$
$$3^{\text{o}}.(1,3)(4,2)=(1,2)(3,3).$$

III. Evolutio classis
qua $a+b=5$.

§.71. In hac igitur classe primo occurrent tres relationes praecedentes, si modo termini priores cujusque characteris unitate augeantur : hinc enim casus exsurgent, quibus est vel $b=2$, vel $b=3$. De novo igitur hic accedent casus, quibus $b=4$ et $a=1$, ubi ergo erit vel $c=1$, vel $c=2$, vel $c=3$, quibus ergo tribus casibus evolutis omnino in hac classe, sex continebuntur relationes, quae erunt

$$1^{\text{o}}.(3,2)(5,1)=(3,1)(4,2),$$
$$2^{\text{o}}.(2,3)(5,1)=(2,1)(3,3),$$
$$3^{\text{o}}.(2,3)(5,2)=(2,2)(4,3),$$
$$4^{\text{o}}.(1,4)(5,1)=(1,1)(2,4),$$
$$5^{\text{o}}.(1,4)(5,2)=(1,2)(3,4),$$
$$6^{\text{o}}.(1,4)(5,3)=(1,3)(4,4).$$

IV. Evolutio classis
qua $a+b=6$.

§. 72. Hic igitur primum occurrent omnes relationes proxime praecedentes, si modo termini priores cujusque characteris unitate augeantur: hi scilicet nascuntur, si fuerit vel $b=2$ vel $b=3$, vel $b=4$. Praeterea vero insuper accedent casus $b=5$ et $a=1$, ubi littera c recipere poterit valores 1, 2, 3, 4, sicque, onmino in hac classe occurrent decem relationes sequentes

$$1^{\text{o}}.(4,2)(6,1)=(4,1)(5,2),$$
$$2^{\text{o}}.(3,3)(6,1)=(3,1)(4,3),$$
$$3^{\text{o}}.(3,3)(6,2)=(3,2)(5,3),$$
$$4^{\text{o}}.(2,4)(6,1)=(2,1)(3,4),$$
$$5^{\text{o}}.(2,4)(6,2)=(2,2)(4,4),$$
$$6^{\text{o}}.(2,4)(6,3)=(2,3)(5,4),$$
$$7^{\text{o}}.(1,5)(6,1)=(1,1)(2,5),$$
$$8^{\text{o}}.(1,5)(6,2)=(1,2)(3,5),$$
$$9^{\text{o}}.(1,5)(6,3)=(1,3)(4,5),$$
$$10^{\text{o}}.(1,5)(6,4)=(1,4)(5,5).$$

V. Evolutio classis qua $a+b=7$.

§. 73. Hic igitur primo occurrent omnes relationes classis IV. postquam scilicet omnes terminos priores singulorum characterum unitate auxerimus, quos igitur hic apposuisse non erit necesse, ac sufficiet eas tantum relationes hic exponere, quae de novo accedunt et ex valore $b=6$ oriuntur, existente $a=1$; ubi pro *c* sumi poterunt numeri 1, 2, 3, 4, 5, ita ut harum numerus sit quinque. Haec ergo relationes sunt

$$(1,6)(7,1)=(1,1)(2,6)$$
$$(1,6)(7,2)=(1,2)(3,6)$$
$$(1,6)(7,3)=(1,3)(4,6)$$
$$(1,6)(7,4)=(1,4)(5,6)$$
$$(1,6)(7,5)=(1,5)(6,6).$$

VI. Evolutio classis qua $a+b=8$.

§. 74. In hac jam classe primo occurrent omnes decem relationes classis IV, dum scilicet omnes termini priores binario augentur; praeterea quoque accedent quinque relationes in classe V allatae, dum partes priores unitate augebuntur; praeter has vera de novo accedent 6 sequentes relationes ex valoribus $a=1$ et $b=7$ oriundae, dum litterae *c* valores 1, 2, 3, 4, 5, & 6 ordine tribuuntur, quae ergo erunt

$$(1,7)(8,1)=(1,1)(2,7)$$
$$(1,7)(8,2)=(1,2)(3,7)$$
$$(1,7)(8,3)=(1,3)(4,7)$$
$$(1,7)(8,4)=(1,4)(5,7)$$
$$(1,7)(8,5)=(1,5)(6,7)$$
$$(1,7)(8,6)=(1,6)(7,7).$$

VII. Evolutio classis
qua $a+b=9$.

§. 75. Ut omnes relationes ad hanc classem pertinentes adipiscamur, notandum est primo hic occurrere decem relationes classis IV, dum partes priores ternario augentur. Secundo adjici oportet quinque relationes in classe V exhibitas, ubi partes priores binario augeri debent. Tertio huc referri debent sex relationes classis VI, partes priores unitate augendo. Insuper vero de novo accedent septem relationes ex valoribus $a=1$ et $b=8$ natae, dum litterae c tribuuntur ordine valores 1, 2, 3, 4, 5, 6, 7. Hae relationes sunt

$$(1,8)(9,1)=(1,1)(2,8)$$
$$(1,8)(9,2)=(1,2)(3,8)$$
$$(1,8)(9,3)=(1,3)(4,8)$$
$$(1,8)(9,4)=(1,4)(5,8)$$
$$(1,8)(9,5)=(1,5)(6,8)$$
$$(1,8)(9,6)=(1,6)(7,8)$$
$$(1,8)(9,7)=(1,7)(8,8).$$

§. 76. Hinc jam ordo progressionis tam clare perspicitur, ut superfluum foret has evolutiones ulterius prosequi; quandoquidem ob ingentem multitudinem relationum, quae in sequentibus classibus occurrerent nimis molestum foret omnes percurrere. Quin etiam nostrum institutum vix permittere videtur, ut in nostra formula generali exponentem *n* ultra sex vel septem augeamus, si quidem omnes relationes ad eum pertinentes enumerare voluerimus. Sin autem animus sit aliquas tantum expendere, classes allatae abunde sufficiunt, dum termini priores cujusque classis quovis numero augebuntur.

§. 77. His jam classibus expeditis, formulam integralem propositam $\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$ secundum diversos valores exponentis *n* pertractemus, dum scilicet successive assumemus $n=3,\ n=4,\ n=5,$ etc. et pro quolibet ordine omnes relationes, quae in eo occurrere possunt, expendamus. Evidens autem est, quicunque numerus exponenti *n* tributaur; formulas omnium classium inferiorum, in quibus scilicet terminus $a+b$ non superet *n*, in usum vocari posse. Ex quo intelligitur, si fuerit $n=3$ unicam relationem locum invenire; statim autem ac *n* magis augetur, numerus onmium relationum mox ita increscit, ut nimis molestum foret omnes recensere. Hos igitur, diversos ordines, ex exponente *n* constituendos, a prima incipiendo, ordine involvamus.

Ordo I.

quo $n=3$ et formula

$$(p,q)=\int \frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-p}}}.$$

§.78. Cum hic sit $n=3$, erit $(3,1)=1$; formulae autem integrales hujus ordinis erunt tres; scilicet 1°. $(1,1)$, 2°. $(1,\ 2\ )$, 3°. $(\ 2,2\ )$, quarum media, ob $1+2=3$, a circulo pendet, quae ergo, quia est cognita, ponatur

$$(1,2)=\frac{\pi}{3\sin.\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}=\mathrm{A}.$$

Hic igitur tantum classis prima locum habet, quae nobis hanc unicam aequationem suppeditat $\mathrm{A}=(1,1)(2,2)$.

§. 79. Hinc ergo patet, productum ex binis formulis transcendentibus (1, 1), et (2, 2) aequari quantitati circulari $\mathrm{A}=\frac{2\pi}{3\sqrt{3}}$, ita ut pro ipsis formulis integralibus habeamus hanc relationem.

$$\int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}\cdot\int \frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)}}=\frac{2\pi}{3\sqrt{3}};$$

unde si altera harum duarum formularum fuerit cognita; etiam valor alterius assignari potest. Spectemus ergo priorem quasi nobis esset cognita; etiamsi sit transcendens, eamque ponamus

$$(1,1)=\int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}=P,$$

eritque $(2,2)=\frac{\mathrm{A}}{\mathrm{P}}$. Sicque nihil praeterea in hoc ordine notandum relinquitur.

Ordo II.
quo $n=4$ et formula

$$\left(p,q\right)=\int \frac{x^{p-1}\partial x}{\sqrt[4]{\left(1-x^4\right)^{4-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[4]{\left(1-x^4\right)^{4-p}}}.$$

§. 80. Cum igitur hic sit $n=4$, erit $(4,1)=1$ et $\left(4,\ 2\right)=\frac{1}{2}$; formulae autem integrales ad hunc ordinem pertinentes erunt sex sequentes:
$1^{\text{o}}.\left(1,1\right), 2°.\left(1,2\right), 3°.\left(1,3\right), 4°.\ \left(2,2\right), 5°.\left(2,3\right), 6°.\left(3,3\right)$, inter quas ergo reperiuntur duae formulae circulares. (1, 3) et (2, 2), quas propterea litteris A et B designemus, ponendo

$$\left(1,3\right)=\frac{\pi}{4\sin.\frac{\pi}{4}}=\frac{2\pi}{2\sqrt{2}}=\text{A, et}$$

$$\left(2,2\right)=\frac{\pi}{4\sin.\frac{2\pi}{4}}=\frac{\pi}{4}=\text{B},$$

ita ut sit $\frac{\text{A}}{\text{B}}=\sqrt{2}.$

§. 81. In hoc ergo ordine aequationes tam primae quam secundae classis locum habere possunt; secunda autem classis nobis has tres praebet aequationes

$$1°.\ \text{B}=\left(2,1\right)\left(3,2\right),\ 2°.\ \text{A}=\left(1,1\right)\left(2,3\right),\ 3°.\ \text{A}=2\left(1,2\right)\left(3,3\right),$$

classis vero prima insuper dat hanc aequationem $\text{A}\left(1,2\right)=\left(1,1\right)\text{B}$,
sive $\frac{\text{A}}{\text{B}}=\frac{(1,1)}{(1,2)}$ quae autem aequatio iam ex duabus prioribus deducitur; namque ob $\left(3,2\right)=\left(2,3\right)$, secunda per primam divisa dabit $\frac{\text{A}}{\text{B}}=\frac{(1,1)}{(1,2)}=\sqrt{2}$, ita ut ratio inter has duas formulas sit algebraica, quae ergo imprimis notari meretur

$$\int \frac{\partial x}{\sqrt[4]{\left(1-x^4\right)^3}}:\int \frac{\partial x}{\sqrt{\left(1-x^4\right)}}=\sqrt{2}.$$

§. 82. Jam in hoc ordine, praeter binas formulas circulares, $\left(1,3\right)=\text{A}$ et $\left(2,\ 2\right)=\text{B}$; tanquam cognitam etiam introducamus formulam (1, 2), quae in ordine praecedente erat circularis, nunc autem est transcendens, eamque ponamus $\left(1,2\right)=\int \frac{\partial x}{\sqrt{\left(1-x^4\right)}}=\text{P};$ ubi caveatur, ne litterae A et P cum iis confundantur, quibus in formulis praecedentibus sumus usi, id quod etiam de ordinibus sequentibus est tenendum. His igitur litteris introductis aequationes nostrae erunt sequentes tres

$$1. \mathrm{B} = \mathrm{P}(3,2),\ 2.\ \mathrm{A} = (1,1)(2,3),\ 3.\ \mathrm{A} = 2\mathrm{P}(3,\ 3),$$

quandoquidem vidimus, quartam in praecederitibus jam contineri.

§. 83. Ope harum trium aequationum ergo ternas formulas integrales etiam nunc incognitas per ternas A, B et P, quas ut datas spectamus, determinare licebit. Ex prima enim fit $(3,2) = \frac{\mathrm{B}}{\mathrm{P}}$; ex tertia autem fit $(3,3) = \frac{\mathrm{A}}{2\mathrm{P}}$; tum vero ex secunda colligitur $(1,1) = \frac{\mathrm{A}}{(3,2)} = \frac{\mathrm{AP}}{\mathrm{B}}$. Cum igitur in hoc ordine omnino sint sex formulae integrales, earum ternae per tres reliquas definiri possunt, quas determinationes igitur ob oculos posuisse iuvabit:

$$1.\ (1,3) = \mathrm{A} = \frac{\pi\sqrt{2}}{2},$$

$$2.\ (2,2) = \mathrm{B}\ = \frac{\pi}{4},$$

$$3.\ (1,2) = \mathrm{P} = \int \frac{\partial x}{\sqrt{\left(1-x^4\right)}},$$

$$4. (1,1) = \frac{\mathrm{AP}}{\mathrm{B}},$$

$$5.\ (2,3) = \frac{\mathrm{B}}{\mathrm{P}},$$

$$6.\ (3,3) = \frac{\mathrm{A}}{2\mathrm{P}}.$$

Ex postremis ergo erit

$$(2,3):(3,3) = 2\mathrm{B} : \mathrm{A} = \sqrt{2} : 1,$$

ita ut etiam hae duae formulae inter se habeant rationem algebraicam, qua est :

$$\int \frac{xx\partial x}{\sqrt{\left(1-x^4\right)}} = \sqrt{2}\int \frac{xx\partial x}{\sqrt[4]{\left(1-x^4\right)}}.$$

Aliis insignibus relationibus, utpote satis cognitis, hic non immoramur.

Ordo III.
quo $n = 5$ et formula

$$(p,q) = \int \frac{x^{p-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-p}}}.$$

§. 84. Hic igitur ob $n = 5$ ante omnia erit

$$(5,1)=1,\ (5,2)=\tfrac{1}{2},\ (5,3)=\tfrac{1}{3};$$

formulae autem integrales hujus ordinis erunt hae decem

$$1.(1,1),\ 2.(1,2),\ 3.(1,3),\ 4.(1,4),\ 5.(2,2),$$
$$6.(2,3), 7.(2,4),\ 8.(3,3),\ 9.(3,4),\ 10.(4,4),$$

inter quas, quarta et sexta sunt circulares, quas ergo ita designemus

$$(1,4)=\frac{\pi}{5\sin.\frac{1}{5}\pi}=\mathrm{A}$$

et

$$(2\ 3)=\frac{\pi}{5\sin.\frac{2\pi}{5}}=B.$$

Praeterea vero binas formulas, quae in ordine praecedenti erant circulares, nunc autem sunt transcendentes, etiam peculiaribus litteris. notemus, scilicet

$$(1,3)=\mathrm{P}\ \text{et}\ (2,2)=\mathrm{Q}.$$

Mox enim patebit, dummodo etiam istae formulae tanquam cognitae spectentur reliquas sex omnes per has quatuor determinari posse.

§. 85. Quoniam hic tres classes priores locum habere possunt, consideremus primo aequationes, quas tertia classis suppeditat, et quae introductis his valoribus erunt

$$\begin{aligned}
&1.\ \ \mathrm{B}=\mathrm{P}(4,2),\\
&2.\ \ \mathrm{B}=(2,1)(3,3),\\
&3.\ \ \mathrm{B}=2\mathrm{Q}(4,3),\\
&4.\ \ \mathrm{A}=(1,1)(2,4),\\
&5.\ \ \mathrm{A}=2(1,2)(3,4),\\
&6.\ \ \mathrm{A}=3\mathrm{P}(4,4).
\end{aligned}$$

Quas hoc modo succinctius repraesentare licet

$$\mathrm{A}=(1,1)(2,4)=2(1,2)(3,4)=3\mathrm{P}(4,4),$$
$$\mathrm{B}=\mathrm{P}(4,2)=(2,1)(3,3)=2\mathrm{Q}(4,3),$$

ubi sex occurrunt producta ex binis formulis integralibus, quae singula quantitati circulari aequantur, unde totidem egregia theoremata formari possent, nisi hinc jam clare in oculos, incurrerent.

§. 86. Jam videamus, quot formulas integrales incognitas ex quatuor cognitis A, B, P et Q definire queamus, at vero prima dat $(4,2)=\frac{\mathrm{B}}{\mathrm{P}}$, tertia praebet $(4,3)=\frac{\mathrm{B}}{2\mathrm{Q}}$, sexta dat $(4,4)=\frac{\mathrm{A}}{3\mathrm{P}}$; hinc autem porro ex quarta deducimus $(1,1)=\frac{\mathrm{A}}{(2,4)}=\frac{\mathrm{AP}}{\mathrm{B}}$, ex quinta vero deducimus $(1,2)=\frac{\mathrm{A}}{2(3,4)}=\frac{\mathrm{AQ}}{\mathrm{B}}$. Denique ex secunda elicimus $(3,3)=\frac{\mathrm{B}}{(2,1)}=\frac{\mathrm{BB}}{\mathrm{AQ}}$ sicque ex his sex aequationibus sex determinationes sumus adepti; atque ad eo per litteras A, B, P et Q valores omnium reliquarum litterarum assignavimus.

§. 87. Quoniam igitur hactenus tantum classe tertia sumus usi, consideremus etiam aequationes secundae classis, quae sunt

$$1.\ \mathrm{AQ}=\mathrm{B}(2,1),$$

$$2.\ \mathrm{AP}=\mathrm{B}(1,1)$$

et

$$3.\mathrm{P}(4,2)=(1,2)(3,3);$$

verum si hic valores modo inventos substituamus, aequationes mere identicae resultant, ita ut hinc nulla nova determinatio sequatur.
Idem usu venit ex aequatione primae classis, quae erat $(2,1)(3,1)=(1,1)(2,2)$, quae facta substitutione quoque fit identica, ita ut duae priores classes nihil novi involvant. Neque tamen hinc concludere licet, etiam in sequentibus ordinibus classes praecedentes praetermitti posse, siquidem in ordine sequente statim contrarium se manifestabit.

§.88 Cum igitur hic ordo complectatur decem formulas integrales, earum valores per quatuor litteras A, B, P et Q ordine ita aspectui exponamus:

$$1.\ (1,1)=\frac{\mathrm{AP}}{\mathrm{B}},\quad 6.\ (2,3)=\mathrm{B},$$

$$2.\ (1,2)=\frac{\mathrm{AQ}}{\mathrm{B}},\quad 7.\ (2,4)=\frac{\mathrm{B}}{\mathrm{P}},$$

$$3.\ (1,3)=\mathrm{P},\quad 8.\ (3,3)=\frac{\mathrm{BB}}{\mathrm{AQ}},$$

$$4.\ (1,4)=\mathrm{A},\quad 9.\ (3,4)=\frac{\mathrm{B}}{2\mathrm{Q}},$$

$$5.\ (2,2)=\mathrm{Q},\quad 10.\ (4,4)=\frac{\mathrm{A}}{3\mathrm{P}}.$$

§. 89. Cum sit

$$\frac{\mathrm{A}}{\mathrm{B}}=\frac{\sin.\frac{2}{5}\pi}{\sin.\frac{1}{5}\pi}=\cos.\tfrac{1}{5}\pi,$$

tum vero

$$\cos.\tfrac{1}{5}\pi=\frac{1+\sqrt{5}}{2},$$

erit

$$\frac{\mathrm{A}}{\mathrm{B}} = \frac{1+\sqrt{5}}{2}$$

ideoque quantitas algebraica. Hinc igitur aliquot paria formularum integralium exhiberi poterunt, quae inter se teneant rationem algebraicam ; erit enim

$$\frac{(1,1)}{(1,3)} = \frac{1+\sqrt{5}}{2}, \quad \frac{(1,2)}{(2,2)} = \frac{1+\sqrt{5}}{2}, \quad \frac{(3,4)}{(3,3)} = \frac{1+\sqrt{5}}{4}, \quad \frac{(4,4)}{(2,4)} = \frac{1+\sqrt{5}}{6},$$

unde totidem egregia theoremata condi possent, nisi ex his formulis manifesto elucerent.

Ordo IV.
quo $n = 6$ et formula

$$(p,q) = \int \frac{x^{p-1}\partial x}{\sqrt[6]{\left(1-x^6\right)^{6-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[6]{\left(1-x^6\right)^{6-p}}}.$$

§. 90. Quoniam hic est $n = 6$, habebimus ante omnia

$$(6,1) = 1, \ (6,2) = \tfrac{1}{2}, \ (6,3) = \tfrac{1}{3}, \ (6,4) = \tfrac{1}{4};$$

formularum autem integralium in hoc ordine occurrentium numerus est quindecim, quae sunt

$$\begin{array}{lllll} 1. \ (1,1), & 2.(1,2), & 3.(1,3), & 4.(1,4), & 5.(1,5), \\ 6. \ (2,2), & 7. \ (2,3), & 8.(2,4), & 9.(2,5), & 10.(3,3), \\ 11. \ (3,4), & 12. \ (3,5), & 13.(4,4), & 14.(4,5), & 15.(5,5), \end{array}$$

inter quas reperiuntur tres. circulares, quas singulari modo designemus, scilicet sit

$$(1,5) = \frac{\pi}{6\sin.\frac{1}{6}\pi} = \frac{\pi}{3} = \mathrm{A},$$

$$(2,4) = \frac{\pi}{6\sin.\frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = \mathrm{B},$$

et

$$(3,3) = \frac{\pi}{6\sin.\frac{3\pi}{6}} = \frac{\pi}{6} = \mathrm{C},$$

ita ut sit

$$\mathrm{A} = 2\mathrm{C}.$$

Praeterea vero ambas formulas, quae in ordine praecedente erant circulares, nunc vero sunt transcendentes, statuamus

$$(1,4)=\mathrm{P} \text{ et } (2,3)=\mathrm{Q}.$$

His factis denominationibus evolvamus decem aequationes classis quartae, quae sunt

$$\begin{array}{ll} 1.\ \mathrm{B}=\mathrm{P}(5,2), & 6.\mathrm{B}=3\mathrm{Q}(5,4), \\ 2.\ \mathrm{C}=(3,1)(4,3), & 7.\mathrm{A}=(1,1)(5,2), \\ 3.\ \mathrm{C}=2\mathrm{Q}(5,3), & 8.\mathrm{A}=2(1,2)(3,5), \\ 4.\ \mathrm{B}=(2,1)(3,4), & 9.\mathrm{A}=3(1,3)(4,5), \\ 5.\ \mathrm{B}=2(2,2)(4,4), & 10.\mathrm{A}=4\mathrm{P}(5,5), \end{array}$$

quas ita succinctius referre licet

$$\begin{aligned} &\mathrm{A}=(1,1)(5,2)=2(1,2)(3,5)=3(1,3)(4,5)=4\mathrm{P}(5,5), \\ &\mathrm{B}=\mathrm{P}(5,2)=(2,1)(3,4)=2(2,2)(4,4)=3\mathrm{Q}(4,5), \\ &\mathrm{C}=(3,1)(4,3)=2\mathrm{Q}(5,3). \end{aligned}$$

Ecce ergo decem producta ex binis formulis integralibus, quorum singula quantitati circulari aequantur.

§. 91. Cum deinde sit $\frac{\mathrm{A}}{\mathrm{B}}=\sqrt{3}$ et $\frac{\mathrm{A}}{\mathrm{C}}=2$, tum etiam $\frac{\mathrm{B}}{\mathrm{C}}=\frac{2}{\sqrt{3}}$, plura paria binarum formularum integralium exhiberi possunt, quae inter se teneant rationem algebricam; erit enim

$$\begin{aligned} &\frac{\mathrm{A}}{\mathrm{B}}=\sqrt{3}=\frac{(1,1)}{(1,4)}=\frac{2(3,5)}{(3,4)}=\frac{(1,3)}{(2,3)}=\frac{4(5,5)}{(5,2)}, \\ &\frac{\mathrm{A}}{\mathrm{C}}=2=\frac{(1,2)}{(2,3)}=\frac{3(4,5)}{(4,3)}, \\ &\frac{\mathrm{B}}{\mathrm{C}}=\frac{2}{\sqrt{3}}=\frac{(1,2)}{(1,3)}=\frac{3(4,5)}{2(3,5)}. \end{aligned}$$

§. 92. Quodsi iam quinque formulas litteris A, B,C, P et Q designatas tamquam cognitas spectemus, videamus, quomodo reliquae formulae per eas definiri queant. Ac primo quidem percurramus decem aequationes classis quartae supra allatas, quarum prima dabit $(5,2)=\frac{\mathrm{B}}{\mathrm{P}}$, tertia dat $(5,3)=\frac{\mathrm{C}}{2\mathrm{Q}}$, sexta praebet $(5,4)=\frac{\mathrm{B}}{3\mathrm{Q}}$, decima dat $(5,5)=\frac{\mathrm{A}}{4\mathrm{P}}$. Quodsi iam hos valores in reliquis surrogemus, septima praebet $(1,1)=\frac{\mathrm{A}}{(5,2)}=\frac{\mathrm{AP}}{\mathrm{B}}$, octava dat $(1,\ 2)=\frac{\mathrm{A}}{2(3,5)}=\frac{\mathrm{AQ}}{\mathrm{C}}$, nona dat $(3,1)=\frac{\mathrm{A}}{3(4,5)}=\frac{\mathrm{AQ}}{\mathrm{B}}$.. Porro vero quarta dat $(3,4)=\frac{\mathrm{B}}{(2,1)}=\frac{\mathrm{BC}}{\mathrm{AQ}}$, quem valorem etiam secunda praebet. At vero ex aequatione quinta nullum valorem elicere possumus, quia neque formula (2, 2) nec (4, 4) etiamnunc constat. Causa est, quia duae reliquarum aequationum eandem determinationem produxerunt.

§. 93. Coacti igitur sumus ad aequationes praecedentium classium confugere atque adeo ex prima classe

$$(1,2)(8,1)=(1,1)(2,2)$$

statim colligimus

$$(2,2)=\frac{(1,2)(3,1)}{(1,1)}=\frac{\mathrm{AQQ}}{\mathrm{CP}},$$

qui valor in quinta aequatione substitutus suppeditat postremam aequationem, nempe

$$(4,4)=\frac{\mathrm{B}}{2(2,2)}=\frac{\mathrm{BCP}}{2\mathrm{AQQ}}.$$

Omnes igitur hos valores hic ordine referemus:

1. $(1,1)=\frac{\mathrm{AP}}{\mathrm{B}}$,	4. $(1,4)=\mathrm{P}$,	7. $(2,3)=Q$,
2. $(1,2)=\frac{\mathrm{AQ}}{\mathrm{C}}$,	5. $(1,5)=\mathrm{A}$,	8. $(2,4)=\mathrm{B}$,
3. $(1,3)=\frac{\mathrm{AQ}}{\mathrm{B}}$,	6. $(2,2)=\frac{\mathrm{AQQ}}{\mathrm{CP}}$,	9. $(2,5)=\frac{\mathrm{B}}{\mathrm{P}}$,
10. $(3,3)=\mathrm{C}$,	12. $(3,5)=\frac{\mathrm{C}}{2\mathrm{Q}}$,	14. $(4,5)=\frac{\mathrm{B}}{3\mathrm{Q}}$,
11. $(3,4)=\frac{\mathrm{BC}}{\mathrm{AQ}}$,	13. $(4,4)=\frac{\mathrm{BCP}}{2\mathrm{AQQ}}$	15. $(5,5)=\frac{\mathrm{A}}{4\mathrm{P}}$.

§. 94. Cum autem in hoc ordine etiam aequationes tam classis secundae quam tertiae valere debeant, videamus, utrum valores inventi his classibus conveniant an vero forte novam determinationem suppeditent? Facta autem substitutione in tribus aequationibus secundae classis ad identitatem pervenitur, quod idem quoque in aequationibus tertiae classis contingere debet, id quod evolventi mox patebit. Unde memorabile est omnes aequationes in quatuor primis classibus contentas, quarum numerus est 20, tantum decem determinationes in se complecti.

Ordo V.
quo $n=7$ et formula

$$(p,q)=\int \frac{x^{p-1}\partial x}{\sqrt[7]{\left(1-x^{7}\right)^{7-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[7]{\left(1-x^{7}\right)^{7-p}}}.$$

§. 95. Quia hic $n=7$, ante omnia habebimus valores absolutos

$$\left(7,1\right)=1,\ \left(7,2\right)=\tfrac{1}{2},\ \left(7,3\right)=\tfrac{1}{3},\ \left(7,4\right)=\tfrac{1}{4}\ \text{et}\ \left(7,5\right)=\tfrac{1}{5},$$

deinde inter formulas integrales huius ordinis imprimis notari debent circulares, quas hoc modo designemus:

$$\left(1,6\right)=\frac{\pi}{7\sin.\frac{\pi}{7}}=\mathrm{A},$$

$$\left(2,5\right)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=\mathrm{B},$$

$$\left(3,4\right)=\frac{\pi}{7\sin.\frac{3\pi}{7}}=\mathrm{C}.$$

Praeterea vero peculiaribus litteris notentur eae formulae, quae in ordine praecedenti erant circulares, hic autem valores transcendentes sortiuntur, qui sint

$$\left(1,5\right)=\mathrm{P},\quad \left(2,4\right)=\mathrm{Q}\ \ \text{et}\ \ \left(3,3\right)=\mathrm{R};$$

per has enim sex litteras videbimus omnes reliquas formulas huius ordinis determinari posse.

§. 96. Quoniam supra non omnes aequationes quintae classis expressimus, eas hic conjunctim exhibeamus, et ad nostrum casum accommodemus

I.	(1,6)(7,1) = (1,1)(2,6)	A = (1,1)(2,6),
II.	(1,6)(7,2) = (1,2)(3,6)	A = 2 (1,2)(3,6),
III.	(1,6)(7,3) = (1,3)(4,6)	A = 3(1,3)(4,6),
IV.	(1,6)(7,4) = (1,4)(5,6)	A = 4(1,4)(5,6),
V.	(1,6)(7,5) = (1,5)(6,6)	A = 5P (6,6),
VI.	(2,5)(7,1) = (2,1)(3,5)	B = (2,1)(3,5),
VII.	(2,5)(7,2) = (2, 2)(4,5)	B = 2(2,2)(4,5),
VIII.	(2,5)(7,3) = (2,3)(5,5)	B = 3 (2,3)(5,5),
IX.	(2,5)(7,4) = (2,4)(6,5)	B = 4 Q (6,5),
X.	(3,4)(7,1) = (3,1)(4,4)	C = (3,1)(4,4),
XI.	(3,4)(7,2) = (3,2)(5,4)	C = 2 (3,2)(5,4),
XII.	(3,4)(7,3) = (3,3)(6,4)	C = 3 R (6,4),
XIII.	(4,3)(7,1) = (4,1)(5,3)	C = (4,1)(5,3),
XIV.	(4,3)(7,2) = (4,2)(6,3)	C = 2 Q (6,3),
XV.	(5,2)(7,1) = (5,1)(6,2)	B = P(6, 2).

Hic igitur habemus quina producta formulae A aequalia totidemque formulis B et C aequalia.

§. 97. Omnino autem in hoc ordine occurrunt 21 formulae integrales, ex quibus sex litteris A, B, C, P, Q et R designavimus, per quas igitur reliquas quindecim formulas integrales definiri oportet, quae sunt

$$1^o.\ (1,1),\ 2^o.\ (1,2),\ 3^o.\ (1,3),\ 4^o.\ (2,2),\ 5^o.\ (1,4),$$
$$6^o.\ (2,3),\ 7^o.\ (2,6),\ 8^o.\ (3,5),\ 9^o.\ (4,4),\ 10^o.\ (3,6),$$
$$11^o.\ (4,5),\ 12^o.\ (4,6),\ 13^o.\ (5,5),\ 14^o.\ (5,6),\ 15^o.\ (6,6),$$

§. 98. Videamus igitur, quot harum formularum ex superioribus quindecim aequationibus determinare liceat, ac primo quidem ex aequationibus V, IX, XII, XIV et XV, immediate deducuntur sequentes formulae

$$(6,6)=\tfrac{A}{5P},\ (6,5)=\tfrac{B}{4Q},\ (6,4)=\tfrac{C}{3R},\ (6,3)=\tfrac{C}{2Q},\ (6,2)=\tfrac{B}{P}.$$

His iam inventis ex aequationibus I, II, III et IV derivamus has formulas

His jam inventis ex aequationibus I, II, et IV derivamus has formulas

$$(1,1)=\tfrac{AP}{B},\ (1,2)=\tfrac{AQ}{C},\ (1,3)=\tfrac{AR}{C},\ (1,4)=\tfrac{AQ}{B}.$$

Ex his vero valoribus per aequationes VI, X et XIII, colligimus

$$(3,5)=\tfrac{BC}{AQ},\ (4,4)=\tfrac{CC}{AR}\ \text{ et }\ (5,3)=\tfrac{BC}{AQ},$$

ubi notasse iuvabit eundem valorem pro (5, 3) prodiisse ex aequationibus VI et XIII. Ex reliquis autem aequationibus VII, VIII et XI nihil concludere licet, unde istae quatuor formulae (2, 2), (2, 3), (5, 4) et (5, 5) nobis etiamnunc manent incognitae.

§.99. Recurrere ergo coacti sumus ad aequationes praecedentium classium, quippe quae aeque ad nostrum ordinem pertinent atque aequationes classis quintae; hanc ob rem simili modo aequationes classis quartae hic apponamus et ad nostrum casum applicemus:

I. (1,5)(6,1) = (1,1)(2,5)	PA = (1,1)B,
II. (1,5)(6,2) = (1,2)(3,5)	P(6,2) = (1,2)(3,5),
III. (1,5)(6,3) = (1,3)(4,5)	P(6,3) = (1,3)(4,5),
IV. (1,5)(6,4) = (1,4)(5,5)	P(6,4) = (1,4)(5,5),
V. (2,4)(6,1) = (2,1)(3,4)	QA = (2,1)C,
VI. (2,4)(6,2) = (2,2)(4,4)	Q(6,2) = (2,2)(4,4),
VII. (2,4)(6,3) = (2,3)(5,4)	Q(6,3) = (2,3)(5,4)
VIII. (3,3)(6,1) = (3,1)(4,3)	BA = (3,1)C,
IX. (3,3)(6,2) = (3,2)(5,3)	R(6,2) = (3,2)(5,3),
X. (4,2) (6,1) = (4,1)(5,2)	QA = (4,1)B.

§.100. Ex aequationibus I, V, VIII et X immediate concludimus has formulas

$$(1,1)=\frac{\mathrm{PA}}{\mathrm{B}},\ (2,1)=\frac{\mathrm{QA}}{\mathrm{C}},\ (3,1)=\frac{\mathrm{AR}}{\mathrm{C}},\ (4,1)=\frac{\mathrm{AQ}}{\mathrm{B}},$$

quos autem valores iam ante adepti sumus. Secunda aequatio, si formulae iam inventae substituantur, praebet aequationem identicam. Ex tertia autem poterimus definire formulam (4,5), cuius valor hinc colligitur

$$(4,5)=\frac{\mathrm{CCP}}{2\mathrm{AQR}}.$$

Simili modo ex quarta elicimus

$$(5,5)=\frac{\mathrm{BCP}}{3\mathrm{AQR}}.$$

Porro ex aequatione sexta concludimus fore

$$(2,2)=\frac{\mathrm{ABQR}}{\mathrm{CCP}}.$$

Deinde septima aequatio dat

$$(2,3)=\frac{\mathrm{AQR}}{\mathrm{CP}}.$$

Nona vero aequatio etiam praebet $(3,2)=\frac{\mathrm{AQR}}{\mathrm{CP}}$. Sicque omnes quindecim formulas incognitas determinavimus per sex litteras cognitas A, B, C, P, Q et R.

§.101. Valores igitur omnium formularum huius ordinis hic aspectui coniunctim exponamus

$(1,6)=\mathrm{A}$	$(6,2)=\frac{\mathrm{B}}{\mathrm{P}}$	$(1,1)=\frac{\mathrm{AP}}{\mathrm{B}}$	$(3,5)=\frac{\mathrm{BC}}{\mathrm{AQ}}$	$(2,3)=\frac{\mathrm{AQR}}{\mathrm{CP}}$
$(2,5)=\mathrm{B}$	$(6,3)=\frac{\mathrm{C}}{2\mathrm{Q}}$	$(1,2)=\frac{\mathrm{AQ}}{\mathrm{C}}$	$(4,4)=\frac{\mathrm{CC}}{\mathrm{AR}}$	$(4,5)=\frac{\mathrm{CCP}}{2\mathrm{AQR}}$
$(3,4)=\mathrm{C}$	$(6,4)=\frac{\mathrm{C}}{3\mathrm{R}}$	$(1,3)=\frac{\mathrm{AR}}{\mathrm{C}}$		$(5,5)=\frac{\mathrm{BCP}}{3\mathrm{AQR}}$
$(1,5)=\mathrm{P}$	$(6,5)=\frac{4}{4\mathrm{Q}}$	$(1,4)=\frac{\mathrm{AQ}}{\mathrm{B}}$		$(2,2)=\frac{\mathrm{ABQR}}{\mathrm{CCP}}.$
$(2,4)=\mathrm{Q}$	$(6,6)=\frac{\mathrm{A}}{5\mathrm{P}}$			
$(3,3)=\mathrm{R}$				

§. 102. Quoniam autem aequationes primae, secundae ac tertiae classis etiam in hoc ordine valent, si in iis valores hic inventos substituamus, perpetuo in aequationes identicas incidemus. Ita, cum aequatio primae classis sit

$$(1,2)(3,1)=(1,1)(2,2),$$

facta substitutione reperietur

$$(1,2)(3,1)=\frac{\mathrm{AAQR}}{\mathrm{CC}};$$

at vero $(1,1)(2,2)$ fit $=\frac{\mathrm{AAQR}}{\mathrm{CC}}$ haecque identitas etiam deprehendetur in tribus aequationibus secundae classis atque etiam in sex aequationibus tertiae classis, quemadmodum calculum instituenti mox patebit.

§.103. Simili modo haud difficile erit hanc investigationem ad ordines superiores extendere, neque tamen legem observare licet, secundum quam determinationes singularum formularum cuiusque ordinis progrediuntur. Interim tamen observasse iuvabit in ordine sequente sexto, ubi $n=8$ et formulae occurrunt 28, eas omnes primo per quatuor formulas circulares

$$(1,7)=\mathrm{A},(2,6)=\mathrm{B},(3,5)=\mathrm{C},(4,4)=\mathrm{D},$$

praeterea vero per has tres transcendentes

$$(1,\ 6) = P,\ (2,\ 5) = Q \text{ et } (3,\ 4) = R,$$

determinari posse. Cum igitur quovis ordine ,determinatio singularem formularum, praeter formulas, circulares, quae utique pro cognitis haberi possunt, etiam aliquot formulas transcendentes postulat, si saltem valores harum formularum vero proxime, cognoscere voluerimus, methodus adhuc desideratur, istos valores proxime, veluti in fractionibus decimalibus, definiendi. Talem igitur methodum hic coronidis loco subjungemus.

Problema.

Proposita formula integrali cujusque ordinis

$$S=\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}$$

a termino $x=0$ *usque ad* $x=1$ *extendenda, investigare seriem convergentem, quae istum valorem* S *exprimat*.

Solutio

§.104. Cum sit

$$\frac{1}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}=\left(1-x^{n}\right)^{-\frac{(n-q)}{n}},$$

facta evolutione hujus potestatis binomii more solito, reperietur

$$\frac{1}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}=1+\frac{n-q}{n}x^{n}+\frac{n-q}{n}\cdot\frac{2n-q}{2n}x^{2n}+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}x^{3n}+\text{etc.}$$

Si haec series ducatur in $x^{p-1}\partial x$ et integretur, prodibit

$$\begin{aligned}S=&\frac{x^{p}}{p}+\frac{n-q}{n}\frac{x^{n+p}}{n+p}+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{x^{2n+p}}{2n+p}\\&+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{x^{3n+p}}{3n+p}+\text{etc.}\end{aligned}$$

quae series jam evanescit posito $x=0$; unde si ponamus $x=1$, valor quaesitus nostrae formulae fiet

$$\begin{aligned}S=&\frac{1}{p}+\frac{n-q}{n}\frac{1}{n+p}+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\frac{1}{2n+p}\\&+\frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{1}{3n+p}+\text{etc.}\end{aligned}$$

§. 105. Verum ista series, quicumque numeri pro litteris *n*, *p* et *q* accipiantur, nimis lente convergit, quam ut ex ea valores ipsius *S* saltem ad tres quatuorve figuras decimales satis exacte definiri queant; quamobrem aliam evolutionem institui conveniet, dum scilicet valorem quaesitum in duas partes resolvemus. Statuamus igitur

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x^n=\frac{1}{2}\end{bmatrix}=\text{P}$$

et

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}\begin{bmatrix}\text{ab} & x^n=\frac{1}{2}\\ \text{ad} & x=1\end{bmatrix}=\text{Q},$$

atque evidens est fore $\text{S}=\text{P}+\text{Q}$. Nunc autem tam pro P quam pro Q haud difficulter series satis convergentes exhiberi poterunt.

§. 106. Quod primum ad valorem P attinet, eum ex valore generali, quem supra pro S invenimus, facile derivabimus ponendo $x^n=\frac{1}{2}$, ita ut sit $x=\sqrt[n]{\frac{1}{2}}$ et $x^p=\frac{1}{\sqrt[n]{2^p}}$, quo facto pro P obtinebimus hanc seriem

$$\text{P}=\frac{1}{\sqrt[n]{2^p}}\left\{\begin{aligned}&\frac{1}{p}+\frac{n-q}{2n}\frac{1}{n+p}+\frac{n-q}{2n}\cdot\frac{2n-q}{4n}\frac{1}{2n+p}\\ &\qquad+\frac{n-q}{2n}\cdot\frac{2n-q}{4n}\cdot\frac{3n-q}{6n}\cdot\frac{1}{3n+p}+\text{etc.}\end{aligned}\right\}.$$

In qua serie singuli termini plus quam in ratione dupla decrescunt, ita ut verbi gratia terminus decimus iam multo minor futurus sit quam unde $\frac{1}{1024}$, si ad partes millionesimas certi esse velimus, sufficeret calculum ne quidem ad vicesimum usque terminum extendere.

§.107. Cum deinde posuerimus

$$\text{Q}=\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}\begin{bmatrix}\text{ab} & x^n=\frac{1}{2}\\ \text{ad} & x=1\end{bmatrix},$$

statuamus $1-x^n=y^n$, ut sit

$$\text{Q}=\int \frac{x^{p-1}\partial x}{y^{n-q}};$$

tum vero erit $x^n=1-y^n$ ideoque $x^p=\sqrt[n]{\left(1-y^n\right)^p}$, unde differentiando colligitur

$$x^{p-1}\partial x=-y^{n-1}\partial y\left(1-y^n\right)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$Q=-\int y^{n-1}\partial y\left(1-y^{n}\right)^{\frac{p-n}{n}}\begin{bmatrix}\text{ab} & y^{n}=\frac{1}{2}\\ \text{ad} & y=0\end{bmatrix}.$$

Quando enim fit $x^{n}=\frac{1}{2}$, tum etiam erit $y^{n}=\frac{1}{2}$, at facto $x=1$ manifesto fit $y=0$; quare si terminos integrationis permutemus, etiam signum ipsius formulae immutari debet sicque fiet

$$Q=\int y^{q-1}\partial y\left(1-y^{n}\right)^{\frac{p-n}{n}}\begin{bmatrix}\text{ab} & y=0\\ \text{ad} & y^{n}=\frac{1}{2}\end{bmatrix}.$$

§.108. Haec autem formula pro Q inventa omnino similis est illi, quam pro P invenimus, hoc tantum discrimine, quod litterae *p* et *q* inter se sunt permutatae; quocirca, si integratio per seriem instituatur, proveniet sequens

$$Q=\frac{1}{\sqrt[n]{2^{q}}}\left\{\begin{aligned}&\frac{1}{q}+\frac{n-p}{2n}\frac{1}{n+q}+\frac{n-p}{4n}\cdot\frac{2n-p}{2n}\frac{1}{2n+q}\\ &\qquad+\frac{n-p}{6n}\cdot\frac{2n-p}{2n}\cdot\frac{3n-p}{3n}\cdot\frac{1}{3n+q}+\text{etc.}\end{aligned}\right\},$$

quae series aeque converget ac praecedens pro P inventa. His autem duabus seriebus ad calculum revocatis semper erit valor quaesitus

$$S=P+Q.$$

COROLLARIUM 1

§.109. Iste calculus plurimum contrahetur iis casibus, quibus est $p=q$; tum enim fiet $P=Q$ hisque casibus, quibus

$$S=\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-p}}}$$

valor istius formulae ab $x=0$ ad $x=1$ extensae erit

$$S=\frac{2}{\sqrt[n]{2^{p}}}\left\{\begin{aligned}&\frac{1}{p}+\frac{n-p}{2n}\frac{1}{n+p}+\frac{n-p}{2n}\cdot\frac{2n-p}{4n}\frac{1}{2n+p}\\ &\qquad+\frac{n-p}{2n}\cdot\frac{2n-p}{4n}\cdot\frac{3n-p}{6n}\cdot\frac{1}{3n+p}+\text{etc.}\end{aligned}\right\}.$$

COROLLARIUM 2.

§. 110. Quoniam igitur in singulis ordinibus nonnullae huiusmodi formulae (*p*, *p*) occurrunt, statim atque valores aliquot huiusmodi formularum fuerint ad calculum decimalem revocati, quoniam formulae circulares per se sunt notae, ex iis valores omnium reliquarum formularum eiusdem ordinis assignare licebit.

EXEMPLUM

§. 111. Proposita sit formula ordinis primi, ubi $p = q = 2$ et

$$\mathrm{S} = \int \frac{x\partial x}{\sqrt[3]{\left(1-x^{3}\right)}}.$$

Series igitur pro S inventa erit

$$\mathrm{S} = \sqrt[3]{2}\left(\frac{1}{2} + \frac{1}{6}\cdot\frac{1}{5} + \frac{1}{6}\cdot\frac{4}{12}\cdot\frac{1}{8} + \frac{1}{6}\cdot\frac{4}{12}\cdot\frac{7}{18}\cdot\frac{1}{11} + \frac{1}{6}\cdot\frac{4}{12}\cdot\frac{7}{18}\cdot\frac{10}{24}\cdot\frac{1}{14} + \text{etc.}\right).$$

Subducta autem calculo reperitur

$$\mathrm{S} = 0,54326\sqrt[3]{2} = 0,68446,$$

qui ergo est valor formulae (2,2) in ordine primo (§. 22), ubi invenimus $\left(2,2\right) = \frac{\mathrm{A}}{\mathrm{P}}$, ita ut jam sit $\mathrm{P} = \frac{\mathrm{A}}{(2,2)}$. Est vero

$$A = \frac{2\pi}{3\sqrt{3}} = 1,20920,$$

hinc erit $\mathrm{P} = 1,76664 = \left(1,1\right)$, unde in fractionibus decimalibus ternae formulae ordinis primi erunt

$$\left(1,1\right) = 1,76664,\ \left(1,\ 2\right) = 1,20918,\ \left(2,2\right) = 0,68445.$$

Hocque modo etiam omnes formulas sequentium ordinum evolvere licebit.

ADDITAMENTUM AD DISSERTATIONEM DE VALORIBUS FORMULAE INTEGRALIS $\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}$ AB $x = 0$ AD $x = 1$ EXTENSAE

Conventui exhibitum die 17. Octobris 1776
Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 118-129

§. 112. Si methodum in praecedente dissertatione traditam ad altiores ordines quam $n = 7$ transferre vellemus, ob ingentem aequationum considerandarum numerum labor fieret nimis molestus. Quoniam autem vidimus, non omnes istas aequationes concurrere ad valores singularum formularum determinandos, opus non mediocriter sublevabitur, si quovis casu eas tantum aequationes in computum ducamus, quae immediate ad determinationes formularum perducant, quemadmodum hic pro casu $n=10$ sum ostensurus.

Determinatio harum formularum pro casu $n=10$, ubi formula

$$\left(p,q\right)=\int \frac{x^{p-1}\partial x}{\sqrt[10]{\left(1-x^{10}\right)^{10-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[10]{\left(1-x^{10}\right)^{10-p}}}.$$

§. 113 . Hoc casu ergo formulae valorem absolutum recipientes sunt

$$\left(10,1\right)=1,\ \left(10,2\right)=\tfrac{1}{2},\ \left(10,3\right)=\tfrac{1}{3},\text{ et in genere }(10,\alpha)=\tfrac{1}{\alpha}.$$

Deinde omnes formulae , in quibus est $p+q=10$, a circulo pendent, ideoque pro cognitis haberi possunt, quas ergo propriis litteris designemus:

$$\left(1,9\right)=\frac{\pi}{10\sin.\frac{1}{10}\pi}=\text{A},\qquad \left(6,4\right)=\frac{\pi}{10\sin.\frac{6}{10}\pi}=\text{D},$$
$$\left(2,8\right)=\frac{\pi}{10\sin.\frac{2}{10}\pi}=\text{B},\qquad \left(7,3\right)=\frac{\pi}{10\sin.\frac{7}{10}\pi}=\text{C},$$
$$\left(3,7\right)=\frac{\pi}{10\sin.\frac{3}{10}\pi}=\text{C},\qquad \left(8,2\right)=\frac{\pi}{10\sin.\frac{8}{10}\pi}=\text{B},$$
$$\left(4,6\right)=\frac{\pi}{10\sin.\frac{4}{10}\pi}=\text{D},\qquad \left(9,1\right)=\frac{\pi}{10\sin.\frac{9}{10}\pi}=\text{A}.$$
$$\left(5,5\right)=\frac{\pi}{10\sin.\frac{5}{10}\pi}=\text{E},$$

§. 114. Per has autem formulas circulares reliquas in forma generali contentas neutiquam determinare licet, sed insuper aliquot formulas transcendentes in subsidium vocari oportet, ex quibus cum circularibus illis coniunctis reliquarum omnium valores assignare licebit. Nostro autem casu, quo $n=10$, sequentes formulas tamquam cognitas spectari conveniet, quae in ordine praecedenti, ubi $n=9$; erant circulares, nunc autem in ordinem transcendentium transeunt. Eas igitur sequenti modo designemus

$$(1,8)=\text{P},\ (2,7)=\text{Q},\ (3,6)=\text{B},\ (4,5)=\text{S},$$
$$(5,4)=\text{S},\ (6,3)=\text{R},\ (7,2)\ =\text{Q},\ (8,1)=\text{P}.$$

Scilicet si valores harum litterarum quoque tamquam cognitos spectemus, per eos cum circularibus iunctos reliquas formulas omnes in hoc ordine contentas determinare poterimus. Cum igitur numerus omnium formularum integralium in hoc ordine $n=10$

contentarum sit 45, ex iis autem novem ut cognitae spectentur, reliquae 36 per has litteras maiusculas determinari debebunt.

§. 115. Istas autem determinationes ex aequatione generali supra [§.10] demonstrata peti oportet, quae hac forma continetur

$$(a,b)(a+b,c)=(a,c)(a+c,b),$$

ubi assumere licebit, semper esse $b>c$, quoniam, si foret $c=b$, aequatio foret identica. Primo igitur ut hinc aequationes, quae immediate determinationes praebeant, nanciscamur, sumamus $a+b=10$, ut sit $(10,c)=\frac{1}{c}$; tum vero capiatur $c=b-1$, quo facto pro *a* ordine scribendo numeros 1, 2, 3, etc. sequentes prodibunt determinationes

$$(1,9)(10,8)=(1,8)(9,9) \quad \text{sive} \quad \tfrac{1}{8}\mathrm{A}=\mathrm{P}(9,9), \quad \text{ergo } (9,9)=\tfrac{\mathrm{A}}{8\mathrm{P}},$$
$$(2,8)(10,7)=(2,7)(9,8) \quad \text{sive} \quad \tfrac{1}{7}\mathrm{B}=\mathrm{Q}(9,8), \quad \text{ergo } (9,8)=\tfrac{B}{7Q},$$
$$(3,7)(10,6)=(3,6)(9,7) \quad \text{sive} \quad \tfrac{1}{6}\mathrm{C}=\mathrm{R}(9,7), \quad \text{ergo } (9,7)=\tfrac{\mathrm{C}}{6\mathrm{R}},$$
$$(4,6)(10,5)=(4,5)(9,6) \quad \text{sive} \quad \tfrac{1}{5}\mathrm{D}=\mathrm{S}(9,6), \quad \text{ergo } (9,6)=\tfrac{\mathrm{D}}{5\mathrm{S}},$$
$$(5,5)(10,4)=(5,4)(9,5) \quad \text{sive} \quad \tfrac{1}{4}\mathrm{E}=\mathrm{S}(9,5), \quad \text{ergo } (9,5)=\tfrac{\mathrm{E}}{4S},$$
$$(6,4)(10,3)=(6,3)(9,4) \quad \text{sive} \quad \tfrac{1}{3}\mathrm{D}=\mathrm{R}(9,4), \quad \text{ergo } (9,4)=\tfrac{\mathrm{D}}{3R},$$
$$(7,3)(10,2)=(7,2)(9,3) \quad \text{sive} \quad \tfrac{1}{2}\mathrm{D}=\mathrm{Q}(9,3), \quad \text{ergo } (9,3)=\tfrac{\mathrm{C}}{2\mathrm{Q}},$$
$$(8,2)(10,1)=(8,1)(9,2) \quad \text{sive} \quad \mathrm{B}=\mathrm{P}(9,2), \quad \text{ergo } (9,2)=\tfrac{\mathrm{B}}{\mathrm{P}}.$$

§. 116. Ex formulis igitur incognitis illis numero 36 iam octo determinavimus quae nobis viam sternent ad novas determinationes, quas primo derivabimus ex aequatione generali sumendo $a=1,\ b=9$ et pro *c* scribendo ordine numeros 1, 2, 3, ... 8, unde calculus ita se habebit:

$(1,9)(10,1)=(1,1)(2,9)$	$\mathrm{A}=(1,1)\frac{\mathrm{B}}{\mathrm{P}}$, ergo $(1,1)=\frac{\mathrm{AP}}{\mathrm{B}}$,
$(1,9)(10,2)=(1,2)(3,9)$	$\frac{1}{2}\mathrm{A}=(1,2)\ \frac{\mathrm{C}}{2\mathrm{Q}}$, ergo $(1,2)=\frac{\mathrm{AQ}}{\mathrm{C}}$,
$(1,9)(10,3)\ =(1,3)(4,9)$	$\frac{1}{3}\mathrm{A}=(1,3)\frac{\mathrm{D}}{3\mathrm{R}}$, ergo $(1,3)=\frac{\mathrm{AR}}{\mathrm{D}}$,
$(1,9)(10,4)=(1,4)(5,9)$	$\frac{1}{4}\mathrm{A}=(1,4)\frac{\mathrm{E}}{4\mathrm{S}}$, ergo $(1,4)=\frac{\mathrm{AS}}{\mathrm{E}}$,
$(1,9)(10,5)=(1,5)(6,9)$	$\frac{1}{5}\mathrm{A}=(1,5)\frac{\mathrm{D}}{5S}$, ergo $(1,5)=\frac{\mathrm{AS}}{\mathrm{D}}$,
$(1,9)(10,6)=(1,6)\ (7,9)$	$\frac{1}{6}\mathrm{A}=(1,6)\frac{\mathrm{C}}{6\mathrm{R}}$, ergo $(1,6)=\frac{\mathrm{AR}}{\mathrm{C}}$,

$(1,9)(10,7) = (1,7)(8,9)$	$\frac{1}{7}\mathrm{A} = (1,7)\frac{\mathrm{B}}{7\mathrm{Q}}$, ergo $(1,7) = \frac{\mathrm{AQ}}{\mathrm{B}}$,	
$(1,9)(10,8) = (1,8)(9,9)$	$\frac{1}{8}\mathrm{A} = (1,8)\frac{\mathrm{A}}{8\mathrm{P}}$, ergo $(1,8) = \frac{\mathrm{AP}}{\mathrm{A}}$;	

hocque modo septem novas determinationes sumus adepti.

§.117. His autem inventis consideremus aequationes ex valoribus $a=1,\ b=8,\ c=1,\ 2,\ 3,\ \dots\ 7$ ortas eritque

$(1,8)(9,1) = (1,1)(2,8)$	$\mathrm{AP} = (1,l)\mathrm{B}$	identica,
$(1,8)(9,2) = (1,2)(3,8)$	$\mathrm{B} = (3,8)\frac{\mathrm{AQ}}{\mathrm{C}}$	$(3,8) = \frac{\mathrm{BC}}{\mathrm{AQ}}$,
$(1,8)(9,3) = (1,3)(4,8)$	$\frac{\mathrm{CP}}{2\mathrm{Q}} = (4,8)\frac{\mathrm{AR}}{\mathrm{D}}$	$(4,8) = \frac{\mathrm{CDP}}{2\mathrm{AQR}}$,
$(1,8)(9,4) = (1,4)(5,8)$	$\frac{\mathrm{DP}}{3\mathrm{R}} = (5,8)\frac{\mathrm{AS}}{\mathrm{E}}$	$(5,8) = \frac{\mathrm{DEP}}{3\mathrm{ARS}}$,
$(1,8)(9,5) = (1,5)(6,8)$	$\frac{\mathrm{EP}}{4S} = (6,8)\frac{\mathrm{AS}}{\mathrm{D}}$	$(6,8) = \frac{\mathrm{DEP}}{4ASS}$,
$(1,8)(9,6) = (1,6)(7,8)$	$\frac{\mathrm{DP}}{5\mathrm{S}} = (7,8)\frac{\mathrm{AR}}{\mathrm{C}}$	$\frac{\mathrm{AR}}{\mathrm{C}}(7,8) = \frac{\mathrm{CDP}}{5\mathrm{ARS}}$,
$(1,8)(9,7) = (1,7)(8,8)$	$\frac{\mathrm{CP}}{6\mathrm{R}} = (8,8)\frac{\mathrm{AQ}}{\mathrm{B}}$	$(8,8) = \frac{\mathrm{BCP}}{6\mathrm{AQR}}$.

§.118. Novas determinationes reperiemus ponendo $a=1,\ b=7,\ c=3,\ 4,\ 5,\ 6;$ hinc enim nanciscimur sequentes determinationes

$(1,7)(8,3) = (1,3)(4,7)$	$\mathrm{D} = (4,7)\frac{\mathrm{AR}}{\mathrm{D}}$	$(4,7) = \frac{\mathrm{CD}}{\mathrm{AR}}$,
$(1,7)(8,4) = (1,4)(5,7)$	$\frac{\mathrm{CDP}}{2\mathrm{BR}} = (5,7)\frac{\mathrm{AS}}{\mathrm{E}}$	$(5,7) = \frac{\mathrm{CDEP}}{2\mathrm{ABRS}}$,
$(1,7)(8,5) = (1,5)(6,7)$	$\frac{\mathrm{DEPQ}}{3\mathrm{BRS}} = (6,7)\frac{\mathrm{AS}}{\mathrm{D}}$	$(6,7) = \frac{\mathrm{DDEPQ}}{3\mathrm{ABRSS}}$,
$(1,7)(8,6) = (1,6)(7,7)$	$\frac{\mathrm{DEPQ}}{4\mathrm{BSS}} = (7,7)\frac{\mathrm{AR}}{\mathrm{C}}$	$(7,7) = \frac{\mathrm{CDEPQ}}{4\mathrm{ABRSS}}$.

§.119. Sumamus nunc: $a=1,\ b=6,\ c=4,\ 5$ eritque

$(1,6)(7,4) = (1,4)(5,6)$	$\mathrm{D} = (5,6)\frac{\mathrm{AS}}{\mathrm{E}}$	$(5,6) = \frac{\mathrm{DE}}{\mathrm{AS}}$,
$(1,6)(7,5) = (1,5)(6,6)$	$\frac{\mathrm{DEP}}{2\mathrm{BS}} = (6,6)\frac{\mathrm{AS}}{\mathrm{D}}$	$(6,6) = \frac{\mathrm{DDEP}}{2\mathrm{ABSS}}$,

Hactenus igitur omnes formulas (p, q) determinavimus, in quibus $p+q>10$.

Ex reliquis autem, ubi $p+q<9$, iam nacti sumus istas

$$(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (1,7),$$

ita ut adhuc determinandae relinquantur istae

$$\begin{aligned}&(2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6),\\&(3,3),\ (3,4),\ (3,5),\\&(4,4).\end{aligned}$$

§. 120. Pro his inveniendis sumamus $a=1$et $c=1$, pro b autem ordine capiamus numeros 2, 3 etc. atque consequemur has aequationes

$$\begin{array}{l|l|l}
(1,2)(3,1)=(1,1)(2,2) & \frac{\text{AAQR}}{\text{CD}}=(2,2)\frac{\text{AP}}{\text{B}} & (2,2)=\frac{\text{ABQR}}{\text{CDP}},\\
(1,3)(4,1)=(1,1)(2,3) & \frac{\text{AARS}}{\text{DE}}=(2,3)\frac{\text{AP}}{\text{B}} & (2,3)=\frac{\text{ABRS}}{\text{DEP}}\\
(1,4)(5,1)=(1,1)(2,4) & \frac{\text{AASS}}{\text{DE}}=(2,4)\frac{\text{AP}}{\text{B}} & (2,4)=\frac{\text{ABSS}}{\text{DEP}},\\
(1,5)(6,1)=(1,1)(2,5) & \frac{\text{AARS}}{\text{CD}}=(2,5)\frac{\text{AP}}{\text{B}} & (2,5)=\frac{\text{ABRS}}{\text{CDP}},\\
(1,6)(7,1)=(1,1)(2,6) & \frac{\text{AAQR}}{\text{BC}}=(2,6)\frac{\text{AP}}{\text{B}} & (2,6)=\frac{\text{ABQR}}{\text{BCP}}.
\end{array}$$

sicque etiamnunc determinandae restant formulae $(3,3)$, $(3,4)$, $(3,5)$ et $(4,4)$.

§. 121. Pro his sumatur $a=1,\ c=2$ et $b=3,\ 4,\ 5$ etc.; tum enim prodibunt hae aequationes

$$\begin{array}{l|l|l}
(1,3)(4,2)=(1,2)(3,3) & \frac{\text{AABRSS}}{\text{DDEP}}=(3,3)\frac{\text{AQ}}{\text{C}} & (3,3)=\frac{\text{ABCRSS}}{\text{DDEPQ}},\\
(1,4)(5,2)=(1,2)(3,4) & \frac{\text{AABRSS}}{\text{CDEP}}=(3,4)\frac{\text{AQ}}{\text{C}} & (3,4)=\frac{\text{ABRSS}}{\text{DEPQ}}\\
(1,5)(6,2)=(1,2)(3,5) & \frac{\text{AAQRS}}{\text{CDP}}=(3,5)\frac{\text{AQ}}{\text{C}} & (3,5)=\frac{\text{ARS}}{\text{DP}}.
\end{array}$$

Unica ergo formula restat determinanda, scilicet (4,4), quae ex hac aequatione

$$(1,4)(5,3)=(1,3)(4,4)$$

definietur; erit enim $\frac{\text{AARSS}}{\text{DEP}}=(4,4)\frac{\text{AR}}{\text{D}}$, ideoque $(4,4)=\frac{\text{ASS}}{\text{EPP}}$.

§.122. Ut nunc omnes has determinationes simul aspectui exponamus, quoniam in hoc ordine $n = 10$ omnino 45 formulae integrales occurrunt , si ex iis ut cognitae spectentur novem sequentes

$$(1,9) = \mathrm{A},\ (2,8) = \mathrm{B},\ (3,7) = \mathrm{C},\ (4,6) = \mathrm{D},\ (5,5) = \mathrm{E},$$
$$(1,8) = \mathrm{P},\ (2,7) = \mathrm{Q},\ (3,6) = \mathrm{R},\ (4,6) = \mathrm{S},$$

reliquae triginta sex ex his sequenti modo determinabuntur :

1.	$(9,9) = \frac{\mathrm{A}}{8\mathrm{P}}$	8.	$(9,2) = \frac{\mathrm{B}}{\mathrm{P}}$
2.	$(9,8) = \frac{\mathrm{B}}{7\mathrm{Q}}$	9.	$(1,1) = \frac{\mathrm{AP}}{\mathrm{B}}$
3.	$(9,7) = \frac{\mathrm{C}}{6\mathrm{R}}$	10.	$(1,2) = \frac{\mathrm{AQ}}{\mathrm{C}}$
4.	$(9,6) = \frac{\mathrm{D}}{5\mathrm{S}}$	11.	$(1,3) = \frac{\mathrm{AR}}{\mathrm{D}}$
5.	$(9,5) = \frac{\mathrm{E}}{4\mathrm{S}}$	12.	$(1,4) = \frac{\mathrm{AS}}{\mathrm{E}}$
6.	$(9,4) = \frac{\mathrm{D}}{3\mathrm{R}}$	13.	$(1,5) = \frac{\mathrm{AS}}{\mathrm{D}}$
7.	$(9,3) = \frac{\mathrm{C}}{2\mathrm{Q}}$	14.	$(1,6) = \frac{\mathrm{AR}}{\mathrm{C}}$
15.	$(1,7) = \frac{\mathrm{AQ}}{\mathrm{B}}$	26.	$(8,8) = \frac{\mathrm{BCP}}{6\mathrm{AQR}}$
16.	$(3,8) = \frac{\mathrm{BC}}{\mathrm{AQ}}$	27.	$(2,2) = \frac{\mathrm{ABQR}}{\mathrm{CDP}}$
17.	$(4,7) = \frac{\mathrm{CE}}{\mathrm{AR}}$	28.	$(2,3) = \frac{\mathrm{ABRS}}{\mathrm{DEP}}$
18.	$(5,6) = \frac{\mathrm{DE}}{\mathrm{AS}}$	29.	$(2,4) = \frac{\mathrm{ABSS}}{\mathrm{DEP}}$
19.	$(2,6) = \frac{\mathrm{AQR}}{\mathrm{CP}}$	30.	$(2,5) = \frac{\mathrm{ABRS}}{\mathrm{CDP}}$
20.	$(3,5) = \frac{\mathrm{ARS}}{\mathrm{DP}}$	31.	$(5,7) = \frac{\mathrm{CDEP}}{2\mathrm{ABRS}}$
21.	$(4,4) = \frac{\mathrm{ASS}}{\mathrm{EP}}$	32.	$(6,6) = \frac{\mathrm{DDEP}}{2\mathrm{ABSS}}$
22.	$(4,8) = \frac{\mathrm{CDP}}{2\mathrm{AQR}}$	33.	$(3,4) = \frac{\mathrm{ABRSS}}{\mathrm{DEPQ}}$
23.	$(5,8) = \frac{\mathrm{DEP}}{3\mathrm{ARS}}$	34.	$(6,7) = \frac{\mathrm{DDEPQ}}{3\mathrm{ABRSS}}$
24.	$(6,8) = \frac{\mathrm{DEP}}{4\mathrm{ASS}}$	35.	$(7,7) = \frac{\mathrm{CDEPQ}}{4\mathrm{ABRSS}}$
25.	$(7,8) = \frac{\mathrm{CDP}}{5\mathrm{ARS}}$	36.	$(3,3) = \frac{\mathrm{ABCRSS}}{\mathrm{DDEPQ}}$.

§.123. Eadem methodo, qua hic usi sumus pro casu $n = 10$, haud difficile erit ordines altiores evolvere; neque tamen hinc adhuc elucet, quanam lege omnes determinationes progrediantur, quandoquidem valores certarum formularum continuo magis evadunt complicati. Ceterum valores, quos hic invenimus, omnibus aequationibus in forma generali

$$(a,b)(a+b,c)=(a,c)(a+c,b)$$

contentis satisfacere deprehenduntur, ita ut perpetuo aequatio identica resultet neque idcirco inde ulla nova relatio inter litteras nostras maiusculas deduci queat. Tandem probe hic notasse iuvabit, quod in omnibus ordinibus praeter formulas a circulo pendentes commodissime eae formulae, quae in ordine proxime praecedente erant circulares, hic etiam tamquam cognitae accipi queant, quippe quibus determinationes omnes optimo successu perfici possunt.

3. (cont'd)

Methodus generalis determinandi valores formulae

$$(p,q)=\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-p}}},$$

a termino $x=0$ usque ad $x=1$ extensa: ubi praeter formulas circulum involventes, in quibus est $p+q=n$, etiam illae pro cognitis accipiuntur; in quibus est $p+q=n-1$.

I. Cum aequatio generalis, unde omnes hae determinationes sunt petendae, sit

$$(a,b)(a+b,c)=(a,c)(a+c,b),$$

sumatar primo $a=n-\alpha,\ b=\alpha,$ et $c=\alpha-1$, eritque aequatio

$$(n-\alpha,\alpha)(n,\alpha-1)=(n-\alpha,\alpha-1)(n-1,\alpha),$$

ubi est

$$(n,\alpha-1)=\tfrac{1}{\alpha-1}.$$

In primo autem factore ob $p=n-\alpha$ et $q=\alpha$ est $p+q=n$ ideoque datur. In tertio autem factore, ubi $p=n-\alpha$ et $q=\alpha-1$, est $p+q=n-1$, ideoque pariter datur. Hinc ergo colligimus

$$(n-1,\alpha)=\tfrac{1}{\alpha-1}\cdot\tfrac{(n-\alpha,\alpha)}{(n-\alpha,\alpha-1)},$$

ubi esse debet $\alpha>1$, ita ut pro α accipi queant omnes numeri a 2 usque ad $n-1$; at vero casu $\alpha=1$ valor formulae per se est notus.

II. In aequatione generali iam sumatur $a=\beta,\ b=n-\beta-1$ et $c=1$
eritque nostra aequatio

$$(\beta,\ n-\beta-1)(n-1,1)=(\beta,1)(\beta+1,n-\beta-1),,$$

ex qua aequatione colligitur

$$(\beta,1)=\tfrac{(\beta,\ n-\beta-1)(n-1,1)}{(\beta+1,n-\beta-1)},$$

ubi esse debet $\beta < n-1$, ita ut hinc omnes formulae $(\beta,1)$ definiantur a valore $\beta = 1$ usque ad $\beta = n-1$, quo posteriore casu formula $(n-1,1)$ per se cognoscitur.

III. Ut hinc etiam alias formas eliciamus, sumamus $a=1,\ b=n-2,\ c=\gamma$ ut oriatur haec aequatio

$$(1,n-2)(n-1,\gamma) = (1,\gamma)(1+\gamma,n-2),$$

ubi primus factor ac tertius dantur per II, secundus vero per I; unde quartus derivatur, scilicet

$$(1+\gamma,n-2) = \frac{(1,n-2)(n-1,\gamma)}{(1,\gamma)},$$

ubi valores ipsius $1+\gamma$ a 2 usque ad $n-2$ augeri possunt. Cum igitur per I sit

$$(n-1,\gamma) = \frac{1}{\gamma-1}\cdot\frac{(n-\gamma,\gamma)}{(n-\gamma,\gamma-1)},$$

tum vero per II sit

$$(\gamma,1) = \frac{(\gamma,n-\gamma-1)(n-1,1)}{(\gamma+1,n-\gamma-1)},$$

his valoribus substitutis fiet

$$(n-2,1+\gamma) = \frac{1}{\gamma-1}\cdot\frac{(1,n-2)(n-\gamma,\gamma)(\gamma+1,n-\gamma-1)}{(n-\gamma,\gamma-1)(\gamma,n-\gamma-1)(n-1,1)}.$$

IV. Sumamus nunc $a=1,\ b=n-3,\ c=\delta$ prodibitque haec aequatio

$$(1,n-3)(n-2,\delta) = (1,\delta)(1+\delta,n-3),$$

unde colligitur

$$(n-3,1+\delta) = \frac{(n-3,1)(n-2,\delta)}{(\delta,1)},$$

ubi ergo $1+\delta$ continet numeros $2,\ 3,\ 4,\ \ldots\ n-3$, ita ut hinc excludatur $(n-3,1)$, quae autem per II datur. At si valores ante reperti substituantur, fiet

$$(n-3,1+\delta) = \frac{1}{\delta-2}\cdot\frac{(n-3,2)(n-2,1)(n-\delta+1,\delta-1)(\delta,n-\delta)(\delta+1,n-\delta-1)}{(n-2,2)(n-\delta+1,\delta-2)(\delta-1,n-\delta)(n-1,1)(\delta,n-\delta-1)},$$

unde patet esse debere $\delta > 2$ eodemque modo pro praecedente formula $\gamma > 1$, ita ut hic excludantur casus $(n-3,1),(n-3,2)$, quorum quidem prior per II datur, alter vero per se.

V. Statuamus nunc $a=1, b=n-4$ et $c=\varepsilon$ prodibitque haec aequatio

$$\left(1,n-4\right)(n-3,\varepsilon)=\left(1,\varepsilon\right)\left(1+\varepsilon,n-4\right),$$

unde concluditur

$$\left(n-4,1+\varepsilon\right)=\frac{(n-4,1)(n-3,\varepsilon)}{(1,\varepsilon)};$$

ubi si loco $(n-3,\varepsilon)$ valor ante inventus substitueretur, factor absolutus ingrederetur $\frac{1}{\varepsilon-3}$, ita ut esse debeat $\varepsilon > 3$ ideoque $1+\varepsilon > 4$, unde hic excluduntur casus $(n-4,1),\ (n-4,2),\ (n-4,3)$, quorum quidem primus ex II, tertius autem per se datur, medius vero revera manet incognitus.

VI. Statuamus porro $a=1,\ b=n-5,\ c=\zeta$ et aequatio erit

$$\left(1,n-5\right)(n-4,\zeta)=(1,\zeta)(1+\zeta,n-5),$$

unde fit

$$(n-5,1+\zeta)=\frac{(n-5,1)(n-4,\zeta)}{(1,\zeta)},$$

ubi ob formulam $(n-4,\zeta)$ debet esse $\zeta > 4$ ideoque $1+\zeta > 5$, unde hinc excluduntur casus $(n-5,1),(n-5,2),(n-5,3),(n-5,4)$, quorum quidem primus ex II constat, quartus vero per se datur, ita ut hic occurrant duo casus etiamnunc incogniti $(n-5,2)$ et $(n-5,3)$.

VII. Simili modo si ulterius sumamus $a=1, b=n-6$ et $c=\eta$, prodibit

$$(n-6,1+\eta)=\frac{(n-6,1)(n-5,\eta)}{(1,\eta)},$$

ubi revera occurrunt tres sequentes casus $(n-6,2),(n-6,3),(n-6,4)$, qui adhuc manent incogniti, atque hoc modo progredi licebit, quousque necesse fuerit; unde patet numerum casuum incognitorum continuo augeri, ita ut terminorum *p* et *q* alter futurus sit vel 2 vel 3 vel 4 etc., qui igitur casus adhuc definiendi restant.

VIII. Sumamus nunc primo $a=1, b=\theta,\ c=1$, ut aequatio nostra fiat

$$(1,\theta)\left(1+\theta,1\right)=\left(1,1\right)(2,\theta),$$

unde concludimus

$$(2,\theta)=\frac{(1,\theta)(1+\theta,1)}{(1,1)},$$

quae formula iam omnes casus exclusos suppeditat, in quibus alter terminus erat 2.

IX. Deinde sumamus $a=2, b=\chi$ et $c=1$, ut aequatio prodeat

$$(2,\chi)(2+\chi,1)=(2,1)(3,\chi),$$

unde fit

$$(3,\chi)=\frac{(2,\chi)(2+\chi,1)}{(2,1)};$$

ubi cum $(2,\chi)$ per praecedentem numerum detur, nunc etiam ii casus innotescunt, ubi alter terminus erat 3.

X. Sumatur porro $a=3, b=\chi,\ c=1$ eritque

$$(3,\chi)(3+\chi,1)=(3,1)(4,\chi),$$

unde fit

$$(4,\chi)=\frac{(3,\chi)(3+\chi,1)}{(3,1)},$$

unde igitur ii casus eliciuntur, ubi alter terminus erat 4.

Eodem modo pro reliquis proceditur sicque omnes plane casus in formula proposita contenti plene sunt determinati.