SUPPLEMENT V*a*

TO CH. VIII, BOOK I.

CONCERNING THE VALUES OF INTEGRALS WHICH CAN TAKE ONLY CERTAIN CASES.

1.) A new method for determining the values of integrals.

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Pag. 66 -102. [E464 Series I, vol. 17, *O.O.*]

§.1. Since frequently differential formulas came to mind, which were to be divided by the logarithm of a variable quantity, such as $\frac{P\partial z}{lz}$, I was unable to see what kind of quantities the integrals of these should be referred to, since also indeed the values of these could perhaps only be seen to be assigned approximately. Which indeed is readily apparent, pertaining to the simplest integral formula of this kind, $\int\frac{\partial z}{lz}$, if I may consider that to be integrated thus, so that it may vanish on putting $z=0$, then truly a quantity of infinite magnitude is going to be produced if there may be put $z=1$; because if the variable z now may approach close to unity, so that $z=1-u$, with u becoming an infinitely small quantity, then on account of

$$\partial z=-\partial u \text{ and } lz=l(1-u)=-u,$$

this formula will be $\int\frac{\partial u}{u}$, of which the value certainly shall become infinite. But truly generally integral formulas of this kind $\int\frac{P\partial z}{lz}$ are given for that which, even if there may be put $z=1$, still give rise to quantities of finite magnitude: because with that determined it may seem to be worth all the more effort, as there is no known way of investigating these same values at present.

§. 2. We will consider this simple enough formula $\int\frac{(z-1)\partial z}{lz}$ as an example, which by the rule mentioned can be readily shown to have a finite value to be integrated. Indeed on putting $\frac{z-1}{lz}=y$, so that our formula may become $\int y\partial z$, and thus expresses the area of the curve for the abscissa z corresponding to the applied line $=y$, the same area extended from the limit $z=0$ as far as to the limit $z=1$ certainly will represent a finite value not much greater than $\frac{1}{2}$; indeed with the abscissa put $z=0$, also the applied line will become $y=0$, but with $z=1$, for the applied line $y=\frac{z-1}{lz}$ both the numerator as well as the denominator vanish, therefore with their differentials substituted in place of these, there becomes $y = z = 1$. But for the intermediate abscissas we may put $z = e^{-n}$, with the number *e* present, the hyperbolic logarithm of which is one; there will become

$$y = \frac{e^{-n}-1}{-n} = \frac{e^{n}-1}{ne^{n}},$$

which, if *n* were a very large number, so that the *z* may become minimal, the applied line will become approximately $y = \frac{1}{n}$; which value therefore will be much greater than the abscissa *z* ; evidently the form of this curve will be similar to the added figure, where A P may denote the abscissa *z* and P M the applied line *y,* truly the applied line $\mathrm{B\,C} = 1$ will correspond to the abscissa $\mathrm{A\,B} = 1$; from which curve described (Fig. l.) , from which describe curve its area A M C B will not exceed by much the area of the triangle A B C, which is $= \frac{1}{2}$.

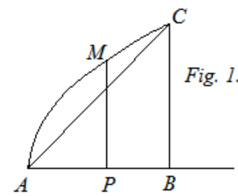

Fig. 1.

§. 3. But recently, occupied with other investigations [*vide* E463], besides the expected value found, this area is equal to the hyperbolic logarithm of 2, thus so that this shall be $l\,2 = 0,6931471805$, expressed as a decimal fraction ; moreover here I have been led by the following reasoning. Actually since there shall be $l\,z = \frac{z^{0}-1}{0}$, because on differentiation of each side there becomes $\frac{\partial z}{z} = \frac{\partial z}{z}$, and by taking $z = 1$ each expression will vanish, in place of 0 I will write $\frac{1}{i}$, with *i* denoting an infinite number, and there will become $l\,z = i\left(z^{\frac{1}{i}} - 1\right)$, and hence the applied line

$$y = \frac{z-1}{i\left(z^{\frac{1}{i}}-1\right)} = \frac{1-z}{i\left(1-z^{\frac{1}{i}}\right)},$$

and the integral formula

$$\int \frac{(1-z)\partial z}{i\left(1-z^{\frac{1}{i}}\right)}$$

Therefore now on putting $z^{\frac{1}{i}} = x$, so that there shall become $z = x^{i}$, where it may be observed, for each limit of the integration, $z = 0$ and $z = 1$ there shall also become $x = 0$ and $x = 1$; therefore since hence there shall become $\partial z = ix^{i-1}\partial x$, the integral formula appears

$$\int \frac{x^{i-1}\partial x\left(1-x^{i}\right)}{(1-x)},$$

which therefore will be required to be integrated from the limit $x = 0$ as far as to the limit $x = 1$.

§. 4. We will now regard *i* as a very large number, and the fraction $\frac{1-x^{i}}{1-x}$ is resolved into this geometric progression

$$1+x+x^2+x^3+x^4+x^5+x^6+\dots x^{i-1},$$

the individual terms of which multiplied and integrated at $x^{i-1}\partial x$ produce this series

$$\frac{x^i}{1}+\frac{x^{i+1}}{i+1}+\frac{x^{i+2}}{i+2}+\frac{x^{i+3}}{i+3}+\dots\dots\frac{x^{2i-1}}{2i-1},$$

which certainly will vanish on making $x=0$. Therefore now there may be taken $x=1$, and the value sought of our integral will be

$$\frac{1}{1}+\frac{1}{i+1}+\frac{1}{i+2}+\frac{1}{i+3}+\dots\dots\frac{1}{2i-1},$$

where indeed the letter *i* denotes an infinitely large number, thus so that the number of these terms actually shall be infinite. Truly with nothing less, because the individual terms are infinitely small, this series will have a finite sum, as it may be permitted to be reduced to an ordinary series in the following manner.

§. 5. The series found can be considered as the difference between the two following harmonic progressions:

$$\mathrm{A}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots\dots\frac{1}{2i-1}$$
$$\mathrm{B}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots\dots\frac{1}{i-1}$$

since the difference $\mathrm{A}-\mathrm{B}$ shows the same series found ; but because the number of terms of the series A is $2i-1$, truly of the series $\mathrm{B}=i-1$, that one is twice as great as this one, on account of which, so that we may obtain a regular series, we may take the individual terms of series B from series A by jumping to the second, fourth, sixth, eighth term, etc, of series A, with which agreed on likewise at the end each will be come upon, and there will be

$$\mathrm{A}-\mathrm{B}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\text{etc.}$$

to infinity, of which the value therefore is $l\,2$, thus so that now indeed certainly it may be shown, the value of the integral of the proposed formula $\int\frac{(z-1)\partial z}{lz}$, in the case $z=1$, actually to be $=l2$.

§. 6. By similar reasoning it is possible also to accommodate the more general formula $\int\frac{\left(z^m-1\right)\partial z}{lz}$, and finally its value will be found, in the case $z=1$, to become $l(m+1)$; as in a similar manner there will be

$$\int\frac{\left(z^n-1\right)\partial z}{lz}=l\left(n+1\right),$$

if we may subtract the one from the other, the integration will produce the following

$$\int \frac{\left(z^m - z^n\right)\partial z}{lz} = l\,\frac{m+1}{n+1},$$

evidently if the integration may be extended from the limit $z = 0$ as far as to the limit $z = 1$.

§. 7. But because this demonstration proceeds by infinitely large or small quantities, we would like another method both plane and customary, which may prevail to lead to these same sums; which investigation indeed will be seen to be a most arduous task. Yet meanwhile, since recently the consideration by me of functions involving two variables for the integration of singular differential formulas may have led directly, which by other methods were attempted in vain, from the same principle also I understood the integrations being derived here were shown here. Hence therefore the method will be set out as a new straight forwards source, from which integrations are allowed to be performed, inaccessible by other methods, which I will set out clearly and in a transparent manner, and which I have resolved especially to be the task of this same investigation.

Lemma I.

§. 8. If P were some function of the two variables *z* and *u*, and there may be put $\int \mathrm{P}\partial z = \mathrm{S}$, so that S also shall be a function of the two variables *z* and *u*, then there will be

$$\int \partial z \left(\frac{\partial \mathrm{P}}{\partial u}\right) = \left(\frac{\partial \mathrm{S}}{\partial u}\right)$$

Demonstration.

Since in the integration of the formula $\int \mathrm{P}\partial z$ *z* alone shall be regarded to be the variable, there will be $\left(\frac{\partial \mathrm{S}}{\partial z}\right) = \mathrm{P}$, which formula differentiated anew with *u* alone variable, presents $\left(\frac{\partial\partial \mathrm{S}}{\partial u \partial z}\right) = \left(\frac{\partial \mathrm{P}}{\partial u}\right)$, which multiplied by ∂z and integrated will produce $\left(\frac{\partial \mathrm{S}}{\partial u}\right) = \int \partial z \left(\frac{\partial \mathrm{P}}{\partial u}\right)$, since from the principles of integral calculus there becomes :

$$\int \partial z \left(\frac{\partial\partial \mathrm{S}}{\partial z \partial u}\right) = \left(\frac{\partial \mathrm{S}}{\partial u}\right) \text{ q. e. d.}$$

Corollary I.

§. 9. In the same manner through differentiations of this kind, where *u* only is considered for the variable, it will be allowed to progress further, from which the following integrations arise :

$$\left(\frac{\partial\partial S}{\partial u^2}\right)=\int\partial z\left(\frac{\partial\partial P}{\partial u^2}\right)$$

and

$$\left(\frac{\partial^3 S}{\partial u^3}\right)=\int\partial z\left(\frac{\partial^3 P}{\partial u^3}\right)$$

etc. etc.

Corollary II.

§. 10. But if the formula $\int P\partial z$ therefore were integrable, thus so that its integral S was able to be shown, then also all these integral formulas

$$\int\partial z\left(\frac{\partial P}{\partial u}\right),\ \int\partial z\left(\frac{\partial\partial P}{\partial u^2}\right),\ \int\partial z\left(\frac{\partial^3 P}{\partial u^3}\right)\text{ etc.}$$

will be allowed to be integrated, and thus the integrals themselves will be able to be shown.

Scholium.

§. 11. From these formulas, if they may be treated in general, little of use will found in the calculation of integrals. But if the function P were to be prepared thus, so that the integral $\int P\partial z$, at least in a particular case, where after integration a certain reliable value is attributed to the variable *z,* for example $z=a$, it may be shown conveniently, that in this case the quantity S shall become a simple enough function of the variable *u*, so that the integrations likewise mentioned above can be performed: if indeed after the individual integrations there may be put $z=a$, and hence a generally principle arises for integrations of this kind , which indeed hardly ever or indeed never could be completed by other methods, and hence there arises :

First Principle of the Integration.

§. 12. If P were a function of this kind of the two variables *z* and *u*, so that the value of the integral $\int P\partial z$ at any rate may be able to be expressed with certainty in the case $z=a$, which value shall be $=S$, clearly a function of *u* only; then also the following integrals will be able to be shown conveniently, if indeed $z=a$ may equally be put in place after the integration, evidently

$$\int P\partial z = S$$
$$\int \partial z\left(\frac{\partial P}{\partial u}\right) = \left(\frac{\partial S}{\partial u}\right)$$
$$\int \partial z\left(\frac{\partial\partial P}{\partial u^2}\right) = \left(\frac{\partial\partial S}{\partial u^2}\right)$$
$$\int \partial z\left(\frac{\partial^3 P}{\partial u^3}\right) = \left(\frac{\partial^3 S}{\partial u^3}\right)$$
$$\int \partial z\left(\frac{\partial^4 P}{\partial u^2}\right) = \left(\frac{\partial^4 S}{\partial u^2}\right)$$
etc. etc.

Example I.

§. 13. If there were $P = z^u$, in general indeed there will be:

$$\int P\partial z = \frac{z^{u+1}}{u+1};$$

from which in the case $z = 1$ this simple enough value arises $\frac{1}{u+1}$ thus so that there shall be $S = \frac{1}{u+1}$; since then by continual differentiations, while *u* alone will be had variable, there may be produced $\left(\frac{\partial P}{\partial u}\right) = z^u lz$, then truly $\left(\frac{\partial\partial P}{\partial u^2}\right) = z^u \left(lz\right)^2$, again

$$\left(\frac{\partial^3 P}{\partial u^3}\right) = z^u \left(lz\right)^3, \ \left(\frac{\partial^4 P}{\partial u^4}\right) = z^u \left(lz\right)^4, \text{ etc.}$$

hence the following integral values are obtained, if indeed after the individual integrations there my be put $z = 1$

$\int z^u \partial z = +\frac{1}{u+1}$	$\int z^u \partial z\left(lz\right)^4 = +\frac{1.2.3.4}{(u+1)^5}$
$\int z^u \partial z lz = -\frac{1}{(u+1)^2}$	$\int z^u \partial z\left(lz\right)^5 = -\frac{1.2.3.4.5}{(u+1)^6}$
$\int z^u \partial z\left(lz\right)^2 = +\frac{1\cdot 2}{(u+1)^3}$	$\int z^u \partial z\left(lz\right)^6 = +\frac{1.....6}{(u+1)^7}$
$\int z^u \partial z\left(lz\right)^3 = -\frac{1\cdot 2\cdot 3}{(u+1)^4}$	$\int z^u \partial z\left(lz\right)^7 = -\frac{1.....7}{(u+1)^8}$

from which we may conclude to become, generally :

$$\int z^u \partial z\left(lz\right)^n = \pm\frac{1.2.3.4....n}{(u+1)^{n+1}},$$

where the sign + prevails if n shall be an even number, the other truly − if n shall be odd. Certainly these integrations are known now well enough from elsewhere, which is not to be wondered at, since we have assumed a very simple formula for P : therefore we may repeat these cases, which I have worked on quite recently [see E463].

Example 2.

§.14. If there were

$$\mathrm{P}=\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}},$$

now recently I have shown, the value of the integral formula $\int \mathrm{P}\partial z$, where after the integration there may be $z=1$, to be

$$\mathrm{S}=\frac{\pi}{2n\cos.\frac{\pi u}{2n}}.$$

Hence therefore since there shall be

$$\left(\frac{\partial \mathrm{P}}{\partial u}\right)=\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}lz,$$

then truly

$$\left(\frac{\partial \mathrm{P}}{\partial u}\right)=\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}lz, \text{ et}$$

$$\left(\frac{\partial\partial \mathrm{P}}{\partial u^2}\right)=\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}(lz)^2,$$

$$\left(\frac{\partial^3 \mathrm{P}}{\partial u^3}\right)=\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}(lz)^3$$

etc. etc.

from the known value of S the following integrations have been obtained :

$$\text{I.}\quad \int\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z=\mathrm{S}=\frac{\pi}{2n\cos.\frac{\pi u}{2n}}$$

$$\text{II.}\quad \int\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z lz=\left(\frac{\partial \mathrm{S}}{\partial u}\right)$$

$$\text{III.}\quad \int\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z(lz)^2=\left(\frac{\partial\partial \mathrm{S}}{\partial u^2}\right)$$

$$\text{IV.}\quad \int\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z(lz)^3=\left(\frac{\partial^3 \mathrm{S}}{\partial u^3}\right)$$

$$\text{V.}\quad \int\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z(lz)^4=\left(\frac{\partial^4 \mathrm{S}}{\partial u^4}\right)$$

etc. etc.

Example 3.

§.15. If there were

$$P = \frac{z^{n-u-1} - z^{n+u-1}}{1-z^{2n}},$$

I have shown in a similar manner, the value of the integral formula $\int P\partial z$, in the case where $z = 1$ after the integration, to become

$$S = \frac{\pi}{2n} \operatorname{tang.} \frac{\pi u}{2n};$$

and hence the following integrations were deduced for the same case $z = 1$:

$$\text{I.} \quad \int \frac{z^{n-u-1} - z^{n+u-1}}{1-z^{2n}} \partial z = S = \frac{\pi}{2n} \operatorname{tang.} \frac{\pi u}{2n}$$

$$\text{II.} \quad \int \frac{-z^{n-u-1} - z^{n+u-1}}{1-z^{2n}} \partial z lz = \left(\frac{\partial S}{\partial u}\right)$$

$$\text{III.} \quad \int \frac{z^{n-u-1} - z^{n+u-1}}{1-z^{2n}} \partial z \left(lz\right)^2 = \left(\frac{\partial\partial S}{\partial u^2}\right)$$

$$\text{IV.} \quad \int \frac{-z^{n-u-1} - z^{n+u-1}}{1-z^{2n}} \partial z \left(lz\right)^3 = \left(\frac{\partial^3 S}{\partial u^3}\right)$$

$$\text{V.} \quad \int \frac{z^{n-u-1} - z^{n+u-1}}{1-z^{2n}} \partial z \left(lz\right)^4 = \left(\frac{\partial^4 S}{\partial u^4}\right)$$

etc. etc.

Scholium.

§. 16. Therefore so that we can expect more benefits to emerge from this principle, the main concern is reduced here to this, that we may investigate for P more functions of this kind of the two variables z and u , thus so that the value of the integral formula may be assigned at any rate certainly ready, for example in the case $z = 1$, in order that it may be used with the examples brought forwards. But since this principle has been deduced from continued differentiation, thus in the same manner by continued integration it will be able to be adapted to our use.

Lemma II.

§. 17. *If P were a function of the two variables z and u, and there may be put* $\int P\partial z = S$ *, so that S also shall be a function of the two variables z and u, then there will be* $\int S\partial u = \int \partial z \int P\partial u$ *, where in the integrable formulas* $\int P\partial u$ *and* $\int S\partial u$ *only u will be had as variable, but in the formula* $\int \partial z \int P\partial u$ *only z.*

Demonstration.

There may be put $\int P\partial u = V$ and there shall become $S = \left(\frac{\partial V}{\partial u}\right)$ and thus $\left(\frac{\partial V}{\partial u}\right) = \int P\partial z$, and there will become $\left(\frac{\partial\partial V}{\partial z\partial u}\right)$; from which on multiplying by ∂u and by integrating there will become $\left(\frac{\partial V}{\partial z}\right) = \int Pdu$, from which it follows that $V = \int \partial z \int P\partial u = \int S\partial u$. q.e.d.

Corollary 1.

§.18. The integral can also be returned in this manner, from which the equation of such may arise

$$\int \partial u \int S\partial u = \int \partial z \int \partial u \int P\partial u \text{ ;}$$

hence moreover it can be expected of little use generally, unless perhaps these integrations follow conveniently.

Corollary 2.

§.19. Because if the formula $\int P\partial z$ were integrable, clearly $= S$, hence the other deduced $\int \partial z \int P\partial u$ will be able to be integrated only to the extent, in as much as the integral $\int S\partial u$ will be allowed to be integrated.

Second principle of the integration.

§. 20. If P were a function of the two variables *z* and *u* of this kind, so that certainly the value of the integral formula $\int P\partial z$ in the case at least, by putting $z = a$, may be able to be shown conveniently, thus so that in this case the quantity S may become a function of the variable *u* only; then also for the same case $z = a$ the value of this integral formula $\int \partial z \int P\partial u$ will be able to be assigned, but only if the formula $\int S\partial u$ were allowed to be integrated.

Example I.

§. 21. We may take $P = z^u$, and there will become $\int P\partial z = \frac{z^{u+1}}{u+1}$; which formula in the case $z = 1$ will be changed into $\frac{1}{u+1}$, which therefore may be written in place of S. Then truly since there is

$$\int P\partial u = \int z^u \partial u = \frac{z^u}{lz}$$

and because

$$\int S\partial u = l(u+1)\text{, there becomes}$$

$$\int \frac{z^u \partial z}{lz} = l(u+1)$$

if indeed after that integration there may be put $z=1$. But because all integration demands the addition of a constant, here rather it will be required to put in place

$$\int \frac{z^u \partial z}{lz} = l(u+1) + C;$$

and here indeed it is easily understood, this constant C must become infinite, since in the formula of the integral the fraction $\frac{z^u}{lz}$ on putting $z=1$ shall become infinite, thus so that hence little may be seen to follow from our setting up of the integral.

Corollary 1.

§. 22. But because this constant C does not depend on the variable *u*, that will retain the same value whatever the numbers determined for *u* may be taken. Therefore in the first place we may take $u=m$, then truly also $u=n$, so that we may have these values

$$\text{I. } \int \frac{z^m \partial z}{lz} = l(m+1) + C \text{ and}$$

$$\text{II. } \int \frac{z^n \partial z}{lz} = l(n+1) + C,$$

of which the one taken from the other leaves the same most noteworthy integration

$$\int \frac{(z^m - z^n)\partial z}{lz} = l\,\frac{m+1}{n+1},$$

just as now we have shown above long since from other principles.

Corollary 2.

§. 23. If we may rise to the other integration, which is $\int P\partial u = \frac{z^u}{lz}$, there will become $\int \partial u \int P\partial u = \frac{z^u}{(lz)^2}$; then truly on account of $\int S\partial u = l(u+1) + C$, there will become

$$\int \partial u \int S\partial u = (u+1)\left[l(u+1) - 1\right] + Cu + D,$$

and thus we will have

$$\int \frac{z^u \partial z}{(lz)^2} = (u+1)\left[l(u+1) - 1\right] + Cu + D,$$

where the constants C and D do not depend on *u* pendent: whereby so that we may eliminate these we will expand three determined cases :

$$\text{I. } \int \frac{z^m \partial z}{(lz)^2} = (m+1)l(m+1) - m - 1 + \mathrm{C}m + \mathrm{D},$$

$$\text{II. } \int \frac{z^n \partial z}{(lz)^2} = (n+1)l(n+1) - n - 1 + \mathrm{C}n + \mathrm{D}, \quad .$$

$$\text{III. } \int \frac{z^k \partial z}{(lz)^2} = (k+1)l(k+1) - k - 1 + \mathrm{C}k + \mathrm{D},$$

and there will be :

$$\text{I} - \text{III} = (m+1)l(m+1) - (k+1)l(k+1) + k - m + \mathrm{C}(m-k)$$

and

$$\text{II} - \text{III} = (n+1)l(n+1) - (k+1)l(k+1) + k - n + \mathrm{C}(n-k),$$

and hence we deduce:

$$(\text{I} - \text{III})(n-k) - (\text{II} - \text{III})(m-k) = \begin{cases} +(m+1)(n-k)l(m+1) \\ -(k+1)(n-k)l(k+1) + (k-m)(n-k) \\ -(n+1)(m-k)l(n+1) - (k-n)(m-k) \\ +(k+1)(m-k)l(k+1) \end{cases}$$

and hence we arrive at the following integration

$$\int \frac{\partial z\left[(n-k)z^m - (m-k)z^n + (m-n)z^k\right]}{(lz)^2} = +(m+1)(n-k)l(m+1)$$
$$-(n+1)(m-k)l(n+1)$$
$$+(k+1)(m-n)l(k+1).$$

Corollary 3.

§. 24. It will be worth the effort to set out some cases, where indeed it is agreed to take the numbers *m, n* and *k* to be unequal to each other, because otherwise all the terms become zero.

I. Therefore let $m = 2,\ n = 1$ and $k = 0$; there will become:

$$\int \frac{(z-1)^2 \partial z}{(\partial z)^2} = 3l3 - 4l2 = l\,\tfrac{27}{16}.$$

II. Let $m=3,\ n=1$ and $k=0$, and there will become

$$\int\frac{\left(z^3-3z+2\right)\partial z}{(\partial z)^2}=\int\frac{(z-1)^2(z+2)\partial z}{(\partial z)^2}=4l4-6l2=2l2=l4.$$

III. Let $m=3,\ n=2$ and $k=0$, and there becomes

$$\int\frac{\left(2z^3-3zz+1\right)\partial z}{(\partial z)^2}=\int\frac{(z-1)^2(2z+1)\partial z}{(\partial z)^2}=8l4-9l3=l\,\tfrac{4^8}{3^9}.$$

IV. Let $m=3,\ n=2$ and $k=1$, and there is produced

$$\int\frac{\left(z^3-3zz+z\right)\partial z}{(\partial z)^2}=\int\frac{z\partial z\left(z-1\right)^2}{(\partial z)^2}=4l4-6l3+2l2=l\,\tfrac{2^{10}}{3^6}.$$

Corollary 4.

§. 25. In these cases it comes to mind to be noteworthy, that the numerator in the integral formulas has the factor $(z-1)^2$, which thus arises by necessity to be used , lest the values of the integrals may become infinite. Because indeed the denominator $(lz)^2$ vanishes in the case $z=1$, if we may put $z=1-\omega$ with ω being infinitely small, there will be

$$lz=-\omega \text{ and } (lz)^2=+\omega\omega.$$

Therefore it is necessary that a factor shall be present in the numerator, which in the case $z=1-\omega$ likewise presents the same $\omega\omega$, which arises if the factor there were $(z-1)^2$.

Scholium.

§. 26. The integration which we have come upon in the first corollary, therefore may be considered worthy of every attention, because the integral values thence obtained in the case $z=1$ I would not be able to assign in any other way at present, even if they may be expressed much simpler by logarithms. But truly the integrations found in the second corollary, even if they may appear to be much harder, yet are able to be derived without difficulty from the previous ones with the aid of the known reductions ; that which may suffice to be shown for a single case. We may put

$$\int\frac{(z-1)^2\partial z}{(lz)^2}=\frac{p}{lz}+\int\frac{q\partial z}{lz}$$

and by differentiation there will become :

$$\frac{(z-1)^2\partial z}{(lz)^2}=\frac{\partial p}{lz}-\frac{p\partial z}{z(lz)^2}+\frac{q\partial z}{lz},$$

from which, with the equality of the terms themselves either divided by $(lz)^2$ or by lz, we will have these two equations :

$$(z-1)^2=-\tfrac{p}{z}\text{ and }\partial p=-q\partial z,$$

from the first of which there arises $p=-z(z-1)^2$ and hence

$$\tfrac{\partial p}{\partial z}=-3zz+4z-1\text{ and thus }q=3zz-4z+1,$$

thus so that there shall become

$$\int\frac{(z-1)^2\partial z}{(lz)^2}=\frac{-z(z-1)^2}{lz}+\int\frac{(3zz-4z+1)\partial z}{lz};$$

but here the first term vanishes at once on putting $z=1$; indeed on putting $z=1-\omega$, so that there shall be $lz=-\omega$, there will become $p=-\omega\omega(1-\omega)$, and thus $\frac{p}{lz}=\omega(1-\omega)=0$, on account of $\omega=0$: truly the latter term can be split up into these parts :

$$3\int\frac{(zz-z)\partial z}{lz}-\int\frac{(z-1)\partial z}{lz}.$$

But the integral of the first part is $3l\frac{3}{2}$, of the latter truly $-1l2$; and thus the total integral will become

$$3l\tfrac{3}{2}-l2=3l3-4l2=l\tfrac{27}{16},$$

just as we have found. Therefore in this manner, if in general we may put :

$$\int\frac{\mathrm{V}\partial z}{(lz)^2}=\frac{p}{lz}+\int\frac{q\partial z}{lz},$$

by differentiation there will :

$$\frac{\mathrm{V}\partial z}{(lz)^2}=\frac{\partial p}{lz}-\frac{p\partial z}{z(lz)^2}+\frac{q\partial z}{lz},$$

from which these two equalities emerge :

$$p=-\mathrm{V}z,\text{ and }q=-\tfrac{\partial p}{\partial z}.$$

Now so that the term $\frac{p}{lz}$ may vanish on putting $z=1$, the numerator p must have the factor $(z-1)^2$; which therefore also must be a factor of the quantity V. Therefore there shall become :

$\mathrm{V}=\frac{\mathrm{U}(z-1)^2}{z}$, and there will be $p=-\mathrm{U}(z-1)^2$,
from which there becomes

$$\partial p=\partial \mathrm{U}(z-1)^2-2\mathrm{U}\partial z(z-1)=(z-1)\left[\partial \mathrm{U}(z-1)-2\mathrm{U}\partial z\right],$$

and hence there shall be

$$q\partial z=(z-1)[2\mathrm{U}\partial z-\partial \mathrm{U}(z-1)];$$

therefore because q has the factor $z-1$, the formula $\int\frac{q\partial z}{lz}$ can always be resolved into two parts, the integrals of which are allowed to be assigned by the first corollary, but only if U were collected together from certain powers of *z*; from which the following theorem is deduced.

Theorem.

§. 27. If there were

$$\mathrm{P}=\mathrm{A}z^{\alpha}+\mathrm{B}z^{\beta}+\mathrm{C}z^{\gamma}+\mathrm{D}z^{\delta}+\text{etc.}$$

thus so that the sum of the coefficients

$$\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\text{etc.}=0,$$

then there will be

$$\int\frac{\mathrm{P}\partial z}{lz}=\mathrm{A}l(\alpha+1)+\mathrm{B}l(\beta+1)+\mathrm{C}l(\gamma+1)+\mathrm{D}l(\delta+1)+\text{etc.}.$$

if indeed after the integration there may be put $z=1$.

Demonstration.

Since this case itself, where after the integration there is put $z=1$, shall become

$$\int\frac{z^n\partial z}{lz}=l(n+1)+\Delta$$

with Δ denoting that infinite integration constant introduced, there will be

$$\mathrm{A}\int\frac{z^{\alpha}\partial z}{lz}=\mathrm{A}l(\alpha+1)+\mathrm{A}\Delta$$

and in the same manner there will be :

$$\mathrm{B}\int\frac{z^{\alpha}\partial z}{lz}=\mathrm{B}l(\beta+1)+\mathrm{B}\Delta$$

etc; etc.

if now all these integrations may be gathered together into one sum, there will become, on account of

$$\left(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\text{etc.}\right)\Delta=0,$$

the integration sought:

$$\int\tfrac{\mathrm{P}\partial z}{lz}=\mathrm{A}l(\alpha+1)+\mathrm{B}l\left(\beta+1\right)+\mathrm{C}l\left(\gamma+1\right)+\mathrm{D}l\left(\delta+1\right)+\text{etc.}$$

q. e. d .

Corollary 1.

§. 28. Because we assume

$$\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\text{etc.}=0,$$

it is evident, the formula

$$\mathrm{P}=\mathrm{A}z^{\alpha}+\mathrm{B}z^{\beta}+\mathrm{C}z^{\gamma}+\mathrm{D}z^{\delta}+\text{etc.}$$

must have the factor $z-1$, just as we have observed above.

Corollary 2.

§. 29. Because there is

$$\left(z-1\right)^{n}=z^{n}-\tfrac{n}{1}z^{n-1}+\tfrac{n(n-1)}{1\cdot2}z^{n-2}-\tfrac{n(n-1)(n-2)}{1\cdot2\cdot3}z^{n-3}+\text{etc.},$$

with this value in place of P there will become

$$\mathrm{A}=1\ \text{ and }\ \alpha=n,$$

then

$$\mathrm{B}=-\tfrac{n}{1}\ \text{ and }\beta=n-1,$$

again

$$\mathrm{C}=\tfrac{n(n-1)}{1\cdot2}\ \text{ and }\gamma=n-2\quad\text{etc.}$$

hence there becomes therefore:

$$\int\tfrac{(z-1)^{n}\partial z}{lz}=l\left(n+1\right)-\tfrac{n}{1}ln+\tfrac{n(n-1)}{1\cdot2}l\left(n-1\right)-\tfrac{n(n-1)(n-2)}{1\cdot2\cdot3}l\left(n-2\right)$$
$$+\tfrac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}l\left(n-3\right)-\text{etc.},$$

but if the exponent *n* at least were no greater nor less than unity, because otherwise in the case $z=1$ the fraction would become infinite ; but this does not prevent the region considered above to become finite, thus so that it may suffice, provided there shall be $n>0$.

Example 2

§. 30. Let

$$\mathrm{P} = \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}; \text{there will be}$$

$$\int \mathrm{P}\partial z = \frac{\pi}{2n}\cos.\frac{\pi u}{2n},$$

if indeed there may be put $z=1$ after the integration, which value therefore we attribute to the letter S . Now with *z* considered as a constant:

$$\int \mathrm{P}\partial u = \frac{1}{1+z^{2n}}\left(\int z^{n-u-1}\partial u + \int z^{n+u-1}\partial u\right)$$

and thus

$$\int \mathrm{P}\partial u = \frac{-z^{n-u-1}+z^{n+u-1}}{\left(1+z^{2n}\right)lz},$$

from which there becomes:

$$\int \mathrm{S}\partial u = \int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\cdot\frac{\partial z}{lz};$$

but since there shall be $\cos.\frac{\pi u}{2n} = \sin.\frac{\pi(n-u)}{2n}$, the equation becomes :

$$\int \mathrm{S}\partial u = \int \frac{\pi\partial u}{2n\sin.\frac{\pi(n-u)}{2n}};$$

hence, if we may put $\frac{\pi(n-u)}{2n} = \varphi$, there will be $\partial\varphi = -\frac{\pi\partial u}{2n}$ and thus

$$\int \mathrm{S}\partial u = -\int \frac{\partial\varphi}{\sin.\varphi} = -l\text{tang.}\tfrac{1}{2}\varphi,$$

on account of which we will have:

$$\int \mathrm{S}\partial u = -l\text{ tang.}\frac{\pi(n-u)}{4n},$$

thus so that on putting $z=1$ after the integration we shall have reached this integration

$$\int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\cdot\frac{\partial z}{lz} = -l\text{ tang.}\frac{\pi(n-u)}{4n} = +l\text{ tang.}\frac{\pi(n+u)}{4n}.$$

Example 3

§. 31. Let

$$P = \frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}; \text{ there will become}$$

$$\int P\partial z = \frac{\pi}{2n}\,\text{tang.}\frac{\pi u}{2n} = S,$$

from which there becomes :

$$\int S\partial u = -l\cos.\frac{\pi u}{2n};$$

hence, since there shall be

$$\int P\partial u = \frac{-z^{n-u-1}-z^{n+u-1}}{\left(1-z^{2n}\right)lz},$$

we obtain the following integration, if indeed the integral may be extended from the limit $z = 0$ as far as to the limit $z = 1$,

$$\int \frac{z^{n-u-1}+z^{n+u-1}}{1-z^{2n}}\cdot\frac{\partial z}{lz} = +l\cos.\frac{\pi u}{2n}.$$

Now indeed I have already established these two latter examples fully [*vide* E463]; so that I shall not tarry longer setting these out, but I shall progress to the following problem.

Problem.

§. 32. *If these two series may be proposed* :

$$P = z\cos.u + z^2\cos.2u + z^3\cos.3u + z^4\cos.4u + z^5\cos.5u + \text{ etc.}$$

and

$$Q = z\sin.u + z^2\sin.2u + z^3\sin.3u + z^4\sin.4u + z^5\sin.5u + \text{etc.}$$

which depend on the two variables z and u, to find the relations between the integral formulas $\int\frac{P\partial z}{z}, \int P\partial u$ *and* $\int\frac{Q\partial z}{z}, \int Q\partial u$, *and with integral formulas thence arising from continued integrations.*

Solution.

Since each series shall be recurring, there will be found by the finite formulas :

$$P = \frac{z\cos.u - zz}{1-2z\sin.u+zz} \text{ et } Q = \frac{z\sin.u}{1-2z\sin.u+zz},$$

from which there shall become

$$\int \frac{P\partial z}{z} = \frac{\partial z\cos.u - z\partial z}{1-2z\sin.u+zz} = -l\sqrt{(1-2z\cos.u+zz)}$$

and

$$\int Q\partial u = \int \frac{z\partial u\sin.u}{1-2z\sin.u+zz} = +l\sqrt{(1-2z\cos.u+zz)},$$

thus so that there shall be :

$$\int \frac{P\partial z}{z} = -\int Q\partial u \ ;$$

while truly there will become also

$$\int \frac{Q\partial z}{z} = \int \frac{\partial z\sin.u}{1-2z\sin.u+zz} = \text{arc.tang.}\frac{z\sin.u}{1-z\cos.u} \ ;$$

but if this arc may be differentiated with only the angle *u* taken to be variable, there will become

$$\frac{1}{\partial u}\text{arc.tang.}\frac{z\sin.u}{1-z\cos.u} = \frac{z\cos.u-zz}{1-2z\sin.u+zz},$$

thus so that there shall become

$$\int \frac{Q\partial z}{z} = \int P\partial u.$$

§. 33. Truly these same relations are derived more easily from these series themselves: for since there shall be

$$P = z\cos.u + z^2\cos.2u + z^3\cos.3u + z^4\cos.4u + \text{etc.},$$

there will be

$$\int \frac{P\partial z}{z} = \frac{z\cos.u}{1} + \frac{zz\cos.2u}{2} + \frac{z^3\cos.3u}{3} + \text{etc.}$$

and

$$\int P\partial u = \frac{z\sin.u}{1} + \frac{zz\sin.2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc.}$$

and because there is

$$Q = z\sin.u + zz\sin.2u + z^3\sin.3u + \text{etc.},$$

there will be

$$\int \frac{Q\partial z}{z} = \frac{z\sin.u}{2} + \frac{zz\sin.2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc.}$$

and

$$\int Q\partial u = -\frac{z\cos.u}{2} - \frac{zz\cos.2u}{2} - \frac{z^3\cos.3u}{3} - \text{etc.}$$

from which clearly there shall become

$$\int \frac{P\partial z}{z} = -\int Q\partial u \text{ and } \int \frac{Q\partial z}{z} = \int P\partial u.$$

§. 34. So that it may be allowed to progress further in this manner, we may put in place for brevity :

$$P' = \frac{z\cos.u}{1} + \frac{zz\cos.2u}{2} + \frac{z^3\cos.3u}{3} + \text{etc. and } Q' = \frac{z\sin.u}{1} + \frac{zz\sin.2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc.}$$

$$P'' = \frac{z\cos.u}{1^2} + \frac{zz\cos.2u}{2^2} + \frac{z^3\cos.3u}{3^2} + \text{etc. and } Q'' = \frac{z\sin.u}{1^2} + \frac{zz\sin.2u}{2^2} + \frac{z^3\sin.3u}{3^2} + \text{etc.}$$

$$P''' = \frac{z\cos.u}{1^3} + \frac{zz\cos.2u}{2^3} + \frac{z^3\cos.3u}{3^3} + \text{etc. and } Q''' = \frac{z\sin.u}{1^3} + \frac{zz\sin.2u}{2^3} + \frac{z^3\sin.3u}{3^3} + \text{etc.}$$

$$P'''' = \frac{z\cos.u}{1^4} + \frac{zz\cos.2u}{2^4} + \frac{z^3\cos.3u}{3^4} + \text{etc. and } Q''' = \frac{z\sin.u}{1^4} + \frac{zz\sin.2u}{2^4} + \frac{z^3\sin.3u}{3^4} + \text{etc.}$$

etc. etc. etc. etc.

and hence the comparisons found before will be continued :

$$P' = \int \frac{P\partial z}{z} = -\int Q\partial u, \quad Q' = \int \frac{Q\partial z}{z} = \int P\partial u,$$

$$P'' = \int \frac{P'\partial z}{z} = -\int Q'\partial u, \quad Q'' = \int \frac{Q'\partial z}{z} = \int P'\partial u,$$

$$P''' = \int \frac{P''\partial z}{z} = -\int Q''\partial u, \quad Q''' = \int \frac{Q''\partial z}{z} = \int P''\partial u,$$

$$P'''' = \int \frac{P'''\partial z}{z} = -\int Q'''\partial u, \quad Q'''' = \int \frac{Q'''\partial z}{z} = \int P'''\partial u,$$

etc. etc. etc. etc.

from which many conspicuous relations can be deduced.

§. 35. But these relations are the especially noteworthy and adapted according to our manner of presentation, where integral formulas, in which *z* alone is variable, are reduced to other integral formulas, in which *u* only is variable; the following are of this kind :

$$P' = \int \frac{P\partial z}{z} = -\int Q\partial u,$$

$$P'' = \int \frac{\partial z}{z}\int \frac{P\partial z}{z} = +\int \partial u\int P\partial u,$$

$$P''' = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{P\partial z}{z} = +\int \partial u\int \partial u\int Q\partial u,$$

$$P'''' = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{P\partial z}{z} = -\int \partial u\int \partial u\int \partial u\int Q\partial u,$$

$$P^{V} = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{P\partial z}{z} = -\int \partial u\int \partial u\int \partial u\int \partial u\int Q\partial u,$$

etc. etc. etc.

And again in a similar manner for the other kind :

$$\mathrm{Q'} = \int \frac{\mathrm{Q}\partial z}{z} = +\int \mathrm{P}\partial u,$$
$$\mathrm{Q''} = \int \frac{\partial z}{z}\int \frac{\mathrm{Q}\partial z}{z} = -\int \partial u\int \mathrm{Q}\partial u,$$
$$\mathrm{Q'''} = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{Q}\partial z}{z} = -\int \partial u\int \partial u\int \mathrm{P}\partial u,$$
$$\mathrm{Q''''} = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{Q}\partial z}{z} = +\int \partial u\int \partial u\int \partial u\int \mathrm{Q}\partial u,$$
$$\mathrm{Q^{V}} = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{Q}\partial z}{z} = +\int \partial u\int \partial u\int \partial u\int \partial u\int \mathrm{P}\partial u,$$

etc. etc. etc.

§. 36. But if now we may wish only the values of our series, or what amounts to the same thing, the quantities

$$\mathrm{P, P', P'', P''', P''''}, \text{etc. and } \mathrm{Q, Q', Q'', Q''', Q''''}, \text{etc.},$$

which are obtained on putting $z = 1$, this we may pursue conveniently, so that in the integral formulas, where only the angle *u* may be had for the variable, immediately before the integration there may be put $z = 1$, moreover with this done there will be

$$\mathrm{P} = \frac{\cos.u-1}{2-2\cos.u} = -\frac{1}{2} \text{ and } \mathrm{Q} = \frac{\sin.u}{2-2\cos.u} = \frac{1}{2}\cot.\tfrac{1}{2}u,$$

then again truly

$$\int \mathrm{P}\partial u = \mathrm{A} - \tfrac{1}{2}u,$$
$$\int \partial u\int \mathrm{P}\partial u = \mathrm{B} + \mathrm{A}u - \tfrac{1}{4}uu,$$
$$\int \partial u\int \partial u\int \mathrm{P}\partial u = \mathrm{C} + \mathrm{B}u + \tfrac{1}{2}\mathrm{A}uu - \tfrac{1}{12}u^3,$$
$$\int \partial u\int \partial u\int \partial u\int \mathrm{P}\partial u = \mathrm{D} + \mathrm{C}u + \tfrac{1}{2}\mathrm{B}uu + \tfrac{1}{6}\mathrm{A}u^3 - \tfrac{1}{48}u^4;$$

but with the formulas, where Q is present, the calculation does not occur quite so neatly; indeed there will be

$$\mathrm{Q} = \tfrac{1}{2}\cot.\tfrac{1}{2}u,$$
$$\int \mathrm{Q}\partial u = l\sin.\tfrac{1}{2}u,$$
$$\int \partial u\int \mathrm{Q}\partial u = \int \partial u l\sin.\tfrac{1}{2}u,$$

since which formula rejecting all integration, scarcely may be able to be integrated further ; yet meanwhile there will be

$$\int \partial u\int \partial u\int \mathrm{Q}\partial u = \int \partial u\int \partial u l\sin.\tfrac{1}{2}u,$$
$$\int \partial u\int \partial u\int \partial u\int \mathrm{Q}\partial u = \int \partial u\int \partial u\int \partial u l\sin.\tfrac{1}{2}u.$$

§. 37. So that the method may be extended to the first formulas involving the variable *z*,

$$\int \frac{\mathrm{P}'\partial z}{z} = \int \frac{\partial z}{z}\int \frac{\mathrm{P}\partial z}{z} = lz\int \frac{\mathrm{P}\partial z}{z} - \int \frac{\mathrm{P}\partial z}{z} lz,$$

will be elicited through known reductions where the first member $lz\int \frac{\mathrm{P}\partial z}{z}$ vanishes on putting $z=1$, then truly

$$\int \frac{\partial z}{z}\int \frac{\mathrm{P}'\partial z}{z} = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{P}\partial z}{z} = +\int \frac{\mathrm{P}\partial z}{z}\cdot \frac{(lz)^2}{2},$$

from which expressions shown we may deduce further to become:

$\mathrm{P}' = +\int \frac{\mathrm{P}\partial z}{z}$	$\mathrm{Q}' = +\int \frac{\mathrm{Q}\partial z}{z}$
$\mathrm{P}'' = -\int \frac{\mathrm{P}\partial z}{z} lz,$	$\mathrm{Q}'' = -\int \frac{\mathrm{Q}\partial z}{z} lz,$
$\mathrm{P}''' = +\int \frac{\mathrm{P}\partial z}{z}\frac{(lz)^2}{1\cdot 2},$	$\mathrm{Q}''' = +\int \frac{\mathrm{Q}\partial z}{z}\frac{(lz)^2}{1\cdot 2},$
$\mathrm{P}^{\mathrm{IV}} = -\int \frac{\mathrm{P}\partial z}{z}\frac{(lz)^3}{1\cdot 2\cdot 3}$	$\mathrm{Q}^{\mathrm{IV}} = -\int \frac{\mathrm{Q}\partial z}{z}\frac{(lz)^3}{1\cdot 2\cdot 3}.$

§. 38. From these there we will be able to assign the values of the following integral formulas, in the case where $z=1$,

$$\mathrm{P} = -\tfrac{1}{2}$$
$$\mathrm{P}' = \int \frac{\mathrm{P}\partial z}{z} = -l\sin.\tfrac{1}{2}u,$$
$$\mathrm{P}'' = -\int \frac{\mathrm{P}\partial z}{z} lz = -\mathrm{B} - \mathrm{A}u + \tfrac{1}{4}uu,$$
$$\mathrm{P}''' = +\int \frac{\mathrm{P}\partial z}{z}\frac{(lz)^2}{1\cdot 2} = \int \partial u \int \partial u l\sin.\tfrac{1}{2}u,$$
$$\mathrm{P}^{\mathrm{IV}} = -\int \frac{\mathrm{P}\partial z}{z}\frac{(lz)^3}{1\cdot 2\cdot 3} = \mathrm{D} + \mathrm{C}u + \tfrac{1}{2}\mathrm{B}uu + \tfrac{1}{6}\mathrm{A}u^3 - \tfrac{1}{48}u^4,$$
$$\mathrm{P}^{\mathrm{V}} = +\int \frac{\mathrm{P}\partial z}{z}\frac{(lz)^4}{1\cdot 2\cdot 3\cdot 4} = \int \partial u\int \partial u\int \partial u\int \partial u l\sin.\tfrac{1}{2}u,$$

etc.

In the same way,

$$Q=-\tfrac{1}{2}\cot.\tfrac{1}{2}u$$
$$Q'=\int\tfrac{Q\partial z}{z}=A-\tfrac{1}{2}u,$$
$$Q''=-\int\tfrac{Q\partial z}{z}\cdot\tfrac{lz}{1}=-\int\partial ul\sin.\tfrac{1}{2}u,$$
$$Q'''=+\int\tfrac{Q\partial z}{z}\tfrac{(lz)^2}{1\cdot 2}=-C-Bu-\tfrac{1}{2}Auu+\tfrac{1}{12}u^3,$$
$$Q^{IV}=-\int\tfrac{Q\partial z}{z}\tfrac{(lz)^3}{6}=\int\partial u\int\partial u\int\partial ul\sin.\tfrac{1}{2}u,$$
$$Q^{V}=+\int\tfrac{Q\partial z}{z}\tfrac{(lz)^4}{24}=E+Du+\tfrac{1}{2}Cuu+\tfrac{1}{6}Bu^3+\tfrac{1}{24}Au^4-\tfrac{1}{240}u^5$$
etc.

§. 39. Therefore since there shall be

$$P=\tfrac{z\cos.u-zz}{1-2z\cos.u+zz}\text{ and }Q=\tfrac{z\sin.u}{1-2z\cos.u+zz},$$

we have understood so far, so that of the two integral formulas

$$\int\tfrac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}(lz)^n\text{ et }\int\tfrac{\partial z\sin.u}{1-2z\cos.u+zz}(lz)^n,$$

in the case $z=1$, we may prevail to assign the angle conveniently by *u*, but only if it may be agreed, by what means the quantities A, B, C, D, etc. shall be required to be determined, that which scarcely by any other manner may be seen to be able to happen except by the series of these, from which these quantities have arisen.

§. 40. Therefore with all the integral formulas omitted, which will involve the quantity Q, evidently the integration of which is less successful, we will consider only the others, and on putting $z=1$ at once, where there shall be $P=-\tfrac{1}{2}$, thus so that there shall become :

$$\cos.u+\cos.2u+\cos.3u+\cos.4u+\text{etc.}=-\tfrac{1}{2},$$

if we may multiply by ∂u and integrate, we will have

$$Q'=\tfrac{\sin.u}{1}+\tfrac{\sin.2u}{2}+\tfrac{\sin.3u}{3}+\tfrac{\sin.4u}{4}+\tfrac{\sin.5u}{5}+\text{etc.}=A-\tfrac{1}{2}u,$$

which constant can be seen to be equal to zero, because on putting $u=0$ the sum of the series may be seen to vanish; but with the angle *u* taken infinitely small the series will be produced $u+u+u+u+u+u+\text{etc.}$ to infinity ; moreover it is observed, such a series can have a finite sum, from which with the omitted case we may put $u=\pi$, or rather $u=\pi+\omega$, and this series will be produced with ω being an infinitely small angle,

$$-\omega+\omega-\omega+\omega-\omega+\omega-\omega+\text{etc.};$$

where, since the signs are alternating, there is no doubt, why the sum of the series may not vanish, which since it shall become $\mathrm{A}-\frac{\pi}{2}$, it is evident the constant A must become $=\frac{\pi}{2}$, thus, so that now we shall have :

$$\mathrm{Q}'=\frac{\sin.u}{1}+\frac{\sin.2u}{2}+\frac{\sin.3u}{3}+\frac{\sin.4u}{4}+\frac{\sin.5u}{5}+\text{etc.}=\frac{\pi-u}{2}.$$

The illustrious *Daniel Bernoulli* was the first to use this method in determining the constant, who in addition noted many outstanding properties regarding the nature of these series.
[*De indole singulari serierum infinitarum*,…. Novi Comment. acad. sc. Petrop. 17 (1772), p. 3.]

§. 41. We may multiply this final series again by ∂u and on integrating it will give :

$$\mathrm{P}''=\frac{\cos.u}{1^2}+\frac{\cos.2u}{2^2}+\frac{\cos.3u}{3^2}+\frac{\cos.4u}{4^2}+\frac{\cos.5u}{5^2}+\text{etc.}=\mathrm{B}-\frac{\pi u}{2}+\frac{uu}{4},$$

towards finding which constant in the first place we may put $u=0$, and there becomes

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}=\mathrm{B}.$$

I first showed the sum of this series some time ago to be $=\frac{\pi\pi}{6}$; truly if this truth were unknown to us, by using that outstanding method employed by the great *Bernoulli* , and putting $u=\pi$, there will become :

$$-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{6^2}-\text{etc.}=\mathrm{B}-\frac{\pi\pi}{2}+\frac{\pi\pi}{4}=\mathrm{B}-\frac{\pi\pi}{4};$$

both these series added will give :

$$\frac{2}{2^2}+\frac{2}{4^2}+\frac{2}{6^2}+\frac{2}{8^2}+\text{etc.}=2\mathrm{B}-\frac{\pi\pi}{4},$$

the double of which provides

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}=4\mathrm{B}-\frac{\pi\pi}{2}=\mathrm{B},$$

from which there is deduced $\mathrm{B}=\frac{\pi\pi}{6}$, thus so that there shall become :

$$\mathrm{P}''=\frac{\cos.u}{1^2}+\frac{\cos.2u}{2^2}+\frac{\cos.3u}{3^2}+\frac{\cos.4u}{4^2}+\text{etc.}=\frac{\pi\pi}{6}-\frac{\pi u}{2}+\frac{uu}{4}.$$

§. 42. We may progress further by the same method, and on multiplying anew by *du* and integrating we arrive at

$$\mathrm{Q}'' = \frac{\sin.u}{1^3} + \frac{\sin.2u}{2^3} + \frac{\sin.3u}{3^3} + \frac{\sin.4u}{4^3} + \text{etc.} = \mathrm{C} + \frac{\pi\pi u}{6} - \frac{\pi uu}{4} + \frac{u^3}{12};$$

where if we may put $u = 0$, the sum of the series clearly vanishes, and indeed on putting $u = \omega$ there will emerge:

$$\frac{\omega}{1^2} + \frac{\omega}{2^2} + \frac{\omega}{3^2} + \frac{\omega}{4^2} + \text{etc.} = \frac{\omega\pi\pi}{6},$$

which on account of $\omega = 0$ becomes $= 0$, and thus there will become $\mathrm{C} = 0$, and therefore

$$\mathrm{Q}''' = \frac{\sin.u}{1^3} + \frac{\sin.2u}{2^3} + \frac{\sin.3u}{3^3} + \frac{\sin.4u}{4^3} + \text{etc.} = \frac{\pi\pi u}{6} - \frac{\pi uu}{4} + \frac{u^3}{12}.$$

§. 43. This series may be multiplied by $-\partial u$, and by integrating it will provide :

$$\mathrm{P}^{\mathrm{IV}} = \frac{\cos.u}{1^4} + \frac{\cos.2u}{2^4} + \frac{\cos.3u}{3^4} + \frac{\cos.4u}{4^4} + \text{etc.} = \mathrm{D} - \frac{\pi\pi uu}{12} - \frac{\pi u^3}{12} + \frac{u^3}{48};$$

hence by taking $u = 0$ it will become

$$\mathrm{P}^{\mathrm{IV}} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = \mathrm{D};$$

now truly there may be made also $u = \pi$, and there will become :

$$-\frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \text{etc.} = \mathrm{D} - \frac{\pi^4}{48};$$

moreover both these series added give :

$$\frac{2}{2^4} + \frac{2}{4^4} + \frac{2}{6^4} + \frac{2}{8^4} + \text{etc.} = 2\mathrm{D} - \frac{\pi^4}{48},$$

which taken eight times so that the numerators shall become $= 2^4$, will produce

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = 16\mathrm{D} - \frac{\pi^4}{6},$$

from which there arises $\mathrm{D} = \frac{\pi^4}{90}$, which is the same sum of the series

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.},$$

which now I have found some time ago ; now we will have :

$$\mathrm{P''''}=\frac{\cos.u}{1^4}+\frac{\cos.2u}{2^4}+\frac{\cos.3u}{3^4}+\frac{\cos.4u}{4^4}+\text{etc.}=\frac{\pi^4}{90}-\frac{\pi\pi u^2}{12}+\frac{\pi u^3}{12}-\frac{u^4}{48}.$$

§. 44. On being multiplied again by du and integrating, we follow up with

$$\mathrm{Q}^{\mathrm{V}}=\frac{\sin.u}{1^5}+\frac{\sin.2u}{2^5}+\frac{\sin.3u}{3^5}+\frac{\sin.4u}{4^5}+\text{etc.}=\mathrm{E}+\frac{\pi^4 u}{90}-\frac{\pi\pi u^3}{36}+\frac{\pi u^4}{48}-\frac{u^5}{240},$$

where, as in the penultimate case, again the constant E becomes $=0$, thus so that we shall have

$$\mathrm{Q}^{\mathrm{V}}=\frac{\sin.u}{1^5}+\frac{\sin.2u}{2^5}+\frac{\sin.3u}{3^5}+\frac{\sin.4u}{4^5}+\text{etc.}=\frac{\pi^4 u}{90}-\frac{\pi\pi u^3}{36}+\frac{\pi u^4}{48}-\frac{u^5}{240}.$$

45. We may multiply anew by $-du$ and on integrating there will emerge :

$$\mathrm{P}^{\mathrm{VI}}=\frac{\cos.u}{1^6}+\frac{\cos.2u}{2^6}+\frac{\cos.3u}{3^6}+\frac{\cos.4u}{4^6}+\text{etc.}$$
$$=\mathrm{F}-\frac{\pi^4}{90}\cdot\frac{uu}{2}+\frac{\pi\pi}{6}\cdot\frac{u^4}{24}-\frac{\pi}{2}\cdot\frac{u^5}{120}+\frac{1}{2}\cdot\frac{u^6}{720},$$

where towards determining the constant there may be put $u=0$, and there will become

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=\mathrm{F};$$

then truly there may be taken $u=\pi$ and there will become :

$$-\frac{1}{1^6}+\frac{1}{2^6}-\frac{1}{3^6}+\frac{1}{4^6}-\text{etc.}=\mathrm{F}-\frac{\pi^6}{480},$$

which added together gives :

$$\frac{2}{1^6}+\frac{2}{2^6}+\frac{2}{3^6}+\frac{2}{4^6}+\text{etc.}=2\mathrm{F}-\frac{\pi^6}{480},$$

which may be multiplied by 32, so that all the numerators may become $64=2^6$, and there will arise :

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=64\mathrm{F}-\frac{\pi^6}{15}=\mathrm{F},$$

from which $\mathrm{F}=\frac{\pi^6}{945}$ is deduced, thus so that there shall become :

$$\mathrm{P}^{\mathrm{VI}} = \frac{\cos.u}{1^6} + \frac{\cos.2u}{2^6} + \frac{\cos.3u}{3^6} + \frac{\cos.4u}{4^6} + \text{etc.}$$
$$= \frac{\pi^6}{945} - \frac{\pi^4}{90}\cdot\frac{u^2}{2} + \frac{\pi^2}{6}\cdot\frac{u^4}{24} - \frac{\pi}{2}\cdot\frac{u^5}{120} + \frac{1}{2}\cdot\frac{u^6}{720},$$

§. 46. It may appear superfluous to continue this series further, since the law of the progression shall now be evident enough, particularly if as an aid the sums of the reciprocal even powers, which at one time I had counted up as far as the third power. [*Introduction to the analysis of the infinite,* Book I chap. 15; to be found translated on this website.] In order that it may be more apparent, we have represented these sums in the following manner :

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.} = \alpha\pi\pi, \text{ so that there shall be } \alpha = \frac{1}{6},$$
$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} = \beta\pi^4, \text{ so that there shall be } \beta = \frac{1}{90},$$
$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \text{etc.} = \gamma\pi^6, \text{ so that there shall be } \gamma = \frac{1}{945},$$
$$\frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \text{etc.} = \delta\pi^4, \text{ so that there shall be } \delta = \frac{1}{9450},$$
etc.

and, with these in place, we will have the following integrations evidently for the case $z = 1$,

$$\mathrm{Q}' = \int \frac{\partial z \sin.u}{1-2z\sin.u+zz} = \frac{1}{2}\pi - \frac{1}{2}u = \text{arc.tang.}\frac{\sin.u}{1-\cos.u},$$
$$\mathrm{P}'' = -\int \frac{\partial z(\cos.u-z)}{1-2z\sin.u+zz}\cdot\frac{lz}{1} = \alpha\pi\pi - \frac{1}{2}\pi u + \frac{1}{2}\cdot\frac{uu}{2},$$
$$\mathrm{Q}''' = +\int \frac{\partial z\sin.u}{1-2z\sin.u+zz}\cdot\frac{(lz)^2}{2} = \alpha\pi\pi\,\frac{u}{1} - \frac{1}{2}\pi\,\frac{uu}{2} + \frac{1}{2}\cdot\frac{u^3}{6},$$
$$\mathrm{P}^{\mathrm{IV}} = -\int \frac{\partial z(\cos.u-z)}{1-2z\sin.u+zz}\cdot\frac{(lz)^3}{6} = \beta\pi^4 - \alpha\pi\pi\,\frac{uu}{2} + \frac{1}{2}\pi\,\frac{u^3}{6} - \frac{1}{2}\cdot\frac{u^4}{24},$$
$$\mathrm{Q}^{\mathrm{V}} = +\int \frac{\partial z\sin.u}{1-2z\sin.u+zz}\cdot\frac{(lz)^4}{24} = \beta\pi^4\,\frac{u}{1} - \alpha\pi\pi\,\frac{u^3}{6} + \frac{1}{2}\pi\,\frac{u^4}{24} - \frac{1}{2}\cdot\frac{u^5}{120},$$
$$\mathrm{P}^{\mathrm{VI}} = -\int \frac{\partial z(\cos.u-z)}{1-2z\sin.u+zz}\cdot\frac{(lz)^5}{120} = \gamma\pi^6\,\frac{u}{1} - \beta\pi^4\,\frac{uu}{2} + \alpha\pi\pi\,\frac{u^4}{24} - \frac{1}{2}\pi\,\frac{u^5}{120} + \frac{1}{2}\cdot\frac{u^6}{720},$$
$$\mathrm{Q}^{\mathrm{VII}} = +\int \frac{\partial z\sin.u}{1-2z\sin.u+zz}\cdot\frac{(lz)^6}{720} = \gamma\pi^6\,\frac{u}{1} - \beta\pi^4\,\frac{u^3}{6} + \alpha\pi\pi\,\frac{u^5}{120} - \frac{1}{2}\pi\,\frac{u^6}{720} + \frac{1}{2}\cdot\frac{u^7}{5040}$$
etc.

§. 47. There will be a need for us to examine some cases, in which a given value may be attributed to the angle *u*. Therefore we may put $u = 0$, in which case our formula vanish alternately, the remained truly will give rise to :

$$-\int \frac{\partial z}{1-z} lz = \alpha\pi\pi = \frac{\pi\pi}{6},$$

$$-\int \frac{\partial z}{1-z} \frac{(lz)^3}{6} = \beta\pi^4 = \frac{\pi^4}{90},$$

$$-\int \frac{\partial z}{1-z} \frac{(lz)^5}{120} = \gamma\pi^6 = \frac{\pi^6}{945};$$

to these are the related formulas, which arise from putting $u = \pi$, where again the alternate ones involving sine *u* vanish and the following will remain:

$$\int \frac{\partial z}{1+z} \cdot lz = -\frac{\pi\pi}{12} = -\frac{1}{2}\alpha\pi\pi,$$

$$\int \frac{\partial z}{1+z} \cdot \frac{(lz)^3}{6} = -\frac{7\pi^4}{720} = -\frac{7}{8}\beta\pi^4,$$

$$\int \frac{\partial z}{1+z} \cdot \frac{(lz)^5}{120} = -\frac{31}{32}\gamma\pi^6,$$

$$\int \frac{\partial z}{1+z} \cdot \frac{(lz)^7}{5040} = -\frac{127}{128}\delta\pi^8.$$

§ 48. Here it is noteworthy, since the alternate values, which we have omitted here, also shall vanish on putting $u = \pi$; then the same formulas which vanish also on putting $u = 2\pi$ are no less noteworthy, with the first only excepted, which also certainly does not vanish on putting $u = 0$; truly the remainder, evidently the third, fifth, seventh, etc, do in fact vanish in the cases $u = 0$ and $u = \pi$, and also when $u = 2\pi$. In order that it may be made clearer, we will represent these formulas by factors and the value of the third will be

$$= \tfrac{1}{12} u(\pi - u)(2\pi - u),$$

truly of the fifth, the value is found

$$= \tfrac{1}{720}(\pi - u)(2\pi - u)(4\pi\pi + 6\pi u - 3uu),$$

which also finds a use in the following. But in general it deserves to be mentioned that all our formulas with the first excepted are allotted the same values, there may be put either $u = 0$ or $u = 2\pi$, clearly for which both the sine as well as the cosine correspond. Indeed it may be seen that the same agreement takes place, if there may be put $u = 4\pi$ and $u = 6\pi$; truly the illustrious *Bernoulli* has shown the angle *u* cannot be increased in its values beyond four right angles. But an anomaly of this kind always occurs in all common series in which arcs are expressed, and thus occur in the works of *Leibniz* [*Mathematische Schriften :*Correspondence vol.1 (1849), p. 62, 114, 144; also vol. 5 (1858), p. 81: *De quadratura arithmetica circuli, ellipseos et hyperbolae.*] , in which

$$u=\frac{\text{tang.}u}{1}-\frac{(\text{tang.}u)^3}{3}+\frac{(\text{tang.}u)^5}{5}-\frac{(\text{tang.}u)^7}{7}+\frac{(\text{tang.}u)^9}{9}-\text{etc.},$$

and the angle u may not be allowed to increase beyond 180°. If indeed we may put $u=180°+u$, there will become everywhere $\text{tang.}u=\text{tang.}u$ and neither yet will that series express the arc $\pi+u$, but only the arc u, examples of this kind can also be found in other similar series. But because the first series generally may be removed, the reason for that has been put in place, because in the formula of the integral on putting $u=0$ the denominator becomes $1-z$, which in the case $z=1$ vanishes, and thus the formula increases indefinitely, that which in the following, which are multiplied by lz, no longer happens, because $\frac{lz}{1-z}$ in the case $z=1$ no longer becomes infinite, but only $=-1$, and if a greater power of the logarithm shall be present, thus it becomes $=0$.

§. 49. Now we may also put $u=90°$ or $u=\frac{\pi}{2}$, so that there shall be $\cos.u=0$ and $\sin.u=1$, and in this case all the following general formulas will obtain the values

$$\int\frac{\partial z}{1+zz}=\frac{\pi}{4},$$

$$\int\frac{z\partial z}{1+zz}\cdot lz=-\frac{\pi\pi}{48},$$

$$\int\frac{\partial z}{1+zz}\cdot\frac{(lz)^2}{2}=\frac{\pi^3}{32},$$

$$\int\frac{z\partial z}{1+zz}\cdot\frac{(lz)^3}{6}=-\frac{7\pi^4}{90\cdot128}.$$

§.50. We will consider also the case $u=60°$, or $u=\frac{\pi}{3}$, so that there shall be $\cos.u=\frac{1}{2}$ and $\sin.u=\frac{\sqrt{3}}{2}$, and the general formulas will lead to the following integrals :

$$\frac{\sqrt{3}}{2}\int\frac{\partial z}{1-z+zz}=\frac{\pi}{3},$$

$$-\frac{1}{2}\int\frac{\partial z(1-2z)}{1-z+zz}\cdot lz=\frac{\pi\pi}{36},$$

$$\frac{\sqrt{3}}{2}\int\frac{\partial z}{1-z+zz}\cdot\frac{(lz)^2}{2}=\frac{5\pi^3}{162}.$$

In a similar manner if we may put $u=120°$, so that there becomes $\cos.u=-\frac{1}{2}$ and $\sin.u=\frac{\sqrt{3}}{2}$, and the following related integrations related to these same ones will be produced :

$$\frac{\sqrt{3}}{2}\int\frac{\partial z}{1+z+zz}=\frac{\pi}{6},$$
$$\frac{1}{2}\int\frac{\partial z(1+2z)}{1+z+zz}\cdot lz=-\frac{\pi\pi}{18},$$
$$\frac{\sqrt{3}}{2}\int\frac{\partial z}{1+z+zz}\cdot\frac{(lz)^2}{2}=\frac{2\pi^3}{81};$$

and thus the number of special integrations can be increased as it pleases.

§. 51. Just as these same integrations mentioned have been deduced from our first series P on putting $z=1$, thus we may treat the other series Q in the same manner. Therefore since there shall be

$$\mathrm{Q}=\sin.u+\sin.2u+\sin.3u+\sin.4u+\text{etc.}=\tfrac{1}{2}\cot.\tfrac{1}{2}u,$$

if we may multiply by $-\partial u$ and integrate, the series is found :

$$\mathrm{P}'=\tfrac{\cos.u}{1}+\tfrac{\cos.2u}{2}+\tfrac{\cos.3u}{3}+\tfrac{\cos.4u}{4}+\text{etc.}=-l\sin.\tfrac{1}{2}u+\mathrm{A},$$

for which with the constant required to be determined there may be put $u=\pi$, so that there shall be

$$-1+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{6}-\text{etc.}=\mathrm{A},$$

on account of which $\mathrm{A}=-l2$, thus so that we may have :

$$\mathrm{P}'=\tfrac{\cos.u}{1}+\tfrac{\cos.2u}{2}+\tfrac{\cos.3u}{3}+\tfrac{\cos.4u}{4}+\text{etc.}=-l2\sin.\tfrac{1}{2}u,$$

for which value of brevity we may write $\varDelta:u$, if indeed we may regard that as a certain function of *u* itself, thus so that there shall be $\mathrm{P}'=\Delta:u$.

§. 52. Again, on multiplying by $-\partial u$ and integrating, we arrive at this series :

$$\mathrm{Q}''=\tfrac{\sin.u}{1^2}+\tfrac{\sin.2u}{2^2}+\tfrac{\sin.3u}{3^2}+\tfrac{\sin.4u}{4^2}+\text{etc.}=\int\partial u\Delta{:}u=\Delta':u,$$

where this formula of the integral will involve a certain constant, which is allowed to be defined easily from the case $u=0$; indeed because the series vanishes, there must become $\Delta':u=0$ and thus the integration is determined fully.

§. 53. If we may progress further in the same manner by multiplying by $-\partial u$, this series will be produced :

$$\mathrm{P}''' = \frac{\cos.u}{1^3} + \frac{\cos.2u}{2^3} + \frac{\cos.3u}{3^3} + \frac{\cos.4u}{4^3} + \text{etc.} = -\int \partial u \Delta' {:} u = \Delta'' : u.$$

Now for the constant, which is contained in this expression, there shall be required to be defined

I° $u = 0$ and there will be

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} = \Delta'' : 0,$$

II° $u = \pi$ and there becomes

$$-\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \frac{1}{5^3} + \text{etc.} = \Delta'' : \pi,$$

with which added there is produced

$$\frac{2}{2^3} + \frac{2}{4^3} + \frac{1}{6^3} + \frac{1}{8^3} + \text{etc.} = \Delta'' : 0 + \Delta'' : \pi,$$

and this taken four times will become :

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.} = 4\Delta'' : 0 + 4\Delta'' : \pi = \Delta'' : 0,$$

from which there arises

$$3\Delta'' : 0 + 4\Delta'' : \pi = 0,$$

from which the constant entering into our integral formula

$$\Delta'' : u = -\int \partial u \Delta' : u$$

must be determined.

§. 54. We will multiply anew by $-\partial u$, and integrate, and there will be produced

$$\mathrm{Q}^{\mathrm{IV}} = \frac{\sin.u}{1^4} + \frac{\sin.2u}{2^4} + \frac{\sin.3u}{3^4} + \frac{\sin.4u}{4^4} + \text{etc.} = \int \partial u \Delta'' : u = \Delta''' {:} u$$

and this function $\Delta''' {:} u$ must be determined thus, so that it may vanish on taking $u = 0$, or so that there may become $\Delta''' {:} u = 0.$ On progressing further in the same manner there will become

$$\mathrm{P}^{\mathrm{V}} = \frac{\cos.u}{1^5} + \frac{\cos.2u}{2^5} + \frac{\cos.3u}{3^5} + \frac{\cos.4u}{4^5} + \text{etc.} = -\int \partial u \Delta''' : u = \Delta^{\mathrm{IV}} {:} u$$

the nature of this function will be determined in the following manner. Clearly there may be put as at present $u = 0$ and $u = \pi$, and there will be

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} = \Delta^{\mathrm{IV}} {:} 0$$

and

$$-\frac{1}{1^5}+\frac{1}{2^5}-\frac{1}{3^5}+\frac{1}{4^5}-\frac{1}{5^5}+\text{etc.}=\Delta^{\text{IV}}:\pi,$$

hence on adding

$$\frac{2}{2^5}+\frac{2}{4^5}+\frac{2}{6^5}+\frac{2}{8^5}+\text{etc.}=\Delta^{\text{IV}}:0+\Delta^{\text{IV}}:\pi$$

and multiplying by 16

$$\frac{1}{2^5}+\frac{1}{4^5}+\frac{1}{6^5}+\frac{1}{8^5}+\text{etc.}=16\Delta^{\text{IV}}:0+16\Delta^{\text{IV}}:\pi=\Delta^{\text{IV}}:0,$$

and thus there will become

$$15\Delta^{\text{IV}}:0+16\Delta^{\text{IV}}:\pi=0$$

§. 55. Hence we will therefore arrive at the following integrations for the case $z=1$

$$\text{I.}\quad -\int\frac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}=-l2\sin.\tfrac{1}{2}u=\Delta:u,$$

$$\text{II.}\quad \int\frac{\partial z\sin.u}{1-2z\cos.u+zz}\cdot lz=\int\partial u\Delta:u=\Delta':u,$$

$$\text{III.}\quad -\int\frac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{(lz)^2}{2}=-\int\partial u\Delta':u=\Delta'':u,$$

$$\text{IV.}\quad \int\frac{\partial z\sin.u}{1-2z\cos.u+zz}\cdot\frac{(lz)^3}{6}=\int\partial u\Delta'':u=\Delta''':u,$$

$$\text{V.}\quad -\int\frac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{(lz)^4}{24}=-\int\partial u\Delta''':u=\Delta^{\text{IV}}:u,$$

$$\text{VI.}\quad \int\frac{\partial z\sin.u}{1-2z\cos.u+zz}\cdot\frac{(lz)^5}{120}=\int\partial u\Delta^{\text{IV}}:u=\Delta^{\text{V}}:u,$$

etc. etc. etc. etc.

But these expressions may be allowed to continue easily as far as wished, but only if the integration of each integral may be set up properly ; but the conditions , which it will be required to fulfill, can be referred to in the following manner.

$\Delta':0=0$	$3\Delta'':0+\quad 4\Delta'':\pi=0$
$\Delta''':0=0$	$15\Delta^{\text{IV}}:0+\quad 16\Delta^{\text{IV}}:\pi=0$
$\Delta^{\text{V}}:0=0$	$63\Delta^{\text{VI}}:0+\quad 64\Delta^{\text{VI}}:\pi=0$
$\Delta^{\text{VII}}:0=0$	$255\Delta^{\text{VIII}}:0+\ 256\Delta^{\text{VI}}:\pi=0$
etc.	etc.

the rest because the later integrations cannot be resolved, hence we can expect to be of little usefulness.

§. 56. The other method, which we have used here for the constants requiring to be determined through some integration arising, was used first by the most celebrated *Bernoulli* and that is esteemed to be worthy of the greatest attention, because with its help the summations of my series of reciprocal powers were able to be obtained, since I had believed these only able to be shown in no other way apart from the consideration of infinite arcs which are used by the same sine or cosine.

SUPPLEMENTUM V

AD TOM. I. CAP. VIII.

DE VALORIBUS lNTEGRALIUM QUOS CERTIS TANTUM CASIBUS RECIPIUNT.

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i) Nova Methodus quantitates integrales determinandi.

Novi Commentarii Academiae Scient. Petropolitanae Tom. XIX. Pag. 66 -102. [E464 Series I, vol. 17.]

§.1. Cum mihi saepius occurrissent formulae differentiales, quae per logarithmum quantitatis variabilis erant divisae, veluti $\frac{P\partial z}{lz}$, nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda, quin etiam maxime difficile videbatur eorum valores saltem vero proxime assignare. Quod quidem ad formulam integralem simplicissimam hujus generis $\int\frac{\partial z}{lz}$ attinet, facile patet, si eam ita integrari concipiam, ut evanescat posito $z=0$, tum vero statuatur $z=1$, quantitatem infinite magnam esse prodituram; Quodsi enim variabilis z jam proxime ad unitatem accesserit, ut sit $z=1-u$, existente u quantitate infinite parva, tum ob

$$\partial z=-\partial u \text{ et } lz=l(1-u)=-u,$$

haec formulae erit $\int\frac{\partial u}{u}$, cujus valor utique fit infinitus. At vero dantur omnino hujusmodi formulae integral eo $\int\frac{P\partial z}{lz}$ quae, etiamsi ponatur $z=1$, tamen valores finitae magnitudinis sortiuntur: quod determinasse eo magis operae pretium videtur, quod nulla adhuc cognita est via istos valores investigandi.

§. 2. Consideremus exempli gratia hanc formulam satis simplicem $\int\frac{(z-1)\partial z}{lz}$, quae memorata lege integrata valorem finitum habere facile ostendi potest. Posito enim $\frac{z-1}{lz}=y$, ut formula nostra fiat $\int y\partial z$, ideoque exprimat aream curvae, pro abscissa z applicatam habentis $=y$, ista area a termino $z=0$ usque ad terminum $z=1$ extensa utique valorem finitum non multo majorem quam $\frac{1}{2}$ repraesentabit; posita enim abscissa $z=0$, fiet etiam applicata $y=0$, at sumta $z=1$, pro applicata $y=\frac{z-1}{lz}$ tam numerator quam denominator evanescit, ergo eorum loco substitutis suis differentialibus, fiet $y=z=1$. Pro abscissis autem mediis ponamus $z=e^{-n}$, existente e numero, cujus logarithmus hyperbolicus est unitas, erit

$$y=\frac{e^{-n}-1}{-n}=\frac{e^{n}-1}{ne^{n}},$$

quae, si n fuerit numerus valde magnus, ut abscissa z fiat minima, applicata erit proxime $y=\frac{1}{n}$; qui ergo valor multo major erit quam abscissa z; forma scilicet hujus curvae similis erit figurae adjectae, ubi A P denotat abscissam z et P M applicatam y, abscissae vero $\mathrm{AB}=1$ respondet applicata $\mathrm{BC}=1$, qua curva descripta, Fig. l. ejus area A M C B non multum superabit aream trianguli A B C quae est $=\frac{1}{2}$.

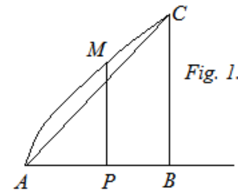

Fig. 1.

§. 3. Nuper autem, in aliis investigationibus occupatus, praeter exspectationem inveni, hanc aream aequalem esse logarithmo hyperbolico binarii , ita ut ea per fractiones decimales sit $l\,2=0,6931471805$; sequenti autem ratiocinio huc sum perductus. Cum revera sit $l\,z=\frac{z^0-1}{0}$, quia differentiando utrinque prodit $\frac{\partial z}{z}=\frac{\partial z}{z}$, et sumto $z=1$ utraque expressio evanescit, loco 0 scribo $\frac{1}{i}$, denotante i numerum infinitum, eritque $l\,z=i\left(z^{\frac{1}{i}}-1\right)$, hincque applicata

$$y=\frac{z-1}{i\left(z^{\frac{1}{i}}-1\right)}=\frac{1-z}{i\left(1-z^{\frac{1}{i}}\right)},$$

et formula integralis

$$\int\frac{(1-z)\partial z}{i\left(1-z^{\frac{1}{i}}\right)}$$

Nunc igitur statuo $z^{\frac{1}{i}}=x$, ut fiat $z=x^i$, ubi notetur, pro utroque integrationis termino, $z=0$ et $z=1$ etiam fore $x=0$ et $x=1$; quia igitur hinc fit $\partial z=ix^{i-1}\partial x$, formula integralis evadit

$$\int\frac{x^{i-1}\partial x\left(1-x^i\right)}{(1-x)},$$

quam ergo integrari oportet a termino $x=0$ usque ad terminum $x=1$.

§. 4. Spectemus nunc i ut numerum valde magnum, et fractio $\frac{1-x^i}{1-x}$ resolvitur in hanc progressionem geometricam

$$1+x+x^2+x^3+x^4+x^5+x^6+\ldots\ x^{i-1},$$

cujus singuli termini in $x^{i-1}\partial x$ ducti et integrati praebent hanc seriem

$$\frac{x^i}{1}+\frac{x^{i+1}}{i+1}+\frac{x^{i+2}}{i+2}+\frac{x^{i+3}}{i+3}+\ldots\ \ldots\frac{x^{2i-1}}{2i-1},$$

quae utique evanescit facto $x=0$. Nunc igitur sumatur $x=1$, et valor quaesitus nostrae formulae integralis erit

$$\frac{1}{1}+\frac{1}{i+1}+\frac{1}{i+2}+\frac{1}{i+3}+\ldots\ \ldots\frac{1}{2i-1},$$

ubi quidem littera *i*, denotat numerum infinite magnum, ita ut numerus horum terminorum sit revera infinitus. Nihilo vera minus, quia singuli termini sunt infinite parvi, haec series summam habebit finitam, quam sequenti modo ad seriem ordinariam reducere licet.

§. 5. Series inventa spectari potest tanquam differentia inter binas sequentes progressiones harmonicas

$$\mathrm{A} = \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots \ldots \tfrac{1}{2i-1}$$
$$\mathrm{B} = \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots \ldots \tfrac{1}{i-1}$$

quandoquidem differentia $\mathrm{A}-\mathrm{B}$ ipsam seriem inventam exhibet ; quia autem numerus terminorum seriei A est $2i-1$, seriei vera $\mathrm{B}=i-1$, ille duplo major est quam hic, quocirca, ut seriem regularem obtineamus, singulos terminos seriei B per saltum a seriei A termino secunda, quarto, sexto, octavo etc. auferamus, quo pacto simul ad finem utriusque pervenietur, eritque

$$\mathrm{A}-\mathrm{B} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.}$$

·in infinitum, cujus ergo valor est $l\,2$, ita ut nunc quidem solide sit demonstratum, formulae integralis propositae $\int \frac{(z-1)\partial z}{lz}$, casu $z=1$, valorem revera esse $=l2$.

§. 6. Simile ratiocinium etiam ad formulam integralem generaliorem $\int \frac{(z^m-1)\partial z}{lz}$ acccommodari potest, ac tandem reperietur, casu $z=1$ ejus valorem fore $l(m+1)$; quia igitur pari modo erit

$$\int \frac{(z^n-1)\partial z}{lz} = l(n+1),$$

si hanc ab illa subtrahamus, prodit sequens integratio

$$\int \frac{(z^m-z^n)\partial z}{lz} = l\,\frac{m+1}{n+1},$$

si scilicet integratio a termino $z=0$ usque ad terminum $z=1$ extendatur.

§. 7. Quia autem haec demonstratio per quantitates infinitas et infinite parvas procedit, merito aliam methodum planam et consuetam desideramus, quae ad easdem summas perducere valeat; quae quidem investigato maxime ardua videbitur. Interim tamen, cum nuper consideratio functionum duas variabiles involventium me ad integrationem formularum differentialium prorsus singularium perduxisset, quae aliis methodis frustra tentantur, ex eadem principio quoque integrationes hic exhibitas derivandas esse intellexi. Hanc igitur methodum tanquam fontem prorsus novum, ex quo integrationes, aliis methodis inaccessas, haurire liceat, clare et perspicue explicabo, cui negotio istam disquisitionem praecipue destinavi.

Lemma I.

§. 8. Si P fuerit functio quaecunque duarum variabilium z et u, ac ponatur $\int P\partial z = S$, ut etiam S sit functio binarum variabilium z et u, tum erit

$$\int \partial z \left(\frac{\partial P}{\partial u}\right) = \left(\frac{\partial S}{\partial u}\right)$$

Demonstratio.

Cum in integratione formulae $\int P\partial z$ sola z ut variabilis spectetur, erit $\left(\frac{\partial S}{\partial z}\right) = P$, quae formula denuo sola u ut variabilis differentiata, sola u pro variabili habita, praebet $\left(\frac{\partial\partial S}{\partial u\partial z}\right) = \left(\frac{\partial P}{\partial u}\right)$, quae in ∂z ducta et integrata producit $\left(\frac{\partial S}{\partial u}\right) = \int \partial z \left(\frac{\partial P}{\partial u}\right)$, quandoquidem ex principiis calculi integralis est

$$\int \partial z \left(\frac{\partial\partial S}{\partial z\partial u}\right) = \left(\frac{\partial S}{\partial u}\right) \text{ q. e. d.}$$

Corollarium I.

§. 9. Eodem modo per hujusmodi differentialia, ubi tantum u pro variabili spectatur, ulterius progredi licet, unde sequentes oriuntur integrationes

$$\left(\frac{\partial\partial S}{\partial u^2}\right) = \int \partial z \left(\frac{\partial\partial P}{\partial u^2}\right) \text{ et}$$

$$\left(\frac{\partial^3 S}{\partial u^3}\right) = \int \partial z \left(\frac{\partial^3 P}{\partial u^3}\right)$$

etc. etc.

Corollarium II.

§. 10. Quodsi ergo formula $\int P\partial z$ fuerit integrabilis, ita ut ejus integrale S exhiberi possit, tum etiam omnes istae formulae integrales

$$\int \partial z \left(\frac{\partial P}{\partial u}\right), \ \int \partial z \left(\frac{\partial\partial P}{\partial u^2}\right), \ \int \partial z \left(\frac{\partial^3 P}{\partial u^3}\right) \text{ etc.}$$

integrationem admittent, atque adeo ipsa integralia exhiberi poterunt.

Scholion.

§. 11. Ex his quidem formulis si in genere tractentur, parum utilitatis in calculum integralem redundat. At si functio P ita fuerit comparata, ut integrale $\int P\partial z$, casu saltem particulari, quo post integrationem variabili z certus quidam valor puta $z=a$ tribuitur, commode exhiberi potest, ut hoc casu quantitas S abeat in functionem solius variabilis u satis simplicem, tum integrationes memoratae perinde locum habebunt, si quidem post singulas integrationes ponatur $z=a$, atque hinc ad ejusmodi integrationes plerumque pervenitur quas aliis methodis vix, ac ne vix quidem perficere liceat: atque hinc oritur.

Primum principium integrationum.

§. 12. Si P ejusmodi fuerit functio binarum variabilium z et u, ut valor integralis $\int P\partial z$ saltem casu certo $z=a$ commode exprimi queat, qui valor sit $=S$, functio scilicet ipsius u tantum; tum etiam sequentia integralia, si quidem post integratione pariter statuatur $z=a$, commode exhiberi poterunt, scilicet

$$\int P\partial z = S$$
$$\int \partial z\left(\frac{\partial P}{\partial u}\right)=\left(\frac{\partial S}{\partial u}\right)$$
$$\int \partial z\left(\frac{\partial\partial P}{\partial u^2}\right)=\left(\frac{\partial\partial S}{\partial u^2}\right)$$
$$\int \partial z\left(\frac{\partial^3 P}{\partial u^3}\right)=\left(\frac{\partial^3 S}{\partial u^3}\right)$$
$$\int \partial z\left(\frac{\partial^4 P}{\partial u^2}\right)=\left(\frac{\partial^4 S}{\partial u^2}\right)$$
$$\text{etc.} \quad \text{etc.}$$

Exemplum I.

§. 13. Si fuerit $P=z^u$, erit quidem in genere

$$\int P\partial z=\frac{z^{u+1}}{u+1};$$

unde casu $z=1$ hic valor satis simplex nascitur $\frac{1}{u+1}$ ita ut sit $S=\frac{1}{u+1}$; cum deinde per differentiationes continuas, dum sola u pro variabili habetur, prodeat $\left(\frac{\partial P}{\partial u}\right)=z^u lz$, tum vera $\left(\frac{\partial\partial P}{\partial u^2}\right)=z^u\left(lz\right)^2$, porro

$$\left(\frac{\partial^3 P}{\partial u^3}\right)=z^u\left(lz\right)^3,\ \left(\frac{\partial^4 P}{\partial u^4}\right)=z^u\left(lz\right)^4,\ \text{etc.}$$

hinc sequentes obtinentur valores integrales, si quidem post singulas integrationes statuatur $z=1$

$$\int z^u \partial z = +\frac{1}{u+1} \qquad \int z^u \partial z (lz)^4 = +\frac{1.2.3.4}{(u+1)^5}$$

$$\int z^u \partial z lz = -\frac{1}{(u+1)^2} \qquad \int z^u \partial z (lz)^5 = -\frac{1.2.3.4.5}{(u+1)^6}$$

$$\int z^u \partial z (lz)^2 = +\frac{1\cdot 2}{(u+1)^3} \qquad \int z^u \partial z (lz)^6 = +\frac{1.....6}{(u+1)^7}$$

$$\int z^u \partial z (lz)^3 = -\frac{1\cdot 2\cdot 3}{(u+1)^4} \qquad \int z^u \partial z (lz)^7 = -\frac{1.....7}{(u+1)^8}$$

unde concludimus generaliter fore

$$\int z^u \partial z (lz)^n = \pm\frac{1.2.3.4....n}{(u+1)^{n+1}}$$

ubi signum + valet si *n* sit numerus par, alterum vero − si *n* sit numerus impar. Hae quidem integrationes jam aliunde satis sunt notae, id quod mirum non est, quoniam tam simplicem formulam pro P assumsimus: breviter igitur repetamus eos casus, quos jam nuper expedivi.

Exemplum 2.

§.14. Si fuerit

$$\mathrm{P} = \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}},$$

jam dudum demonstravi, formula $\int \mathrm{P}\partial z$ valorem integralem casu post integrationem ponitur $z=1$, esse

$$\mathrm{S} = \frac{\pi}{2n\cos.\frac{\pi u}{2n}}.$$

Hinc ergo cum sit

$$\left(\frac{\partial \mathrm{P}}{\partial u}\right) = \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}} lz,$$

tum vero

$$\left(\frac{\partial \mathrm{P}}{\partial u}\right) = \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}} lz, \text{ et}$$

$$\left(\frac{\partial\partial \mathrm{P}}{\partial u^2}\right) = \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}} (lz)^2,$$

$$\left(\frac{\partial^3 \mathrm{P}}{\partial u^3}\right) = \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}} (lz)^3$$

etc. etc.

ex cognito valore S sequentes nacti sumus integrationes

$$\text{I.}\quad \int \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z = \mathrm{S} = \frac{\pi}{2n\cos.\frac{\pi u}{2n}}$$

$$\text{II.}\quad \int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z lz = \left(\frac{\partial \mathrm{S}}{\partial u}\right)$$

$$\text{III.}\quad \int \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z\left(lz\right)^2 = \left(\frac{\partial\partial \mathrm{S}}{\partial u^2}\right)$$

$$\text{IV.}\quad \int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z\left(lz\right)^3 = \left(\frac{\partial^3 \mathrm{S}}{\partial u^3}\right)$$

$$\text{V.}\quad \int \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\partial z\left(lz\right)^4 = \left(\frac{\partial^4 \mathrm{S}}{\partial u^4}\right)$$

etc. etc.

Exemplum 3.

§.15. Si fuerit

$$\mathrm{P} = \frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}},$$

simili modo demonstravi, valorem formulae integralis $\int \mathrm{P}\partial z$, casu quo post integrationem ponitur $z=1$, fore

$$\mathrm{S} = \frac{\pi}{2n}\,\text{tang.}\frac{\pi u}{2n};$$

atque hinc sequentes integrationes pro eodem casu $z=1$ fuerunt deductae

$$\text{I.}\quad \int \frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\partial z = \mathrm{S} = \frac{\pi}{2n}\,\text{tang.}\frac{\pi u}{2n}$$

$$\text{II.}\quad \int \frac{-z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\partial z lz = \left(\frac{\partial \mathrm{S}}{\partial u}\right)$$

$$\text{III.}\quad \int \frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\partial z\left(lz\right)^2 = \left(\frac{\partial\partial \mathrm{S}}{\partial u^2}\right)$$

$$\text{IV.}\quad \int \frac{-z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\partial z\left(lz\right)^3 = \left(\frac{\partial^3 \mathrm{S}}{\partial u^3}\right)$$

$$\text{V.}\quad \int \frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\partial z\left(lz\right)^4 = \left(\frac{\partial^4 \mathrm{S}}{\partial u^4}\right)$$

etc. etc.

Scholion.

§. 16. Quo igitur uberiores fructus ex hoc principio, exspectare queamus, praecipuum negotium huc redit, ut ejusmodi functiones binarum variabilium z et u pro P investigemus, ita ut valor formulae integralis saltem certo quodam casu puta $z=1$ succincte assignari possit, quemadmodum in allatis exemplis fieri licuit. Quemadmodum autem hoc principium ex continua differentiatione est deductum, ita eodem modo continua integratio ad usum nostrum accommodari poterit.

Lemma II.

§. 17. *Si P fuerit functio duarum variabilium z et u, ac ponatur* $\int P\partial z = S$ *, ut etiam S sit functio duarum variabilium z et u, tum erit* $\int S\partial u = \int \partial z \int P\partial u$ *, ubi in integralibus formulis* $\int P\partial u$ *et* $\int S\partial u$ *sola u pro variabili habetur, in formula autem* $\int \partial z \int P\partial u$ *sola z.*

Demonstratio.

Ponatur $\int \mathrm{P}\partial u = \mathrm{V}$ et sit $\mathrm{S} = \left(\frac{\partial \mathrm{V}}{\partial u}\right)$ ideoque $\left(\frac{\partial \mathrm{V}}{\partial u}\right) = \int \mathrm{P}\partial z$, eritque $\left(\frac{\partial\partial \mathrm{V}}{\partial z\partial u}\right)$; unde per ∂u multiplicando et integrando erit $\left(\frac{\partial \mathrm{V}}{\partial z}\right) = \int \mathrm{P}du$, ex quo sequitur $\mathrm{V} = \int \partial z \int \mathrm{P}\partial u = \int \mathrm{S}\partial u$.
q.e.d.

Corollarium 1.

§.18. Hoc modo etiam integratio repeti potest, unde orietur talis aequatio

$$\int \partial u \int \mathrm{S}\partial u = \int \partial z \int \partial u \int \mathrm{P}\partial u \,;$$

hinc autem plerumque parum utilitatis exspectari potest, nisi forte istae integrationes commode succedant.

Corollarium 2.

§.19. Quodsi ergo formula $\int \mathrm{P}\partial z$ fuerit integrabilis, scilicet $=\mathrm{S}$, altera hinc deducta $\int \partial z \int \mathrm{P}\partial u$ eatenus tantum integrali poterit, quatenus integrale $\int \mathrm{S}\partial u$ integrare licet.

Secundum principium integrationum.

§. 20. Si P ejusmodi fuerit functio duarum variabilium z et u, ut formulae integralis $\int \mathrm{P}\partial z$ valor certo saltem casu, puta $z=a$, commode exhiberi queat, ita ut hoc casu quantitas S fiat functio solius variabilis *u;* tum etiam pro eadem casu $z = a$ hujus formulae integralis $\int \partial z \int \mathrm{P}\partial u$ valor assignari poterit, si modo formulam $\int \mathrm{S}\partial u$ integrare licuerit.

Exemplum I.

§. 21. Sumamus $\mathrm{P} = z^{u}$, eritque $\int \mathrm{P}\partial \mathrm{z} = \frac{z^{u+1}}{u+1}$; quae formula casu $z = 1$ abit in $\frac{1}{u+1}$, quod ergo loco S scribatur. Tum vero quia est

$$\int \mathrm{P}\partial \mathrm{u} = \int z^{u}\partial u = \frac{z^{u}}{lz}$$

et quia

$$\int \mathrm{S}\partial u = l\left(u+1\right), \text{ erit}$$

$$\int \frac{z^{u}\partial z}{lz} = l\left(u+1\right)$$

si quidem post illam integrationem ponatur $z = 1$. Quia autem omnis integratio additionem constantis postulat, hic potius statui oportebit

$$\int \frac{z^{u}\partial z}{lz} = l\left(u+1\right) + \mathrm{C};$$

atque hic quidem facile intelligitur, hanc constantem C esse debere infinitam, quoniam in formula integrali fractio $\frac{z^{u}}{lz}$ posito $z = 1$ fit infinita, ita ut hinc parum pro instituto nostro sequi videatur.

Corollarium 1.

§. 22. Quoniam autem haec constans C non a variabili *u* pendet, ea retinebit eundem valorem, quicunque numeri determinati pro *u* accipiantur. Sumamus igitur primo $u = m$, tum vero etiam $u = n$, ut habeamus istos valores

$$\text{I. } \int \frac{z^{m}\partial z}{lz} = l\left(m+1\right) + \mathrm{C} \text{ et}$$

$$\text{II. } \int \frac{z^{n}\partial z}{lz} = l\left(n+1\right) + \mathrm{C},$$

quarum altera ab altera subtracta relinquet istam integrationem notatu dignissimam

$$\int \frac{\left(z^{m}-z^{n}\right)\partial z}{lz} = l\,\frac{m+1}{n+1},$$

quemadmodum jam supra ex longe aliis principiis demonstravimus.

Corollarium 2.

§. 23. Si ad alteram integrationem ascendamus, quia est $\int P\partial u = \frac{z^u}{lz}$, erit $\int \partial u \int P\partial u = \frac{z^u}{(lz)^2}$;
tum vero ob
$\int S\partial u = l(u+1) + C$, erit

$$\int \partial u \int S\partial u = (u+1)\left[l(u+1)-1\right] + Cu + D,,$$

sicque habebimus

$$\int \frac{z^u \partial z}{(lz)^2} = (u+1)\left[l(u+1)-1\right] + Cu + D,$$

ubi constantes C et D non ab *u* pendent: quare ut eas eliminemus tres casus determinatos evolvamus

$$\text{I. } \int \frac{z^m \partial z}{(lz)^2} = (m+1)l(m+1) - m - 1 + Cm + D,$$

$$\text{II. } \int \frac{z^n \partial z}{(lz)^2} = (n+1)l(n+1) - n - 1 + Cn + D, \quad .$$

$$\text{III. } \int \frac{z^k \partial z}{(lz)^2} = (k+1)l(k+1) - k - 1 + Ck + D,$$

eritque

$$\text{I} - \text{III} = (m+1)l(m+1) - (k+1)l(k+1) + k - m + C(m-k) \text{ et}$$
$$\text{II} - \text{III} = (n+1)l(n+1) - (k+1)l(k+1) + k - n + C(n-k),$$

hincque deducimus

$$(\text{I}-\text{III})(n-k) - (\text{II}-\text{III})(m-k) = \begin{cases} +(m+1)(n-k)l(m+1) \\ -(k+1)(n-k)l(k+1) + (k-m)(n-k) \\ -(n+1)(m-k)l(n+1) - (k-n)(m-k) \\ +(k+1)(m-k)l(k+1) \end{cases}$$

atque hinc pervenimus ad sequentem integrationem

$$\int \frac{\partial z\left[(n-k)z^m - (m-k)z^n + (m-n)z^k\right]}{(lz)^2} = +(m+1)(n-k)l(m+1)$$
$$-(n+1)(m-k)l(n+1)$$
$$+(k+1)(m-n)l(k+1).$$

Corollarium 3.

§. 24. Operae pretium erit aliquot casus evolvere, ubi quidem numeros *m, n* et *k* inter se inaequales accipi convenit, quia aliter omnes termini se destruerent.

I. Sit igitur $m=2,\ n=1$ et $k=0$; erit

$$\int\frac{(z-1)^2\partial z}{(\partial z)^2}=3l3-4l2=l\,\tfrac{27}{16}.$$

II. Sit $m=3, n=1$ et $k=0$, eritque

$$\int\frac{\left(z^3-3z+2\right)\partial z}{(\partial z)^2}=\int\frac{(z-1)^2(z+2)\partial z}{(\partial z)^2}=4l4-6l2=2l2=l4.$$

III. Sit $m=3, n=2$ et $k=0$, et erit

$$\int\frac{\left(2z^3-3zz+1\right)\partial z}{(\partial z)^2}=\int\frac{(z-1)^2(2z+1)\partial z}{(\partial z)^2}=8l4-9l3=l\,\tfrac{4^8}{3^9}.$$

IV. Sit $m=3, n=2$ et $k=1$, et prodit

$$\int\frac{\left(z^3-3zz+z\right)\partial z}{(\partial z)^2}=\int\frac{z\partial z(z-1)^2}{(\partial z)^2}=4l4-6l3+2l2=l\,\tfrac{2^{10}}{3^6}.$$

Corollarium 4.

§. 25. In his casibus notatu dignum occurrit, quod numerator in formulis integralibus factorem habet $(z-1)^2$, quod ideo necessario usu venit, ne valores integralium evadant infiniti. Quia enim denominator $(lz)^2$ evanescit casu $z=1$, si ponamus $z=1-\omega$ existente ω infinite parvo, erit

$$lz=-\omega \text{ et } (lz)^2=+\omega\omega.$$

Necesse ergo est, ut in numeratore adsit factor, qui casu $z=1-\omega$ itidem praebeat $\omega\omega$, quod evenit si ibi factor fuerit $(z-1)^2$.

Scholion.

§. 26. Integratio, quam in corollario primo sumus nacti, ideo omni digna videtur attentione, quod valores integrales inde nati casu $z=1$ nullo adhuc modo assignare potuerim, etiamsi tam simpliciter per logarithmos exprimantur. At vera integrationes in corollario secunda inventae, etiamsi multo magis arduae, videantur, tamen ex prioribus ope reductionum cognitarum non difficulter derivari possunt; id quod pro unico casu ostendisse sufficiet. Ponamus

$$\int\frac{(z-1)^2\partial z}{(lz)^2}=\frac{p}{lz}+\int\frac{q\partial z}{lz}$$

eritque differentiando

$$\frac{(z-1)^2\partial z}{(lz)^2}=\frac{\partial p}{lz}-\frac{p\partial z}{z(lz)^2}+\frac{q\partial z}{lz},$$

unde aequatis terminis seorsim vel per $(\partial z)^2$ vel per lz divisis habebimus has duas aequalitates

$$(z-1)^2=-\frac{p}{z}\text{ et }\partial p=-q\partial z,$$

ex quarum priore oritur $p=-z(z-1)^2$ hincque

$$\frac{\partial p}{\partial z}=-3zz+4z-1\text{ ideoque }q=3zz-4z+1,$$

ita ut sit

$$\int\frac{(z-1)^2\partial z}{(lz)^2}=\frac{-z(z-1)^2}{lz}+\int\frac{(3zz-4z+1)\partial z}{lz}$$

hic autem prius membrum posito $z=1$ sponte evanescit ; posito enim $z=1-\omega$, ut sit $lz=-\omega$, erit $p=-\omega\omega(1-\omega)$, ideoque $\frac{p}{lz}=\omega(1-\omega)=0$, ob $\omega=0$: posteriorus vero membrum in has partes discerpi potest

$$3\int\frac{(zz-z)\partial z}{lz}-\int\frac{(z-1)\partial z}{lz}.$$

Prioris autem partis integrale est $3l\frac{3}{2}$, posterioris vero $-1l2$; sicque totum hoc integrale erit

$$3l\frac{3}{2}-l2=3l3-4l2=l\frac{27}{16},$$

prorsus uti invenimus. Hoc igitur modo si in genere statuamus

$$\int\frac{\mathrm{V}\partial z}{(lz)^2}=\frac{p}{lz}+\int\frac{q\partial z}{lz}$$

erit differentiando

$$\frac{\mathrm{V}\partial z}{(lz)^2}=\frac{\partial p}{lz}-\frac{p\partial z}{z(lz)^2}+\frac{q\partial z}{lz},$$

unde istae duae fluunt aequalitates

$$p=-\mathrm{V}z,\text{ et }q=-\frac{\partial p}{\partial z}.$$

Jam ut terminus $\frac{p}{lz}$ evanescat posito $z=1$, numerator p factorem habere debet $(z-1)^2$; qui ergo etiam factor esse debet quantitatis V. Sit igitur

$$\mathrm{V}=\frac{\mathrm{U}(z-1)^2}{z}\text{, eritque } p=-\mathrm{U}(z-1)^2,$$

unde fit

$$\partial p=\partial\mathrm{U}(z-1)^2-2\mathrm{U}\partial z(z-1)=(z-1)\left[\partial\mathrm{U}(z-1)-2\mathrm{U}\partial z\right],$$

hincque

$$q\partial z=(z-1)[2\mathrm{U}\partial z-\partial\mathrm{U}(z-1)];$$

quia ergo q factorem habet $z-1$, formula $\int\frac{q\partial z}{lz}$, semper in partes resolvi potest, quarum integralia per corollarium primum assignare licet, si modo U fuerit aggregatum ex quotcunque potestatibus ipsius z; unde sequens deducitur theorema.

Theorema.

§. 27. Si fuerit

$$\mathrm{P}=\mathrm{A}z^{\alpha}+\mathrm{B}z^{\beta}+\mathrm{C}z^{\gamma}+\mathrm{D}z^{\delta}+\text{etc.}$$

ita ut summa coefficientium

$$\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\text{etc.}=0,$$

tum erit

$$\int\frac{\mathrm{P}\partial z}{lz}=\mathrm{A}l(\alpha+1)+\mathrm{B}l(\beta+1)+\mathrm{C}l(\gamma+1)+\mathrm{D}l(\delta+1)+\text{etc.}.$$

si quidem post integrationem statuatur $z=1$.

Demonstratio.

Cum hoc ipso casu, quo post integrationem ponitur $z=1$, sit

$$\int\frac{z^n\partial z}{lz}=l(n+1)+\Delta$$

denotante Δ illam constantem infinitam integratione ingressam, erit

$$\mathrm{A}\int\frac{z^{\alpha}\partial z}{lz}=\mathrm{A}l(\alpha+1)+\mathrm{A}\Delta$$

eodemque modo

$$\mathrm{B}\int\frac{z^{\alpha}\partial z}{lz}=\mathrm{B}l(\beta+1)+\mathrm{B}\Delta$$

etc; etc.

si nunc haec integralia omnia in unam summam colligantur, erit ob

$$(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\text{etc.})\Delta=0$$

integrate quaesitum

$$\int\frac{\mathrm{P}\partial z}{lz}=\mathrm{A}l(\alpha+1)+\mathrm{B}l(\beta+1)+\mathrm{C}l(\gamma+1)+\mathrm{D}l(\delta+1)+\text{etc.}$$

q. e. d .

Corollarium 1.

§. 28. Quia supponimus

$$\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\text{etc.}=0,$$

evidens est, formulam

$$\mathrm{P}=\mathrm{A}z^{\alpha}+\mathrm{B}z^{\beta}+\mathrm{C}z^{\gamma}+\mathrm{D}z^{\delta}+\text{etc.}$$

factorem habere $z-1$, quaemadmodum jam ante notavimus.

Corollarium 2.

§. 29. Quia est

$$(z-1)^{n}=z^{n}-\tfrac{n}{1}z^{n-1}+\tfrac{n(n-1)}{1\cdot 2}z^{n-2}-\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}z^{n-3}+\text{etc.},$$

hoc valore loco P posito erit

$$\mathrm{A}=1 \text{ et } \alpha=n,$$

deinde

$$\mathrm{B}=-\tfrac{n}{1} \text{ et } \beta=n-1,$$

porro

$$\mathrm{C}=\tfrac{n(n-1)}{1\cdot 2} \quad \text{et } \gamma=n-2 \quad \text{etc.}$$

hinc igitur erit

$$\int\tfrac{(z-1)^{n}\partial z}{lz}=l(n+1)-\tfrac{n}{1}ln+\tfrac{n(n-1)}{1\cdot 2}l(n-1)-\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}l(n-2)$$
$$+\tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}l(n-3)-\text{etc.},$$

si modo exponens n fuerit nihilo maior vel saltem unitate non minor, quia alioquin casu $z=1$ fractio fieret infinita; hoc autem non obstante area supra considerata fiet finita, ita ut sufficiat, dummodo sit $n>0$.

Exemplum 2

§. 30. Sit

$$\mathrm{P}=\tfrac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}};\text{erit}$$

$$\int \mathrm{P}\partial z=\tfrac{\pi}{2n}\cos.\tfrac{\pi u}{2n},$$

siquidem post integrationem ponatur $z=1$, quem ergo valorem litterae S tribuimus. Nunc spectata z ut constante erit

$$\int \mathrm{P}\partial u=\tfrac{1}{1+z^{2n}}\left(\int z^{n-u-1}\partial u+\int z^{n+u-1}\partial u\right)$$

ideoque

$$\int P\partial u = \frac{-z^{n-u-1}+z^{n+u-1}}{\left(1+z^{2n}\right)lz},$$

unde fiet

$$\int S\partial u = \int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}} \cdot \frac{\partial z}{lz};$$

cum igitur sit $\cos.\frac{\pi u}{2n} = \sin.\frac{\pi(n-u)}{2n}$, erit

$$\int S\partial u = \int \frac{\pi\partial u}{2n\sin.\frac{\pi(n-u)}{2n}};$$

hinc, si ponamus $\frac{\pi(n-u)}{2n} = \varphi$, erit $\partial\varphi = -\frac{\pi\partial u}{2n}$ ideoque

$$\int S\partial u = -\int \frac{\partial\varphi}{\sin.\varphi} = -l\text{tang.}\frac{1}{2}\varphi,$$

quocirca habebimus

$$\int S\partial u = -l\,\text{tang.}\frac{\pi(n-u)}{4n},$$

ita ut posito post integrationem $z = 1$ assecuti simus hanc integrationem

$$\int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}} \cdot \frac{\partial z}{lz} = -l\,\text{tang.}\frac{\pi(n-u)}{4n} = +l\,\text{tang.}\frac{\pi(n+u)}{4n}.$$

Exemplum 3

§. 31. Sit

$$P = \frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}};\text{erit}$$

$$\int P\partial z = \frac{\pi}{2n}\text{tang.}\frac{\pi u}{2n} = S,$$

unde fit

$$\int S\partial u = -l\cos.\frac{\pi u}{2n};$$

hinc, cum sit

$$\int P\partial u = \frac{-z^{n-u-1}-z^{n+u-1}}{\left(1-z^{2n}\right)lz},$$

nanciscimur sequentem integrationem, siquidem integrale a termino $z = 0$ usque ad terminum $z = 1$ extendatur,

$$\int \frac{z^{n-u-1}+z^{n+u-1}}{1-z^{2n}} \cdot \frac{\partial z}{lz} = +l\cos.\frac{\pi u}{2n}.$$

Haec quidem duo posteriora exempla iam ante fusius expedivi [E463]; unde iis magis evolvendis non immoror, sed ad sequens problema progredior.

Problema.

§. 32. *Si proponantur hae duae series infinitae*

$$P = z\cos.u + z^2\cos.2u + z^3\cos.3u + z^4\cos.4u + z^5\cos.5u + \text{ etc. et}$$
$$Q = z\sin.u + z^2\sin.2u + z^3\sin.3u + z^4\sin.4u + z^5\sin.5u + \text{etc.}$$

quae binas variabiles z et u involvunt, invenere relationes inter formulas integrale; $\int\frac{P\partial z}{z}, \int P\partial u$ *et* $\int\frac{Q\partial z}{z}, \int Q\partial u$ *, aliasque formulas integrales per continuam integrationem inde natas.*

Solutio.

Cum utraque series sit recurrens, reperitur per formulas finitas

$$P = \frac{z\cos.u - zz}{1-2z\sin.u+zz} \text{ et } Q = \frac{z\sin.u}{1-2z\sin.u+zz},$$

unde fit

$$\int\frac{P\partial z}{z} = \frac{\partial z\cos.u - z\partial z}{1-2z\sin.u+zz} = -l\sqrt{(1-2z\cos.u+zz)}$$
$$\text{et } \int Q\partial u = \int\frac{z\partial u\sin.u}{1-2z\sin.u+zz} = +l\sqrt{(1-2z\cos.u+zz)},$$

ita ut sit

$$\int\frac{P\partial z}{z} = -\int Q\partial u \;;$$

tum vero etiam erit

$$\int\frac{Q\partial z}{z} = \int\frac{\partial z\sin.u}{1-2z\sin.u+zz} = \text{arc.tang.}\frac{z\sin.u}{1-z\cos.u} \;;$$

at si iste arcus differentietur sumto solo angulo u variabili, erit

$$\frac{1}{\partial u}\partial.\text{arc.tang.}\frac{z\sin.u}{1-z\cos.u} = \frac{z\cos.u - zz}{1-2z\sin.u+zz},$$

ita ut sit

$$\int\frac{Q\partial z}{z} = \int P\partial u.$$

§. 33. Verum eaedem relationes facilius ex ipsis seriebus derivantur: cum enim sit

$$P = z\cos.u + z^2\cos.2u + z^3\cos.3u + z^4\cos.4u + \text{etc.}$$

erit

$$\int \frac{P\partial z}{z} = \frac{z\cos.u}{1} + \frac{zz\cos.2u}{2} + \frac{z^3\cos.3u}{3} + \text{etc. et}$$
$$\int P\partial u = \frac{z\sin.u}{1} + \frac{zz\sin.2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc.}$$

et quia est

$$Q = z\sin.u + zz\sin.2u + z^3\sin.3u + \text{etc. erit}$$
$$\int \frac{Q\partial z}{z} = \frac{z\sin.u}{2} + \frac{zz\sin.2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc. et}$$
$$\int Q\partial u = -\frac{z\cos.u}{2} - \frac{zz\cos.2u}{2} - \frac{z^3\cos.3u}{3} - \text{etc.}$$

unde, manifestum est fore

$$\int \frac{P\partial z}{z} = -\int Q\partial u \text{ et } \int \frac{Q\partial z}{z} = \int P\partial u.$$

§. 34. Quo hoc modo ulterius progredi liceat, statuamus brevitatis gratia

$$P' = \frac{z\cos.u}{1} + \frac{zz\cos.2u}{2} + \frac{z^3\cos.3u}{3} + \text{etc. et } Q' = \frac{z\sin.u}{1} + \frac{zz\sin.2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc.}$$
$$P'' = \frac{z\cos.u}{1^2} + \frac{zz\cos.2u}{2^2} + \frac{z^3\cos.3u}{3^2} + \text{etc. et } Q'' = \frac{z\sin.u}{1^2} + \frac{zz\sin.2u}{2^2} + \frac{z^3\sin.3u}{3^2} + \text{etc.}$$
$$P''' = \frac{z\cos.u}{1^3} + \frac{zz\cos.2u}{2^3} + \frac{z^3\cos.3u}{3^3} + \text{etc. et } Q''' = \frac{z\sin.u}{1^3} + \frac{zz\sin.2u}{2^3} + \frac{z^3\sin.3u}{3^3} + \text{etc.}$$
$$P'''' = \frac{z\cos.u}{1^4} + \frac{zz\cos.2u}{2^4} + \frac{z^3\cos.3u}{3^4} + \text{etc. et } Q''' = \frac{z\sin.u}{1^4} + \frac{zz\sin.2u}{2^4} + \frac{z^3\sin.3u}{3^4} + \text{etc.}$$
etc. etc. etc. etc.

et hinc comparationes ante inventae continuabuntur

$$P' = \int \frac{P\partial z}{z} = -\int Q\partial u, \; Q' = \int \frac{Q\partial z}{z} = \int P\partial u,$$
$$P'' = \int \frac{P'\partial z}{z} = -\int Q'\partial u, \; Q'' = \int \frac{Q'\partial z}{z} = \int P'\partial u,$$
$$P''' = \int \frac{P''\partial z}{z} = -\int Q''\partial u, \; Q''' = \int \frac{Q''\partial z}{z} = \int P''\partial u,$$
$$P'''' = \int \frac{P'''\partial z}{z} = -\int Q'''\partial u, \; Q'''' = \int \frac{Q'''\partial z}{z} = \int P'''\partial u,$$
etc. etc. etc. etc.

unde plures insignes relationes deduci possunt.

§. 35. Maxime autem notatu dignae et ad nostrum institutum accommodatae sunt eae relationes, ubi formulae integrales, in quibus sola *z* est variabilis, reducuntur ad alias formulas integrales, in quibus sola *u* est variabilis; cujusmodi sunt, quae sequuntur

$$\mathrm{P}' = \int \frac{\mathrm{P}\partial z}{z} = -\int \mathrm{Q}\partial u,$$
$$\mathrm{P}'' = \int \frac{\partial z}{z}\int \frac{\mathrm{P}\partial z}{z} = +\int \partial u\int \mathrm{P}\partial u,$$
$$\mathrm{P}''' = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{P}\partial z}{z} = +\int \partial u\int \partial u\int \mathrm{Q}\partial u,$$
$$\mathrm{P}'''' = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{P}\partial z}{z} = -\int \partial u\int \partial u\int \partial u\int \mathrm{Q}\partial u,$$
$$\mathrm{P}^{\mathrm{V}} = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{P}\partial z}{z} = -\int \partial u\int \partial u\int \partial u\int \partial u\int \mathrm{Q}\partial u,$$
etc. etc. etc.

Similique modo pro altero genere

$$\mathrm{Q}' = \int \frac{\mathrm{Q}\partial z}{z} = +\int \mathrm{P}\partial u,$$
$$\mathrm{Q}'' = \int \frac{\partial z}{z}\int \frac{\mathrm{Q}\partial z}{z} = -\int \partial u\int \mathrm{Q}\partial u,$$
$$\mathrm{Q}''' = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{Q}\partial z}{z} = -\int \partial u\int \partial u\int \mathrm{P}\partial u,$$
$$\mathrm{Q}'''' = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{Q}\partial z}{z} = +\int \partial u\int \partial u\int \partial u\int \mathrm{Q}\partial u,$$
$$\mathrm{Q}^{\mathrm{V}} = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{Q}\partial z}{z} = +\int \partial u\int \partial u\int \partial u\int \partial u\int \mathrm{P}\partial u,$$
etc. etc. etc.

§. 36. Quodsi jam nostrarum serierum, sive quod eadem redit, quantitatum

$$\mathrm{P},\ \mathrm{P}',\mathrm{P}'',\ \mathrm{P}''',\ \mathrm{P}'''',\text{etc. et } \mathrm{Q},\ \mathrm{Q}',\mathrm{Q}'',\mathrm{Q}''',\ \mathrm{Q}'''',\text{etc.}$$

tantum valores desideremus, quos adipiscuntur posito $z=1$, hoc commodi assequimur, ut in formulis integralibus, ubi solus angulus *u* pro variabili habetur, statim ante integrationes ponere liceat $z=1$, hoc autem facto erit

$$\mathrm{P} = \frac{\cos.u-1}{2-2\cos.u} = -\frac{1}{2} \text{ et } \mathrm{Q} = \frac{\sin.u}{2-2\cos.u} = \frac{1}{2}\cot.\frac{1}{2}u,$$

tum vera porro

$$\int \mathrm{P}\partial u = \mathrm{A} - \frac{1}{2}u,$$
$$\int \partial u\int \mathrm{P}\partial u = \mathrm{B} + \mathrm{A}u - \frac{1}{4}uu,$$
$$\int \partial u\int \partial u\int \mathrm{P}\partial u = \mathrm{C} + \mathrm{B}u + \frac{1}{2}\mathrm{A}uu - \frac{1}{12}u^3,$$
$$\int \partial u\int \partial u\int \partial u\int \mathrm{P}\partial u = \mathrm{D} + \mathrm{C}u + \frac{1}{2}\mathrm{B}uu + \frac{1}{6}\mathrm{A}u^3 - \frac{1}{48}u^4,$$
$$\mathrm{P}^{\mathrm{V}} = \int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\partial z}{z}\int \frac{\mathrm{P}\partial z}{z} = -\int \partial u\int \partial u\int \partial u\int \partial u\int \mathrm{Q}\partial u,$$
etc. etc. etc.

at pro formulis, ubi est Q, calculus non tam concinne succedit; erit enim

$$\mathrm{Q} = \tfrac{1}{2}\cot.\tfrac{1}{2}u,$$
$$\int \mathrm{Q}\partial u = l\sin.\tfrac{1}{2}u,$$
$$\int \partial u \int \mathrm{Q}\partial u = \int \partial u l\sin.\tfrac{1}{2}u,$$

quae formula cum omnem integrationem respuat, vix ulterius progredi licet; interim tamen erit

$$\int \partial u \int \partial u \int \mathrm{Q}\partial u = \int \partial u \int \partial u l\sin.\tfrac{1}{2}u,$$
$$\int \partial u \int \partial u \int \partial u \int \mathrm{Q}\partial u = \int \partial u \int \partial u \int \partial u l\sin.\tfrac{1}{2}u.$$

§. 37. Quod ad priores formulas variabilem z involventes attinet, per notas reductiones elicitur

$$\int \tfrac{\mathrm{P}'\partial z}{z} = \int \tfrac{\partial z}{z}\int \tfrac{\mathrm{P}\partial z}{z} = lz\int \tfrac{\mathrm{P}\partial z}{z} - \int \tfrac{\mathrm{P}\partial z}{z}lz,$$

ubi prius membrum $lz\int \tfrac{\mathrm{P}\partial z}{z}$ evanescit posito $z=1$, tum vero

$$\int \tfrac{\partial z}{z}\int \tfrac{\mathrm{P}'\partial z}{z} = \int \tfrac{\partial z}{z}\int \tfrac{\partial z}{z}\int \tfrac{\mathrm{P}\partial z}{z} = +\int \tfrac{\mathrm{P}\partial z}{z}\cdot\frac{(lz)^2}{2},$$

quibus expressionibus ulterius exhibitis colligimus fore

$\mathrm{P}' = \int \tfrac{\mathrm{P}\partial z}{z}$	$\mathrm{Q}' = \int \tfrac{\mathrm{Q}\partial z}{z}$
$\mathrm{P}'' = -\int \tfrac{\mathrm{P}\partial z}{z}lz,$	$\mathrm{Q}'' = -\int \tfrac{\mathrm{Q}\partial z}{z}lz,$
$\mathrm{P}''' = +\int \tfrac{\mathrm{P}\partial z}{z}\frac{(lz)^2}{1\cdot 2},$	$\mathrm{Q}''' = +\int \tfrac{\mathrm{Q}\partial z}{z}\frac{(lz)^2}{1\cdot 2},$
$\mathrm{P}^{\mathrm{IV}} = -\int \tfrac{\mathrm{P}\partial z}{z}\frac{(lz)^3}{1\cdot 2\cdot 3}$	$\mathrm{Q}^{\mathrm{IV}} = -\int \tfrac{\mathrm{Q}\partial z}{z}\frac{(lz)^3}{1\cdot 2\cdot 3}.$

§. 38. Ex his igitur sequentium formularum integralium valores assignare possumus, casu quo $z=1$,

$$\mathrm{P} = -\tfrac{1}{2}$$
$$\mathrm{P}' = \int \tfrac{\mathrm{P}\partial z}{z} = -l\sin.\tfrac{1}{2}u,$$
$$\mathrm{P}'' = -\int \tfrac{\mathrm{P}\partial z}{z}lz = -\mathrm{B} - \mathrm{A}u + \tfrac{1}{4}uu,$$
$$\mathrm{P}''' = +\int \tfrac{\mathrm{P}\partial z}{z}\frac{(lz)^2}{1\cdot 2} = \int \partial u \int \partial u l\sin.\tfrac{1}{2}u,$$
$$\mathrm{P}^{\mathrm{IV}} = -\int \tfrac{\mathrm{P}\partial z}{z}\frac{(lz)^3}{1\cdot 2\cdot 3} = \mathrm{D} + \mathrm{C}u + \tfrac{1}{2}\mathrm{B}uu + \tfrac{1}{6}\mathrm{A}u^3 - \tfrac{1}{48}u^4,$$
$$\mathrm{P}^{\mathrm{V}} = +\int \tfrac{\mathrm{P}\partial z}{z}\frac{(lz)^4}{1\cdot 2\cdot 3\cdot 4} = \int \partial u \int \partial u \int \partial u \int \partial u l\sin.\tfrac{1}{2}u,$$

etc.

Eodem modo

$$\mathrm{Q} = -\tfrac{1}{2}\cot.\tfrac{1}{2}u$$
$$\mathrm{Q}' = \int \tfrac{\mathrm{Q}\partial z}{z} = A - \tfrac{1}{2}u,$$
$$\mathrm{Q}'' = -\int \tfrac{\mathrm{Q}\partial z}{z}\cdot\tfrac{lz}{1} = -\int \partial u l \sin.\tfrac{1}{2}u,$$
$$\mathrm{Q}''' = +\int \tfrac{\mathrm{Q}\partial z}{z}\tfrac{(lz)^2}{1\cdot 2} = -\mathrm{C} - \mathrm{B}u - \tfrac{1}{2}\mathrm{A}uu + \tfrac{1}{12}u^3,$$
$$\mathrm{Q}^{\mathrm{IV}} = -\int \tfrac{\mathrm{Q}\partial z}{z}\tfrac{(lz)^3}{6} = \int \partial u \int \partial u \int \partial u l \sin.\tfrac{1}{2}u,$$
$$\mathrm{Q}^{\mathrm{V}} = +\int \tfrac{\mathrm{Q}\partial z}{z}\tfrac{(lz)^4}{24} = E + \mathrm{D}u + \tfrac{1}{2}\mathrm{C}uu + \tfrac{1}{6}\mathrm{B}u^3 + \tfrac{1}{24}\mathrm{A}u^4 - \tfrac{1}{240}u^5$$

etc.

§. 39. Cum igitur sit

$$\mathrm{P} = \tfrac{z\cos.u - zz}{1-2z\cos.u+zz} \text{ et } \mathrm{Q} = \tfrac{z\sin.u}{1-2z\cos.u+zz},$$

hactenus id sumus assecuti, ut harum duarum formularum integralium

$$\int \tfrac{\partial z(\cos.u - z)}{1-2z\cos.u+zz}(lz)^n \text{ et } \int \tfrac{\partial z\sin.u}{1-2z\cos.u+zz}(lz)^n$$

valores casu $z=1$ commode per angulmn u assignare valeamus, si modo constaret, quo facto quantitates A, B, C, D, etc. determinari oporteat, id quod vix alio modo nisi per ipsas series, unde hae quantitates sunt natae, fieri posse videtur.

§. 40. Omissis igitur formulis integralibus, quae quantitatem Q involvent, quippe quarum integratio minus succedit, alteras tantum consideremus, et posito statim $z=1$, ubi sit $\mathrm{P} = -\tfrac{1}{2}$, ita ut sit

$$\cos.u + \cos.2u + \cos.3u + \cos.4u + \text{etc.} = -\tfrac{1}{2},$$

si per ∂u multiplicemus et integremus, habebimus

$$\mathrm{Q}' = \tfrac{\sin.u}{1} + \tfrac{\sin.2u}{2} + \tfrac{\sin.3u}{3} + \tfrac{\sin.4u}{4} + \tfrac{\sin.5u}{5} + \text{etc.} = \mathrm{A} - \tfrac{1}{2}u,$$

quae constans nihilo aequalis videri potest, quia posito $u=0$ summa seriei evanescere videtur; at sumto angulo u infinite parvo series praebebit $u+u+u+u+u+u+\text{etc.}$ in infinitum; notum autem est, talem seriem summam finitam habere posse, unde hoc casu omisso statuamus $u=\pi$, seu potius $u=\pi+\omega$, prodibitque haec series existente ω angulo infinite parvo,

$$-\omega+\omega-\omega+\omega-\omega+\omega-\omega+\text{etc.};$$

ubi, quia signa alternantur, nullum est dubium, quin summa seriei evanescat, quae cum esse debeat $\mathrm{A}-\frac{\pi}{2}$, evidens est fieri constantem $\mathrm{A}=\frac{\pi}{2}$, ita, ut jam habeamus

$$Q'=\frac{\sin.u}{1}+\frac{\sin.2u}{2}+\frac{\sin.3u}{3}+\frac{\sin.4u}{4}+\frac{\sin.5u}{5}+\text{etc.}=\frac{\pi-u}{2}.$$

Hoc modo constantem determinandi Illustr. *Daniel Bernoulli* primus est usus, qui praeterea multa praeclara circa indolem harum serierum annotavit.

§. 41. Multiplicemus porro hanc ultimam seriem per ∂u et integratio dabit

$$P''=\frac{\cos.u}{1^2}+\frac{\cos.2u}{2^2}+\frac{\cos.3u}{3^2}+\frac{\cos.4u}{4^2}+\frac{\cos.5u}{5^2}+\text{etc.}=\mathrm{B}-\frac{\pi u}{2}+\frac{uu}{4},$$

ad quam constantem inveniendam ponamus primo $u=0$, fietque

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}=\mathrm{B}.$$

Cujus seriei summam iam pridem primus demonstravi esse $=\frac{\pi\pi}{6}$; verum si haec veritas nobis esset ignota, egregia illa methodo a magno *Bernoullio* adhibita utamur, ac ponamus $u=\pi$ eritque

$$-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{6^2}-\text{etc.}=\mathrm{B}-\frac{\pi\pi}{2}+\frac{\pi\pi}{4}=\mathrm{B}-\frac{\pi\pi}{4};$$

ambae hae series additae dabunt

$$\frac{2}{2^2}+\frac{2}{4^2}+\frac{2}{6^2}+\frac{2}{8^2}+\text{etc.}=2\mathrm{B}-\frac{\pi\pi}{4},$$

cujus duplum praebet

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}=4\mathrm{B}-\frac{\pi\pi}{2}=\mathrm{B},$$

unde colligitur $\mathrm{B}=\frac{\pi\pi}{6}$, ita ut sit

$$P''=\frac{\cos.u}{1^2}+\frac{\cos.2u}{2^2}+\frac{\cos.3u}{3^2}+\frac{\cos.4u}{4^2}+\text{etc.}=\frac{\pi\pi}{6}-\frac{\pi u}{2}+\frac{uu}{4}.$$

§. 42. Eodem modo ulterius progrediamur et denuo per *du* multiplicando et integrando adipiscimur

$$Q''=\frac{\sin.u}{1^3}+\frac{\sin.2u}{2^3}+\frac{\sin.3u}{3^3}+\frac{\sin.4u}{4^3}+\text{etc.}=\mathrm{C}+\frac{\pi\pi u}{6}-\frac{\pi uu}{4}+\frac{u^3}{12};$$

ubi si statuatur $u=0$, summa seriei manifesto evanescit, prodiret enim posito $u=\omega$

$$\frac{\omega}{1^2}+\frac{\omega}{2^2}+\frac{\omega}{3^2}+\frac{\omega}{4^2}+\text{etc.}=\frac{\omega\pi\pi}{6},$$

quae ob $\omega=0$ fit $=0$, sicque erit $\mathrm{C}=0$, ideoque

$$\mathrm{Q}'''=\frac{\sin.u}{1^3}+\frac{\sin.2u}{2^3}+\frac{\sin.3u}{3^3}+\frac{\sin.4u}{4^3}+\text{etc.}=\frac{\pi\pi u}{6}-\frac{\pi uu}{4}+\frac{u^3}{12}.$$

§. 43. Ducatur haec series in $-\partial u$, et integratio praebebit

$$\mathrm{P}^{\mathrm{IV}}=\frac{\cos.u}{1^4}+\frac{\cos.2u}{2^4}+\frac{\cos.3u}{3^4}+\frac{\cos.4u}{4^4}+\text{etc.}=\mathrm{D}-\frac{\pi\pi uu}{12}-\frac{\pi u^3}{12}+\frac{u^3}{48};$$

hinc sumto $u=0$ fiet

$$\mathrm{P}^{\mathrm{IV}}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=\mathrm{D};$$

nunc vero fiat etiam $u=\pi$, fietque

$$-\frac{1}{1^4}+\frac{1}{2^4}-\frac{1}{3^4}+\frac{1}{4^4}-\frac{1}{5^4}+\text{etc.}=\mathrm{D}-\frac{\pi^4}{48};$$

hae autem ambae series additae dant

$$\frac{2}{2^4}+\frac{2}{4^4}+\frac{2}{6^4}+\frac{2}{8^4}+\text{etc.}=2\mathrm{D}-\frac{\pi^4}{48},$$

quae octies sumta ut numeratores fiant $=2^4$, praebebit

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=16\mathrm{D}-\frac{\pi^4}{6},$$

unde oritur $\mathrm{D}=\frac{\pi^4}{90}$, quae est eadem summa seriei

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.},$$

quam jam dudum inveneram; habebimus jam

$$\mathrm{P}''''=\frac{\cos.u}{1^4}+\frac{\cos.2u}{2^4}+\frac{\cos.3u}{3^4}+\frac{\cos.4u}{4^4}+\text{etc.}=\frac{\pi^4}{90}-\frac{\pi\pi u^2}{12}+\frac{\pi u^3}{12}-\frac{u^4}{48}.$$

§. 44. Multiplicando iterum per *du* et integrando consequimur

$$\mathrm{Q}^{\mathrm{V}}=\frac{\sin.u}{1^5}+\frac{\sin.2u}{2^5}+\frac{\sin.3u}{3^5}+\frac{\sin.4u}{4^5}+\text{etc.}=\mathrm{E}+\frac{\pi^4 u}{90}-\frac{\pi\pi u^3}{36}+\frac{\pi u^4}{48}-\frac{u^5}{240},$$

ubi uti in casu penultimo constans E iterum fit $=0$, ita ut habeamus

$$\mathrm{Q}^{\mathrm{V}}=\frac{\sin.u}{1^5}+\frac{\sin.2u}{2^5}+\frac{\sin.3u}{3^5}+\frac{\sin.4u}{4^5}+\text{etc.}=\frac{\pi^4u}{90}-\frac{\pi\pi u^3}{36}+\frac{\pi u^4}{48}-\frac{u^5}{240}.$$

45. Multiplicemus denuo per $-du$ prodibitque integrando

$$\mathrm{P}^{\mathrm{VI}}=\frac{\cos.u}{1^6}+\frac{\cos.2u}{2^6}+\frac{\cos.3u}{3^6}+\frac{\cos.4u}{4^6}+\text{etc.}$$
$$=\mathrm{F}-\frac{\pi^4}{90}\cdot\frac{uu}{2}+\frac{\pi\pi}{6}\cdot\frac{u^4}{24}-\frac{\pi}{2}\cdot\frac{u^5}{120}+\frac{1}{2}\cdot\frac{u^6}{720},$$

ubi ad constantem determinandam ponatur $u=0$ eritque

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=\mathrm{F};$$

tum vero sumatur $u=\pi$ et fiet

$$-\frac{1}{1^6}+\frac{1}{2^6}-\frac{1}{3^6}+\frac{1}{4^6}-\text{etc.}=\mathrm{F}-\frac{\pi^6}{480},$$

quae additae dant

$$\frac{2}{1^6}+\frac{2}{2^6}+\frac{2}{3^6}+\frac{2}{4^6}+\text{etc.}=2\mathrm{F}-\frac{\pi^6}{480},$$

quae multiplicetur per 32, ut omnes numeratores fiant $64=2^6$, et orietur

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=64\mathrm{F}-\frac{\pi^6}{15}=\mathrm{F},$$

unde colligitur $\mathrm{F}=\frac{\pi^6}{945}$, ita ut sit

$$\mathrm{P}^{\mathrm{VI}}=\frac{\cos.u}{1^6}+\frac{\cos.2u}{2^6}+\frac{\cos.3u}{3^6}+\frac{\cos.4u}{4^6}+\text{etc.}$$
$$=\frac{\pi^6}{945}-\frac{\pi^4}{90}\cdot\frac{u^2}{2}+\frac{\pi^2}{6}\cdot\frac{u^4}{24}-\frac{\pi}{2}\cdot\frac{u^5}{120}+\frac{1}{2}\cdot\frac{u^6}{720},$$

§. 46. Has series ulterius continuare superfluum foret, cum lex progressionis iam satis sit manifesta, praecipue si in subsidium vocentur summationes potestatum reciprocarum parium, quas olim usque ad potestatem trigesimam supputatas dedi. Quod quo clarius perspiciatur, istas summas sequenti modo repraesentemus

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}=\alpha\pi\pi,\text{ut sit }\alpha=\tfrac{1}{6},$$
$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}=\beta\pi^4,\text{ut sit }\beta=\tfrac{1}{90},$$
$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\text{etc.}=\gamma\pi^6,\text{ut sit }\gamma=\tfrac{1}{945},$$
$$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{6^8}+\text{etc.}=\delta\pi^4,\text{ut sit }\delta=\tfrac{1}{9450},$$
etc.

atque, his positis sequentes habebimus integrationes pro casu scilicet $z=1$,

$$\mathrm{Q}'=\int\frac{\partial z\sin.u}{1-2z\sin.u+zz}=\tfrac{1}{2}\pi-\tfrac{1}{2}u=\text{arc.tang.}\frac{\sin.u}{1-\cos.u},$$
$$\mathrm{P}''=-\int\frac{\partial z(\cos.u-z)}{1-2z\sin.u+zz}\cdot\frac{lz}{1}=\alpha\pi\pi-\tfrac{1}{2}\pi u+\tfrac{1}{2}\cdot\tfrac{uu}{2},$$
$$\mathrm{Q}'''=+\int\frac{\partial z\sin.u}{1-2z\sin.u+zz}\cdot\frac{(lz)^2}{2}=\alpha\pi\pi\,\frac{u}{1}-\tfrac{1}{2}\pi\,\frac{uu}{2}+\tfrac{1}{2}\cdot\frac{u^3}{6},$$
$$\mathrm{P}^{\mathrm{IV}}=-\int\frac{\partial z(\cos.u-z)}{1-2z\sin.u+zz}\cdot\frac{(lz)^3}{6}=\beta\pi^4-\alpha\pi\pi\,\frac{uu}{2}+\tfrac{1}{2}\pi\,\frac{u^3}{6}-\tfrac{1}{2}\cdot\frac{u^4}{24},$$
$$\mathrm{Q}^{\mathrm{V}}=+\int\frac{\partial z\sin.u}{1-2z\sin.u+zz}\cdot\frac{(lz)^4}{24}=\beta\pi^4\,\frac{u}{1}-\alpha\pi\pi\,\frac{u^3}{6}+\tfrac{1}{2}\pi\,\frac{u^4}{24}-\tfrac{1}{2}\cdot\frac{u^5}{120},$$
$$\mathrm{P}^{\mathrm{VI}}=-\int\frac{\partial z(\cos.u-z)}{1-2z\sin.u+zz}\cdot\frac{(lz)^5}{120}=\gamma\pi^6\,\frac{u}{1}-\beta\pi^4\,\frac{uu}{2}+\alpha\pi\pi\,\frac{u^4}{24}-\tfrac{1}{2}\pi\,\frac{u^5}{120}+\tfrac{1}{2}\cdot\frac{u^6}{720},$$
$$\mathrm{Q}^{\mathrm{VII}}=+\int\frac{\partial z\sin.u}{1-2z\sin.u+zz}\cdot\frac{(lz)^6}{720}=\gamma\pi^6\,\frac{u}{1}-\beta\pi^4\,\frac{u^3}{6}+\alpha\pi\pi\,\frac{u^5}{120}-\tfrac{1}{2}\pi\,\frac{u^6}{720}+\tfrac{1}{2}\cdot\frac{u^7}{5040}$$
etc.

§. 47. Operae pretium erit aliquos casus, quibus angulo *u* datus valor tribuitur, ob oculos exponere. Ponamus igitur $u=0$, quo casu formulae nostrae alternatim evanescunt, reliquae vero praebebunt

$$-\int\frac{\partial z}{1-z}lz=\alpha\pi\pi=\frac{\pi\pi}{6},$$
$$-\int\frac{\partial z}{1-z}\frac{(lz)^3}{6}=\beta\pi^4=\frac{\pi^4}{90},$$
$$-\int\frac{\partial z}{1-z}\frac{(lz)^5}{120}=\gamma\pi^6=\frac{\pi^6}{945};$$

his affines sunt formulae, quae oriuntur ex positione $u=\pi$, ubi iterum abeunt alternae sinum *u* involventes et remanebunt sequentes

$$\int \frac{\partial z}{1+zz} = \frac{\pi}{4},$$
$$\int \frac{z\partial z}{1+zz} \cdot lz = -\frac{\pi\pi}{48},$$
$$\int \frac{\partial z}{1+zz} \cdot \frac{(lz)^2}{2} = \frac{\pi^3}{32},$$
$$\int \frac{z\partial z}{1+zz} \cdot \frac{(lz)^3}{6} = -\frac{7\pi^4}{90\cdot 128}.$$

§ 48. Hic notatu dignum occurrit, quod valores alterni, quos hic omisimus, etiam evanescant posito $u=\pi$ *;* deinde non minus notatu dignum est easdem formulas quoque evanescere posito $u=2\pi$ *,* sola prima excepta, quippe quae etiam non evanescit posito $u=0$; reliquae vero, scilicet tertia, quinta, septima etc., certe evanescunt casibus $u=0$ et $u=\pi$ *,* quin etiam $u=2\pi$ *.* Quod quo clarius appareat, has formulas per factores repraesentemus eritque tertiae valor

$$=\tfrac{1}{12}u(\pi-u)(2\pi-u),$$

quintae vero valor reperitur

$$=\tfrac{1}{720}(\pi-u)(2\pi-u)(4\pi\pi+6\pi u-3uu),$$

quod etiam in sequentibus usu venit. In genere autem observari meretur omnes nostras formulas sola prima excepta eosdem sortiri valores, sive ponatur $u=0$ sive $u=2\pi$, quippe quibus tam idem sinus quam cosinus respondet. Videtur quidem eundem consensum locum habere debere, si ponatur $u=4\pi$ et $u=6\pi$ *;* verum lllustr. *Bernoullius* iam luculenter ostendit angulum u in his valoribus non ultra quatuor rectos augeri posse. Huiusmodi autem anomalia etiam in omnibus vulgaribus seriebus, quibus arcus exprimuntur, occurrit atque adeo in *Leibniziana*, in qua est.

$$u=\frac{\text{tang.}u}{1}-\frac{(\text{tang.}u)^3}{3}+\frac{(\text{tang.}u)^5}{5}-\frac{(\text{tang.}u)^7}{7}+\frac{(\text{tang.}u)^9}{9}-\text{etc.},$$

angulum u non ultra 180° augere licet. Si enim poneremus $u=180°+u$, foret utique $\text{tang.}u=\text{tang.}u$ neque tamen series illa exprimeret arcum $\pi+u$, sed tantum arcum u, cuiusmodi phaenomena etiam in aliis similibus seriebus locum habent. Quod autem prima series hinc plerumque excipi debeat, ratio in eo est sita, quod in formula integrali posito $u=0$ denominator fiat $1-z$, qui casu $z=1$ evanescit, ideoque formula in infinitum excrescit, id quod in sequentibus, quae per lz sunt multiplicatae, non amplius evenit, quia $\frac{lz}{1-z}$ casu $z=1$ non amplius fit infinitus, sed tantum $=-1$, et si maior potestas logarithmi adsit, fit adeo $=0$.

§. 49. Ponamus nunc etiam $u=90°$ seu $u=\frac{\pi}{2}$, ut sit $\cos.u=0$ et $\sin.u=1$, hocque casu omnes formulae generales sequentes obtinebunt valores

$$\int \frac{\partial z}{1+zz} = \frac{\pi}{4},$$
$$\int \frac{\partial z}{1+zz} \cdot lz = -\frac{\pi\pi}{48},$$
$$\int \frac{\partial z}{1+zz} \cdot \frac{(lz)^2}{2} = \frac{\pi^3}{32},$$
$$\int \frac{\partial z}{1+zz} \cdot \frac{(lz)^3}{6} = -\frac{7\pi^4}{90 \cdot 128}.$$

§.50. Consideremus etiam casum $u = 60°$, sive $u = \frac{\pi}{3}$, ut sit $\cos.u = \frac{1}{2}$ et $\sin.u = \frac{\sqrt{3}}{2}$, et formulae generales perducent ad sequentia integralia.

$$\frac{\sqrt{3}}{2} \int \frac{\partial z}{1-z+zz} = \frac{\pi}{3},$$
$$-\frac{1}{2} \int \frac{\partial z(1-2z)}{1-z+zz} \cdot lz = \frac{\pi\pi}{36},$$
$$\frac{\sqrt{3}}{2} \int \frac{\partial z}{1-z+zz} \cdot \frac{(lz)^2}{2} = \frac{5\pi^3}{162}.$$

Simili modo si ponamus $u = 120°$, ut sit $\cos.u = -\frac{1}{2}$ et $\sin.u = \frac{\sqrt{3}}{2}$, et sequentes integrationes istis affines prodibunt

$$\frac{\sqrt{3}}{2} \int \frac{\partial z}{1+z+zz} = \frac{\pi}{6},$$
$$\frac{1}{2} \int \frac{\partial z(1+2z)}{1+z+zz} \cdot lz = -\frac{\pi\pi}{18},$$
$$\frac{\sqrt{3}}{2} \int \frac{\partial z}{1+z+zz} \cdot \frac{(lz)^2}{2} = \frac{2\pi^3}{81};$$

sicque pro lubitu numerus huismodi integrationum specialium augeri poterit.

§. 51. Quemadmodum istae integrationes memorabiles ex priore serie nostra P posito $z = 1$ sunt deductae, ita eodem modo alteram seriem Q pertractemus. Cum igitur sit

$$\mathrm{Q} = \sin.u + \sin.2u + \sin.3u + \sin.4u + \text{etc.} = \frac{1}{2}\cot.\frac{1}{2}u,$$

si per $-\partial u$ multiplicemus et integremus, reperitur series

$$\mathrm{P}' = \frac{\cos.u}{1} + \frac{\cos.2u}{2} + \frac{\cos.3u}{3} + \frac{\cos.4u}{4} + \text{etc.} = -l\sin.\frac{1}{2}u + \mathrm{A},$$

pro qua constante determinanda ponatur $u = \pi$, ut sit

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \text{etc.} = \mathrm{A},$$

quocirca fit $\mathrm{A} = -l2$, ita ut habeamus

$$\mathrm{P}' = \frac{\cos.u}{1} + \frac{\cos.2u}{2} + \frac{\cos.3u}{3} + \frac{\cos.4u}{4} + \text{etc.} = -l2\sin.\tfrac{1}{2}u,$$

pro quo valore scribamus brevitatis gratia $\Delta : u$, si quidem eum spectamus tanquam certam ipsius u functionem, ita ut sit $\mathrm{P}' = \Delta : u$.

§. 52. Multiplicando porro per $-\partial u$ et integrando, nancisimur hanc seriem

$$\mathrm{Q}'' = \frac{\sin.u}{1^2} + \frac{\sin.2u}{2^2} + \frac{\sin.3u}{3^2} + \frac{\sin.4u}{4^2} + \text{etc.} = \int \partial u \Delta{:}u = \Delta' : u,$$

ubi haec formula integralis involvet certam constantem, quam facile definire licet ex casu $u = 0$; quia enim series evanescit, fieri debet $\Delta' : u = 0$ sicque integratio plene determinatur.

§. 53. Si eodem modo ulterius progrediamur multiplicando per $-\partial u$ prodibit haec series

$$\mathrm{P}''' = \frac{\cos.u}{1^3} + \frac{\cos.2u}{2^3} + \frac{\cos.3u}{3^3} + \frac{\cos.4u}{4^3} + \text{etc.} = -\int \partial u \Delta'{:}u = \Delta'' : u.$$

Iam ad constantem, quae in hac expressione continetur, definiendam sit
I° $u = 0$ eritque

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} = \Delta'' : 0,$$

II° $u = \pi$ et fiet

$$-\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \frac{1}{5^3} + \text{etc.} = \Delta'' : \pi,$$

quibus additis prodit

$$\frac{2}{2^3} + \frac{2}{4^3} + \frac{1}{6^3} + \frac{1}{8^3} + \text{etc.} = \Delta'' : 0 + \Delta'' : \pi,$$

hacque quater sumta erit

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.} = 4\Delta'' : 0 + 4\Delta'' : \pi = \Delta'' : 0,$$

unde oritur

$$3\Delta'' : 0 + 4\Delta'' : \pi = 0,$$

ex qua constans in formulam nostram integralem

$$\Delta'' : u = -\int \partial u \Delta' : u$$

ingressa determinari debet.

§. 54. Multiplicemus denuo per $-\partial u$, et integremus, prodibitque

$$\mathrm{Q}^{\mathrm{IV}}=\frac{\sin.u}{1^4}+\frac{\sin.2u}{2^4}+\frac{\sin.3u}{3^4}+\frac{\sin.4u}{4^4}+\text{etc.}=\int\partial u\Delta'':u=\Delta''':u$$

atque haec functio $\Delta''':u$ ita debet determinari, ut evanescat sumto $u=0$, sive ut fiat $\Delta''':u=0.$ Eadem modo ulterius progrediendo fiet

$$\mathrm{P}^{\mathrm{V}}=\frac{\cos.u}{1^5}+\frac{\cos.2u}{2^5}+\frac{\cos.3u}{3^5}+\frac{\cos.4u}{4^5}+\text{etc.}=\int\partial u\Delta''':u=\Delta^{\mathrm{IV}}:u$$

huiusque functionis indoles sequenti modo determinabitur. Ponatur scilicet ut hactenus $u=0$ et $u=\pi$ eritque

$$\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\text{etc.}=\Delta^{\mathrm{IV}}:0$$

et

$$-\frac{1}{1^5}+\frac{1}{2^5}-\frac{1}{3^5}+\frac{1}{4^5}-\frac{1}{5^5}+\text{etc.}=\Delta^{\mathrm{IV}}:\pi,$$

hinc addendo

$$\frac{2}{2^5}+\frac{2}{4^5}+\frac{2}{6^5}+\frac{2}{8^5}+\text{etc.}=\Delta^{\mathrm{IV}}:0+\Delta^{\mathrm{IV}}:\pi$$

et multiplicando per 16

$$\frac{1}{2^5}+\frac{1}{4^5}+\frac{1}{6^5}+\frac{1}{8^5}+\text{etc.}=16\Delta^{\mathrm{IV}}:0+16\Delta^{\mathrm{IV}}:\pi=\Delta^{\mathrm{IV}}:0,$$

sicque fieri debebit

$$15\Delta^{\mathrm{IV}}:0+16\Delta^{\mathrm{IV}}:\pi=0$$

§. 55. Hinc igitur sequentes adipiscemur integrationes pro casu $z=1$

$$\text{I.}\quad -\int\frac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}=-l2\sin.\tfrac{1}{2}u=\Delta:u,$$

$$\text{II.}\quad \int\frac{\partial z\sin.u}{1-2z\cos.u+zz}\cdot lz=\int\partial u\Delta:u=\Delta':u,$$

$$\text{III.}\quad -\int\frac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{(lz)^2}{2}=-\int\partial u\Delta':u=\Delta'':u,$$

$$\text{IV.}\quad \int\frac{\partial z\sin.u}{1-2z\cos.u+zz}\cdot\frac{(lz)^3}{6}=\int\partial u\Delta'':u=\Delta''':u,$$

$$\text{V.}\quad -\int\frac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{(lz)^4}{24}=-\int\partial u\Delta''':u=\Delta^{\mathrm{IV}}:u,$$

$$\text{VI.}\quad \int\frac{\partial z\sin.u}{1-2z\cos.u+zz}\cdot\frac{(lz)^5}{120}=\int\partial u\Delta^{\mathrm{IV}}:u=\Delta^{\mathrm{V}}:u,$$

etc. etc. etc. etc.

Has autem expressiones facile, quousque libuerit, continuare licet, si modo integratio cuiusque integralis rite instituatur ; conditiones autem, quas impleri oportet, sequenti modo referri possunt.

$\Delta':0 = 0$	$3\Delta'':0 + 4\Delta'':\pi = 0$
$\Delta''':0 = 0$	$15\Delta^{IV}:0 + 16\Delta^{IV}:\pi = 0$
$\Delta^{V}:0 = 0$	$63\Delta^{VI}:0 + 64\Delta^{VI}:\pi = 0$
$\Delta^{VII}:0 = 0$	$255\Delta^{VIII}:0 + 256\Delta^{VI}:\pi = 0$
etc.	etc.

caeterum quia posteriores integrationes absolvere non licet, hinc parum utilitatis exspectare possumus.

§. 56. Caeterum methodus, qua hic sumus usi ad constantes per quamque integrationem ingressas determinandas, a celeberrimo *Bernoullio* primum est adhibita atque eo maiori attentione digna est aestimanda, quod eius ope summationes meae serierum reciprocarum potestatum obtineri possunt, quandoquidem credideram eas non aliter nisi ex consideratione infinitorum arcuum, qui vel eodem sinu vel cosinu gaudent, demonstrari posse.