SUPPLEMENT IV*b*.

TO BOOK I. CHAP. V.

ON THE INTEGRATION OF FORMULAS INVOLVING ANGLES OR THE SINES OF ANGLES.

3) An enquiry concerning a conjecture about the formula of the integral

$$\int \frac{\partial\phi \cos.i\phi}{(\alpha+\beta\cos.\phi)^n}.$$

M. S. of the Academy exhibited on the 31st *of August* 1778.

§. 40. We shall begin from the simplest case where $i=0$ and $n=1$, and this is the formula being proposed to integrate $\int \frac{\partial\phi}{\alpha+\beta\cos.\phi}$, towards which this outstanding substitution $\text{tang.}\frac{1}{2}\phi = t;$ most conveniently may be called into help , so that at once there becomes $\partial\phi = \frac{2\partial t}{1+tt}$: truly again since hence there shall become

$$\sin.\tfrac{1}{2}\phi = \frac{t}{\sqrt{1+tt}} \text{ and } \cos.\tfrac{1}{2}\phi = \frac{1}{\sqrt{1+tt}},$$

there will be $\cos.\phi = \frac{1-tt}{1+tt}$, and thus the denominator of our formula

$$\alpha+\beta\cos.\phi = \frac{\alpha+\beta+(\alpha-\beta)tt}{1+tt}$$

and thus our formula to be integrated will become

$$\int \frac{2\partial t}{\alpha+\beta+(\alpha-\beta)tt}.$$

§. 41. Moreover it is agreed from elementary considerations, that

$$\int \frac{\partial t}{f+gtt} = \frac{1}{\sqrt{fg}}\text{Arc.tang}\, t\sqrt{\frac{g}{f}}$$

Whereby since for our case there shall be $f=\alpha+\beta$ and $g=\alpha-\beta$, we will have this integration

$$\int\frac{\partial\phi}{\alpha+\beta\cos.\phi}=\frac{2}{\sqrt{(\alpha\alpha-\beta\beta)}}\text{Arc.tang }t\sqrt{\frac{\alpha-\beta}{\alpha+\beta}},$$

with $t=\text{tang}.\frac{1}{2}\phi$; which integral hence vanishes in the case $t=0$, and thus in the case $\phi=0$. But if hence we may wish the integral to extend from the limit $\phi=0$ as far as to the limit $\phi=180^\circ$, there shall become $t=\infty$, this same integral will be $\frac{2}{\sqrt{(\alpha\alpha-\beta\beta)}}\cdot\frac{\pi}{2}$, with π denoting the semi-periphery of the circle, of which the radius $=1$.

§. 42. Therefore, because the integral of our formula from the limit $\phi=0$ as far as to the limit $\phi=180^\circ$ is expressed neatly and simpler, also generally in this dissertation I am going to investigate there only integrals of the proposed general formulas

$$\int\frac{\partial\phi\cos.i\phi}{(\alpha+\beta\cos.\phi)^n},$$

which are taken between the limits $\phi=0$ and $\phi=180^\circ$. But because in the case treated the formula the irrational formula $\sqrt{(\alpha\alpha-\beta\beta)}$ is involved, in order to remove this inconvenience, in the following we will assume always that $\alpha=1+aa$ and $\beta=-2a$, from which there becomes $\sqrt{(\alpha\alpha-\beta\beta)}=1-aa$, and thus our inquiries will be concerned with the integrations of the general formulas:

$$\int\frac{\partial\phi\cos.i\phi}{(1+aa-2a\cos.\phi)^n},$$

for which for brevity we may put everywhere to become :

$$1+aa-2a\cos.\phi=\Delta,$$

so that our general formula now shall become

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^n},$$

so that now it is to be observed, that only the values of the integrals are to be investigated, which may be held between the limits $\phi=0$ and $\phi=180^\circ$, which value we will try to include from the particular cases.

Truly in addition here in general it may be observed, the letter i always by us will indicate no numbers other than integers, and indeed positive ones, since always there is :

$$\cos.-i\phi=\cos.+i\phi.$$

I. Concerning the integration of the formula :

$$\int\frac{\partial\phi\cos.i\phi}{\Delta}\begin{bmatrix}a & \phi=0\\ ad & \phi=180^{\circ}\end{bmatrix}.$$

§. 43. Therefore here the case may be encountered, by putting the exponent $n=1$, which case we may consider to be the most simple, if indeed the case $n=0$ may be produced without difficulty at once , since there shall become

$$\int\partial\phi\cos.i\phi=\tfrac{1}{i}\sin.i\phi,$$

as the integral now vanishes in the case $i=0$, and because i now indicates only whole numbers, this integral again will vanish on assuming $\phi=180^{\circ}$, with the case excepted where $i=0$, certainly in which case it becomes $=\phi$, and thus by supposing $\phi=180^{\circ}$ there will be, for the limit of integration put in place, $\int\partial\phi=\pi$.

§. 44. This same latter case contains the property, by which it will be convenient to perform integrations of the form proposed here; since indeed there shall be

$$\partial\phi=\frac{(1+aa)\partial\phi}{\Delta}-\frac{2a\partial\phi\cos.\phi}{\Delta},$$

[Note the use of $\partial\phi=\partial\phi.1=\frac{(1+aa)\partial\phi-2a\partial\phi\cos.\phi}{\Delta}=\frac{(1+aa)\partial\phi}{\Delta}-\frac{2a\partial\phi\cos.\phi}{\Delta}$.]
for the integration being , with the prescribed limits :

$$\pi=(1+aa)\int\frac{\partial\phi}{\Delta}-2a\int\frac{\partial\phi\cos.\phi}{\Delta};$$

but above we have found $\int\frac{\partial\phi}{\Delta}=\frac{\pi}{1-aa}$, with which value substituted we arrive at the integration for the case $i=1$, since indeed there becomes

$$\pi=\frac{(1+aa)\pi}{1-aa}-2a\int\frac{\partial\phi\cos.\phi}{\Delta},\text{ there will become }\int\frac{\partial\phi\cos.\phi}{\Delta}=\frac{\pi a}{1-aa};$$

and thus now we have come upon two cases ; which are

$$\int \frac{\partial \phi}{\Delta} = \frac{\pi}{1-aa}, \text{and} \int \frac{\partial \phi \cos.\phi}{\Delta} = \frac{\pi a}{1-aa}.$$

§. 45. Moreover, from these two cases $i=0$ and $i=1$, it is possible to derive all the following without difficulty with the help of this lemma ; since as we have seen that there shall be $\int \partial \phi \cos.i\phi = 0$, there will be

$$0 = (1+aa)\int \frac{\partial \phi \cos.i\phi}{\Delta} - 2a\int \frac{\partial \phi \cos.\phi \cos.i\phi}{\Delta}.$$

[For $\partial \phi = \frac{(1+aa)\partial \phi}{\Delta} - \frac{2a\partial \phi \cos.\phi}{\Delta}$; hence $\partial \phi \cos.i\phi = \frac{(1+aa)\cos.i\phi\partial \phi}{\Delta} - \frac{2a\partial \phi \cos.\phi \cos.i\phi}{\Delta}$, etc.]

But it is agreed that

$$2\cos.\phi \cos.i\phi = \cos.(i-1)\phi + \cos.(i+1)\varphi,$$

from which we will have this equation

$$\frac{1+aa}{a}\int \frac{\partial \phi \cos.i\phi}{\Delta} = \int \frac{\partial \phi \cos.(i-1)\phi}{\Delta} + \int \frac{\partial \phi \cos.(i+1)\varphi}{\Delta}$$

from which this same lemma arises :

$$\int \frac{\partial \phi \cos.(i+1)\varphi}{\Delta} = \frac{1+aa}{a}\int \frac{\partial \phi \cos.i\phi}{\Delta} - \int \frac{\partial \phi \cos.(i-1)\phi}{\Delta}.$$

Now on taking $i=1$, this same lemma gives rise to this case

$$\int \frac{\partial \phi \cos.2\varphi}{\Delta} = \frac{1+aa}{a}\int \frac{\partial \phi \cos.\phi}{\Delta} - \int \frac{\partial \phi}{\Delta},$$

which therefore was established by the two preceding cases; for there becomes

$$\int \frac{\partial \phi \cos.2\varphi}{\Delta} = \frac{\pi aa}{1-aa}.$$

Now there may be taken $i=2$, and the lemma will give us

$$\int \frac{\partial \phi \cos.3\varphi}{\Delta} = \frac{1+aa}{a}\int \frac{\partial \phi \cos.2\phi}{\Delta} - \int \frac{\partial \phi \cos.\phi}{\Delta}, \text{ or}$$
$$\int \frac{\partial \phi \cos.3\varphi}{\Delta} = \frac{\pi a^3}{1-aa}:$$

in a similar manner by taking $i=3$, the lemma will give

$$\int\frac{\partial\phi\cos.4\varphi}{\varDelta}=\frac{1+aa}{a}\int\frac{\partial\phi\ \cos.3\phi}{\varDelta}-\int\frac{\partial\phi\cos.2\phi}{\varDelta},\ \text{or}$$
$$\int\frac{\partial\phi\cos.4\varphi}{\varDelta}=\frac{\pi a^4}{1-aa}:$$

Again, the case $i=4$ provides :

$$\int\frac{\partial\phi\cos.5\varphi}{\varDelta}=\frac{1+aa}{a}\int\frac{\partial\phi\ \cos.4\phi}{\varDelta}-\int\frac{\partial\phi\cos.3\phi}{\varDelta},\ \text{sive}$$
$$\int\frac{\partial\phi\cos.5\varphi}{\varDelta}=\frac{\pi a^5}{1-aa},\ \text{and thus henceforth.}$$

§.46. Hence therefore it is apparent, those individual cases are to be determined from the two preceding cases with the aid of the scale of the relation $\frac{1+aa}{a},-1$, and the recurrent series hence arising to become a geometric series: indeed if the two last binary terms were now found to become :

$$\frac{\pi a^{\lambda}}{1-aa}\ \text{and}\ \frac{\pi a^{\lambda+1}}{1-aa}$$

the following is found $=\frac{\pi a^{\lambda+2}}{1-aa}$, from which therefore it follows without any doubt, for a particular case to be examined here to become in general:

$$\int\frac{\partial\phi\cos.i\varphi}{\varDelta}=\frac{\pi a^{i}}{1-aa},$$

but where it is to be understood properly, only positive integers must be assumed for i .

II. The integration of the formula

$$\int\frac{\partial\phi\cos.i\phi}{\varDelta^2}\begin{bmatrix}a & \phi=0\\ ad & \phi=180^{\circ}\end{bmatrix}.$$

§. 47. Here the simplest case will occur $\int\frac{\partial\phi}{\varDelta^2}$, the integral of which will be required to be found before all else ; in the end we will consider this finite formula $\frac{\sin.\phi}{\varDelta}=\mathrm{V}$, which vanishes for each term $\phi=0$ and $\phi=180°$; hence moreover there will be

$$\partial V=\frac{\partial\phi\cos.\phi}{\varDelta}-\frac{2a\partial\phi\sin^2.\phi}{\varDelta^2},\ \text{or}$$
$$\partial V=\frac{(1+aa)\partial\phi\cos.\phi-2a\partial\phi}{\varDelta^2};$$

for which, on integrating, we now know to become

$$0=(1+aa)\int\frac{\partial\phi\cos.\phi}{\Delta^2}-2a\int\frac{\partial\phi}{\Delta^2}.$$

Again truly since before we had $\int\frac{\partial\phi}{\Delta}=\frac{\pi}{1-aa}$, by multiplying this formula above and below by Δ, there will be also

$$\frac{\pi}{1-aa}=(1+aa)\int\frac{\partial\phi}{\Delta^2}-2a\int\frac{\partial\phi\cos.\phi}{\Delta^2}.$$

But there is deduced from the preceding :

$$\int\frac{\partial\phi\cos.\phi}{\Delta^2}=\frac{2a}{1+aa}\int\frac{\partial\phi}{\Delta^2},$$

with which value substituted we will have

$$\frac{\pi}{1-aa}=(1+aa)\int\frac{\partial\phi}{\Delta^2}-\frac{4aa}{1+aa}\int\frac{\partial\phi}{\Delta^2}=\frac{(1-aa)^2}{1+aa}\int\frac{\partial\phi}{\Delta^2},$$

on account of which we arrive at this principle integration :

$$\int\frac{\partial\phi}{\Delta^2}=\frac{\pi(1+aa)}{(1-aa)^3},$$

from which the following case is deduced at once from above:

$$\int\frac{\partial\phi\cos.\phi}{\Delta^2}=\frac{2\pi a}{(1-aa)^3}.$$

§. 48. For the following cases we will consider the integration found in the preceding article

$$\int\frac{\partial\phi\cos.i\phi}{\Delta}=\frac{\pi a^i}{1-aa},$$

which formula of the integral on multiplying above and below by Δ, is separated into the two following parts:

$$\frac{\pi a^i}{1-aa}=(1+aa)\int\frac{\partial\phi\cos.i\phi}{\Delta^2}-2a\int\frac{\partial\phi\cos.\phi\cos.i\phi}{\Delta^2},$$

which equation is expanded again into this form

$$\frac{\pi a^i}{1-aa}=(1+aa)\int\frac{\partial\phi\cos.i\phi}{\Delta^2}-a\int\frac{\partial\phi\cos.(i-1)\phi}{\Delta^2}-a\int\frac{\partial\phi\cos.(i+1)\phi}{\Delta^2},$$

thence accordingly there is deduced from this lemma

$$\int \frac{\partial\phi \cos.(i+1)\phi}{\varDelta^2} = \frac{1+aa}{a}\int \frac{\partial\phi \cos.i\phi}{\varDelta^2} - \int \frac{\partial\phi \cos.(i-1)\phi}{\varDelta^2} - \frac{\pi a^{i-1}}{1-aa}.$$

§. 49. Now we may assume at once $i=1$, and this same lemma gives us :

$$\int \frac{\partial\phi \cos.2\phi}{\varDelta^2} = \frac{1+aa}{a}\int \frac{\partial\phi \cos.\phi}{\varDelta^2} - \int \frac{\partial\phi}{\varDelta^2} - \frac{\pi}{1-aa};$$

now both the values found [*i.e.* $\int \frac{\partial\phi}{\varDelta^2} = \frac{\pi(1+aa)}{(1-aa)^3}$ and $\int \frac{\partial\phi \cos.\phi}{\varDelta^2} = \frac{2\pi a}{(1-aa)^3}$.] are substituted, and there will be found :

$$\int \frac{\partial\phi \cos.2\phi}{\varDelta^2} = \frac{\pi(1+aa)-\pi(1-aa)^2}{(1-aa)^3},$$

hence it follows to become :

$$\int \frac{\partial\phi \cos.2\phi}{\varDelta^2} = \frac{\pi\left(3aa-a^4\right)}{(1-aa)^3} = \frac{\pi aa(3-aa)}{(1-aa)^3}.$$

Now $i=2$ is assumed for the proposed lemma, and there will become

$$\int \frac{\partial\phi \cos.3\phi}{\varDelta^2} = \frac{1+aa}{a}\int \frac{\partial\phi \cos.2\phi}{\varDelta^2} - \int \frac{\partial\phi \cos.\phi}{\varDelta^2} - \frac{\pi a}{1-aa}, \text{ or}$$

$$\int \frac{\partial\phi \cos.3\phi}{\varDelta^2} = \frac{(1+aa)\pi a(3-aa)-2\pi a-\pi a(1-aa)^2}{(1-aa)^3},$$

which expression is contracted into this :

$$\int \frac{\partial\phi \cos.3\phi}{\varDelta^2} = \frac{\pi a^3(4-2aa)}{(1-aa)^3}.$$

Now $i=3$ may be prescribed in the lemma, and there will become :

$$\int \frac{\partial\phi \cos.4\phi}{\varDelta^2} = \frac{1+aa}{a}\int \frac{\partial\phi \cos.3\phi}{\varDelta^2} - \int \frac{\partial\phi \cos.2\phi}{\varDelta^2} - \frac{\pi aa}{1-aa}, \text{ or}$$

$$\int \frac{\partial\phi \cos.4\phi}{\varDelta^2} = \frac{(1+aa)\pi aa(4-2aa)-\pi aa(3-aa)-\pi aa(1-aa)^2}{(1-aa)^3},$$

Which expression is contracted into this form :

$$\int \frac{\partial\phi \cos.4\phi}{\Delta^2} = \frac{\pi a^4(5-3aa)}{(1-aa)^3}.$$

Now in our lemma there shall be $i = 4$, and there will become

$$\int \frac{\partial\phi \cos.5\phi}{\Delta^2} = \frac{1+aa}{a}\int \frac{\partial\phi \cos.4\phi}{\Delta^2} - \int \frac{\partial\phi \cos.3\phi}{\Delta^2} - \frac{\pi a^3}{1-aa}, \text{ or}$$

$$\int \frac{\partial\phi \cos.5\phi}{\Delta^2} = \frac{(1+aa)\pi a^3(5-3aa)-\pi a^3(4-2aa)-\pi a^3(1-aa)^2}{(1-aa)^3},$$

which expression is contracted into this

$$\int \frac{\partial\phi \cos.5\phi}{\Delta^2} = \frac{\pi a^5(6-4aa)}{(1-aa)^3}.$$

Now in our lemma there shall become $i = 5$, and there will be

$$\int \frac{\partial\phi \cos.6\phi}{\Delta^2} = \frac{1+aa}{a}\int \frac{\partial\phi \cos.5\phi}{\Delta^2} - \int \frac{\partial\phi \cos.4\phi}{\Delta^2} - \frac{\pi a^4}{1-aa}, \text{ or}$$

$$\int \frac{\partial\phi \cos.6\phi}{\Delta^2} = \frac{(1+aa)\pi a^4(6-4aa)-\pi a^4(5-3aa)-\pi a^4(1-aa)^2}{(1-aa)^3},$$

which expression is contracted into this

$$\int \frac{\partial\phi \cos.6\phi}{\Delta^2} = \frac{\pi a^6(7-5aa)}{(1-aa)^3}.$$

§. 50. Anyone who has considered these formulas more carefully, certainly beyond any doubt, will arrive thence at this conclusion, that the proposed case presented here shall become

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^2} = \frac{\pi a^i\left[i+1-(i-1)aa\right]}{(1-aa)^3}$$

the rule of which shall be just as evident, as in the preceding case, and thence we may put all the formulas together to be seen at once :

$$\int\frac{\partial\phi}{\Delta^2}=\frac{\pi(1+aa)}{(1+aa)^3}$$

$$\int\frac{\partial\phi\cos.\phi}{\Delta^2}=\frac{\pi a(2-0aa)}{(1-aa)^3}$$

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^2}=\frac{\pi aa(3-aa)}{(1-aa)^3}$$

$$\int\frac{\partial\phi\cos.3\phi}{\Delta^2}=\frac{\pi a^3(4-2aa)}{(1-aa)^3}$$

$$\int\frac{\partial\phi\cos.4\phi}{\Delta^2}=\frac{\pi a^4(5-3aa)}{(1-aa)^3}$$

$$\int\frac{\partial\phi\cos.5\phi}{\Delta^2}=\frac{\pi a^5(6-4aa)}{(1-aa)^3}$$

$$\int\frac{\partial\phi\cos.6\phi}{\Delta^2}=\frac{\pi a^6(7-5aa)}{(1-aa)^3}.$$

III. The integration of the formula

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^3}\begin{bmatrix}\text{from} & \phi=0\\ \text{to} & \phi=180^{\circ}\end{bmatrix}.$$

§. 51. For the simplest case to be elicited $\int\frac{\partial\phi}{\Delta^3}$, we may use this formula

$V=\frac{\sin.\phi}{\Delta^2}$, and there will become $\partial V=\frac{\partial\phi\cos.\phi}{\Delta^2}-\frac{2\partial\phi\sin^2.\phi}{\Delta^3}$, or

$$\partial V=\frac{(1+aa)\partial\phi\cos.\phi-2a\partial\phi\cos^2.\phi-4a\partial\phi\sin^2.\phi}{\Delta^3};$$

Here in place of $\sin.^2\phi$ there may be written $1-\cos.^2\phi$, and by integrating, on account of $V=0$, we will have this equation

$$0=(1+aa)\int\frac{\partial\phi\cos.\phi}{\Delta^3}-4a\int\frac{\partial\phi}{\Delta^3}+2a\int\frac{\partial\phi\cos^2.\phi}{\Delta^3}.$$

§. 52. Here we may add this indefinite form

$$s=A\int\frac{\partial\phi}{\Delta}+B\int\frac{\partial\phi}{\Delta^2}$$

the differential of which shall lead to the denominator Δ^3, truly the letters A and B may be defined thus, so that the terms $\partial\phi\cos.\phi$ and $\partial\phi\cos.^2\phi$ shall vanish, and with the differential formulas added

$$\frac{\Delta^3(\partial V+\partial s)}{\partial\phi}=-4a+(1+aa)\cos.\phi\ +2a\partial\phi\cos.^2\phi$$
$$+\mathrm{A}(1+aa)^2-4\mathrm{A}a(1+aa)\cos.\phi+4\mathrm{A}aa\cos.^2\phi$$
$$+\mathrm{B}(1+aa)\ -2\mathrm{B}a\cos.\phi.$$

Now therefore so that the terms $\cos.^2\phi$ may be removed, there may be put

$$2a+4Aaa=0,\text{ and thus }A=\tfrac{-1}{2a}.$$

Now also the terms $\cos.\phi$ may be removed from within; and there will become

$$1+aa-4Aa(1+aa)-2Ba=0,$$

from which there becomes

$$B=\tfrac{3(1+aa)}{2a}.$$

From which values we come upon :

$$\frac{\Delta^3(\partial V+\partial s)}{\partial\phi}=\frac{(1-aa)^2}{a};$$

hence therefore by integrating in turn we will have

$$V+s=\frac{(1-aa)^2}{a}\int\frac{\partial\phi}{\Delta^3}.$$

§. 53. Therefore since, as we have observed, there shall be $\mathrm{V}=0$, and from the cases now examined [*i.e.* $s=A\int\frac{\partial\phi}{\Delta}+B\int\frac{\partial\phi}{\Delta^2}$] :

$$s=-\frac{1}{2a}\cdot\frac{\pi}{1-aa}+\frac{3(1+aa)}{2a}\cdot\frac{\pi(1+aa)}{(1-aa)^3},$$

we will have this equation:

$$\frac{(1-aa)^2}{a}\int\frac{\partial\phi}{\Delta^3}=\frac{3\pi(1+aa)^2-\pi(1-aa)^2}{2a(1-aa)^3},$$

from which it is deduced:

$$\int\frac{\partial\phi}{\Delta^3}=\frac{\pi\left(1+4aa+a^4\right)}{(1-aa)^5}.$$

§. 54. Since there shall be $\int\frac{\partial\phi}{\Delta^2}=\frac{\pi(1+aa)}{(1-aa)^3}$, by the reduction used so far there will become :

$$\frac{\pi(1+aa)}{(1-aa)^3}=(1+aa)\int\frac{\partial\phi}{\Delta^3}-2a\int\frac{\partial\phi\cos.\phi}{\Delta^3},$$

from which we may conclude :

$$\int \frac{\partial\phi\cos.\phi}{\Delta^3} = \frac{1+aa}{2a}\int\frac{\partial\phi}{\Delta^3} - \frac{\pi(1+aa)}{2a(1-aa)^3}, \text{ and thus}$$

$$\int \frac{\partial\phi\cos.\phi}{\Delta^3} = \frac{1+aa}{2a}\cdot\frac{\pi\left(1+4aa+a^4\right)}{(1-aa)^5} - \frac{\pi(1+aa)}{2a(1-aa)^3}$$

$$= \frac{3\pi a(1+aa)}{(1-aa)^5} = \frac{\pi a(3+3aa)}{(1-aa)^5}.$$

§. 55. Therefore since we have found in the preceding article :

$$\int \frac{\partial\phi\cos.i\phi}{\Delta^2} = \frac{\pi a^i\left[i+1-(i-1)aa\right]}{(1-aa)^3}$$

on multiplying this formula above and below by Δ, we will have :

$$\frac{\pi a^i\left[i+1-(i-1)aa\right]}{(1-aa)^3} = (1+aa)\int\frac{\partial\phi\cos.i\phi}{\Delta^3} - 2a\int\frac{\partial\phi\cos.i\phi\cos.\phi}{\Delta^3}, \text{ or}$$

$$\frac{\pi a^i\left[i+1-(i-1)aa\right]}{(1-aa)^3} = (1+aa)\int\frac{\partial\phi\cos.i\phi}{\Delta^3}$$

$$-a\int\frac{\partial\phi\cos.(i-1)\phi}{\Delta^3} - a\int\frac{\partial\phi\cos.(i+1)\phi}{\Delta^3};$$

from which this apparent lemma is deduced :

$$\int\frac{\partial\phi\cos.(i+1)\phi}{\Delta^3} = \frac{1+aa}{a}\int\frac{\partial\phi\cos.i\phi}{\Delta^3} - \int\frac{\partial\phi\cos.(i-1)\phi}{\Delta^3} - \frac{\pi a^{i-1}\left[i+1-(i-1)aa\right]}{(1-aa)^3}$$

§. 56. Now at once we may assume $i=1$, and thus this same lemma gives us :

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^3} = \frac{1+aa}{2a}\int\frac{\partial\phi\cos.\phi}{\Delta^3} - \int\frac{\partial\phi}{\Delta^3} - \frac{2\pi}{2(1-aa)^3};$$

now here both the values now found may be substituted, and there will be found

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^3} = \frac{1+aa}{a}\cdot\frac{\pi a(3+3aa)}{(1-aa)^5} - \frac{\pi\left(1+4aa+a^4\right)}{(1-aa)^5} - \frac{\pi(1-aa)^2}{(1-aa)^5} = \frac{6\pi aa}{(1-aa)^5};$$

on taking $i=2$, there will become:

$$\int\frac{\partial\phi\cos.3\phi}{\Delta^3} = \frac{\pi a^3\left(10-5aa+a^4\right)}{(1-aa)^5};$$

on assuming $i=3$, we come upon

$$\int\frac{\partial\phi\cos.4\phi}{\Delta^3}=\frac{\pi a^4\left(15-12aa+3a^4\right)}{(1-aa)^5};$$

on taking $i=4$, there is produced:

$$\int\frac{\partial\phi\cos.5\phi}{\Delta^3}=\frac{\pi a^5\left(21-21aa+6a^4\right)}{(1-aa)^5};$$

putting $i=5$, there will be

$$\int\frac{\partial\phi\cos.6\phi}{\Delta^3}=\frac{\pi a^6\left(28-32aa+10a^4\right)}{(1-aa)^5};$$

and generally

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^3}=\frac{\pi a^i}{(1-aa)^5}\left[\frac{i(i+3)+2}{2}-2\left(ii-4\right)aa+\left[\frac{i(i-3)+2}{2}\right]a^4\right],$$

which form is transformed easily into this :

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^3}=\frac{\pi a^i}{(1-aa)^5}\left[\frac{(i+1)(i+2)}{2}-2\left(i+2\right)\left(i-2\right)aa+\left[\frac{(i-1)(i-2)}{2}\right]a^4\right]$$

§. 57. It will be possible to proceed to the following formulas, in which the denominator is Δ^4, Δ^5, Δ^6, etc. truly the forms of the integral thus continually will become more complicated, as scarcely any order in these may be able to be observed, on account of which it will be more convenient to enter in another way, where we assume the number i given, and we proceed continually from smaller to greater numbers *n*. Therefore initially we take $i=0$, and we will find the value of the formula $\int\frac{\partial\phi}{\Delta^{n+1}}$.

The integration of the formula

$$\int\frac{\partial\phi}{\Delta^{n+1}}\begin{bmatrix} a & \phi=0 \\ ad & \phi=180^{\circ}\end{bmatrix}.$$

with $\Delta=1+aa-2a\cos.\phi$ being given.

§. 58. It is possible to deduce any case of the exponent $n+1$ depending on the two preceding cases, thus so that there shall be under the prescribed limits of the integration

$$\int\frac{\partial\phi}{\Delta^{n+1}}=\alpha\int\frac{\partial\phi}{\Delta^{n}}+\beta\int\frac{\partial\phi}{\Delta^{n-1}};;$$

where the whole undertaking returns to this, so that the coefficients α and β duly will be determined; these finally we may put in general to be

$$\int\frac{\partial\phi}{\Delta^{n+1}}=\alpha\int\frac{\partial\phi}{\Delta^{n}}+\beta\int\frac{\partial\phi}{\Delta^{n-1}}+\gamma\frac{\sin.\phi}{\Delta^{n}},$$

clearly whereby the last term vanished for each limit of the integral.

§.59. This same equation may now be differentiated, and divided by $\partial\phi$, the following equation will arise

$$\frac{1}{\Delta^{n+1}}=\frac{\alpha}{\Delta^{n}}+\frac{\beta}{\Delta^{n-1}}+\frac{\gamma\cos.\phi(1+aa-2a\cos.\phi)-2\gamma\alpha n\sin^{2}.\phi}{\Delta^{n+1}},$$

and with this equation multiplied by Δ^{n+1}, it will change into this form

$$\begin{aligned}1=\alpha\left(1+aa-2a\cos.\phi\right)+\beta\left(1+aa\right)^{2}-2\beta a\left(1+aa\right)\\+4\beta aa\cos^{2}.\phi+\gamma\cos.\phi\left(1+aa-2a\cos.\phi\right)-2\gamma an\sin^{2}.\phi.\end{aligned}$$

Now since there shall be

$$2\cos.^{2}\phi=1+\cos.2\phi \text{ and } 2\sin.^{2}\phi=1-\cos.2\phi,$$

by this reduction it will be possible to arrive at the following equation :

$$\begin{aligned}1=\alpha\left(1+aa\right)-2aa\cos.\phi+2\beta aa\cos.2\phi\\+\beta\left(1+aa\right)^{2}-4\beta a\left(1+aa\right)\cos.\phi-\gamma a\cos.2\phi\\+2\beta aa+\gamma\left(1+aa\right)\cos.\phi+\gamma na\cos.2\phi-\gamma a-\gamma na.\end{aligned}$$

§.60. Now so that we may resolve this equation, it is necessary, that both the terms involving $\cos.\phi$, as well as $\cos.2\phi$, themselves may be rendered to zero ; accordingly from the last term we may deduce

$$2\beta aa-\gamma a+\gamma na=0;$$

and thus

$$\beta=\frac{\gamma(1-n)}{2a}=-\frac{\gamma(n-1)}{2a}$$

which value substituted for the terms affected in accordance with $\cos.\phi$ leads to this equation

$$-2\alpha a+2\gamma(n-1)(1+aa)+\gamma(1+aa)=0,$$

from which there becomes

$$2\alpha a=2\gamma n(1+aa)-\gamma(1+aa);$$

and thus there will be

$$\alpha=\frac{2\gamma n(1+aa)-\gamma(1+aa)}{2a}.$$

Now here the values found in the first part may be substituted in place of α and β, and we are led to this equation

$$1=\frac{\gamma n(1+aa)^2}{a}-\frac{\gamma(n-1)(1+aa)^2}{2a}-\gamma a(n-1)-\gamma a-\gamma na,\text{ or}$$

$$2a=2\gamma n(i+aa)^2-\gamma(n-1)(1+aa)^2-2\gamma aa(n-1)-2\gamma aa-2\gamma naa,$$

$$\text{or } 2a=\gamma(n+1)(1+aa)^2-4\gamma naa,$$

from which there becomes : $\gamma=\frac{2a}{n(1-aa)^2}$.

§. 61. Now with this value found for γ, hence we may elicit :

$$\alpha=\frac{(2n-1)(1+aa)}{n(1-aa)^2}\text{ and }\beta=\frac{-(n-1)}{n(1-aa)^2},$$

and hence on multiplying by $n(1-aa)^2$, we arrive at

$$n(1-aa)^2\int\frac{\partial\phi}{\Delta^{n+1}}=(2n-1)(1+aa)\int\frac{\partial\phi}{\Delta^{n}}-(n-1)\int\frac{\partial\phi}{\Delta^{n-1}},$$

now the benefit from the two known cases will be able to assign the following cases.

§. 62 . Moreover we have found now before to be $\int\frac{\partial\phi}{\Delta}=\frac{\pi}{1-aa}$.
Therefore for the following we may put

$$\int\frac{\partial\phi}{\Delta^2}=\frac{\pi A}{(1-aa)^3};\int\frac{\partial\phi}{\Delta^3}=\frac{\pi B}{(1-aa)^5};\int\frac{\partial\phi}{\Delta^4}=\frac{\pi C}{(1-aa)^7};$$

$$\int\frac{\partial\phi}{\Delta^5}=\frac{\pi D}{(1-aa)^9};\ \int\frac{\partial\phi}{\Delta^6}=\frac{\pi E}{(1-aa)^{11}};\text{etc.}$$

Now before we have found everywhere $\mathrm{A}=1+aa$ and $\mathrm{B}=1+4aa+a^4$; from which all the following values C, D, E, etc. will be able to be found with the help of the reduction found.

§. 63. Therefore we may introduce these values, and we arrive at the following equations :

$$\text{I. } \mathrm{A}=1+aa,$$
$$\text{II. } 2\mathrm{B}=3(1+aa)\mathrm{A}-(1-aa)^2,$$
$$\text{III. } 3\mathrm{C}=5(1+aa)\mathrm{B}-2(1-aa)^2\,\mathrm{A},$$
$$\text{IV. } 4\mathrm{D}=7(1+aa)\mathrm{C}-3(1-aa)^2\,\mathrm{B},$$
$$\text{V. } 5\mathrm{E}=9(1+aa)\mathrm{D}-4(1-aa)^2\,\mathrm{C},$$
$$\text{VI. } 6\mathrm{F}=11(1+aa)\mathrm{E}-5(1-aa)^2\,\mathrm{D},$$
$$\text{VII. } 7\mathrm{G}=13(1+aa)\mathrm{F}-6(1-aa)^2\,\mathrm{E},$$
$$\text{VIII. } 8\mathrm{H}=15(1+aa)\mathrm{G}-7(1-aa)^2\,\mathrm{F},$$
etc.

§. 64. Of these the first equation gives the value found before $\mathrm{A}=1+aa$; truly the second gives

$$2\mathrm{B}=\begin{cases}\ \ 3+6aa+3a^4\\ -1+2aa+a^4\end{cases}$$

and there becomes:

$$\mathrm{B}=1+4aa+a^4$$

Then truly the third equation gives

$$3\mathrm{C}=\begin{cases}\ \ 5+25aa+25a^4+5a^6\\ -1+\ 2aa+\ 2a^4-2a^6\end{cases}$$

from which there is elicited :

$$\mathrm{C}=1+9aa+9a^4+a^6.$$

Again the fourth equation:

$$4\mathrm{D}=\begin{cases}\ \ 7+70aa+126a^4+70a^6+7a^8\\ -3-\ 6aa+\ 18a^4\ -\ 6a^6-3a^8\end{cases}$$

from which there is deduced :

$$D = 1 + 16aa + 36a^4 + 16a^6 + a^8.$$

In a similar manner we gather from the fifth equation:

$$5E = \begin{cases} 9 + 153aa + 468a^4 + 468a^6 + 153a^8 + 9a^{10} \\ -4 - 28aa \quad + 32a^4 + 32a^6 \quad - 28a^8 \quad - 4a^{10} \end{cases}$$

and there is deduced

$$E = 1 + 25aa + 100a^4 + 100a^6 + 25a^4 + a^{10}.$$

We may expand out the sixth equation which provides :

$$6F = \begin{cases} 11 + 286aa + 1375a^4 + 2200a^6 + 1375a^8 + 286a^{10} + 11a^{12} \\ -5 \quad - 70aa \; - \; 25a^4 \; + \; 200a^6 - \; 25a^8 \; - \; 70a^{10} - \; 5a^{12} \end{cases}$$

and hence it is concluded :

$$F = 1 + 36aa + 225a^2 + 400a^6 + 225a^8 + 36a^{10} + a^{12}.$$

§. 65. Here not without wonder we have been surprised, that all the coefficients of these formulas to be square numbers, the roots of which occur in corresponding powers of the binomial $1 + aa$, and thus for the following letter we will have

$$G = 1 + 7^2 aa + 21^2 a^4 + 35^2 a^6 + 35^2 a^3 + 21^2 a^{10} + 7^2 a^{12} + a^{14},$$

which letter corresponds to the integral formula $\int \frac{\partial\phi}{\Delta^{7+1}}$, thus so that here there shall be $n = 7$. Therefore if we may put the integral of the general form $\int \frac{\partial\phi}{\Delta^{n+1}}$ to be $= \frac{\pi V}{(1-aa)^{in+1}}$, the value of the letter will become :

$$V = 1 + \left(\tfrac{n}{1}\right)^2 aa + \left(\tfrac{n}{2}\right)^2 a^4 + \left(\tfrac{n}{3}\right)^2 a^6 + \left(\tfrac{n}{4}\right)^2 a^8 + \left(\tfrac{n}{5}\right)^2 a^{10} \; + \text{etc.}$$

evidently with these characters used by which we have become accustomed to indicate the coefficients of the powers of the binomial, while clearly there shall be

$$\left(\tfrac{n}{1}\right) = n; \; \left(\tfrac{n}{2}\right) = \tfrac{n}{1} \cdot \tfrac{n-1}{2}; \; \left(\tfrac{n}{3}\right) = \tfrac{n}{1} \cdot \tfrac{n-1}{2} \cdot \tfrac{n-2}{3} \text{ etc.}$$

§. 66. Indeed this conclusion has been deduced only by induction as if from conjecture; indeed scarcely anything will be found, by which the same conjecture may appear to be in doubt, though it shall not yet be corroborated by a rigorous demonstration ; indeed in the fortuitous case it is by no means certain that it can eventuate, so that all these coefficients can be shown to produce square numbers, and indeed of these same coefficients which arise in the expansion of the power $(1+aa)^n$, yet meanwhile we will provide next a solid demonstration for this truth to become more evident.

§. 67. Therefore by this rule established, we have been led to the values of the letters A, B, C, D etc., which are obtained in the following manner in the expressions of the integrals

$$\mathrm{A}=1^2+1^2aa,$$
$$\mathrm{B}=1^2+2^2aa+1^2a^4,$$
$$\mathrm{C}=1^2+3^2aa+3^2a^4+1^2a^6,$$
$$\mathrm{D}=1^2+4^2aa+6^2a^4+4^2a^6+1^2a^8,$$
$$\mathrm{E}=1^2+5^2aa+10^2a^4+10^2a^6+5^2a^8+1^2a^{10},$$
$$\mathrm{F}=1^2+6^2aa+15^2a^4+20^2a^6+15^2a^8+6^2a^{10}+1^2a^{12},$$
$$\mathrm{G}=1^2+7^2aa+21^2a^4+35^2a^6+35^2a^8+21^2a^{10}+7^2a^{12}+1^2a^{14},$$

etc. etc.

The integration of the general formula

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^{n+1}}\begin{bmatrix}a & \phi=0\\ ad & \phi=180^0\end{bmatrix},$$

on being given $\Delta=1+aa-2a\cos.\phi.$

§. 68. This general formula can be treated thence as in the preceding, while the value of the integral of each case depends on the two previous cases, thus so that we are able to put

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^{n+1}}=\alpha\int\frac{\partial\phi\cos.i\phi}{\Delta^{n}}+\beta\int\frac{\partial\phi\cos.i\phi}{\Delta^{n-1}},$$

in as much as clearly the integrations are referred to two established terms of the integration; but because it is necessary, that we may put in place the general equation free from that same condition, it requires several extra members to be taken together, which vanish for each limit, nor indeed does it suffice here, so that before a single term may be added [to the integrals], truly three terms of this kind must be added on, the ratio of which will be elicited soon from the same calculation; on this account we may put in place the following equation

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^{n+1}} = \alpha \int \frac{\partial\phi \cos.i\phi}{\Delta^{n}} + \beta \int \frac{\partial\phi \cos.i\phi}{\Delta^{n-1}},$$
$$+\gamma \frac{\sin.i\phi}{\Delta^{n}} + \delta \frac{\sin.(i-1)\phi}{\Delta^{n}} + \varepsilon \frac{\sin.(i+1)\phi}{\Delta^{n}},$$

which latter members, since i is a whole number, vanish for each term of the integration.

§. 69. Now therefore the same equation may be differentiated, and on putting for brevity $1+aa=b$, so that there shall be $\Delta = b-2a\cos.\phi,$ the denominators shall be ignored, which will be Δ^{n+1} together with the element $\partial\phi.$ Initially it may be observed :

$$\Delta \cos.i\phi = b\cos.i\phi - a\cos.(i-1)\phi - a\cos.(i+1)\phi,$$

then truly on account of

$$\Delta^2 = bb - 4ab\cos.\phi + 4aa\cos^2.\phi = 2aa + bb$$
$$-4ab\cos.\phi + 2aa\cos.2\phi, \text{ there will be}$$
$$\Delta^2\cos.i\phi = (bb+2aa)\cos.i\phi - 2ab\cos.(i-1)\phi - 2ab\cos.(i+1)\phi$$
$$+aa\cos.(i-2)\phi + aa\cos.(i+2)\phi.$$

From which truly there will become

$$\partial.\frac{\sin.i\phi}{\Delta^{n}} = i\Delta\cos.i\phi - 2na\sin.i\phi\sin.\phi = ia\cos.i\phi$$
$$+ia\cos.(i-1)\phi - ia\cos.(i+1)\phi - na\cos.(i-1)\phi + na\cos.(i+1)\phi.$$

In a similar manner there will be

$$\partial.\frac{\sin.(i-1)\phi}{\Delta^{n}} = (i-1)b\cos.(i-1)\phi - (i-1)a\cos.(i-2)\phi$$
$$-(i-1)a\cos.i\phi - na\cos.(i-2)\phi + na\cos.i\phi,$$

and finally

$$\partial.\frac{\sin.(i+1)\phi}{\Delta^{n}} = (i+1)b\cos.(i+1)\phi - (i+1)a\cos.i\phi$$
$$-(i+1)a\cos.(i+2)\phi - na\cos.i\phi + na\cos.(i+2)\phi.$$

§. 70. Therefore here clearly five angles occur

$$\cos.i\phi,\ \cos.(i-1)\phi,\ (i+1)\phi,\ (i-2)\phi \text{ et } (i+2)\phi,$$

from which the account is apparent, why three absolute terms shall be added above ; therefore the differential made from the setting out of the individual terms, will represented by five columns in the following manner, thus so that the left member, which is $\cos.i\phi$, is equal to the following expression

$\cos.i\phi$	$\cos.(i-1)\phi$	$\cos.(i+1)\phi$	$\cos.(i-2)\phi$	$\cos.(i+2)\phi$
$+\alpha b$	$-\alpha a$	$-\alpha a$		
$+\beta(bb+2aa)$	$-2\beta ab$	$-2\beta ab$	$+\beta aa$	$+\beta aa$
$+\gamma ib$	$-\gamma ia$	$-\gamma ia$		
	$-\gamma na$	$+\gamma na$		
$-\delta(i-1)a$	$+\delta(i-1)b$	$+\varepsilon(i+1)b$	$-\delta(i-1)a$	$-\varepsilon(i+1)a$
$+\delta na$			$-\delta na$	$+\varepsilon na$
$-\varepsilon(i+1)a$				
$-\varepsilon na$				

§. 71 . Therefore here all four latter columns must be reduced to zero, on account of which the differential can be equal only to the left member in the first column ; we may begin therefore with the two final columns, from which we deduce

$$\delta=\frac{\beta a}{i+n-1} \text{ et } \varepsilon=\frac{\beta a}{i-n+1}.$$

With these values introduced, there will be for the second column

$$-2\beta ab+\varepsilon(i-1)b=\frac{\beta ab(1-i-2n)}{i+n-1}=-\frac{\beta ab(i+2n-1)}{i+n-1};$$

Truly for the third column, there will be

$$-2\beta ab+\varepsilon(i+1)b=-\frac{\beta ab(i-2n+1)}{i-n+1};$$

From which these two columns give us these two equations :

$$-\alpha a-\gamma(i+n)a-\frac{\beta ab(i+2n-1)}{i+n-1}=0,$$
$$-\alpha a-\gamma(i-n)a-\frac{\beta ab(i-2n+1)}{i-n+1}=0,$$

§. 72. Of these two equations, the latter is subtracted from the former, and there will be produced

$$-2\gamma na-\frac{2\beta inab}{ii-(n-1)^2}=0,$$

from which we deduce

$$\gamma = -\frac{\beta ib}{ii-(n-1)^2}.$$

And hence again the value of α itself can be deduced from the second , since there shall be

$$\alpha a = -\gamma(i+n)a = \frac{\beta ab(i+2n-1)}{i+n-1},$$

there will be

$$\alpha = \frac{\beta i(i+n)b}{ii-(n-1)^2} - \frac{\beta(i+2n-1)b}{i+n-1} = \frac{\beta(2nn-3n+1)b}{ii-(n-1)^2}$$
$$= \frac{\beta(n-1)(2n-1)b}{ii-(n-1)^2}.$$

§.73. Now these values may be substituted into the first column, and the following equation, and the following equation will arise :

$$\left.\begin{array}{r} \frac{\beta(n-1)(2n-1)bb}{ii-(n-1)^2} + 2\beta aa \\ +\beta bb - \frac{\beta(i-n-1)aa}{i+n-1} \\ -\frac{\beta iibb}{ii-(n-1)^2} - \frac{\beta(i+n+1)aa}{i-n+1} \end{array}\right\} = 1.$$

Therefore on multiplying by $ii-(n-1)^2$, this equation will be produced :

$$ii-(n-1)^2 = 2\beta aa\left[ii-(n-1)^2\right] + \beta bb(n-1)(2n-1)$$
$$-\beta aa(i-n-1)(i-n+1) + \beta bb\left[ii-(n-1)^2\right]$$
$$-\beta aa(i+n+1)(i+n-1) - \beta iibb.$$

Moreover, with the reduction made, the term βaa will be multiplied by

$$2\left[ii-(n-1)^2\right] - (i-n)^2 + 1 - (i+n)^2 + 1,$$

or by $-4n(n-1)$; but truly βbb will be multiplied by

$$(n-1)(2n-1) + ii - (n-1)^2 - ii,$$

or by $n(n-1)$, and thus there will become:

$$ii(n-1)^2 = -4\beta n(n-1)aa + \beta n(n-1)bb$$
$$= \beta n(n-1)(bb-4aa).$$

Therefore since we may put $b = 1+aa$, there will become :

$$bb-4aa = (1-aa)^2,$$

hence consequently we find :

$$\beta = \frac{ii-(n-1)^2}{n(n-1)(1-aa)^2}.$$

§. 74. Now with the value of the letter β found, from that we deduce the value $\alpha = \frac{(2n-1)b}{n(1-aa)^2}$: but the values of the letters γ, δ, and ε no more come into consideration, and the reduction that we seek will be

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^{n+1}} = \alpha \int \frac{\partial\phi \cos.i\phi}{\Delta^{n}} + \beta \int \frac{\partial\phi \cos.i\phi}{\Delta^{n-1}}$$

or with the fractions removed this equation will be had :

$$n(n-1)(1-aa)^2 \int \frac{\partial\phi \cos.i\phi}{\Delta^{n+1}} = (n-1)(2n-1)(1+aa) \int \frac{\partial\phi \cos.i\phi}{\Delta^{n}} + \left[ii-(n-1)^2\right] \int \frac{\partial\phi \cos.i\phi}{\Delta^{n-1}},$$

which equation in the case $i = 0$ returns to the reduction of the preceding section.

§. 75. With this general reduction found, since there shall be for its application

$$\int \frac{\partial\phi \cos.i\phi}{\Delta} = \frac{\pi a^i}{1-aa}, \text{ when } n = 0,$$

we may put for the following :

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^2} = \frac{\pi a^i}{(1-aa)^3} \mathrm{A}, \text{ when } n = 1,$$
$$\int \frac{\partial\phi \cos.i\phi}{\Delta^3} = \frac{\pi a^i}{(1-aa)^5} \mathrm{B}, \text{ when } n = 2,$$
$$\int \frac{\partial\phi \cos.i\phi}{\Delta^4} = \frac{\pi a^i}{(1-aa)^7} \mathrm{C}, \text{ when } n = 3,$$
$$\int \frac{\partial\phi \cos.i\phi}{\Delta^5} = \frac{\pi a^i}{(1-aa)^9} \mathrm{D}, \text{ when } n = 4,$$
$$\int \frac{\partial\phi \cos.i\phi}{\Delta^6} = \frac{\pi a^i}{(1-aa)^{11}} \mathrm{E}, \text{ when } n = 5,$$

and thus in general there shall be :

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^{n+1}}=\frac{\pi a^{i}}{(1-aa)^{2n+1}}\mathrm{V}:$$

but now above we have found for i :

$$\mathrm{A}=i+1-(i+1)aa\,,$$

or, being represented by the positive term

$$\mathrm{A}=i+1+(i-1)aa.$$

§. 76. But if in the reduction we may put $n=1$ into our formula

$$[i.e.\;\; n(n-1)(1-aa)^{2}\int\frac{\partial\phi\cos.i\phi}{\Delta^{n+1}}=(n-1)(2n-1)(1+aa)\int\frac{\partial\phi\cos.i\phi}{\Delta^{n}}+\left[ii-(n-1)^{2}\right]\int\frac{\partial\phi\cos.i\phi}{\Delta^{n-1}}],$$

that will give, $ii\int\partial\phi\cos.i\phi=0$, because indeed in the first case $i=0$, as well as on account of $\int\partial\phi\cos.i\phi=\frac{1}{2}\sin.i\phi=0,$ which indeed is apparent by itself. Therefore we may begin from the case $n=2$, and by preceding through the following values $n=3,\ n=4,\ n=5,$ etc. we will come upon the following equations :

I. If $n=2$, there will be

$$2.1\mathrm{B}=1.3(1+aa)\mathrm{A}+(ii-1)(1-aa)^{2}.$$

II. If $n=3,$ there will be

$$3.2\mathrm{C}=2.5(1+aa)\mathrm{B}+(ii-4)(1-aa)^{2}\mathrm{A}.$$

III. If $n=4,$ there will be

$$4.3\mathrm{D}=3.7(1+aa)\mathrm{C}+(ii-9)(1-aa)^{2}\mathrm{B}.$$

IV. If $n=5,$ there will be

$$4\mathrm{E}=4.9(1+aa)\mathrm{D}+(ii-16)(1-aa)^{2}\mathrm{C}.$$

V. If $n=6,$ there will be

$$6.5\mathrm{F}=5.11(1+aa)\mathrm{E}+(ii-25)(1-aa)^{2}\mathrm{D}.$$

etc. etc.

§. 77. Therefore since there shall be

$$\mathrm{A}=1+i+(1-i)aa,$$

there shall become for the first equation

$$(1+aa)\mathrm{A} = 1+i+2aa+(i-i)a^4,$$

to the triple of which there is required to be added

$$(ii-1)(1-aa)^2 = ii-1-2(ii-1)aa+(ii-1)a^4,$$

$$[i.e., \text{ since } 2.1\mathrm{B} = 1.3(1+aa)\mathrm{A}+(ii-1)(1-aa)^2,]$$

for which the first absolute term appears $=(2+i)(1+i)$, hence the coefficient of aa will be $8-2ii$, truly the coefficient of a^4 will be $(2-i)(1-i)$, from which we may conclude the letter $\mathrm{B} = \frac{(2+i)(1+i)}{1.\ \ 2.} + (2+i)(2-i)aa + \frac{(2-i)(1-i)}{1.\ \ 2.}a^4.$

§. 78. This same formula leads us to the coefficients of the binomial powers, which as now we have indicated we have represented by peculiar characters, and thus by such characters there will become :

$$\mathrm{A} = \left(\tfrac{1+i}{1}\right) + \left(\tfrac{1-i}{1}\right)aa, \text{ then truly}$$
$$\mathrm{B} = \left(\tfrac{2+i}{2}\right) + \left(\tfrac{2+i}{1}\right)\left(\tfrac{2-i}{1}\right)aa + \left(\tfrac{2-i}{2}\right)a^4.$$

Moreover we may see how this rule is going to be used with the following values.

§. 79. Therefore we expand the following equation, for which it is required to put in place the two following multiplications:

$$10\left[\tfrac{2+3i+ii}{2} + (4-ii)aa + \tfrac{2-3i+ii}{2}a^4\right] \text{ by } 1+aa,$$

moreover the final member requires this multiplication :

$$(ii-4)\left(1-2aa+a^4\right) \text{ by } 1+i+(1-i)aa;$$

hence from the first this absolute term arises

$$10+15i+5ii+(ii-4)(1+i),$$

which is reduced to this form $(2+i)(1+i)(3+i)$. But for the term aa there will become

$$40-10ii+5(2+i)(1+i)+(ii-4)[-2\ (1+i)+1-i]$$
$$=(4-ii)(11+3i)+5(2+i)(1+i),$$

which expression is reduced to

$$(2+i)(27-3ii) = 3(2+i)(3+i)(3-i).$$

Again, the coefficient of a^4 will become

$$(2-i)(27-3ii)=3(2-i)(3+i)(3-i).$$

Finally the coefficient of a^6 will become $(2-i)(1-i)(3-i)$.

§. 80. Therefore with this calculation carried out, we will have

$$3.2\mathrm{C}=(3+i)(2+i)(1+i)+3(3+i)(2+i)(3-i)aa$$
$$+3(3+i)(2-i)(3-i)a^4+(3-i)(2-i)(1-\ i)\ a^6,$$

which form is conveniently reduced to the same by the characters of binomial coefficients

$$\mathrm{C}=\left(\tfrac{3+i}{3}\right)+\left(\tfrac{3+i}{2}\right)\left(\tfrac{3-i}{1}\right)aa+\left(\tfrac{3+i}{1}\right)\left(\tfrac{3-i}{2}\right)a^4+\left(\tfrac{3-i}{3}\right)a^6.$$

This order confirms especially the conjecture deduced from the preceding cases, nor can there be any doubt, why the following letters may not draw out these values

$$\mathrm{D}=\left(\tfrac{4+i}{4}\right)+\left(\tfrac{4+i}{3}\right)\left(\tfrac{4-i}{1}\right)aa+\left(\tfrac{4+i}{2}\right)\left(\tfrac{4-i}{2}\right)a^4$$
$$+\left(\tfrac{4+i}{1}\right)\left(\tfrac{4-i}{3}\right)a^6+\left(\tfrac{4-i}{4}\right)a^8.$$
$$\mathrm{E}=\left(\tfrac{5+i}{5}\right)+\left(\tfrac{5+i}{4}\right)\left(\tfrac{5-i}{1}\right)aa+\left(\tfrac{5+i}{3}\right)\left(\tfrac{5-i}{2}\right)a^4$$
$$+\left(\tfrac{5+i}{2}\right)\left(\tfrac{5-i}{3}\right)a^6+\left(\tfrac{5+i}{1}\right)\left(\tfrac{5-i}{4}\right)a^8+\left(\tfrac{5-i}{5}\right)a^{10}.$$

etc. etc.

Yet meanwhile it must be admitted, this outstanding order to have been advanced by us by conjecture; therefore a more rigorous demonstration is desired at this stage.

§. 81. Therefore since in general we may have put

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^{n+1}}\begin{bmatrix}a & \phi=0\\ ad & \phi=180^0\end{bmatrix}=\frac{\pi a^i}{(1-aa)^{2n+1}}\mathrm{V},$$

now there will be

$$\mathrm{V}=\left(\tfrac{n+i}{n}\right)+\left(\tfrac{n+i}{n-1}\right)\left(\tfrac{n-i}{1}\right)aa+\left(\tfrac{n+i}{n-2}\right)\left(\tfrac{n-i}{2}\right)a^4$$
$$+\left(\tfrac{n+i}{n-3}\right)\left(\tfrac{n-i}{3}\right)a^6+\left(\tfrac{n+i}{n-4}\right)\left(\tfrac{n-i}{4}\right)a^8+\text{etc.}$$

from which the form is deduced at once concluded in the previous article, where there was $i = 0$. For indeed in this case there will be

$$V = \left(\tfrac{n}{n}\right) + \left(\tfrac{n}{n-1}\right)\left(\tfrac{n}{1}\right)aa + \left(\tfrac{n}{n-2}\right)\left(\tfrac{n}{2}\right)a^4 + \left(\tfrac{n}{n-3}\right)\left(\tfrac{n}{3}\right)a^6 + \text{etc.}$$

But since with characters of this kind there shall be always $\left(\tfrac{n}{p}\right) = \left(\tfrac{n}{n-p}\right)$, therefore just as we have conjectured above

$$V = \left(\tfrac{n}{0}\right) + \left(\tfrac{n}{1}\right)^2 aa + \left(\tfrac{n}{2}\right)^2 a^4 + \left(\tfrac{n}{3}\right)^2 a^6 + \left(\tfrac{n}{4}\right)^2 a^8 + \text{etc.}$$

Hence there is therefore a need to put the following theorem in place.

General Theorem.

[*Opera Omnia*: Series 1, Volume 19, pp. 197 – 216:
A dissertation with the Eneström index 674 .]

§. 82. If the formula of the integral

$$\int \frac{\partial\phi \cos.i\phi}{(1+aa-2a\cos.\phi)^{n+1}},$$

may be extended from the limit $\phi = 0$ as far as to the limit $\phi = 180°$, the value of the integral always will have such a form

$$\frac{\pi a^i}{(1-aa)^{2n+1}}\text{V}, \text{ with V given by :}$$

$$V = \left(\tfrac{n+i}{i}\right) + \left(\tfrac{n+i}{i+1}\right)\left(\tfrac{n-i}{1}\right)aa + \left(\tfrac{n+i}{i+2}\right)\left(\tfrac{n-i}{2}\right)a^4$$
$$+\left(\tfrac{n+i}{i+3}\right)\left(\tfrac{n-i}{3}\right)a^6 + \left(\tfrac{n+i}{i+4}\right)\left(\tfrac{n-i}{4}\right)a^8 + \text{etc.}$$

provided *i* were a whole number, and thus positive as well as negative; also since the same form for the latter case is taken agreeing to be true, thus so that the same expression may be more widely apparent, as all the special cases are to be taken jointly, from which we conclude that through conjecture; and if in all the special cases the letter *i* by necessity will be denoting only positive integers.

4) Demonstration of the Theorem designated by conjecture to be elicited, concerning the integration of the formula

$$\int \frac{\partial\phi \cos.i\phi}{(1+aa-2a\cos.\phi)^{n+1}}.$$

M. S. of the Academy, shown on the 10*th* *of September* 1778.

§. 83. Since I have treated this integral formula recently, and chiefly I have enquired into its value which it may take, if the integral may be extended from the limit $\phi=0$ as far as to the limit $\phi=180°$; from several cases, which it may be allowed to show, to be concluded its integral in general is going to be expressed thus

$$\frac{\pi a^i}{(1-aa)^{2n+1}}\mathrm{V},$$

where V denotes the sum of this series

$$\mathrm{V}=\left(\tfrac{n-i}{0}\right)\left(\tfrac{n+i}{i}\right)+\left(\tfrac{n-i}{1}\right)\left(\tfrac{n+i}{i+1}\right)a^2+\left(\tfrac{n-i}{2}\right)\left(\tfrac{n+i}{i+2}\right)a^4+\text{etc.}$$

Here evidently the same inclusive characters designate the coefficients of the binomial powers for the conclusions, so that we may put in place

$$(1+x)^m=1+\left(\tfrac{m}{1}\right)x+\left(\tfrac{m}{2}\right)x^2+\left(\tfrac{m}{3}\right)x^3+\left(\tfrac{m}{4}\right)x^4+\text{etc.}$$

§. 84. Before everything else concerning this integral formula it is to be kept in mind, that the letter i always signifies whole numbers, since in the analysis it is assumed constantly, in the case $\phi=180°$, $\sin.i\phi=0$ always; as indeed truly its values always are to be regarded as positive, so that therefore $\cos.(-i\phi)=\cos.(+i\phi)$. Yet meanwhile soon we will show our integral formula also to be agreed to be true, even if negative values may be attributed to the letter i . Towards showing this about the characters the following called in to help are to be noted.

1°). If p and q may designate whole numbers, and indeed with the first positive, because in the setting out of binomial powers all the terms preceding the first are zero, if q were a negative number, then there will be always $\left(\tfrac{p}{q}\right)=0.$

2°). Because the coefficient of both the first term as well as the last always is one, there will be $\left(\tfrac{p}{0}\right)=1$ as well as $\left(\tfrac{p}{p}\right)=1.$

3°). Because the terms following the final one equally are zero, as often as there were $q<p$, the value of the characteristic $\left(\tfrac{p}{q}\right)$ then will be zero.

4°). Because in setting out the coefficients of the binomial powers they maintain the order in reverse, hence there follows to become always $\left(\tfrac{p}{q}\right)=\left(\tfrac{p}{p-q}\right)$.

But if the upper number p were negative, on account of the preceding ratio, there will be always also $\left(\tfrac{-p}{-q}\right)=0.$

5°). But if q may denote positive numbers, then $\left(\tfrac{-p}{q}\right)$ will always give alternating positive and negative numbers ; since there shall become

$$\left(\frac{-p}{0}\right)=1;\ \left(\frac{-p}{1}\right)=-p;\ \left(\frac{-p}{2}\right)=+\frac{p(p+1)}{1.2};\ \left(\frac{-p}{3}\right)=-\frac{p(p+1)(p+2)}{1.2.3}\text{ etc.}$$

$$[\ e.g.\ (1+x)^{-p}=1-px+\frac{p(p+1)}{1.2}x^2-\text{etc.}\ ]$$

And hence :

6°). In general such characters, where the number above is negative, will be able to be made positive, since there shall be $\left(\frac{-p}{q}\right)=\pm\left(\frac{p+q-1}{q}\right)$, where the $+$ ve sign prevails if q were an even number, truly the $-$ve , if it were odd.

§. 85. With these properties concerning characters this is noteworthy, if in our form of the integral in place of *i* we may write $-\ i$, and there becomes

$$\int\frac{\partial\phi\cos.-i\phi}{(1+aa-2a\cos.\phi)^{n+1}}=\frac{\pi a^{-i}}{(1-aa)^{2n+1}}\mathrm{V}$$

with V given by [*cf.* §. 83]

$$\mathrm{V}=\left(\frac{n+i}{0}\right)\left(\frac{n-i}{-i}\right)+\left(\frac{n+i}{1}\right)\left(\frac{n-i}{-i+1}\right)a^2+\left(\frac{n+i}{2}\right)\left(\frac{n-i}{-i+2}\right)a^4$$
$$+\left(\frac{n+i}{3}\right)\left(\frac{n-i}{-i+3}\right)a^6+\text{etc.}$$

where the latter factors vanish, while the denominators are negative : therefore the first term having significance will be $\left(\frac{n+i}{i}\right)\left(\frac{n-i}{-1+i}\right)a^{2i}$, of which the value will be $\left(\frac{n+i}{i}\right)a^{2i}$; moreover the following members will be

$$\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{-i+i+1}\right)a^{2i+2}=\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^{2i+2},\text{then truly }\left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^{2i+4},\text{etc.}$$

Therefore in this manner there will be

$$\mathrm{V}=a^{2i}\left[\left(\frac{n+i}{i}\right)\left(\frac{n-i}{0}\right)+\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^2+\left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^4+\text{etc.}\right]$$

which value multiplied by $\frac{\pi a^{-i}}{(1-aa)^{2n+1}}$ gives this form

$$\frac{\pi a^{i}}{(1-aa)^{2n+1}}\left[\left(\frac{n+i}{i}\right)\left(\frac{n-i}{0}\right)+\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^2+\left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^4+\text{etc.}\right]$$

which agrees at once with our formula corresponding to the positive value of i, which contains outstanding agreement, without decrying the support, for the truth of our integral form.

§. 86. Truly besides, concerning the form of our integral, it must be impressed to be observed, the series for V given above always to be broken off somewhere whenever *n* were a positive whole number, certainly which would arise, when either in the first factor, the form of which is $\left(\frac{n-i}{\lambda}\right)$, it would come to a term where $\lambda > n-i$, or in the latter factor, the form of which is $\left(\frac{n+i}{i+\lambda}\right)$, there would arise $\lambda > n$; which property is being observed more there, if the series V may be extending to infinity, we may have little to be gained to be considering, that which especially is to be noted in these cases, in which n may become a fraction, which cases therefore we remove completely from our setup, that so that we shall be assuming only whole numbers.

§. 87. Therefore we will consider the case also, in which *n* is a negative number, and in the first place indeed now it is itself clear, as long as this should be smaller than *i*, therefore $n+i$ even now is a positive number, then the series given for V thus is going to be broken off quickly ; therefore then at last it may depart to infinity, even if $n+i$ were a positive number. But in these cases the form of the integral given above thus can be transformed, so that a place may be found equally for the breaking off.

§. 88. Towards showing this, we may put $n = -m-1$, so that our integral formula may become

$$\int \partial\phi \cos.i\phi \left(1+aa-2a\cos.\phi\right)^m,$$

and therefore its value $= \pi a^i \left(1-aa\right)^{2m+1} \mathrm{V}$, now with V being

$$\mathrm{V} = \left(\frac{-m-1-i}{0}\right)\left(\frac{-m-1+i}{i}\right) + \left(\frac{-m-1-i}{1}\right)\left(\frac{-m-1+i}{i+1}\right)a^2 + \left(\frac{-m-1-i}{2}\right)\left(\frac{-m-1+i}{i+2}\right)a^4 + \left(\frac{-m-1-i}{3}\right)\left(\frac{-m-1+i}{i+3}\right)a^6 + \text{etc}$$

which series evidently runs off to infinity, but which we will be able to transform with the aid of the following lemma.

Lemma.

§. 89. The same series proceeding by the characters introduced here :

$$\hbar = \left(\frac{f}{0}\right)\left(\frac{h}{e}\right) + \left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right)x + \left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right)x^2 + \left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right)x^3 + \text{etc.}$$

can be transformed into this similar to itself :

$$♂ = \left(\frac{-h-i}{0}\right)\left(\frac{-f-i}{e}\right) + \left(\frac{-h-i}{1}\right)\left(\frac{-f-i}{e+1}\right)x + \left(\frac{-h-i}{2}\right)\left(\frac{-f-i}{e+2}\right)x^2 + \text{etc.}$$

since the same relation can always be found between the values of these ♄ and ♂, which has been shown by me quite recently:

$$\left(\tfrac{e+f}{1}\right)\hbar = \left(\tfrac{e-h-1}{e}\right)(1-x)^{f+h+1}♂,$$

of which the most profound demonstration has been found, while thus it precedes by second order differential equations.

§. 90. Now we will apply the same lemma to our proposed case, and so that the series ♄ may be returned agreeing with our V , as there may be made ♄ = V ,there must be taken $f = -m-1-i$, $h = -m-1+i, e = i$ and $x = aa$, from which the other series ♂ may adopt this form

$$♂ = \left(\tfrac{m-i}{0}\right)\left(\tfrac{m+i}{1}\right) + \left(\tfrac{m-i}{1}\right)\left(\tfrac{m+i}{i+1}\right)aa + \left(\tfrac{m-i}{2}\right)\left(\tfrac{m+i}{i+2}\right)a^4 + \text{etc.}$$

which series certainly now is terminated somewhere, as here *m* therefore may denote a positive whole number: but truly the relation between the further above V = ♄ and this new series ♂ thus itself may be put in place :

$$\left(\tfrac{m-1}{i}\right)\text{V} = \left(\tfrac{m}{i}\right)(1-aa)^{-2m-1}♂ .$$

§. 91. Hence I can set out the integral of our formula of this

$$\int \partial\phi \cos.i\phi\left(1+aa-2a\cos.\phi\right)^m = \frac{\left(\tfrac{m}{i}\right)\pi a^i ♂}{\left(\tfrac{-m-1}{i}\right)},$$

where ♂ denotes the series in the previous manner §.89, which value will vanish since it shall always have a factor $\left(\tfrac{m}{i}\right)$, as long as there were $i > m$, thus so that in these cases the value of the integral shall be equal to zero always. Finally here it will be pleased to note, with the expansion made to be :

$$\left(\tfrac{m}{i}\right) : \left(\tfrac{-m-1}{i}\right) = \pm\frac{m(m-1)\ .\ .\ .\ .\ (m-i+1)}{(m+1)(m+2)\ .\ .\ .\ .\ (m+i)},$$

where the upper sign + prevails if i were an even number, truly the lower − if odd. With these matters noted concerning the nature of our theorem, we may approach the demonstration of this itself, which we may distribute into several parts so that it may emerge more clearly.

Demonstration of the first part.

§. 92. Since we have adapted the value of our integral according to two formulas, these we will designate by the signs ☉ and ☽ for the sake of distinguishing between them, and there shall become :

$$\square = \int \frac{\partial\phi \cos.i\phi}{(1+aa-2a\cos.\varphi)^{n+1}} \begin{bmatrix} \text{from} & \phi = 0 \\ \text{to} & \phi = 180^0 \end{bmatrix},$$

$$☽ = \int \partial\phi \cos.i\phi (1+aa-2a\cos.\varphi)^{m} \begin{bmatrix} \text{from} & \phi = 0 \\ \text{to} & \phi = 180^0 \end{bmatrix},$$

of which the latter ☽ is changed into the former ⊙ if we write $-n-1$ in place of m ; but in the manner we have seen, these two formulas depend on each other in turn, so that we may begin from the latter so that it may become simpler, if indeed it were without the denominator $(1-aa)^{2n+1}$, so that we may render that more simply we may put in place $\frac{a}{1+aa} = b$; thus indeed we will have

$$☽ = (1+aa)^{m} \int \partial\phi \cos.i\phi (1-2b\cos.\varphi)^{m};$$

of which therefore the integral will be investigated by us.

§. 93. Therefore before everything it will be agreed that the power $(1-2b\cos.\varphi)^{m}$ be expanded out, from which it will become :

$$(1-2b\cos.\phi)^{m} = 1-\left(\tfrac{m}{1}\right)2b\cos.\phi + \left(\tfrac{m}{2}\right)4b^{2}\cos.^{2}\phi - \left(\tfrac{m}{3}\right)8b^{3}\cos.^{3}\phi + \text{etc.}$$

any term of which therefore will be $\pm\left(\tfrac{m}{\lambda}\right)2^{\lambda}b^{\lambda}\cos.^{\lambda}\phi$; where the sign + will prevail if λ were an even number, truly the other − if odd. Now because here the powers of $\cos.\phi$ itself occur, it will be necessary to convert these by known precepts into simple cosines, from which there becomes :

$$2^{2}\cos.^{2}\phi = 2\cos.2\phi + 1\left(\tfrac{2}{1}\right),$$
$$2^{3}\cos.^{3}\phi = 2\cos.3\phi + 2\left(\tfrac{3}{1}\right)\cos.\phi,$$
$$2^{4}\cos.^{4}\phi = 2\cos.4\phi + 2\left(\tfrac{4}{1}\right)\cos.2\phi + 1\left(\tfrac{4}{2}\right),$$
$$2^{5}\cos.^{5}\phi = 2\cos.5\phi + 2\left(\tfrac{5}{1}\right)\cos.3\phi + 2\left(\tfrac{5}{2}\right)\cos.\phi,$$
$$2^{6}\cos.^{6}\phi = 2\cos.6\phi + 2\left(\tfrac{6}{1}\right)\cos.4\phi + 2\left(\tfrac{6}{2}\right)\cos.2\phi + 1\left(\tfrac{6}{3}\right),$$
etc. etc. etc.

Where it is to be observed, in powers with the last term even $\cos.0\phi = 1$, with only half the coefficient to be affected. Hence in general therefore, there will be

$$2^{\lambda}\cos.^{\lambda}\phi = 2\cos.\lambda\phi + 2\left(\tfrac{\lambda}{1}\right)\cos.(\lambda+2)\phi + 2\left(\tfrac{\lambda}{2}\right)\cos.(\lambda-4)\phi$$
$$+2\left(\tfrac{\lambda}{3}\right)\cos.(\lambda-6)\phi + \text{etc.}$$

where it may be noted, whenever λ was an even number, on putting $\lambda = 2i$, the final term to be only

$$1.\left(\tfrac{2i}{i}\right)\cos.0\,\phi.$$

§. 94. Therefore after all the powers of cosines were reduced to simple cosines, our integrations always may be rendered according to such a form $\int \partial\phi \cos.i\phi \cos.\lambda\phi$, from which this is to be noted especially, the integral of which from extended from $\phi = 0$ to $\phi = 180°$ is always zero, with the sole case $\lambda = i$ excepted. Since indeed there shall be

$$\cos.i\phi\ \cos.\lambda\phi = \tfrac{1}{2}\cos.(i+\lambda)\phi + \tfrac{1}{2}\cos.(i-\lambda)\phi,$$

there will be that indefinite integral

$$= \frac{\sin.(i+\lambda)\phi}{2(i+\lambda)} + \frac{\sin.(i-\lambda)\phi}{2(i-\lambda)},$$

which for the term $\phi = 0$ clearly vanishes; truly for the other term $\phi = 180° = \pi$, on account of the whole numbers *i* and λ, it is evident, this integral again vanishes, with the sole case excepted where $\lambda = i$. Indeed if $i-\lambda$ may be considered as being infinitely small, on putting $=\omega$, the latter part of this integral will be $\frac{\sin.\omega\phi}{2\omega} = \frac{\pi}{2}$, that which also thence will be apparent, because there shall be

$$\int \partial\phi \cos.^{2} i\phi = \tfrac{1}{2}\phi + \tfrac{1}{4}\sin.2i\phi = \tfrac{1}{2}\pi.$$

§. 95. Therefore according to the integral sought to be obtained, from the power expansion $(1-2b\cos.\phi)^{m}$, it will suffice to have picked out only those terms which contain $\cos.i\phi$, since all the rest clearly will produce zero, which if taken jointly will produce $\mathrm{N}\cos.i\phi$, the total integral for ☽, will be ☽ $=(1+aa)^{m}.\tfrac{1}{2}\mathrm{N}\pi;$ wherefore it falls on us, to inquire into all the parts of the above forms, which will affect the formula $\cos.i\phi$; from which it is evident, as long as in that general term $\pm\left(\tfrac{m}{\lambda}\right)2^{\lambda}b^{\lambda}\cos.^{\lambda}\phi$ the exponent λ were less than *i*, thence clearly nothing will be contributed to the integral.

§. 96. Therefore the first term, which here arises in the computation, will be $\pm\left(\tfrac{m}{i}\right)2^{i}b^{i}\cos.^{i}\phi$, for which the above sign + will prevail if *i* were an even number, truly the lower − if odd. Hence moreover the above even reduction will arise

$$2^{i}\cos.^{i}\phi = 2\cos.\ i\phi,$$

thus so that for N the first part may arise $\pm\left(\frac{m}{i}\right)2^i b^i$, with the term following immediately, which will be

$$\mp\left(\frac{m}{i+1}\right)2^{i+1}b^{i+1}\cos.^{i+1}\phi,$$

no angle $i\phi$ arises, since there shall be

$$2^{i+1}\cos.^{i+1}\phi = 2\cos.(i+1)\phi + 2\left(\frac{i+1}{1}\right)\cos.(i-1)\phi + \text{etc.}$$

But truly the following term

$$\pm\left(\frac{m}{i+2}\right)2^{i+2}b^{i+2}\cos.^{i+2}\phi, \text{ on account of}$$
$$2^{i+2}\cos.^{i+2}\phi = 2\cos.(i+2)\phi + 2\left(\frac{i+2}{1}\right)\cos.i\phi + \text{etc.}$$

hence gives the part into the resulting letter N

$$2\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right)b^{i+2}.$$

In a similar manner nothing arises from the case $\lambda = i+3$. But from the following

$$\pm\left(\frac{m}{i+4}\right)2^{i+4}b^{i+4}\cos.^{i+4}\phi, \text{ on account of}$$
$$2^{i+4}\cos.^{i+4}\phi = 2\cos.(i+4)\phi + 2\left(\frac{i+4}{1}\right)\cos.(i+2)\phi + 2\left(\frac{i+4}{2}\right)\cos.i\phi + \text{etc.}$$

the part to be added to the letter N will be

$$2\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right)b^{i+4}.$$

In the same manner from the case $\lambda = i+6$ the part added to the letter N will be $2\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right)b^{i+6}$, and thus so on.

§. 97. Therefore with all these contributions gathered together, we will obtain the complete value of the letter N, which will be

$$\mathrm{N} = \pm 2b^i\left[\left(\frac{m}{i}\right) + \left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right)b^{i+6} + \left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right)b^{i+4} + \left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right)b^{i+6} + \text{etc.}\right]$$

where it will please to be observed, as follows

$$\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right)=\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)$$
$$\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right)=\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)$$
$$\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right)=\left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right),$$
etc.

From these values there will be :

$$\mathrm{N}=\pm 2b^{i}\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)b^{2}+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)b^{4}+\left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)b^{6}+\text{etc.}\right]$$

with which value found, our integral sought will be

$$☽=\pm\pi(1+aa)^{m}\,b^{i}\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)b^{2}+\text{etc.}\right]$$

which series evidently is interrupted, whenever m should be a positive integer. And indeed immediately the denominator $i+\lambda$ in this character $\left(\frac{m-\lambda}{i+\lambda}\right)$ begins to exceed the numerator $m-\lambda$, its value will go to zero.

The second part of the demonstration.

§. 98. But so that we may recall the expression of this integral to the letter *a*, exactly as has been represented in our theorem above, here in place of *b* we must restore the value assumed $\frac{a}{1+aa}$, and there becomes :

$$☽=\pm\pi a^{i}(1+aa)^{m-i}\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\frac{a^{2}}{(1+aa)^{2}}+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)\frac{a^{4}}{(1+aa)^{4}}+\text{etc.}\right]$$

where, so that we may elicit the form given above, it is necessary to expand the powers of $1+aa$ itself . Hence finally we may put in place $☽=\pm\pi a^{i}\mathrm{A}$, thus so that now there shall be :

$$\mathrm{A}=\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)(1+aa)^{m-i}+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)a^{2}(1+aa)^{m-i-2}$$
$$+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)a^{4}(1+aa)^{m-i-4}\ +\left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)a^{6}(1+aa)^{m-i-6}+\text{etc.}$$

But with the expansion of these powers, there may become

$$\mathrm{A}=\alpha+\beta a^{2}+\gamma a^{4}+\delta a^{6}+\varepsilon a^{8}+\zeta a^{10}+\eta a^{12}+\text{etc}.$$

the values of which letters $\alpha,\beta,\gamma,\delta,$ etc. we may investigate.

§. 99. Therefore there appears at once to be $\alpha = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)$; then truly there may be found

$$\beta = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{1}\right) + \left(\frac{m}{1}\right)\left(\frac{m-i}{i+1}\right),$$

But truly the latter part divided by the former, with the expansion made, gives $\frac{m-i-1}{i+1}$; from which noted there will become :

$$\beta = \frac{m}{i+1}\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{1}\right),$$

which is reduced to $\beta = \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)$. The letter γ will be put together in the same manner from the three parts: indeed there will be

$$\gamma = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{1}\right) + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right),$$

where the second part divided by the first part gives $\frac{2(m-i-2)}{i+1}$. But the third part divided by the first gives $\frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}$, from which there becomes

$$\gamma = 1 + \frac{2(m-i-2)}{i+1} + \frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}.$$

But truly there is

$$1 + \frac{m-i-2}{i+1} = \frac{m-1}{i+1}, \text{ and}$$
$$\left(\frac{m-i-2}{i+1}\right)\left(1 + \frac{m-i-3}{i+2}\right) = \frac{m-1}{i+2} \cdot \frac{m-i-2}{i+1}.$$

from which there is deduced:

$$\gamma = \frac{m-1}{i+1} \cdot \frac{m}{i+2} \cdot \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right),$$

which expression is contracted into this $\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)$.

§. 100. Since there shall become:

$$\alpha = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right), \beta = \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right), \gamma = \left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right),$$

hence now it will be possible to conclude safely enough, to become

$$\delta = \left(\frac{m}{3}\right)\left(\frac{m}{i+3}\right), \varepsilon = \left(\frac{m}{4}\right)\left(\frac{m}{i+4}\right), \text{etc.}$$

Truly here lest we may present some kind of conjecture or induction in general for the value of the letter Λ, we will investigate the coefficient of the power of the indefinite $a^{2\lambda}$, which we will call $=\lambda$, and there will become :

$$\Lambda = \left(\tfrac{m-i}{\lambda}\right)\left(\tfrac{m}{0}\right)\left(\tfrac{m}{i}\right) + \left(\tfrac{m}{1}\right)\left(\tfrac{m-1}{i+1}\right)\left(\tfrac{m-i-2}{\lambda-1}\right) + \left(\tfrac{m}{2}\right)\left(\tfrac{m-2}{i+2}\right)\left(\tfrac{m-i-4}{\lambda-2}\right) + \left(\tfrac{m}{3}\right)\left(\tfrac{m-3}{i+3}\right)\left(\tfrac{m-i-6}{\lambda-3}\right) + \text{etc.}$$

§. 101. The individual terms of this series found for A may be allowed to be included under this general form , which with the following factors $\left(\tfrac{m}{\theta}\right)\left(\tfrac{m-\theta}{i+\theta}\right)\left(\tfrac{m-i-2\theta}{\lambda-\theta}\right)$ expanded out, may be transformed into this form

$$\frac{m(m-1).\ .\ .\ .\ .\ .\ .\ .\ .\ (m-i-\lambda-\theta+1)}{1.2\ .\ .\ .\ .\ .\ \theta\times1.\ .\ .\ .\ .\ .(i+\theta)\times1.2\ .\ .\ (\lambda-\theta)},$$

where the factors with the numbers starting from m continually decrease by one as far as to the final factor $(m-i-\lambda-\theta+1)$; Now the same fraction may be multiplied above and below by this product

$$\lambda(\lambda-1).\ .\ .\ .\ .\ .(\lambda-\theta+1),$$

and this same fraction will be produced

$$\frac{\lambda(\lambda-1).\ .\ .\ .\ (\lambda-\theta+1)\times m(m-1)\ .\ .\ .\ .\ .\ (m-i-\lambda-\theta+1)}{1.2.3\ .\ .\ .\ .\ .\ \theta\times1.2.3\ .\ .\ .\ .\ .(i+\theta)\times1.2.3\ .\ .\ .\lambda},$$

in which in the first place the character $\left(\tfrac{\lambda}{\theta}\right)$ will be contained, then the character $\left(\tfrac{m}{\lambda}\right)$ will also be contained there ; what remains will give the character $\left(\tfrac{m-\lambda}{i+\theta}\right)$, and thus the general form Λ will be had $=\left(\tfrac{\lambda}{\theta}\right)\left(\tfrac{m}{\lambda}\right)\left(\tfrac{m-\lambda}{i+\theta}\right)$. From which if in place of θ we may write successively 0, 1, 2, 3, etc:, because the factor $\left(\tfrac{m}{\lambda}\right)$ is present in the individual terms in the value of the letter Λ will be given by :

$$\Lambda = \left(\tfrac{m}{\lambda}\right)\left[\left(\tfrac{\lambda}{0}\right)\left(\tfrac{m-\lambda}{i}\right) + \left(\tfrac{\lambda}{1}\right)\left(\tfrac{m-\lambda}{i+1}\right) + \left(\tfrac{\lambda}{2}\right)\left(\tfrac{m-\lambda}{i+2}\right) + \text{etc.}\right]$$

Truly some time ago I have demonstrated, the sum of this similar series

$$\left(\tfrac{p}{0}\right)\left(\tfrac{q}{r}\right) + \left(\tfrac{p}{1}\right)\left(\tfrac{q}{r+1}\right) + \left(\tfrac{p}{2}\right)\left(\tfrac{q}{r+2}\right) + \left(\tfrac{p}{3}\right)\left(\tfrac{q}{r+3}\right) + \text{etc.}$$

is always $=\left(\tfrac{p+q}{p+r}\right)=\left(\tfrac{p+q}{q-r}\right)$. Therefore with the application performed, there will be $p=\lambda,\ q=m-\lambda,\ r=i$: and thus in the finally we will have

$$\Lambda=\left(\frac{m}{\lambda}\right)\left(\frac{m}{\lambda+i}\right)=\left(\frac{m}{\lambda}\right)\left(\frac{m}{m-\lambda-i}\right),$$

which is the demonstration of the above conjecture advanced, and concluded from the values $\alpha,\ \beta,\ \gamma$.

§. 102. So that if now here in place of λ we may write successively the numbers 0, 1, 2, 3, etc., we will obtain the true value of the series, which contained under the letter A ; clearly there will be :

$$\mathrm{A}=\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)a^2+\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)a^4 +\left(\frac{m}{3}\right)\left(\frac{m}{i+3}\right)a^6+\text{etc.}$$

and hence the value of the integral indicated under the sign ☽ of the indicated formula will be

$$☽=\pm\pi a^i\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)a^2+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)a^4+\text{etc.}\right]$$

which expression manifestly always is interrupted, whenever m is a positive integer. But here it is required to remember, of the ambiguous sign $\pm$ the upper always is used if i were an even number, truly the lower if it were odd.

Third part of the demonstration.

§. 103. The same form, which here we have gained much for the value of the integral ☽ thus is simpler there, than what our theorem provided for us, clearly which, if in place of ♂ we may write the series that it designates, it will become

$$☽=\frac{\pi a^i\left(\frac{m}{1}\right)}{\left(\frac{-m-1}{i}\right)}\left[\left(\frac{m}{0}\right)\left(\frac{m+i}{i}\right)+\left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)a^2+\left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)a^4+\text{etc.}\right]$$

Therefore it remains, so that we may show perfect consensus between these two expressions with the appearance of many discrepancies between themselves. But here it will be most helpful to have observed, $\left(\frac{-m-1}{i}\right)=\pm\left(\frac{m+i}{i}\right)$, therefore as we have now observed above in §.88, to be in general $\left(\frac{-p}{q}\right)=\pm\left(\frac{p+q-1}{q}\right)$, where the upper sign prevails if q were an even number, truly the lower if odd; with which observed the latter form for ☽ found will become

$$☽ = \frac{\pi a^i\left(\frac{m}{i}\right)}{\left(\frac{m+i}{i}\right)}\left[\left(\frac{m-i}{i}\right)\left(\frac{m+i}{i}\right)+\left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)a^2+\text{etc.}\right].$$

§. 104. Now because both forms are affected by the ambiguous sign $\pm$, it falls on us to show, if we may multiply each expression by $\left(\frac{m+i}{i}\right)$, the two following series are in fact equal to each other :

$$\text{I.}\quad \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m+i}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)\left(\frac{m+i}{i}\right)a^2+\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)\left(\frac{m+i}{i}\right)a^4+\text{etc.}$$

$$\text{II.}\quad \left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right)\left(\frac{m}{i}\right)+\left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)\left(\frac{m}{i}\right)a^2+\left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)\left(\frac{m}{i}\right)a^4+\text{etc.}$$

where the equality of the first terms on account of $\left(\frac{m}{0}\right)$ and $\left(\frac{m-i}{0}\right)=1$ is produced at once: then truly without difficulty the equality itself between the second terms $a\,a$ affected can be shown, and in a similar manner also from the following this equality likewise is held.

§. l05. Truly also here lest here we may be urged to use induction, we will demonstrate the agreement of two terms to the same power $a^{2\lambda}$. Truly in the first series the same power $a^{2\lambda}$ has this coefficient $\left(\frac{m}{\lambda}\right)\left(\frac{m}{i+\lambda}\right)\left(\frac{m+i}{i}\right)$; truly in the other of the same power the coefficient is $\left(\frac{m-i}{\lambda}\right)\left(\frac{m+i}{i+\lambda}\right)\left(\frac{m}{i}\right)$. Therefore each may be expanded out into simple factors, and the first leads to this fraction:

$$\frac{m.\,(m-1)\,.\,.\,.\,(m-\lambda+1)\times m(m-1)\,.\,.\,.\,.\,.\,.(m-i-\lambda+1)\times(m+i)\,.\,.\,.\,.\,.\,.(m+1)}{1.2.\,.\,.\,.\,.\,.\,\lambda\times 1.2.\,.\,.\,.\,.\,.(i+\lambda)\times 1.2.3\,.\,.\,.\,i};$$

truly the latter gives the same

$$\frac{(m-i)\,.\,.\,.\,.\,(m-i-\lambda+1)\times(m+i)\,.\,.\,.\,.\,.\,.(m-\lambda+1)\times m\,.\,.\,.\,.\,.\,.(m-i+1)}{1.2.\,.\,.\,.\,.\,.\,\lambda\times 1.2.\,.\,.\,.\,.\,.(i+\lambda)\times 1.2.3\,.\,.\,.\,.\,.\,i};$$

where the denominators on both sides clearly are the same, thus so that only the equality between the numerators shall be required to be demonstrated.

§. 106. But initially in the first numerator the third general factor combined with the first gives this product

$$(m+i)\,.\,.\,.\,.\,.\,.(m-\lambda+1),$$

which also occurs in the latter form : therefore with these removed it will be required to show the equality between the remaining parts which are,
in the prior formula $m(m-1)\,.\,.\,.\,.\,.\,.(m-i-\lambda+1)$,

in the other $m(m-1)\ .\ .\ .\ .\ (m-i+1)\times(m-i)\ .\ .\ .\ .\ (m-i-\lambda+1)$
which again is now obvious. Therefore the truth of our theorem, which demonstration we have undertaken, now is proposed rigorously to be considered as the formula of the integral

$$☽ = \int \partial\phi \cos.i\phi\left(1+aa-2a\cos.\varphi\right)^n \left[\begin{smallmatrix} from & \phi=0 \\ to\ \phi=\pi \end{smallmatrix}\right].$$

Fourth part of the demonstration.

§. 107. With the value of the formula ☽ found, the whole demonstration has now been completed, since now that duly has been derived from the initial value of the formula ☉. Yet meanwhile this too in turn may be convenient to derive the other value ☉ from the value ☽. Moreover, we may use the simpler form of ☽, to which the same demonstration led us at once, which was

$$☽ = \pm\pi a^i\left[\left(\tfrac{m}{0}\right)\left(\tfrac{m}{i}\right)+\left(\tfrac{m}{1}\right)\left(\tfrac{m}{i+1}\right)a^2+\left(\tfrac{m}{2}\right)\left(\tfrac{m}{i+2}\right)a^4+\text{etc.}\right]$$

where the upper sign prevails if i were an even number, the lower if odd.

§. 108. Now from this value of the formula ☽ the value of the other formula ☉ is deduced, but only if we may write $-n-1$ in place of m, which value therefore will be

$$☉ = \pm\pi a^i\left[\left(\tfrac{-n-1}{0}\right)\left(\tfrac{-n-1}{i}\right)+\left(\tfrac{-n-1}{1}\right)\left(\tfrac{-n-1}{i+1}\right)a^2+\left(\tfrac{-n-1}{2}\right)\left(\tfrac{-n-1}{i+2}\right)a^4+\text{etc.}\right]$$

but which series now progresses to infinity, if indeed n were a positive whole number; on account of which this series is required to be changed into another, which does terminate, whenever n should be a positive whole number, that which can be brought forwards better with the aid of the lemma above.

§. 109. Therefore we will compare the series found here with the series $\hbar$ in the lemma, that which shall be required to be put in place

$$f=-n-1,\ h=-n-1 \text{ et } e=i,$$

thus so that now there shall be $☉=\pm\pi a^i\hbar$. But from these values the other series noted by the sign ♂ will become, on account of

$$-h-i=n,\ -f-i=n,\ \text{and } x=a^2,$$
$$♂ = \left(\tfrac{n}{0}\right)\left(\tfrac{n}{i}\right)+\left(\tfrac{n}{1}\right)\left(\tfrac{n}{i+1}\right)a^2+\left(\tfrac{n}{2}\right)\left(\tfrac{n}{i+2}\right)a^4+\text{etc.}$$

But truly the relation between these two series will be

$$\left(\frac{i-n-1}{i}\right)\hbar = \frac{\left(\frac{n+i}{i}\right)♂}{(1-aa)^{2n+1}};$$

where it may be noted, since now we have observed above to be

$\left(\frac{-p}{q}\right) = \pm\left(\frac{p+q-1}{q}\right)$, here to become $\left(\frac{-n-1+i}{i}\right) = \pm\left(\frac{n}{i}\right)$; where again the upper sign prevails, if *i* were an even number. Hence therefore there will be

$$\hbar = \pm\frac{\left(\frac{n+i}{i}\right)♂}{\left(\frac{n}{i}\right)(1-aa)^{2n+1}}.$$

§. 110. Therefore this same value may be substituted in place of $\hbar$, where the twofold ambiguity of the signs may be removed from within, but in place of ♂ the series may be written in the manner given, and for ⊙ we will obtain the following expression

$$\odot = \frac{\pi a^i\left(\frac{n+i}{i}\right)}{\left(\frac{n}{i}\right)(1-aa)^{2n+1}}.\left[\left(\frac{n}{0}\right)\left(\frac{n}{i}\right)+\left(\frac{n}{1}\right)\left(\frac{n}{i+1}\right)a^2+\left(\frac{n}{2}\right)\left(\frac{n}{i+2}\right)a^4+\text{etc.}\right]$$

which series evidently always terminates, whenever *n* were a positive whole number. But yet truly this labours from a defect, because the cases in which $n < i$, on account of $\left(\frac{n}{i}\right)=0$, will be seen to emerge infinite. Truly it is to be observed, in these cases also all the terms of the series ♂ become zero ; from which it is necessary, so that we may examine its true value and from the whole expression. But truly in the remaining cases, with which $n > i$ this expression thus may be seen to be preferred to that which we have deduced in the theorem.

§. 111. Therefore here it must be shown, all the terms of our series thus are able to be transformed, so that they allow division by the denominator $\left(\frac{n}{i}\right)$. But truly any of our series may be contained in this form $\left(\frac{n}{\lambda}\right)\left(\frac{n}{i+\lambda}\right)$, which multiplied by the common factor $\left(\frac{n+i}{i}\right)$ will become $\left(\frac{n+i}{i}\right)\left(\frac{n}{\lambda}\right)\left(\frac{n}{i+\lambda}\right)$, which expanded into factors is reduced to this fraction

$$\frac{(n+i)\,.\,.\,.\,.\,(n+1)\times(n)\,.\,.\,.\,.\,.\,.(n-\lambda+1)\times n\,.\,.\,.\,.\,.\,.(n-i-\lambda+1)}{1.2.\,.\,.\,.\,.\,.i\times1.2.\,.\,.\,.\,.\,.\lambda\times1.2.\,.\,.\,.\,.\,.(i+\lambda)};$$

where both the numerator as well as the denominator have three principal factors ; but the individual factors in the numerator decrease continually to one, while in the denominator they increase by one. Therefore since $\left(\frac{n}{i}\right)=\frac{n\ .\ .\ .\ .\ .\ .(n-i+1)}{1.2.\ .\ .\ .\ .\ .\ .i}$, the upper fraction divided by this, on account of

$$\frac{n\ .\ .\ .\ .\ .\ .(n-i-\lambda+1)}{n\ .\ .\ .\ .\ .\ .(n-i+1)}=(n-i).\ .\ .\ .\ .(n-i-\lambda+1),$$

will become

$$\frac{(n+i)\ .\ .\ .\ .\ (n+1)\times(n)\ .\ .\ .\ .\ .\ .(n-\lambda+1)\times(n-i)\ .\ .\ .\ .\ .\ .(n-i-\lambda+1)}{1.2.\ 3\ .\ .\ .\ .\ .\lambda\times1.2.3\ .\ .\ .\ .\ .(i+\lambda)};$$

which evidently passes into this (on account of the two prior factors joining together)

$$\frac{(n+i)\ .\ .\ .\ .\ .\ .\ .(n-\lambda+1)\times(n-i)\ .\ .\ .\ .\ .\ .(n-i-\lambda+1)}{1.2.\ 3\ .\ .\ .\ .\ .\lambda\times1.2.3\ .\ .\ .\ .\ .(i+\lambda)},$$

thus so that with everything reduced to charactes, the general form of each term shall become $=\left(\frac{n+i}{i+\lambda}\right)\left(\frac{n-i}{\lambda}\right)$.

§. 112. Therefore now in place of λ we may write successively the values 0, i, 2, 3, etc. and the value of the integral formula ⊙ will be produced, exactly as has been enunciated in the theorem, clearly

$$\odot=\frac{\pi a^{i}}{(1-aa)^{2n+1}}\left(\left(\frac{n-i}{0}\right)\left(\frac{n+i}{i}\right)+\left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right)a^{2}+\left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right)a^{4}+\text{etc}\right).$$

which expression now not only always is terminated, whenever *n* were a positive whole number, nor suffers from any further defect, since for all the cases the value of ⊙ itself may be shown to be determined, and thus indeed our theorem, which before depended on conjecture only, has been confirmed by the most rigorous demonstration.

SUPPLEMENTUM IV*b*.

AD TOM. I. CAP.V.
DE
INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM.

3) Disquisitio conjecturalis super formula integrali

$$\int \frac{\partial\phi \cos.i\phi}{(\alpha+\beta\cos.\phi)^n}.$$

M. S. Academiae exhib. die 31 *Augusti* 1778.

§. 40. Incipiamus a casu simplicissimo quo $i=0$ et $n=1$, et formula integranda proponitur haec $\int\frac{\partial\phi}{\alpha+\beta\cos.\phi}$, ad quod praestandum commodissime in subsidium vocatur haec substitutio $\operatorname{tang}.\frac{1}{2}\phi=t$, unde statim fit $\partial\phi=\frac{2\partial t}{1+tt}$: porro vero cum hinc sit

$$\sin.\tfrac{1}{2}\phi=\frac{t}{\sqrt{1+tt}} \text{ et } \cos.\tfrac{1}{2}\phi=\frac{1}{\sqrt{1+tt}}$$

erit $\cos.\phi=\frac{1-tt}{1+tt}$, eoque denominator nostrae formulae $\alpha+\beta\cos.\phi=\frac{\alpha+\beta+(\alpha-\beta)tt}{1+tt}$

sicque nostra formula integranda erit

$$\int \frac{2\partial t}{\alpha+\beta+(\alpha-\beta)tt}.$$

§. 41. Constat autem ex elementis esse

$$\int \frac{\partial t}{f+gtt} = \frac{1}{\sqrt{fg}} \text{Arc.tang}\, t\sqrt{\frac{g}{f}}$$

Quare cum pro nostro casu sit $f = \alpha + \beta$ et $g = \alpha - \beta$, habebimus hanc integrationem

$$\int \frac{\partial \phi}{\alpha+\beta \cos.\phi} = \frac{2}{\sqrt{(\alpha\alpha-\beta\beta)}} \text{Arc.tang}\, t\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}$$

existente $t = \text{tang.}\frac{1}{2}\phi$; quod ergo integrale evanescit casu $t = 0$, ideoque casu $\phi = 0$, Quodsi ergo hoc integrale extendere velimus a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$, ubi sit $t = \infty$, istud integrale erit $\frac{2}{\sqrt{(\alpha\alpha-\beta\beta)}} \cdot \frac{\pi}{2}$ denotante π semiperipheriam circuli, cujus radius $= 1$.

§. 42. Quoniam igitur integrale nostrae formulae a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$ tam concinne et simpliciter exprimitur, etiam generatim in hac dissertatione in ea tantum integralia formulae generalis propositae

$$\int \frac{\partial \phi \cos.i\phi}{(\alpha+\beta \cos.\phi)^n}.$$

sum inquisiturus, quae comprehenduntur inter terminos $\phi = 0$ et $\phi = 180^\circ$. Quia autem in casu tractato formula inest irrationalis $\sqrt{(\alpha\alpha - \beta\beta)}$, ad hoc incommodum tollendum, in sequentibus perpetuo assumemus $\alpha = 1 + aa$ et $\beta = -2a$, unde fit $\sqrt{(\alpha\alpha - \beta\beta)} = 1 - aa$, sicque nostrae disquisitiones versabuntur circa integrationem hujus formulae generalis

$$\int \frac{\partial \phi \cos.i\phi}{(1+aa-2a\cos.\phi)^n},$$

pro qua brevitatis gratia ubique statuamus

$$1 + aa - 2a\cos.\phi = \varDelta,$$

ut nostra formula generalis jam sit

$$\int \frac{\partial \phi \cos.i\phi}{\varDelta^n},$$

ubi ut jam notatum, eum tantum integralis valorem explorare nobis est propositum, qui intra terminos $\phi=0$ et $\phi=180^\circ$ contineatur, quem valorem ex casibus particularibus concludere conabimur.
Praeterea vero hic in genere notetur, litteram *i* nobis perpetuo alios numeros non designare praeter integros, et quidem positives, quandoquidem semper est

$$\cos.-i\phi=\cos.+i\phi.$$

I. De integratione formulae

$$\int\frac{\partial\phi\cos.i\phi}{\Delta}\begin{bmatrix}a & \phi=0\\ ad & \phi=180^{\text{o}}\end{bmatrix}.$$

§. 43. Hic ergo casus in generali continetur, ponendo exponentem $n=1$, quem casum ut simplicissimum spectamus, siquidem casus $n=0$ nulla prorsus laborat difficultate, cum sit

$$\int\partial\phi\cos.i\phi=\tfrac{1}{i}\sin.i\phi,$$

quod integrale jam evanescit casu $i=0$, et quoniam *i* numeros tantum integros significat, sumto $\phi=180^\circ$ hoc integrale iterum evanescit, solo casu excepto quo $i=0$, quippe quo casu integrale fiet $=\phi$, ideoque sumto $\phi=180^\circ$ erit pro terminis integrationis constitutis $\int\partial\phi=\pi$.

§. 44. Iste postremus casus fundamentum continet, unde integralia formae hic propositae haurire conveniet; cum enim sit $\partial\phi=\frac{(1+aa)\partial\phi}{\Delta}-\frac{2a\partial\phi\cos.\phi}{\Delta}$,
erit integrando pro terminis praescriptis

$$\pi=(1+aa)\int\tfrac{\partial\phi}{\Delta}-2a\int\tfrac{\partial\phi\cos.\phi}{\Delta};$$

supra autem invenimus esse $\int\frac{\partial\phi}{\Delta}=\frac{\pi}{1-aa}$, quo valore substituto adipiscimur integrationem casus $i=1$, cum enim sit

$$\pi=\tfrac{(1+aa)\pi}{1-aa}-2a\int\tfrac{\partial\phi\cos.\phi}{\Delta},\text{ erit }\int\tfrac{\partial\phi\cos.\phi}{\Delta}=\tfrac{\pi a}{1-aa};$$

sicque jam duos casus sumus adepti; qui sunt

$$\int\tfrac{\partial\phi}{\Delta}=\tfrac{\pi}{1-aa},\text{ et }\int\tfrac{\partial\phi\cos.\phi}{\Delta}=\tfrac{\pi a}{1-aa}.$$

§. 45. Ex his autem duobus casibus $i=0$ et $i=1$ sequentes omnes haud difficulter derivare licet ope hujus lemmatis; cum sit ut vidimus $\int\partial\phi\cos.i\phi=0$, erit

$$0=(1+aa)\int\frac{\partial\phi\cos.i\phi}{\Delta}-2a\int\frac{\partial\phi\cos.\phi\cos.i\phi}{\Delta}.$$

Constat autem esse

$$2\cos.\phi\ \cos.i\phi=\cos.(i-1)\phi+\cos.(i+1)\varphi,$$

unde habebimus hanc aequationem

$$\frac{1+aa}{a}\int\frac{\partial\phi\ \cos.i\phi}{\Delta}=\int\frac{\partial\phi\cos.(i-1)\phi}{\Delta}+\int\frac{\partial\phi\cos.(i+1)\varphi}{\Delta}$$

unde oritur istud lemma

$$\int\frac{\partial\phi\cos.(i+1)\varphi}{\Delta}=\frac{1+aa}{a}\int\frac{\partial\phi\ \cos.i\phi}{\Delta}-\int\frac{\partial\phi\cos.(i-1)\phi}{\Delta}.$$

Sumto nunc $i=1$, istud lemma nobis suppeditat hunc casum

$$\int\frac{\partial\phi\cos.2\varphi}{\Delta}=\frac{1+aa}{a}\int\frac{\partial\phi\ \cos.\phi}{\Delta}-\int\frac{\partial\phi}{\Delta},$$

qui ergo per binos praecedentes expeditur; fiet enim

$$\int\frac{\partial\phi\cos.2\varphi}{\Delta}=\frac{\pi aa}{1-aa}.$$

Sumatur nunc $i=2$, et lemma nobis dabit

$$\int\frac{\partial\phi\cos.3\varphi}{\Delta}=\frac{1+aa}{a}\int\frac{\partial\phi\ \cos.2\phi}{\Delta}-\int\frac{\partial\phi\cos.\phi}{\Delta},\ \text{sive}$$
$$\int\frac{\partial\phi\cos.3\varphi}{\Delta}=\frac{\pi a^3}{1-aa}:$$

simil modo sumto $i=3$, lemma dabit

$$\int\frac{\partial\phi\cos.4\varphi}{\Delta}=\frac{1+aa}{a}\int\frac{\partial\phi\ \cos.3\phi}{\Delta}-\int\frac{\partial\phi\cos.2\phi}{\Delta},\ \text{sive}$$
$$\int\frac{\partial\phi\cos.4\varphi}{\Delta}=\frac{\pi a^4}{1-aa}:$$

Porro casus $i=4$ praebet

$$\int\frac{\partial\phi\cos.5\varphi}{\Delta}=\frac{1+aa}{a}\int\frac{\partial\phi\ \cos.4\phi}{\Delta}-\int\frac{\partial\phi\cos.3\phi}{\Delta},\ \text{sive}$$
$$\int\frac{\partial\phi\cos.5\varphi}{\Delta}=\frac{\pi a^5}{1-aa},\ \text{atque ita porro.}$$

§.46. Hinc igitur patet, singulos istos casus ex binis praecedentibus determinari ope scalae relationis $\frac{1+aa}{a}, -1$, atque seriem recurrentem hinc oriundam abire in geometricam: quodsi enim bini termini postremi jam inventi fuerint

$$\frac{\pi a^{\lambda}}{1-aa} \text{ et } \frac{\pi a^{\lambda+1}}{1-aa}$$

sequens reperitur $= \frac{\pi a^{\lambda+2}}{1-aa}$, ex quo ergo sine ullo dubio sequitur, pro casu particulari hoc loco tractati in genere fore

$$\int \frac{\partial\phi \cos.i\varphi}{\Delta} = \frac{\pi a^{i}}{1-aa},$$

ubi autem probe est notandum, loco *i* non nisi numeros integros positives assumi debere.

II. De integratione formulae

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^2} \begin{bmatrix} a & \phi=0 \\ ad & \phi=180^{\circ} \end{bmatrix}.$$

§. 47. Casus simplicissimus hic occurret $\int \frac{\partial\phi}{\Delta^2}$, cujus ergo integrale ante omnia perscrutari oportet; hunc in finem consideremus hanc formulam finitam $\frac{\sin.\phi}{\Delta} = \mathrm{V}$, quae pro utroque termino $\phi = 0$ et $\phi = 180^{\circ}$ evanescit; hinc autem erit

$$\partial V = \frac{\partial\phi \cos.\phi}{\Delta} - \frac{2a\partial\phi \sin^2.\phi}{\Delta^2}, \text{ sive}$$
$$\partial V = \frac{(1+aa)\partial\phi \cos.\phi - 2a\partial\phi}{\Delta^2};$$

unde integrando jam novimus esse

$$0 = (1+aa)\int \frac{\partial\phi \cos.\phi}{\Delta^2} - 2a\int \frac{\partial\phi}{\Delta^2}.$$

Porro vero quoniam ante habuimus $\int \frac{\partial\phi}{\Delta} = \frac{\pi}{1-aa}$, hanc formulam integralem supra et infra per Δ multiplicando, erit quoque

$$\frac{\pi}{1-aa} = (1+aa)\int \frac{\partial\phi}{\Delta^2} - 2a\int \frac{\partial\phi \cos.\phi}{\Delta^2}.$$

Ex praecedente autem colligitur

$$\int \frac{\partial\phi \cos.\phi}{\Delta^2} = \frac{2a}{1+aa} \int \frac{\partial\phi}{\Delta^2},$$

quo valore substituto habebimus

$$\frac{\pi}{1-aa} = (1+aa) \int \frac{\partial\phi}{\Delta^2} - \frac{4aa}{1+aa} \int \frac{\partial\phi}{\Delta^2} = \frac{(1-aa)^2}{1+aa} \int \frac{\partial\phi}{\Delta^2},$$

quamobrem hinc adipiscimur hanc integrationem principalem

$$\int \frac{\partial\phi}{\Delta^2} = \frac{\pi(1+aa)}{(1-aa)^3},$$

ex quo immediate deducitur casus sequens

$$\int \frac{\partial\phi \cos.\phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3}.$$

§. 48. Pro sequentibus casibus consideremus integrationem in articulo praecedente inventam

$$\int \frac{\partial\phi \cos.i\phi}{\Delta} = \frac{\pi a^i}{1-aa},$$

quae formula integralis supra et infra per Δ multiplicando discerpitur in sequentes duas partes

$$\frac{\pi a^i}{1-aa} = (1+aa) \int \frac{\partial\phi \cos.i\phi}{\Delta^2} - 2a \int \frac{\partial\phi \cos.\phi \cos.i\phi}{\Delta^2},$$

quae aequatio porro evolvitur in hanc formam

$$\frac{\pi a^i}{1-aa} = (1+aa) \int \frac{\partial\phi \cos.i\phi}{\Delta^2} - a \int \frac{\partial\phi \cos.(i-1)\phi}{\Delta^2} - a \int \frac{\partial\phi \cos.(i+1)\phi}{\Delta^2},$$

inde deducitur hoc quasi lemma

$$\int \frac{\partial\phi \cos.(i+1)\phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial\phi \cos.i\phi}{\Delta^2} - \int \frac{\partial\phi \cos.(i-1)\phi}{\Delta^2} - \frac{\pi a^{i-1}}{1-aa}.$$

§. 49. Sumamus nunc statim $i=1$, atque istud lemma nobis praebet

$$\int \frac{\partial\phi \cos.2\phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial\phi \cos.\phi}{\Delta^2} - \int \frac{\partial\phi}{\Delta^2} - \frac{\pi}{1-aa};$$

hic jam bini valores jam inventi substituantur, atque reperietur

$$\int\frac{\partial\phi\cos.2\phi}{\varDelta^2}=\frac{\pi(1+aa)-\pi(1-aa)^2}{(1-aa)^2},$$

hinc ergo sequitur fore

$$\int\frac{\partial\phi\cos.2\phi}{\varDelta^2}=\frac{\pi\left(3aa-a^4\right)}{(1-aa)^5}=\frac{\pi aa(3-aa)}{(1-aa)^3}.$$

Sumatur nunc problemate praemisso $i=2$, eritque

$$\int\frac{\partial\phi\cos.3\phi}{\varDelta^2}=\frac{1+aa}{a}\int\frac{\partial\phi\cos.2\phi}{\varDelta^2}-\int\frac{\partial\phi\cos.\phi}{\varDelta^2}-\frac{\pi a}{1-aa},\ \text{sive}$$

$$\int\frac{\partial\phi\cos.3\phi}{\varDelta^2}=\frac{(1+aa)\pi a(3-aa)-2\pi a-\pi a(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int\frac{\partial\phi\cos.3\phi}{\varDelta^2}=\frac{\pi a^3(4-2aa)}{(1-aa)^3}.$$

Sit nunc in lemmate praemisso $i=3$, eritque

$$\int\frac{\partial\phi\cos.4\phi}{\varDelta^2}=\frac{1+aa}{a}\int\frac{\partial\phi\cos.3\phi}{\varDelta^2}-\int\frac{\partial\phi\cos.2\phi}{\varDelta^2}-\frac{\pi aa}{1-aa},\ \text{sive}$$

$$\int\frac{\partial\phi\cos.4\phi}{\varDelta^2}=\frac{(1+aa)\pi aa(4-2aa)-\pi aa(3-aa)-\pi aa(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int\frac{\partial\phi\cos.4\phi}{\varDelta^2}=\frac{\pi a^4(5-3aa)}{(1-aa)^3}.$$

Sit nunc in lemmate nostro $i=4$, eritque

$$\int\frac{\partial\phi\cos.5\phi}{\varDelta^2}=\frac{1+aa}{a}\int\frac{\partial\phi\cos.4\phi}{\varDelta^2}-\int\frac{\partial\phi\cos.3\phi}{\varDelta^2}-\frac{\pi a^3}{1-aa},\ \text{sive}$$

$$\int\frac{\partial\phi\cos.5\phi}{\varDelta^2}=\frac{(1+aa)\pi a^3(5-3aa)-\pi a^3(4-2aa)-\pi a^3(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial\phi \cos.5\phi}{\Delta^2} = \frac{\pi a^5(6-4aa)}{(1-aa)^3}.$$

Sit nunc in lemmate nostro $i = 5$, eritque

$$\int \frac{\partial\phi \cos.6\phi}{\Delta^2} = \frac{1+aa}{a}\int \frac{\partial\phi \cos.5\phi}{\Delta^2} - \int \frac{\partial\phi \cos.4\phi}{\Delta^2} - \frac{\pi a^4}{1-aa}, \text{ sive}$$

$$\int \frac{\partial\phi \cos.6\phi}{\Delta^2} = \frac{(1+aa)\pi a^4(6-4aa) - \pi a^4(5-3aa) - \pi a^4(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial\phi \cos.6\phi}{\Delta^2} = \frac{\pi a^6(7-5aa)}{(1-aa)^3}.$$

§. 50. Qui has formulas earumque generationem attentius perpendit, nullo certe modo dubitabit, inde hanc conclusionem deducere, quin in genere pro casu hic proposito futurum sit

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^2} = \frac{\pi a^i\left[i+1-(i-1)aa\right]}{(1-aa)^3}$$

cujus lex cum non sit tam manifesta , quam in casu praecedente, omnes formulas inventas junctim ante oculos ponamus

$$\int \frac{\partial\phi}{\Delta^2} = \frac{\pi(1+aa)}{(1-aa)^3}$$

$$\int \frac{\partial\phi \cos.\phi}{\Delta^2} = \frac{\pi aa(2-0aa)}{(1-aa)^3}$$

$$\int \frac{\partial\phi \cos.2\phi}{\Delta^2} = \frac{\pi aa(3-aa)}{(1-aa)^3}$$

$$\int \frac{\partial\phi \cos.4\phi}{\Delta^2} = \frac{\pi a^4(5-3aa)}{(1-aa)^3}$$

$$\int \frac{\partial\phi \cos.5\phi}{\Delta^2} = \frac{\pi a^4(6-4aa)}{(1-aa)^3}$$

$$\int \frac{\partial\phi \cos.6\phi}{\Delta^2} = \frac{\pi a^6(7-5aa)}{(1-aa)^3}.$$

III. De integratione formulae

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^3}\begin{bmatrix} a & \phi=0 \\ ad & \phi=180^{\circ} \end{bmatrix}.$$

§. 51. Pro casu simplicissimo $\int\frac{\partial\phi}{\varDelta^3}$ eruendo, utamur hac formula

$$V=\frac{\sin.\phi}{\varDelta^2},\ \text{eritque}\ \partial V=\frac{\partial\phi\cos.\phi}{\varDelta^2}-\frac{2\partial\phi\sin^2.\phi}{\varDelta^3},\ \text{sive}$$

$$\partial V=\frac{(1+aa)\partial\phi\cos.\phi-2a\partial\phi\cos^2.\phi-4a\partial\phi\sin^2.\phi}{\varDelta^3};$$

Hic loco $\sin.^2\phi$ scribatur $1-\cos.^2\phi$, atque integrando, ob $\mathrm{V}=0$ habebimus hanc aequationem

$$0=(1+aa)\int\frac{\partial\phi\cos.\phi}{\varDelta^3}-4a\int\frac{\partial\phi}{\varDelta^3}+2a\int\frac{\partial\phi\cos^2.\phi}{\varDelta^3}.$$

§. 52. Huc addamus hanc formam indefinitam

$$s=A\int\frac{\partial\phi}{\varDelta}+B\int\frac{\partial\phi}{\varDelta^2}$$

cujus differentiale ad denominationem $\varDelta^3$ ducatur, litterae vero A et B ita definiantur, ut membra $\partial\phi\cos.\phi$ et $\partial\phi\cos.^2\phi$ evanescant, eritque formulis differentialibus additis

$$\begin{aligned}\frac{\varDelta^3(\partial V+\partial s)}{\partial\phi}=-4a+(1+aa)\cos.\phi\ &+2a\partial\phi\cos.^2\phi\\ +\mathrm{A}(1+aa)^2-4\mathrm{A}a(1+aa)\cos.\phi&+4\mathrm{A}aa\cos.^2\phi\\ +\mathrm{B}(1+aa)\ -2\mathrm{B}a\cos.\phi&.\end{aligned}$$

Nunc igitur ut termini $\cos.^2\phi$ abigantur, statuatur

$$2a+4Aaa=0,\ \text{ideoque}\ A=\frac{-1}{2a}.$$

Nunc etiam termini $\cos.\phi$ e medio tollantur; eritque

$$1+aa-4Aa(1+aa)-2Ba=0,$$

unde fit

$$B=\frac{3(1+aa)}{2a}.$$

Ex quibus valoribus nanciscimur

$$\frac{\varDelta^3(\partial V+\partial s)}{\partial\phi}=\frac{(1-aa)^2}{a};$$

hinc ergo vicissim integrando habebimus

$$V+s=\frac{(1-aa)^2}{a}\int\frac{\partial\phi}{\Delta^3}.$$

§. 53. Cum igitur, ut jam notavimus, sit $V=0$, atque ex casibus jam tractatis

$$s=-\frac{1}{2a}\cdot\frac{\pi}{1-aa}+\frac{3(1+aa)}{2a}\cdot\frac{\pi(1+aa)}{(1-aa)^3},$$

habebimus hanc aequationem

$$\frac{(1-aa)^2}{a}\int\frac{\partial\phi}{\Delta^3}=\frac{3\pi(1+aa)^2-\pi(1-aa)^2}{2a(1-aa)^3},$$

unde colligitur

$$\int\frac{\partial\phi}{\Delta^3}=\frac{\pi\left(1+4aa+a^4\right)}{(1-aa)^5}.$$

§. 54. Cum sit $\int\frac{\partial\phi}{\Delta^2}=\frac{\pi(1+aa)}{(1-aa)^3}$, erit per reductionem hactenus usitatam

$$\frac{\pi(1+aa)}{(1-aa)^3}=(1+aa)\int\frac{\partial\phi}{\Delta^3}-2a\int\frac{\partial\phi\cos.\phi}{\Delta^3},$$

unde concludimus

$$\int\frac{\partial\phi\cos.\phi}{\Delta^3}=\frac{1+aa}{2a}\int\frac{\partial\phi}{\Delta^3}-\frac{\pi(1+aa)}{2a(1-aa)^3},\ \text{ideoque}$$

$$\int\frac{\partial\phi\cos.\phi}{\Delta^3}=\frac{1+aa}{2a}\cdot\frac{\pi\left(1+4aa+a^4\right)}{(1-aa)^5}-\frac{\pi(1+aa)}{2a(1-aa)^3}$$

$$=\frac{3\pi a(1+aa)}{(1-aa)^5}=\frac{\pi a(3+3aa)}{(1-aa)^5}.$$

§. 55. Cum igitur in articulo praecedente invenimus

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^2}=\frac{\pi a^i\left[i+1-(i-1)aa\right]}{(1-aa)^3}$$

hanc formulam integralem supra et infra per Δ multiplicando habebimus

$$\frac{\pi a^i\left[i+1-(i-1)aa\right]}{(1-aa)^3}=(1+aa)\int\frac{\partial\phi\cos.i\phi}{\Delta^3}-2a\int\frac{\partial\phi\cos.i\phi\cos.\phi}{\Delta^3},\ \text{sive}$$

$$\frac{\pi a^i\left[i+1-(i-1)aa\right]}{(1-aa)^3}=(1+aa)\int\frac{\partial\phi\cos.i\phi}{\Delta^3}$$

$$-a\int\frac{\partial\phi\cos.(i-1)\phi}{\Delta^3}-a\int\frac{\partial\phi\cos.(i+1)\phi}{\Delta^3};$$

unde deducitur hoc quasi lemma

$$\int \frac{\partial\phi \cos.(i+1)\phi}{\varDelta^3} = \frac{1+aa}{a}\int \frac{\partial\phi\cos.i\phi}{\varDelta^3} - \int \frac{\partial\phi\cos.(i-1)\phi}{\varDelta^3} - \frac{\pi a^{i-1}\left[i+1-(i-1)aa\right]}{(1-aa)^3}$$

§. 56. Sumamus nunc statim $i=1$, atque istud lemma nobis praebet

$$\int \frac{\partial\phi\cos.2\phi}{\varDelta^3} = \frac{1+aa}{2a}\int\frac{\partial\phi\cos.\phi}{\varDelta^3} - \int\frac{\partial\phi}{\varDelta^3} - \frac{2\pi}{2(1-aa)^3};$$

hic jam bini valores jam inventi substituantur, reperieturque

$$\int \frac{\partial\phi\cos.2\phi}{\varDelta^3} = \frac{1+aa}{a}\cdot\frac{\pi a(3+3aa)}{(1-aa)^5} - \frac{\pi\left(1+4aa+a^4\right)}{(1-aa)^5} - \frac{\pi(1-aa)^2}{(1-aa)^5} = \frac{\pi aa(6)}{(1-aa)^5};$$

sumto $i=2$, erit

$$\int \frac{\partial\phi\cos.3\phi}{\varDelta^3} = \frac{\pi a^3\left(10-5aa+a^4\right)}{(1-aa)^5};$$

sumto $i=3$, nanciscimur

$$\int \frac{\partial\phi\cos.4\phi}{\varDelta^3} = \frac{\pi a^4\left(15-12aa+3a^4\right)}{(1-aa)^5};$$

sumto $i=4$, prodit

$$\int \frac{\partial\phi\cos.5\phi}{\varDelta^3} = \frac{\pi a^5\left(21-21aa+6a^4\right)}{(1-aa)^5};$$

posito $i=5$, erit

$$\int \frac{\partial\phi\cos.6\phi}{\varDelta^3} = \frac{\pi a^6\left(28-32aa+10a^4\right)}{(1-aa)^5};$$

et in genere

$$\int \frac{\partial\phi\cos.i\phi}{\varDelta^3} = \frac{\pi a^i}{(1-aa)^5}\left[\frac{i(i+3)+2}{2} - 2(ii-4)aa + \left[\frac{i(i-3)+2}{2}\right]a^4\right],$$

quae forma facile transformatur in hanc

$$\int \frac{\partial\phi\cos.i\phi}{\varDelta^3} = \frac{\pi a^i}{(1-aa)^5}\left[\frac{(i+1)(i+2)}{2} - 2(i+2)(i-2)aa + \left[\frac{(i-1)(i-2)}{2}\right]a^4\right]$$

§. 57. Hoc modo procedere liceret ad sequentes formulas, in quibus denominator est $\varDelta^4$, $\varDelta^5$, $\varDelta^6$, etc. verum integralium formae ita continuo magis fierent complicatae, ut vix ullus ordo in iis observari posset, quamobrem aliam viam inire conveniet, quo numerum *i* dato assumimus, et continuo a minoribus ad majores numeros n procedemus. Primo igitur sumamus $i=0$, et investigemus valoren integralem formulae $\int \frac{\partial\phi}{\varDelta^{n+1}}$.

Integratio formulae.

$$\int \frac{\partial\phi}{\varDelta^{n+1}} \begin{bmatrix} a & \phi{=}0 \\ ad & \phi{=}180^{\circ} \end{bmatrix}.$$

existente $\varDelta = 1 + aa - 2a\cos.\phi.$

§. 58. Ex praecedentibus colligere licet, quemlibet casum exponentis $n+1$ a duobus praecedentibus pendere, ita ut sit sub terminis integrationis praescriptis

$$\int \frac{\partial\phi}{\varDelta^{n+1}} = \alpha\int \frac{\partial\phi}{\varDelta^{n}} + \beta\int \frac{\partial\phi}{\varDelta^{n-1}};;$$

ubi totum negotium eo redit, ut coefficientes α et β rite determinentur;
hunc in finem statuamus in genere esse

$$\int \frac{\partial\phi}{\varDelta^{n+1}} = \alpha\int \frac{\partial\phi}{\varDelta^{n}} + \beta\int \frac{\partial\phi}{\varDelta^{n-1}} + \gamma\frac{\sin.\phi}{\varDelta^{n}},$$

quippe qui postremus terminus pro utroque integrationis termino evanescit.

§.59. Differentietur nunc ista aequatio, et facta divisione per $\partial\phi$, orietur sequens aequatio

$$\frac{1}{\varDelta^{n+1}} = \frac{\alpha}{\varDelta^{n}} + \frac{\beta}{\varDelta^{n-1}} + \frac{\gamma\cos.\phi(1+aa-2a\cos.\phi)-2\gamma\alpha n\sin^{2}.\phi}{\varDelta^{n+1}},$$

haecque aequatio multiplicata per $\varDelta^{n+1}$ abibit in hanc formam

$$\begin{aligned}1 &= \alpha\left(1+aa-2a\cos.\phi\right)+\beta\left(1+aa\right)^{2}-2\beta a\left(1+aa\right)\\ &+4\beta aa\cos^{2}.\phi+\gamma\cos.\phi\left(1+aa-2a\cos.\phi\right)-2\gamma an\sin^{2}.\phi.\end{aligned}$$

Cum nunc sit

$$2\cos.^{2}\phi = 1+\cos.2\phi \text{ et } 2\sin.^{2}\phi = 1-\cos.2\phi,$$

hac reductione adhibita pervenietur ad sequentem aequationem

$$1=\alpha(1+aa)-2aa\cos.\phi+2\beta aa\cos.2\phi$$
$$+\beta(1+aa)^2-4\beta a(1+aa)\cos.\phi-\gamma a\cos.2\phi$$
$$+2\beta aa+\gamma(1+aa)\cos.\phi+\gamma na\cos.2\phi-\gamma a-\gamma na.$$

§. 60. Ut nunc hanc aequationem resolvamus, necesse est, ut tam termini involventes $\cos.\phi$, quam $\cos.2\phi$, seorsim ad nihilum redigantur; unde ex postremo termino deducimus

$$2\beta aa-\gamma a+\gamma na=0;$$

ideoque

$$\beta=\frac{\gamma(1-n)}{2a}=-\frac{\gamma(n-1)}{2a}$$

qui valor in terminis $\cos.\phi$ affectis substitutus perducit ad hanc aequationem

$$-2\alpha a+2\gamma(n-1)(1+aa)+\gamma(1+aa)=0,$$

unde fit

$$2\alpha a=2\gamma n(1+aa)-\gamma(1+aa);$$

ideoque erit

$$\alpha=\frac{2\gamma n(1+aa)-\gamma(1+aa)}{2a}.$$

Jam hic valores loco α et β inventi substituantur in prima parte, atque deducemur ad hanc aequationem

$$1=\frac{\gamma n(1+aa)^2}{a}-\frac{\gamma(n-1)(1+aa)^2}{2a}-\gamma a(n-1)-\gamma a-\gamma na,\ \text{sive}$$
$$2a=2\gamma n(i+aa)^2-\gamma(n-1)(1+aa)^2-2\gamma aa(n-1)-2\gamma aa-2\gamma naa,$$
$$\text{vel } 2a=\gamma(n+1)(1+aa)^2-4\gamma naa,$$

unde fit

$$\gamma=\frac{2a}{n(1-aa)^2}.$$

§. 61. Invento jam isto valore γ, hinc eliciemus

$$\alpha=\frac{(2n-1)(1+aa)}{n(1-aa)^2}\ \text{et}\ \beta=\frac{-(n-1)}{n(1-aa)^2},$$

hincque per $n(1-aa)^2$ multiplicando, adipiscimur

$$n(1-aa)^2\int\frac{\partial\phi}{\Delta^{n+1}}=(2n-1)(1+aa)\int\frac{\partial\phi}{\Delta^{n}}-(n-1)\int\frac{\partial\phi}{\Delta^{n-1}},$$

cujus beneficio ex cognitis jam duobus casibus assignari poterit casus sequens.

§. 62. Jam ante autem invenimus esse $\int \frac{\partial\phi}{\Delta} = \frac{\pi}{1-aa}$.
Pro sequentibus vero ponamus

$$\int \frac{\partial\phi}{\Delta^2} = \frac{\pi A}{(1-aa)^3}; \int \frac{\partial\phi}{\Delta^3} = \frac{\pi B}{(1-aa)^5}; \int \frac{\partial\phi}{\Delta^4} = \frac{\pi C}{(1-aa)^7};$$
$$\int \frac{\partial\phi}{\Delta^5} = \frac{\pi D}{(1-aa)^9}; \int \frac{\partial\phi}{\Delta^6} = \frac{\pi E}{(1-aa)^{11}}; \text{etc.}$$

Ubi jam ante invenimus $\mathrm{A} = 1 + aa$ et $\mathrm{B} = 1 + 4aa + a^4$;
unde sequentes valores omnes. C, D, E, etc. ope reductionis inventae definiri poterunt.

§. 63. Introducamus ergo istos valores, atque sequentes nanciscemur aequationes

$$\text{I. } \mathrm{A} = 1 + aa,$$
$$\text{II. } 2\mathrm{B} = 3(1+aa)\mathrm{A} - (1-aa)^2,$$
$$\text{III. } 3\mathrm{C} = 5(1+aa)\mathrm{B} - 2(1-aa)^2\,\mathrm{A},$$
$$\text{IV. } 4\mathrm{D} = 7(1+aa)\mathrm{C} - 3(1-aa)^2\,\mathrm{B},$$
$$\text{V. } 5\mathrm{E} = 9(1+aa)\mathrm{D} - 4(1-aa)^2\,\mathrm{C},$$
$$\text{VI. } 6\mathrm{F} = 11(1+aa)\mathrm{E} - 5(1-aa)^2\,\mathrm{D},$$
$$\text{VII. } 7\mathrm{G} = 13(1+aa)\mathrm{F} - 6(1-aa)^2\,\mathrm{E},$$
$$\text{VIII. } 8\mathrm{H} = 15(1+aa)\mathrm{G} - 7(1-aa)^2\,\mathrm{F},$$
etc.

§. 64 Harum aequationum prima statim dat valorem ante
inventum $\mathrm{A} = 1 + aa$; secunda vero praebet

$$2\mathrm{B} = \begin{cases} \;\;3 + 6aa + 3a^4 \\ -1 + 2aa + a^4 \end{cases}$$

unde fit

$$\mathrm{B} = 1 + 4aa + a^4$$

Deinde vero tertia aequatio praebet

$$3\mathrm{C} = \begin{cases} \;\;5 + 25aa + 25a^4 + 5a^6 \\ -1 + \;\;2aa + \;\;2a^4 - 2a^6 \end{cases}$$

unde elicitur

$$C = 1 + 9aa + 9a^4 + a^6.$$

Porro quarta aequatio

$$4D = \begin{cases} 7 + 70aa + 126a^4 + 70a^6 + 7a^8 \\ -3 - 6aa + 18a^4 - 6a^6 - 3a^8 \end{cases}$$

unde colligitur

$$D = 1 + 16aa + 36a^4 + 16a^6 + a^8.$$

Simili modo ex aequatione quinta colligimus

$$5E = \begin{cases} 9 + 153aa + 468a^4 + 468a^6 + 153a^8 + 9a^{10} \\ -4 - 28aa + 32a^4 + 32a^6 - 28a^8 - 4a^{10} \end{cases}$$

unde colligitur

$$E = 1 + 25aa + 100a^4 + 100a^6 + 25a^4 + a^{10}.$$

Evolvamus etiam sextam aequationem quae praebet

$$6F = \begin{cases} 11 + 286aa + 1375a^4 + 2200a^6 + 1375a^8 + 286a^{10} + 11a^{12} \\ -5 - 70aa - 25a^4 + 200a^6 - 25a^8 - 70a^{10} - 5a^{12} \end{cases}$$

hincque concluditur

$$F = 1 + 36aa + 225a^2 + 400a^6 + 225a^8 + 36a^{10} + a^{12}.$$

§. 65. Hic non sine admiratione deprehendimus, omnes coefficientes harum formarum esse numeros quadrates, quorum radices occurrunt in potestatibus respondentibus binomii $1 + aa$, sicque pro littera sequente habebimus

$$G = 1 + 7^2 aa + 21^2 a^4 + 35^2 a^6 + 35^2 a^3 + 21^2 a^{10} + 7^2 a^{12} + a^{14},$$

quae littera respondet formulae integrali $\int \frac{\partial \phi}{\Delta^{7+1}}$, ita ut hic sit $n = 7$. Quodsi ergo formae generalis $\int \frac{\partial \phi}{\Delta^{n+1}}$ integrale statuamus $= \frac{\pi V}{(1-aa)^{in+1}}$, erit valor litterae

$$V = 1 + \left(\tfrac{n}{1}\right)^2 aa + \left(\tfrac{n}{2}\right)^2 a^4 + \left(\tfrac{n}{3}\right)^2 a^6 + \left(\tfrac{n}{4}\right)^2 a^8 + \left(\tfrac{n}{5}\right)^2 a^{10} + \text{etc.}$$

adhibitis scilicet characteribus quibus coefficientes potestatum binomii designare consuevimus, dum scilicet est

$$\left(\frac{n}{1}\right)=n;\ \left(\frac{n}{2}\right)=\frac{n}{1}\cdot\frac{n-1}{2};\ \left(\frac{n}{3}\right)=\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\ \text{etc.}$$

§. 66. Haec quidem conclusio tantum per inductionem quasi conjectura est deducta; vix enim quisquam reperietur, cui ista conjectura suspecta videatur, quamquam rigorosa demonstratione nondum sit corroborata; casu enim fortuito neutiquam evenire certe potest, ut omnes istos coefficientes prodierint numeri quadrati, atque adeo ipsorum cocfficientium qui in evolutione potestatis $(1+aa)^n$ occurrunt, interim tamen deinceps vidi pro hac veritate solidam demonstrationem adornari posse.

§. 67. Hac igitur lege stabilita, valores litterarum A, B, C, D etc., quas in expressiones integralium induximus, sequenti modo se habebunt

$$\mathrm{A}=1^2+1^2aa,$$
$$\mathrm{B}=1^2+2^2aa+1^2a^4,$$
$$\mathrm{C}=1^2+3^2aa+3^2a^4+1^2a^6,$$
$$\mathrm{D}=1^2+4^2aa+6^2a^4+4^2a^6+1^2a^8,$$
$$\mathrm{E}=1^2+5^2aa+10^2a^4+10^2a^6+5^2a^8+1^2a^{10},$$
$$\mathrm{F}=1^2+6^2aa+15^2a^4+20^2a^6+15^2a^8+6^2a^{10}+1^2a^{12},$$
$$\mathrm{G}=1^2+7^2aa+21^2a^4+35^2a^6+35^2a^8+21^2a^{10}+7^2a^{12}+1^2a^{14},$$
etc. etc.

Integratio formulae generalis

$$\int\frac{\partial\phi\cos.i\phi}{\varDelta^{n+1}}\begin{bmatrix}a & \phi=0\\ ad & \phi=180^0\end{bmatrix}$$

existente

$$\varDelta=1+aa-2a\cos.\phi.$$

§. 68. Haec formula generalis perinde tractari potest ac praecedens, dum valor integralis cujusque casus etiam a duobus casibus praecedentibus pendet, ita ut ponere queamus

$$\int\frac{\partial\phi\cos.i\phi}{\varDelta^{n+1}}=\alpha\int\frac{\partial\phi\cos.i\phi}{\varDelta^{n}}+\beta\int\frac{\partial\phi\cos.i\phi}{\varDelta^{n-1}},$$

quatenus scilicet integralia ad binos terminos integrationis stabilitos referuntur; quia autem necesse est, ut aequationem generalem ob ista conditione liberam constituamus, aliquot membra adjungi oportet, quae pro utroque termino evanescant, neque enim hic sufficit, ut ante unicum terminum adjunxisse, verum ad eo ternos hujusmodi terminos adjungi debebunt, cujus ratio mox ex ipso calculo elucebit;

hanc ob rem constituamus sequentem aequationem

$$\int\frac{\partial\phi\cos.i\phi}{\Delta^{n+1}}=\alpha\int\frac{\partial\phi\cos.i\phi}{\Delta^{n}}+\beta\int\frac{\partial\phi\cos.i\phi}{\Delta^{n-1}},$$
$$+\gamma\frac{\sin.i\phi}{\Delta^{n}}+\delta\frac{\sin.(i-1)\phi}{\Delta^{n}}+\varepsilon\frac{\sin.(i+1)\phi}{\Delta^{n}},$$

quae postrema membra, quoniam *i* est numerus integer, pro utroque termino integrationis evanescunt.

§. 69. Differentietur igitur nunc ista aequatio, ac posito brevitatis gratia $1+aa=b$, ut sit $\Delta=b-2a\cos.\phi,$ negligantur denominatores ; qui erunt Δ^{n+1} una cum elemento $\partial\phi.$ Primo notetur esse

$$\Delta\cos.i\phi=b\cos.i\phi-a\cos.(i-1)\phi-a\cos.(i+1)\phi,$$

tum vero ob

$$\Delta^2=bb-4ab\cos.\phi+4aa\cos^2.\phi=2aa+bb$$
$$-4ab\cos.\phi+2aa\cos.2\phi,\text{ erit}$$
$$\Delta^2\cos.\phi=(bb+2aa)\cos.i\phi-2ab\cos.(i-1)\phi-2ab\cos.(i+1)\phi$$
$$+aa\cos.(i-2)\phi+aa\cos.(i+2)\phi.$$

Deinde vero habebitur

$$\partial.\frac{\sin.i\phi}{\Delta^{n}}=i\Delta\cos.i\phi-2na\sin.i\phi\sin.\phi=ia\cos.i\phi$$
$$+ia\cos.(i-1)\phi-ia\cos.(i+1)\phi-na\cos.(i-1)\phi+na\cos.(i+1)\phi.$$

Simili modo erit

$$\partial.\frac{\sin.(i-1)\phi}{\Delta^{n}}=(i-1)b\cos.(i-1)\phi-(i-1)a\cos.(i-2)\phi$$
$$-(i-1)a\cos.i\phi-na\cos.(i-2)\phi+na\cos.i\phi,$$

ac denique

$$\partial.\frac{\sin.(i+1)\phi}{\Delta^{n}}=(i+1)b\cos.(i+1)\phi-(i+1)a\cos.i\phi$$
$$-(i+1)a\cos.(i+2)\phi-na\cos.i\phi+na\cos.(i+2)\phi.$$

§. 70. Hic igitur occurrunt quinque anguli scilicet

$$\cos.i\phi,\ \cos.(i-1)\phi,\ (i+1)\phi,\ (i-2)\phi\text{ et }(i+2)\phi,$$

unde patet ratio, cur terni termini absoluti sint supra adjuncti; differentiale ergo facta evolutione singulorum terminorum, per quinque columnas sequenti modo repraesentetur, ita ut membrum sinistrum, quod est $\cos.i\phi$, aequetur sequenti expressioni

$\cos.i\phi$	$\cos.(i-1)\phi$	$\cos.(i+1)\phi$	$\cos.(i-2)\phi$	$\cos.(i+2)\phi$
$+\alpha b$	$-\alpha a$	$-\alpha a$		
$+\beta(bb+2aa)$	$-2\beta ab$	$-2\beta ab$	$+\beta aa$	$+\beta aa$
$+\gamma ib$	$-\gamma ia$	$-\gamma ia$		
	$-\gamma na$	$+\gamma na$		
$-\delta(i-1)a$	$+\delta(i-1)b$	$+\varepsilon(i+1)b$	$-\delta(i-1)a$	$-\varepsilon(i+1)a$
$+\delta na$			$-\delta na$	$+\varepsilon na$
$-\varepsilon(i+1)a$				
$-\varepsilon na$				

§. 71 . Hic igitur omnes quatuor posteriores columnae ad nihilum redigi debent, propterea quod sola prima columna membro sinistro aequari potest; incipiamus igitur a binis columnis ultimis, unde deducimus

$$\delta=\frac{\beta a}{i+n-1} \text{ et } \varepsilon=\frac{\beta a}{i-n+1}.$$

His valoribus introductis, pro secunda columna erit

$$-2\beta ab+\varepsilon(i-1)b=\frac{\beta ab(1-i-2n)}{i+n-1}=-\frac{\beta ab(i+2n-1)}{i+n-1};$$

Pro tertia vero columna erit

$$-2\beta ab+\varepsilon(i+1)b=-\frac{\beta ab(i-2n+1)}{i-n+1}$$

unde haec binae columnae nobis praebent has duas aequationes

$$-\alpha a-\gamma(i+1)a-\frac{\beta ab(i+2n-1)}{i+n-1}=0,$$
$$-\alpha a-\gamma(i-1)a-\frac{\beta ab(i-2n+1)}{i-n+1}=0,$$

§. 72. Harum duarum aequationum subtrahatur posterior a priore, ac prodibit

$$-2\gamma na-\frac{2\beta inab}{ii-(n-1)^2}=0,$$

unde colligimus

$$\gamma=-\frac{\beta ib}{ii-(n-1)^2}.$$

Atque hinc porro ex secunda deduci potest valor ipsius α, cum sit

$$\alpha a = -\gamma(i+n)a = \frac{\beta ab(i+2n-1)}{i+n-1},$$

erit

$$\alpha = \frac{\beta i(i+n)b}{ii-(n-1)^2} - \frac{\beta(i+2n-1)b}{i+n-1} = \frac{\beta(2nn-3n+1)b}{ii-(n-1)^2}$$
$$= \frac{\beta(n-1)(2n-1)b}{ii-(n-1)^2}.$$

§.75. Hi jam valores substituantur in prima colunma, atque orietur sequens aequatio

$$\left.\begin{array}{r} \frac{\beta(n-1)(2n-1)bb}{ii-(n-1)^2} + 2\beta aa \\ +\beta bb - \frac{\beta(i-n-1)aa}{i+n-1} \\ -\frac{\beta iibb}{ii-(n-1)^2} - \frac{\beta(i+n+1)aa}{i-n+1} \end{array}\right\} = 1.$$

Multiplicando igitur per $ii-(n-1)^2$, prodibit haec aequatio

$$ii-(n-1)^2 = 2\beta aa\left[ii-(n-1)^2\right] + \beta bb(n-1)(2n-1)$$
$$-\beta aa(i-n-1)(i-n+1) + \beta bb\left[ii-(n-1)^2\right]$$
$$-\beta aa(i+n+1)(i+n-1) - \beta iibb.$$

Facta autem reductione, terminus βaa multiplicabitur per

$$2\left[ii-(n-1)^2\right] - (i-n)^2 + 1 - (i+n)^2 + 1,$$

sive per $-4n(n-1)$; at vero βbb multiplicabitur per

$$(n-1)(2n-1) + ii - (n-1)^2 - ii,$$

sive per $n(n-1)$, sicque erit

$$ii(n-1)^2 = -4\beta n(n-1)aa + \beta n(n-1)bb$$
$$= \beta n(n-1)(bb-4aa).$$

Cum igitur posuerimus $b = 1 + aa$, erit

$$bb - 4aa = (1-aa)^2,$$

consequenter hinc elicimus

$$\beta = \frac{ii-(n-1)^2}{n(n-1)(1-aa)^2}.$$

§. 74. Invento jam valore litterae β, ex eo deducimus valorem $\alpha = \frac{(2n-1)b}{n(1-aa)^2}$: valores autem letterarum γ, δ, et ε non amplius in censum veniunt, et reductio quam quaerimus erit

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^{n+1}} = \alpha \int \frac{\partial\phi \cos.i\phi}{\Delta^{n}} + \beta \int \frac{\partial\phi \cos.i\phi}{\Delta^{n-1}}$$

sive sublatis fractionibus habebitur ista aequatio

$$n(n-1)(1-aa)^2 \int \frac{\partial\phi \cos.i\phi}{\Delta^{n+1}} = (n-1)(2n-1)(1+aa) \int \frac{\partial\phi \cos.i\phi}{\Delta^{n}} + \left[ii-(n-1)^2\right] \int \frac{\partial\phi \cos.i\phi}{\Delta^{n-1}},$$

quae aequatio casu $i = 0$ redit ad reductionem praecedentis sectionis.

§. 75. Inventa hac reductione generali, pro ejus applicatione cum sit

$$\int \frac{\partial\phi \cos.i\phi}{\Delta} = \frac{\pi a^i}{1-aa}, \text{ ubi } n = 0,$$

ponamus pro sequentibus

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^2} = \frac{\pi a^i}{(1-aa)^3} \mathrm{A}, \text{ ubi } n = 1,$$

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^3} = \frac{\pi a^i}{(1-aa)^5} \mathrm{B}, \text{ ubi } n = 2,$$

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^4} = \frac{\pi a^i}{(1-aa)^7} \mathrm{C}, \text{ ubi } n = 3,$$

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^5} = \frac{\pi a^i}{(1-aa)^9} \mathrm{D}, \text{ ubi } n = 4,$$

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^6} = \frac{\pi a^i}{(1-aa)^{11}} \mathrm{E}, \text{ ubi } n = 5,$$

atque adeo in genere sit

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^{n+1}} = \frac{\pi a^i}{(1-aa)^{2n+1}} \mathrm{V}:$$

supra autem jam *i* invenimus esse

$$\mathrm{A} = i + 1 - (i+1)aa,$$

sive terminos positive repraesentando

$$\mathrm{A} = i + 1 + (i-1)aa.$$

§. 76. Quodsi in reductione nostra inventa poneremus $n=1$, ea daret $ii\int\partial\phi\cos.i\phi = 0$, quod primo verum est casu $i=0$, tum vero ob $\int\partial\phi\cos.i\phi = \frac{1}{2}\sin.i\phi = 0$, quod quidem per se patet. Incipiamus igitur a casu $n=2$, et procedendo per sequentes valores $n=3,\ n=4,\ n=5$, etc. nanciscemur sequentes aequationes

I. Si $n=2$, erit

$$2.1\,\mathrm{B} = 1.3(1+aa)\mathrm{A} + (ii-1)(1-aa)^2.$$

II. Si $n=3$, erit

$$3.2\mathrm{C} = 2.5(1+aa)\mathrm{B} + (ii-4)(1-aa)^2\,\mathrm{A}.$$

III. Si $n=4$, erit

$$4.3\mathrm{D} = 3.7(1+aa)\mathrm{C} + (ii-9)(1-aa)^2\,\mathrm{B}.$$

IV. Si $n=5$, erit

$$4\mathrm{E} = 4.9(1+aa)\mathrm{D} + (ii-16)(1-aa)^2\,\mathrm{C}.$$

V. Si $n=6$, erit

$$6.5\mathrm{F} = 5.11(1+aa)\mathrm{E} + (ii-25)(1-aa)^2\,\mathrm{D}.$$

etc. etc.

§. 77. Cum igitur sit

$$\mathrm{A} = 1 + i + (1-i)aa,$$

pro prima aequatione erit

$$(1+aa)\mathrm{A} = 1 + i + 2aa + (i-i)a^4,$$

hujus triplo addi oportet

$$(ii-1)(1-aa)^2 = ii - 1 - 2(ii-1)aa + (ii-1)a^4,$$

unde oritur primo terminus absolutus $=(2+i)(1+i)$, deinde coefficiens ipsius aa erit $8-2ii$, coefficiens vero ipsius a^4 erit $(2-i)(1-i)$, unde concludimus litteram

$$\mathrm{B} = \frac{(2+i)(1+i)}{1.\quad 2.} + (2+i)(2-i)aa + \frac{(2-i)(1-i)}{1.\quad 2.}a^4.$$

§. 78. Ista forma nos manuducit ad coefficientes potestatum binomii, quos ut jam moninus per characteres peculiares repraesentamus, sicque per tales characteres erit

$$\mathrm{A}=\left(\frac{1+i}{1}\right)+\left(\frac{1-i}{1}\right)aa,\ \text{tum vero}$$
$$\mathrm{B}=\left(\frac{2+i}{2}\right)+\left(\frac{2+i}{1}\right)\left(\frac{2-i}{1}\right)aa+\left(\frac{2-i}{2}\right)a^4.$$

Videamus autem, quomodo haec lex in sequentibus valoribus se sit habitura.

§. 79. Evolvamus igitur aequationem secundam, pro qua sequentes duas multiplicationes institui oportet

$$10\left[\frac{2+3i+ii}{2}+(4-ii)aa+\frac{2-3i+ii}{2}a^4\right]\ \text{per}\ 1+aa\,,$$

ultimum autem membrum postulat hanc multiplicationem

$$(ii-4)\left(1-2aa+a^4\right)\ \text{per}\ 1+i+(1-i)aa\,;$$

unde primo oritur iste terminus absolutus

$$10+15i+5ii+(ii-4)(1+i)\,,$$

quae reducitur ad hanc formam $(2+i)(1+i)(3+i)$. Pro termino autem aa erit

$$40-10ii+5(2+i)(1+i)+(ii-4)\left[-2\ (1+i)+1-i\right]$$
$$=(4-ii)(11+3i)+5(2+i)(1+i),$$

quae expressio reducitur ad

$$(2+i)(27-3ii)=3(2+i)(3+i)(3-i)$$

Porro coefficiens ipsius a^4 erit

$$(2-i)(27-3ii)=3(2-i)(3+i)(3-i).$$

Denique coefficiens ipsius a^6 erit $(2-i)(1-i)(3-i)$.

§. 80. Calculo ergo hoc peracto habebimus

$$3.2\mathrm{C}=(3+i)(2+i)(1+i)+3(3+i)(2+i)(3-i)aa$$
$$+3(3+i)(2-i)(3-i)a^4+(3-i)(2-i)(1-\ i)\ a^6,$$

quae forma commode redigitur ad istam per characteres coefficientium binomii

$$C=\left(\frac{3+i}{3}\right)+\left(\frac{3+i}{2}\right)\left(\frac{3-i}{1}\right)aa+\left(\frac{3+i}{1}\right)\left(\frac{3-i}{2}\right)a^4+\left(\frac{3-i}{3}\right)a^6.$$

Hic ordo maxime confirmat conjecturam ex casibus praecedentibus deductam, neque dubium ullum esse potest, quin sequentes litterae istos sortiantur valores

$$\mathrm{D} = \left(\tfrac{4+i}{4}\right) + \left(\tfrac{4+i}{3}\right)\left(\tfrac{4-i}{1}\right)aa + \left(\tfrac{4+i}{2}\right)\left(\tfrac{4-i}{2}\right)a^4$$
$$+\left(\tfrac{4+i}{1}\right)\left(\tfrac{4-i}{3}\right)a^6 + \left(\tfrac{4-i}{4}\right)a^8.$$
$$\mathrm{E} = \left(\tfrac{5+i}{5}\right) + \left(\tfrac{5+i}{4}\right)\left(\tfrac{5-i}{1}\right)aa + \left(\tfrac{5+i}{3}\right)\left(\tfrac{5-i}{2}\right)a^4$$
$$+\left(\tfrac{5+i}{2}\right)\left(\tfrac{5-i}{3}\right)a^6 + \left(\tfrac{5+i}{1}\right)\left(\tfrac{5-i}{4}\right)a^8 + \left(\tfrac{5-i}{5}\right)a^{10}.$$
etc. etc.

Interim tamen fatendum est, hunc ordinem egregium tantum per conjecturam se nobis obtulisse; cujus ergo demonstratio rigorosa adhuc desideratur.

§. 81. Cum igitur supra ingenere posuerimus

$$\int \frac{\partial\phi \cos.i\phi}{\Delta^{n+1}} \begin{bmatrix} a & \phi = 0 \\ ad & \phi = 180^0 \end{bmatrix} = \frac{\pi a^i}{(1-aa)^{2n+1}}\mathrm{V},$$

erit nunc

$$\mathrm{V} = \left(\tfrac{n+i}{i}\right) + \left(\tfrac{n+i}{i+1}\right)\left(\tfrac{n-i}{1}\right)aa + \left(\tfrac{n+i}{i+2}\right)\left(\tfrac{n-i}{2}\right)a^4$$
$$+\left(\tfrac{n+i}{i+3}\right)\left(\tfrac{n-i}{3}\right)a^6 + \left(\tfrac{n+i}{i+4}\right)\left(\tfrac{n-i}{4}\right)a^8 + \text{etc.}$$

unde sponte deducitur forma in articulo praecedenti conclusa, ubi erat $i = 0$. Pro hoc enim casu erit

$$\mathrm{V} = \left(\tfrac{n}{n}\right) + \left(\tfrac{n}{n-1}\right)\left(\tfrac{n}{1}\right)aa + \left(\tfrac{n}{n-2}\right)\left(\tfrac{n}{2}\right)a^4 + \left(\tfrac{n}{n-3}\right)\left(\tfrac{n}{3}\right)a^6 + \text{etc.}$$

Cum autem in hujusmodi characteribus perpetuo sit $\left(\frac{n}{p}\right) = \left(\frac{n}{n-p}\right)$, erit prorsus uti supra conjectavimus

$$\mathrm{V} = \left(\tfrac{n}{0}\right) + \left(\tfrac{n}{1}\right)^2 aa + \left(\tfrac{n}{2}\right)^2 a^4 + \left(\tfrac{n}{3}\right)^2 a^6 + \left(\tfrac{n}{4}\right)^2 a^8 + \text{etc.}$$

Hinc igitur operae pretium erit sequens theorema constituere.

Theorema generale.

[*Opera Omnia*: Series 1, Volume 19, pp. 197 – 216:
Commentatio 674 indicis ENESTROEMIANI.]

§. 82. Si formula integralis

$$\int \frac{\partial\phi \cos.i\phi}{(1+aa-2a\cos.\phi)^{n+1}}.$$

a termino $\phi=0$ usque ad terminum $\phi=180°$ extendatur, valor integralis semper habebit talem formam

$$\frac{\pi a^i}{(1-aa)^{2n+1}}\mathrm{V}, \text{ existente}$$

$$\mathrm{V}=\left(\frac{n+i}{n}\right)+\left(\frac{n+i}{n-1}\right)\left(\frac{n-i}{1}\right)aa+\left(\frac{n+i}{n-2}\right)\left(\frac{n-i}{2}\right)a^4$$
$$+\left(\frac{n+i}{n-3}\right)\left(\frac{n-i}{3}\right)a^6+\left(\frac{n+i}{n-4}\right)\left(\frac{n-i}{4}\right)a^8+\text{etc.}$$

dummodo fuerit *i* numerus integer, atque adeo tam positivus quam negativus; quandoquidem etiam posteriori casu ista forma veritati consentanea deprehenditur, ita ut ista expressio latius pateat, quam omnes casus speciales junctim sumti, unde eam per conjecturam, conclusimus; namque in omnibus casibus specialibus littera *i* necessario denotabat numeros integros tantum positivos.

4) Demonstratio Theorematis insignis per conjecturam eruti, circa integrationem formulae

$$\int\frac{\partial\phi\cos.i\phi}{(1+aa-2a\cos.\phi)^{n+1}}.$$

M. S. Academiae exhib. die 10 *Septembris* 1778.

§. 83. Cum nuper hanc formulam integralem tractassem, ac potissimum in ejus valorem inquisivissem, quem accipit, si integrale a termino $\phi=0$ ad terminum $\phi=180°$ usque extendatur ; ex pluribus casibus, quos evolvere licuit, conclusi ejus integrale in genere ita expressum iri

$$\frac{\pi a^i}{(1-aa)^{2n+1}}\mathrm{V},$$

ubi V denotat summam hujus seriei

$$\mathrm{V}=\left(\frac{n-i}{0}\right)\left(\frac{n+i}{i}\right)+\left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right)a^2+\left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right)a^4+\text{etc.}$$

Hic scilicet isti characteres clausulis inclusi designant coefficientes potestatis binomialis, dum statuimus

$$(1+x)^m=1+\left(\frac{m}{1}\right)x+\left(\frac{m}{2}\right)x^2+\left(\frac{m}{3}\right)x^3+\left(\frac{m}{4}\right)x^4+\text{etc.}$$

§. 84. Circa hanc autem formulam integralem ante omnia tenendum est, litteram *i* perpetuo significare numeros integros, quandoquidem in analysi constanter assumitur, casu $\phi=180°$ semper esse $\sin.i\phi=0$; tum vero etiam ejus valores perpetuo ut positivi spectari possunt, propterea quod $\cos.(-i\phi)=\cos.(+i\phi)$. Interim tamen mox ostendemus nostram formam integralem etiam veritati esse consentaneam, quamvis litterae *i* valores negativi tribuantur. Ad hoc ostendendum circa characteres in subsidium vocatos sequentia sunt observanda.

1°). Si *p* et *q* designent numeros integros, ac primo quidem positives, quoniam in evolutione potestatis binomialis omnes termini primum antecedentes sunt nulli, quoties fuerit *q* numerus negativus, semper erit $\left(\frac{p}{q}\right)=0.$

2°). Quia coefficient tam primi termini quam ultimi semper est unitas, erit tam $\left(\frac{p}{0}\right)=1$ quam $\left(\frac{p}{p}\right)=1.$

3°). Quia termini ultimum sequentes pariter sunt nulli, quoties fuerit $q<p$, valor characteris $\left(\frac{p}{q}\right)$ semper pro nihilo haberi poterit.

4°). Quia in evolutione potestatis binomialis coefficientes ordinem tenent retrogradum, hinc sequitur semper fore $\left(\frac{p}{q}\right)=\left(\frac{p}{p-q}\right).$

Sin autem superior numerus *p* fuerit negativus, ob rationem praecedentem semper etiam erit $\left(\frac{-p}{-q}\right)=0.$

5°). At si *q* denotet numeres positives, character $\left(\frac{-p}{q}\right)$, perpetuo dabit valores alternatim positives et negatives; cum sit

$$\left(\frac{-p}{0}\right)=1;\ \left(\frac{-p}{1}\right)=-p;\ \left(\frac{-p}{2}\right)=+\frac{p(p+1)}{1.2};\ \left(\frac{-p}{3}\right)=-\frac{p(p+1)(p+2)}{1.2.3}\ \text{etc.}$$

Atque hinc

6°). In genere tales characteres, ubi superior numerus est negativus, ad positives reduci poterunt, cum sit $\left(\frac{-p}{q}\right)=\pm\left(\frac{p+q-1}{q}\right)$, ubi signum + valet si *q* fuerit numerus par, inferius $-$vero , si impar.

§. 85. His proprietatibus circa characteres hic adhibitos notatis, in forma nostra integrali loco *i* scribamus $-i$, eritque

$$\int\frac{\partial\phi\cos.-i\phi}{(1+aa-2a\cos.\phi)^{n+1}}=\frac{\pi a^{-i}}{(1-aa)^{2n+1}}\mathrm{V}$$

existente

$$\mathrm{V}=\left(\frac{n+i}{0}\right)\left(\frac{n-i}{-i}\right)+\left(\frac{n+i}{1}\right)\left(\frac{n-i}{-i+1}\right)a^2+\left(\frac{n+i}{2}\right)\left(\frac{n-i}{-i+2}\right)a^4$$
$$+\left(\frac{n+i}{3}\right)\left(\frac{n-i}{-i+3}\right)a^6+\text{etc.}$$

ubi posteriores factores evanescunt, quamdiu denominatores sunt negativi : primum igitur membrum significatum habens. erit $\left(\frac{n+i}{i}\right)\left(\frac{n-i}{-i+i}\right)a^{2i}$, cujus valor erit $\left(\frac{n+i}{i}\right)a^{2i}$; sequentia autem membra erunt

$$\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{-i+i+1}\right)a^{2i+2}=\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^{2i+2}, \text{tum vero } \left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^{2i+4}, \text{etc.}$$

Hoc igitur modo erit

$$\mathrm{V}=a^{2i}\left[\left(\frac{n+i}{i}\right)\left(\frac{n-i}{0}\right)+\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^{2}+\left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^{4}+\text{etc.}\right]$$

qui valor ductus in $\frac{\pi a^{-i}}{(1-aa)^{2n+1}}$ praebet hanc formam

$$\frac{\pi a^{i}}{(1-aa)^{2n+1}}\left[\left(\frac{n+i}{i}\right)\left(\frac{n-i}{0}\right)+\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^{2}+\left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^{4}+\text{etc.}\right]$$

quae prorsus congruit cum nostra formula valori positivo ipsius *i* respondente, qui egregius consensus haud contemnendum firmamentum pro veritate nostrae formae integralis continet.

§. 86. Praeterea vero circa formam nostram integralem imprimi notari debet, seriem pro V supra datam semper alicubi abrumpi quoties *n* fuerit numerus integer positivus, quippe quod eveniet, quando vel in priore factore, cujus forma est $\left(\frac{n-i}{\lambda}\right)$, pervenitur ad terminum quo $\lambda>n-i$, vel in posteriore factore, cujus forma est $\left(\frac{n+i}{i+\lambda}\right)$, evadet $\lambda>n$; quae proprietas eo magis est observanda, quod, si series V in infinitum porrigeretur, parum lucrati essemus censendi, id quod praecipue de iis casibus est notandum, quibus *n* foret numerus fractus, quos ergo casus penitus ab instituto nostro removemus, ita ut pro *n* tantum numeros integros simus assumturi.

§. 87. Consideremus ergo etiam casus, quibus *n* est numerus negativus, ac prima quidem jam per se clarum est, quamdiu is minor fuerit quam *i*, ideoque $n+i$ etiamnum numerus positivus, tum seriem pro V datam adeo citius abruptum iri; tum igitur demum in infinitum excurret, quando etiam $n+i$ fuerit numerus positivus. His autem casibus forma int.egralis supra data ita transformari potest, ut abruptio pariter locum inveniat.

§. 88. Ad hoc ostendendum statuamus $n=-m-1$, ut formula nostra integralis evadat

$$\int\partial\phi\cos.i\phi\left(1+aa-2a\cos.\phi\right)^{m},$$

ejusque igitur valor $=\pi a^{i}\left(1-aa\right)^{2m+1}\mathrm{V}$, existente jam

$$\mathrm{V}=\left(\frac{-m-1-i}{0}\right)\left(\frac{-m-1+i}{i}\right)+\left(\frac{-m-1-i}{1}\right)\left(\frac{-m-1+i}{i+1}\right)a^{2}$$
$$+\left(\frac{-m-1-i}{2}\right)\left(\frac{-m-1+i}{i+2}\right)a^{4}+\left(\frac{-m-1-i}{3}\right)\left(\frac{-m-1+i}{i+3}\right)a^{6}+\text{etc}$$

quae series manifesto in infinitum excurrit, quam autem ope sequentes lemmatis transformare poterimus,

Lemma.

§. 89. Ista series per characteres hic introductos procedens

$$♄ = \left(\frac{f}{0}\right)\left(\frac{h}{e}\right) + \left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right)x + \left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right)x^2 + \left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right)x^3 + \text{etc.}$$

in hanc sui similem transmutari potest

$$♂ = \left(\frac{-h-i}{0}\right)\left(\frac{-f-i}{e}\right) + \left(\frac{-h-i}{1}\right)\left(\frac{-f-i}{e+1}\right)x + \left(\frac{-h-i}{2}\right)\left(\frac{-f-i}{e+2}\right)x^2 + \text{etc.}$$

quandoquidem inter earum valores ♄ et ♂ ista relatio semper locum habere, non ita pridem a me est demonstrata

$$\left(\frac{e+f}{1}\right)♄ = \left(\frac{e-h-1}{e}\right)(1-x)^{f+h+1}♂,$$

cujus demonstratio profundissimae est indaginis, dum ideo per aequationes differentiales secundi gradus procedit.

§. 90. Applicemus jam istud lemma ad casum nostrum propositum, atque ut series ♄ cum nostro V consentiens reddatur,
ut fiat ♄ $=\text{V}$, sumi debet $f=-m-1-i,\ h=-m-1+i, e=i$ et $x=aa$, unde altera series ♂ hanc accipiet formam

$$♂ = \left(\frac{m-i}{0}\right)\left(\frac{m+i}{1}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)aa + \left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)a^4 + \text{etc.}$$

quae series jam certe abrumpitur alicubi, propterea quod hic m denotat numerum integrum positivum: at vero relatio inter superiorem $\text{V} =$ ♄ ; et novam hanc seriem ♂ ita se habebit

$$\left(\frac{m-1}{i}\right)\text{V} = \left(\frac{m}{i}\right)(1-aa)^{-2m-1}♂ .$$

§. 91. Hinc igitur formulae nostrae integralis hujus

$$\int \partial\phi \cos .i\phi\left(1+aa-2a\cos .\phi\right)^m = \frac{\left(\frac{m}{i}\right)\pi a^i ♂}{\left(\frac{-m-1}{i}\right)},$$

ubi ♂ denotat seriem modo ante §.89. expositam, qui valor cum factorem habeat $\left(\frac{m}{i}\right)$ semper evanescet, quamdiu fuerit $i > m$, ita ut his casibus valor integralis semper nihilo sit aequalis. Ceterum hic notasse juvabit, facta evolutione esse

$$\left(\tfrac{m}{i}\right):\left(\tfrac{-m-1}{i}\right)=\pm\frac{m(m-1)\ .\ .\ .\ .\ (m-i+1)}{(m+1)(m+2)\ .\ .\ .\ .\ (m+i)},$$

ubi signum superius + valet si i fuerit numerus par, inferius $-$ vero si impar. His circa indolem nostri theorematis notatis, ipsam ejus demonstrationem aggrediamur, quam quo clarior evadat in varias partes distribuamus.

Demonstrationis pars prima.

§. 92. Quoniam valorem nostrum integralem ad duas formulas accommodavimus, eas distinctionis gratia signis ☉ et ☽ designemus, sitque

$$\square=\int\frac{\partial\phi\cos.i\phi}{(1+aa-2a\cos.\varphi)^{n+1}}\begin{bmatrix}a & \phi=0\\ ad & \phi=180^0\end{bmatrix},$$

$$☽=\int\partial\phi\cos.i\phi(1+aa-2a\cos.\varphi)^{m}\begin{bmatrix}a & \phi=0\\ ad & \phi=180^0\end{bmatrix},$$

quarum posterior ☽ in priorem ☉ convertitur si loco m scribamus $-n-1$; modo autem vidimus, has duas formulas a se invicem pendere, unde a posteriori tanquam simpliciori, siquidem denominatore $(1-aa)^{2n+1}$ caret, incipiamus, quam quo simpliciorem reddamus statuamus $\frac{a}{1+aa}=b$; sic enim habebimus

$$☽=(1+aa)^m\int\partial\phi\cos.i\phi(1-2b\cos.\varphi)^m;$$

cujus ergo integrale nobis erit investigandum.

§. 93; Ante omnia igitur conveniet potestatem $(1-2b\cos.\varphi)^m$ evolvi, unde fiet

$$(1-2b\cos.\phi)^m=1-\left(\tfrac{m}{1}\right)2b\cos.\phi+\left(\tfrac{m}{2}\right)4b^2\cos.^2\phi-\left(\tfrac{m}{3}\right)8b^3\cos.^3\phi+\text{etc.}$$

cujus ergo terminus quicunque erit $\pm\left(\frac{m}{\lambda}\right)2^\lambda b^\lambda\cos.^\lambda\phi$; ubi signum + valet si λ fuerit numerus par, alterum vero $-$ si impar. Jam quia hic potestates ipsius $\cos.\phi$ occurrunt, eas per praecepta satis cognita in cosinus simplices converti oportet, quibus fit

$$2^2\cos.^2\phi=2\cos.2\phi+1\left(\tfrac{2}{1}\right),$$
$$2^3\cos.^3\phi=2\cos.3\phi+2\left(\tfrac{3}{1}\right)\cos.\phi,$$
$$2^4\cos.^4\phi=2\cos.4\phi+2\left(\tfrac{4}{1}\right)\cos.2\phi+1\left(\tfrac{4}{2}\right),$$
$$2^5\cos.^5\phi=2\cos.5\phi+2\left(\tfrac{5}{1}\right)\cos.3\phi+2\left(\tfrac{5}{2}\right)\cos.\phi,$$
$$2^6\cos.^6\phi=2\cos.6\phi+2\left(\tfrac{6}{1}\right)\cos.4\phi+2\left(\tfrac{6}{2}\right)\cos.2\phi+1\left(\tfrac{6}{3}\right),$$

etc. etc. etc.

Ubi notandum, in potestatibus paribus postremum membrum $\cos.0\phi=1$ dimidio tantum coefficiente esse affectum. Hinc igitur in genere erit

$$2^{\lambda}\cos.^{\lambda}\phi=2\cos.\lambda\phi+2\left(\tfrac{\lambda}{1}\right)\cos.(\lambda+2)\phi+2\left(\tfrac{\lambda}{2}\right)\cos.(\lambda-4)\phi$$
$$+2\left(\tfrac{\lambda}{3}\right)\cos.(\lambda-6)\phi+\text{etc.}$$

ubi notetur, quoties fuerit λ numerus par, puta $\lambda=2i$, ultimum membrum fore tantum

$$1.\left(\tfrac{2i}{i}\right)\cos.0\phi.$$

§. 94 Postquam igitur omnes cosinuum potestates ad cosinus simplices fuerint reductae, integrationes nostrae semper ad talem formam redigentur $\int\partial\phi\cos.i\phi\cos.\lambda\phi$, de qua forma hic imprimis est notandum, ejus integrale a $\phi=0$ ad $\phi=180°$ extensum semper esse nullum, solo casu $\lambda=i$ excepto. Cum enim sit

$$\cos.i\phi\ \cos.\lambda\phi=\tfrac{1}{2}\cos.(i+\lambda)\phi+\tfrac{1}{2}\cos.(i-\lambda)\phi,$$

erit illud integrale indefinitum

$$=\frac{\sin.(i+\lambda)\phi}{2(i+\lambda)}+\frac{\sin.(i-\lambda)\phi}{2(i-\lambda)},$$

quod pro termino $\phi=0$ manifesto evanescit; pro altere vero termino $\phi=180°=\pi$, ob *i* et λ numeros integros, manifestum est, hoc integrale denuo evanescere, solo casu excepto quo $\lambda=i$. Si enim $i-\lambda$ ut infinite parvum spectetur, puta $=\omega$, pars posterior hujus integralis erit $\frac{\sin.\omega\phi}{2\omega}=\frac{\pi}{2}$, id quod inde patet, quod sit $\int\partial\phi\cos.^{2}i\phi=\tfrac{1}{2}\phi+\tfrac{1}{4}\sin.2i\phi=\tfrac{1}{2}\pi.$

§. 95. Ad integrale igitur quaesitum obtinendum, ex potestate $(1-2b\cos.\phi)^{m}$ evoluta, eos tantum terminos, qui $\cos.i\phi$ continent, excerpsisse sufficiet, cum reliqui omnes nihil plane producant, qui si junctim sumti praebeant, $\mathrm{N}\cos.i\phi$, totum nostrum integrale pro ☽, erit ☽ $=(1+aa)^{m}.\tfrac{1}{2}\mathrm{N}\pi;$ quocirca nobis incumbet, in omnes superioris formae partes inquirere, quae formula $\cos.i\phi$ erunt affectae; unde evidens est, quamdiu in illo termino generali $\pm\left(\tfrac{m}{\lambda}\right)2^{\lambda}b^{\lambda}\cos.^{\lambda}\phi$ exponens λ minor fuerit quam *i*, inde nihil plane in integrale inferri.

§. 96. Primus igitur terminus, qui hic in computum venit, erit $\pm\left(\tfrac{m}{i}\right)2^{i}b^{i}\cos.^{i}\phi$, pro quo signum superius + valebit si *i* fuerit numerus par, inferius − vero si impar. Hinc autem par superiorem reductionem proveniet

$$2^{i}\,cos.^{i}\phi=2cos.\ i\phi,$$

ita ut pro N oriatur pars prima $\pm\left(\frac{m}{i}\right)2^i b^i$, termino immediate sequente, qui erit

$$\mp\left(\frac{m}{i+1}\right)2^{i+1}b^{i+1}\cos.^{i+1}\phi,$$

nullus angulus $i\phi$ oritur, cum sit

$$2^{i+1}\cos.^{i+1}\phi = 2\cos.(i+1)\phi + 2\left(\frac{i+1}{1}\right)\cos.(i-1)\phi + \text{etc.}$$

At vero terminus sequens

$$\pm\left(\frac{m}{i+2}\right)2^{i+2}b^{i+2}\cos.^{i+2}\phi, \text{ ob}$$

$$2^{i+2}\cos.^{i+2}\phi = 2\cos.(i+2)\phi + 2\left(\frac{i+2}{1}\right)\cos.i\phi + \text{etc.}$$

partem hinc in litteram N resultantem dat

$$2\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right)b^{i+2}.$$

Simili modo ex. casu $\lambda = i+3$ nihil nascitur. At ex sequente

$$\pm\left(\frac{m}{i+4}\right)2^{i+4}b^{i+4}\cos.^{i+4}\phi, \text{ ob}$$

$$2^{i+4}\cos.^{i+4}\phi = 2\cos.(i+4)\phi + 2\left(\frac{i+4}{1}\right)\cos.(i+2)\phi + 2\left(\frac{i+4}{2}\right)\cos.i\phi + \text{etc.}$$

pars ad litteram N accedens erit

$$2\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right)b^{i+4}.$$

Eodem modo ex casu $\lambda = i+6$ pars ad litteram N accedens erit
$2\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right)b^{i+6}$, et ita porro.

§. 97. His igitur omnibus partibus colligendis, nanciscemur valorem completum litterae N, qui erit

$$\text{N} = \pm 2b^i\left[\left(\frac{m}{i}\right) + \left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right)b^{i+6} + \left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right)b^{i+4} + \left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right)b^{i+6} + \text{etc.}\right]$$

ubi notasse juvabit esse, ut sequitur

$$\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right) = \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)$$

$$\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right) = \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)$$

$$\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right) = \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right),$$

etc.

Per hos igitur valores erit

$$\text{N} = \pm 2b^i\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)b^2 + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)b^4 + \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)b^6 + \text{etc.}\right]$$

quo valore invento, erit integrale nostrum quaesitum

$$☽ = \pm\pi\left(1+aa\right)^{m} b^{i}\left[\left(\tfrac{m}{0}\right)\left(\tfrac{m}{i}\right)+\left(\tfrac{m}{1}\right)\left(\tfrac{m-1}{i+1}\right)b^{2}+\text{etc.}\right]$$

quae series manifesto abrumpitur, quoties fuerit *m* numerus integer positivus. Statim enim atque in hoc charactere $\left(\frac{m-\lambda}{i+\lambda}\right)$ denominator $i+\lambda$ superare incipit numeratorem $m-\lambda$, valor ejus in nihilum abit.

Demonstrationis pars secunda.

§. 98. Ut autem hanc integralis expressionem ad solam litteram *a* revocamus, prouti in nostro theoremate supra est repraesentata, hic loco *b* restituamus valorem assumtum $\frac{a}{1+aa}$, fietque

$$☽ = \pm\pi a^{i}\left(1+aa\right)^{m-i}\left[\left(\tfrac{m}{0}\right)\left(\tfrac{m}{i}\right)+\left(\tfrac{m}{1}\right)\left(\tfrac{m-1}{i+1}\right)\frac{a^{2}}{\left(1+aa\right)^{2}}+\left(\tfrac{m}{2}\right)\left(\tfrac{m-2}{i+2}\right)\frac{a^{4}}{\left(1+aa\right)^{4}}+\text{etc.}\right]$$

ubi, ut formam supra datam eliciamus, potestates ipsius $1+aa$ evolvi oportet. Hunc in finem statuamus $☽ = \pm\pi a^{i}\mathrm{A}$, ita ut jam sit

$$\mathrm{A} = \left(\tfrac{m}{0}\right)\left(\tfrac{m}{i}\right)\left(1+aa\right)^{m-i}+\left(\tfrac{m}{1}\right)\left(\tfrac{m-1}{i+1}\right)a^{2}\left(1+aa\right)^{m-i-2}$$
$$+\left(\tfrac{m}{2}\right)\left(\tfrac{m-2}{i+2}\right)a^{4}\left(1+aa\right)^{m-i-4}+\left(\tfrac{m}{3}\right)\left(\tfrac{m-3}{i+3}\right)a^{6}\left(1+aa\right)^{m-i-6}+\text{etc.}$$

Facta autem harum potestatum evolutione, fiat

$$\mathrm{A} = \alpha+\beta a^{2}+\gamma a^{4}+\delta a^{6}+\varepsilon a^{8}+\zeta a^{10}+\eta a^{12}+\text{etc}.$$

quarum litterarum $\alpha, \beta, \gamma, \delta,$ etc. valores investigemus.

§. 99. Prima igitur statim patet esse $\alpha = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)$; deinde vero reperietur

$$\beta = \left(\tfrac{m}{0}\right)\left(\tfrac{m}{i}\right)\left(\tfrac{m-i}{1}\right)+\left(\tfrac{m}{1}\right)\left(\tfrac{m-i}{i+1}\right),$$

At vero pars posterior per priorem divisa, facta evolutione, praebet $\frac{m-i-1}{i+1}$; quo observato erit

$$\beta=\frac{m}{i+1}\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{1}\right),$$

quod reducitur ad $\beta=\left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)$. Simili modo littera γ constabit ex tribus partibus : erit enim

$$\gamma=\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{1}\right)+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right),$$

ubi pars secunda per primam divisa dat $\frac{2(m-i-2)}{i+1}$. At tertius terminus per primum divisus praebet $\frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}$, unde fit

$$\gamma=1+\frac{2(m-i-2)}{i+1}+\frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}.$$

An vero est

$$1+\frac{m-i-2}{i+1}=\frac{m-1}{i+1},\ \text{et}$$
$$\left(\frac{m-i-2}{i+1}\right)\left(1+\frac{m-i-3}{i+2}\right)=\frac{m-1}{i+2}\cdot\frac{m-i-2}{i+1}.$$

unde colligitur

$$\gamma=\frac{m-1}{i+1}\cdot\frac{m}{i+2}\cdot\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right),$$

quae expressio contrahitur in hanc $\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)$.

§. 100. Cum igitur sit

$$\alpha=\left(\frac{m}{0}\right)\left(\frac{m}{i}\right),\beta=\left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right),\gamma=\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right),$$

hinc jam satis tuto concludere liceret, fore

$$\delta=\left(\frac{m}{3}\right)\left(\frac{m}{i+3}\right),\varepsilon=\left(\frac{m}{4}\right)\left(\frac{m}{i+4}\right),\text{etc.}$$

Verum ne hic quicquam conjecturae vel inductioni tribuamus in genere pro valore litterae Λ investigemus coefficientem potestatis indefinitae $a^{2\lambda}$, quem vocemus $=\lambda$, eritque

$$\boldsymbol{\Lambda}=\left(\frac{m-i}{\lambda}\right)\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{\lambda-1}\right)+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)\left(\frac{m-i-4}{\lambda-2}\right)$$
$$+\left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)\left(\frac{m-i-6}{\lambda-3}\right)+\text{etc.}$$

§. 101. Hujus seriei pro Λ inventae singulos terminos sub hac forma generali complecti licet $\left(\frac{m}{\theta}\right)\left(\frac{m-\theta}{i+\theta}\right)\left(\frac{m-i-2\theta}{\lambda-\theta}\right)$, quae secundum factores evoluta transmutatur in hanc formam

$$\frac{m(m-1).\ .\ .\ .\ .\ .\ .\ .\ .\ .(m-i-\lambda-\theta+1)}{1.2\ .\ .\ .\ .\ .\theta\times1.\ .\ .\ .\ .\ .(i+\theta)\times1.2\ .\ .\ .(\lambda-\theta)},$$

ubi numeratoris factores ab *m* incipientes continuo unitate decrescent usque ad ultimum $(m-i-\lambda-\theta+1)$; Jam ista fractio supra et infra multiplicetur per hoc productum
$\lambda(\lambda-1).\ .\ .\ .\ .\ .(\lambda-\theta+1)$,
ac prodibit ista fractio

$$\frac{\lambda(\lambda-1).\ .\ .\ .(\lambda-\theta+1)\times m(m-1)\ .\ .\ .\ .\ .\ .(m-i-\lambda-\theta+1)}{1.2.3\ .\ .\ .\ .\ .\theta\times 1.2.3\ .\ .\ .\ .\ .(i+\theta)\times 1.2.3\ .\ .\ .\lambda},$$

in qua primo continetur character $\left(\frac{\lambda}{\theta}\right)$, deinde etiam ibi continetur character $\left(\frac{m}{\lambda}\right)$; quod restat dabit characterem $\left(\frac{m-\lambda}{i+\theta}\right)$, sicque habebitur forma Λ generalis $=\left(\frac{\lambda}{\theta}\right)\left(\frac{m}{\lambda}\right)\left(\frac{m-\lambda}{i+\theta}\right)$.
Unde si loco θ successive scribamus 0, 1, 2, 3, etc:, quia in singulis terminis communis inest factor $\left(\frac{m}{\lambda}\right)$; erit valor litterae

$$\boldsymbol{\Lambda}=\left(\frac{m}{\lambda}\right)\left[\left(\frac{\lambda}{0}\right)\left(\frac{m-\lambda}{i}\right)+\left(\frac{\lambda}{1}\right)\left(\frac{m-\lambda}{i+1}\right)+\left(\frac{\lambda}{2}\right)\left(\frac{m-\lambda}{i+2}\right)+\text{etc.}\right]$$

Verum ante aliquod tempus demonstravi, hujus similis seriei

$$\left(\frac{p}{0}\right)\left(\frac{q}{r}\right)+\left(\frac{p}{1}\right)\left(\frac{q}{r+1}\right)+\left(\frac{p}{2}\right)\left(\frac{q}{r+2}\right)+\left(\frac{p}{3}\right)\left(\frac{q}{r+3}\right)+\text{etc.}$$

summam semper esse $=\left(\frac{p+q}{p+r}\right)=\left(\frac{p+q}{q-r}\right)$. Facta ergo applicatione erit
$p=\lambda,\ q=m-\lambda,\ r=i$: sicque finito modo habebimus

$$\boldsymbol{\Lambda}=\left(\frac{m}{\lambda}\right)\left(\frac{m}{\lambda+i}\right)=\left(\frac{m}{\lambda}\right)\left(\frac{m}{m-\lambda-i}\right),$$

quae est demonstratio conjecturae supra allatae et ex valoribus $\alpha,\ \beta,\ \gamma$ conclusae.

§. 102. Quod si jam hic loco λ successive scribamus numeros 0, 1, 2, 3, etc., nanciscemur verum valorem seriei, quam, sub littera A complexi; erit scilicet

$$\text{A}=\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)a^2+\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)a^4$$
$$+\left(\frac{m}{3}\right)\left(\frac{m}{i+3}\right)a^6+\text{etc.}$$

atque hinc valor integralis sub signo ☽ indicatae formulae erit

$$☽=\pm\pi a^i\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)a^2+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)a^4+\text{etc.}\right]$$

quae expressio manifesto semper abrumpitur, quoties *m* est numerus integer positivus. Hic autem meminisse oportet, signi ambigui $\pm$ superius locum habere quando *i* fuerit numerus par, inferius vero si impar.

Demonstrationis pars tertia.

§. 103. Ista forma, quam pro valore integrali ☽ hic sumus adepti multo ad eo est simplicior ea, quam theorema nostrum nobis suppeditaverat, quippe quae, si loco ♂ seriem quam designat scribamus, erit

$$☽ = \frac{\pi a^i \left(\frac{m}{1}\right)}{\left(\frac{-m-1}{i}\right)} \left[\left(\frac{m}{0}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right) a^2 + \left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right) a^4 + \text{etc.} \right]$$

Superest igitur, ut perfectum consensum inter has duas expressiones specie multum a se invicem discrepantes ostendamus. Hic autem plurimum notasse juvabit, esse $\left(\frac{-m-1}{i}\right) = \pm\left(\frac{m+i}{i}\right)$, propterea quod supra §.88. jam observavimus, esse in genere $\left(\frac{-p}{q}\right) = \pm\left(\frac{p+q-1}{q}\right)$, ubi signum superius valet si fuerit q numerus par, inferius vero si impar; quo notato posterior forma pro ☽ inventa erit

$$☽ = \frac{\pi a^i \left(\frac{m}{i}\right)}{\left(\frac{m+i}{i}\right)} \left[\left(\frac{m-i}{i}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right) a^2 + \text{etc.} \right].$$

§. 104. Quoniam nunc ambae formae affectae sunt signo ambiguo $\pm$, demonstrandum nobis incumbit, si utramque expressionem per $\left(\frac{m+i}{i}\right)$ multiplicemus, duas sequentes series inter se prorsus esse aequales

$$\text{I.}\quad \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)\left(\frac{m+i}{i}\right) a^2 + \left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)\left(\frac{m+i}{i}\right) a^4 + \text{etc.}$$

$$\text{II.}\quad \left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right)\left(\frac{m}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)\left(\frac{m}{i}\right) a^2 + \left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)\left(\frac{m}{i}\right) a^4 + \text{etc.}$$

ubi aequalitas primorum terminorum ob $\left(\frac{m}{0}\right)$ et $\left(\frac{m-i}{0}\right) = 1$ sponte se prodit: deinde vero non difficulter aequalitas inter terminos secundos ipso *a a* affectos ostendi poterit, similique modo etiam de sequentibus hoc idem est tenendum.

§. l05. Verum ne etiam hic inductione uti cogamur, convenientiam binorum terminorum eadem potestate $a^{2\lambda}$ demonstremus. In priore vero serie ista potestas $a^{2\lambda}$ hunc habet coefficientem $\left(\frac{m}{\lambda}\right)\left(\frac{m}{i+\lambda}\right)\left(\frac{m+i}{i}\right)$; in altera vero ejusdem coefficiens est $\left(\frac{m-i}{\lambda}\right)\left(\frac{m+i}{i+\lambda}\right)\left(\frac{m}{i}\right)$. Evolvatur igitur uterque in factores simplices, ac prior deducit ad hanc fractionem

$$\frac{m.(m-1)\ .\ .\ .\ (m-\lambda+1)\times m(m-1)\ .\ .\ .\ .\ .\ .(m-i-\lambda+1)\times(m+i)\ .\ .\ .\ .\ .\ .(m+1)}{1.2.\ .\ .\ .\ .\ .\ .\lambda\times 1.2.\ .\ .\ .\ .\ .\ .(i+\lambda)\times 1.2.3\ .\ .\ .i};$$

posterior vero praebet istam

$$\frac{(m-i)\;.\;.\;.\;.\;(m-i-\lambda+1)\times(m+i)\;.\;.\;.\;.\;.\;.(m-\lambda+1)\times m\;.\;.\;.\;.\;.\;.(m-i+1)}{1.2.\;.\;.\;.\;.\;.\;\lambda\times1.2.\;.\;.\;.\;.\;.\;(i+\lambda)\times1.2.3\;.\;.\;.\;.\;.\;i};$$

ubi denominatores utrinque manifesto sunt iidem, ita ut tantum aequalitas inter numeratores sit demonstranda.

§. 106. Prima autem in priore numeratore tertius factor generalis cum prima conjunctus praebet hoc productum

$$(m+i)\;.\;.\;.\;.\;.\;.(m-\lambda+1),$$

quod etiam in forma posteriori occurrit: his igitur sublatis aequalitatem monstrari oportet inter partes residuas quae sunt,

in priori forma $m(m-1)\;.\;.\;.\;.\;.\;.(m-i-\lambda+1)$,

in altera $m(m-1)\;.\;.\;.\;.\;(m-i+1)\times(m-i)\;.\;.\;.\;.\;(m-i-\lambda+1)$

quae nunc iterum est manifesta. Sic igitur veritas nostri theorematis, quod demonstrandum suscepimus, jam rigide est ob oculos posita pro formula integrali

$$☽ = \int \partial\phi \cos.i\phi(1+aa-2a\cos.\varphi)^n \begin{bmatrix} a & \phi=0 \\ ad & \phi=\pi \end{bmatrix}.$$

Demonstrationis pars quarta.

§. 107. Invento valore formulae ☽, tota demonstratio jam confecta est censenda, quandoquidem jam initio ex valore formulae ☉ ille rite est derivatus. Interim tamen hic quoque vicissim ex valore ☽ alternum valorem ☉ derivari conveniet. Utamur autem forma simpliciori ipsius ☽, ad quem nos ipsa demonstratio immediate perduxit, qui erat

$$☽ = \pm\pi a^i\left[\left(\tfrac{m}{0}\right)\left(\tfrac{m}{i}\right)+\left(\tfrac{m}{1}\right)\left(\tfrac{m}{i+1}\right)a^2+\left(\tfrac{m}{2}\right)\left(\tfrac{m}{i+2}\right)a^4+\text{etc.}\right]$$

ubi signum superius valet si i fuerit numerus par, inferius si impar.

§. 108. Ex hoc jam valore formulae ☽ alterius formulae ☉ valor deducitur, si modo loco *m* scribamus $-n-1$, qui ergo valor hinc erit

$$☉ = \pm\pi a^i\left[\left(\tfrac{-n-1}{0}\right)\left(\tfrac{-n-1}{i}\right)+\left(\tfrac{-n-1}{1}\right)\left(\tfrac{-n-1}{i+1}\right)a^2+\left(\tfrac{-n-1}{2}\right)\left(\tfrac{-n-1}{i+2}\right)a^4+\text{etc.}\right]$$

quae autem series nunc in infinitum progreditur, siquidem *n* fuerit numerus integer positivus; quamobrem hanc seriem in aliam converti oportet, quae abrumpatur, quoties *n* fuerit numerus integer positivus, id quod ope lemmatis supra initio allati praestari poterit.

§. 109. Seriem igitur hic inventam cum serie $\hbar$ in lemmate comparemus id quod fit statuendo

$$f = -n-1,\ h = -n-1 \text{ et } e = i,$$

ita ut jam sit $\odot = \pm\pi a^i \hbar$. Ex his autem valoribus altera series signo ♂ notata fiet, ob

$$-h-i = n,\ -f-i = n,\ \text{et } x = a^2,$$

$$♂ = \left(\tfrac{n}{0}\right)\left(\tfrac{n}{i}\right) + \left(\tfrac{n}{1}\right)\left(\tfrac{n}{i+1}\right)a^2 + \left(\tfrac{n}{2}\right)\left(\tfrac{n}{i+2}\right)a^4 + \text{etc.}$$

At vero relatio inter has duas series erit

$$\left(\tfrac{i-n-1}{i}\right)\hbar = \frac{\left(\tfrac{n+i}{i}\right)♂}{(1-aa)^{2n+1}};$$

ubi notetur, cum supra jam observaverimus esse

$\left(\tfrac{-p}{q}\right) = \pm\left(\tfrac{p+q-1}{q}\right)$, hic fore $\left(\tfrac{-n-1+i}{i}\right) = \pm\left(\tfrac{n}{i}\right)$; ubi iterum signum superius valet, si i fuerit numerus par. Hinc igitur erit

$$\hbar = \pm\frac{\left(\tfrac{n+i}{i}\right)♂}{\left(\tfrac{n}{i}\right)(1-aa)^{2n+1}}.$$

§. 110. Substituatur igitur iste valor loco $\hbar$, quo ipso duplex signorum ambiguitas e media tolletur, loco ♂ autem series modo data scribatur, atque pro $\odot$ sequentem nanciscemur expressionem

$$\odot = \frac{\pi a^i\left(\tfrac{n+i}{i}\right)}{\left(\tfrac{n}{i}\right)(1-aa)^{2n+i}}.\left[\left(\tfrac{n}{0}\right)\left(\tfrac{n}{i}\right) + \left(\tfrac{n}{1}\right)\left(\tfrac{n}{i+1}\right)a^2 + \left(\tfrac{n}{2}\right)\left(\tfrac{n}{i+2}\right)a^4 + \text{etc.}\right]$$

quae series manifesto semper abrumpitur, quoties n fuerit numerus integer positivus. Verumtamen hoc laborat defectu, quod casibus quibus $n < i$, $\left(\tfrac{n}{i}\right) = 0$, infinita evadere videtur. Verum notandum est, his casibus etiam omnes terminos seriei ♂ in nihilum abire ; ex quo necesse est, ut in ejus verum valorem totiusque expressionis inquiramus. At vero reliquis casibus, quibus $n > i$ haec expressio adeo illi quam in theoremate dedimus praeferenda videtur.

§. 111. Ostendi ergo hic debet, omnes terminos nostrae seriei ita transformari posse, ut per denominatorem $\left(\tfrac{n}{i}\right)$ divisionem admittant. At vero quilibet nostrae seriei terminus sub hac forma continetur $\left(\tfrac{n}{\lambda}\right)\left(\tfrac{n}{i+\lambda}\right)$, quae per factorem comunem $\left(\tfrac{n+i}{i}\right)$ multiplicata fit $\left(\tfrac{n+i}{i}\right)\left(\tfrac{n}{\lambda}\right)\left(\tfrac{n}{i+\lambda}\right)$, quae in factores evoluta ad hanc fractionem reducitur

$$\frac{(n+i)\ldots(n+1)\times(n)\ldots(n-\lambda+1)\times n\ldots(n-i-\lambda+1)}{1.2.\ldots i\times 1.2.\ldots\lambda\times 1.2.\ldots(i+\lambda)};$$

ubi tam numerator quam denominator tres habet factores principales; factores autem singulares in numeratore continuo unitate decrescunt, in denominatore unitate increscunt. Cum igitur sit $\left(\frac{n}{i}\right)=\frac{n\ldots(n-i+1)}{1.2.\ldots i}$, superior fractio per hanc divisa, ob

$$\frac{n\ldots(n-i-\lambda+1)}{n\ldots(n-i+1)}=(n-i).\ \ldots(n-i-\lambda+1),$$

proveniet

$$\frac{(n+i)\ldots(n+1)\times(n)\ldots(n-\lambda+1)\times(n-i)\ldots(n-i-\lambda+1)}{1.2.3\ldots\lambda\times 1.2.3\ldots(i+\lambda)};$$

quae manifesto in hanc transit (ob duo priores factores cohaerentes)

$$\frac{(n+i)\ldots(n-\lambda+1)\times(n-i)\ldots(n-i-\lambda+1)}{1.2.3\ldots\lambda\times 1.2.3\ldots(i+\lambda)},$$

ita ut omnibus ad characteres reductis, sit forma generalis cujusque termini $=\left(\frac{n+i}{i+\lambda}\right)\left(\frac{n-i}{\lambda}\right)$.

§. 112. Nunc igitur loco λ successive scribantur valores 0, i, 2, 3, etc. atque valor integralis formulae ⊙ prodibit, prorsus uti in Theoremate est enunciatus, scilicet

$$\odot=\frac{\pi a^{i}}{(1-aa)^{2n+1}}\left(\left(\frac{n-i}{0}\right)\left(\frac{n+i}{i}\right)+\left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right)a^{2}+\left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right)a^{4}+\text{etc}\right).$$

quae expressio jam non solum semper abrumpitur, quoties *n* fuerit numerus integer positivus , nec ullo amplius laborat defectu, cum omnibus casibus valorem ipsius ⊙ determinatum exhibeat, sicque adeo nostrum theorema, quod antea sola conjectura innitebatur; solidissima demonstratione est confirmatum.