SUPPLEMENT IV*a*.

TO BOOK. I. CH.V.

ON
THE INTEGRATION OF FORMULAS INVOLVING ANGLES
OR THE SINES OF ANGLES.

1) Concerned especially with irrational angular differential formulas , which nevertheless can be integrated by logarithms or circular arcs.

M.S. shown to the Academy on the 5th *of May*, 1777.

§. 1. Now I have often discussed formulas with irrational differentials, which cannot be returned to rationality by any substitution, are allowed to be integrated by nothing less than logarithms or circular arcs: also they can be transformed into angular formulas of this kind, which involve the sine or cosine of some angle. Moreover the general form of differentials of this kind, which can be treated in this manner, can be represented in the following way: with ϕ denoting some angle, Φ may designate some rational function of tang.$n\phi$, and this same formula:

$$\frac{\Phi\cdot\partial\phi(f\sin.\lambda\phi+g\cos.\lambda\phi)}{\sqrt[n]{(a\sin.n\phi+b\cos.n\phi)^{\lambda}}}$$

always found able to be integrated by logarithms and circular arcs, that which I have decided to show by following from the simplest cases in the following problems .

Problem I.

§2. *For the proposed differential formula* $\frac{\partial\phi\cos.\phi}{\sqrt[n]{\cos.n\phi}}$, *to investigate its integration by logarithms and circular arcs.*

Solution.

Because indeed to me at this time no other way appears to be excelling that same one, except by proceeding by imaginary numbers, and I will designate in the following the formula $\sqrt{-1}$ by the letter i , thus so that there shall become $ii=-1$, and thus $\frac{1}{i}=-i.$ Now first of all in the numerator of our formulas we will substitute in place of $\cos.\varphi$ these two parts :

$$\tfrac{1}{2}(\cos.\phi+i\sin.\phi)+\tfrac{1}{2}(\cos.\phi-i\sin.\phi),$$

and we will represent the same proposed formula by the two parts of this kind, which shall be :

$$\partial p = \frac{\partial\phi(\cos.\phi + i\sin.\phi)}{\sqrt[n]{\cos.n\phi}} \text{ and } \partial q = \frac{\partial\phi(\cos.\phi - i\sin.\phi)}{\sqrt[n]{\cos.n\phi}}$$

thus so that our proposed formula itself shall be $\frac{1}{2}\partial p + \frac{1}{2}\partial q$, and thus its integral $\frac{p+q}{2}$.

§. 3. Now we will examine both these parts separately in the following manner. Clearly for the first formula

$$\partial p = \frac{\partial\phi(\cos.\phi + i\sin.\phi)}{\sqrt[n]{\cos.n\phi}} \text{ we may put } \frac{\cos.\phi + i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = x,$$

so that there shall be $\partial p = x\partial\phi$, and with the n^{th} powers of the exponent taken, we will have :

$$x^n = \frac{(\cos.\phi + i\sin.\phi)^n}{\cos.n\phi}.$$

But it is to be agreed,

$$(\cos.\phi + i\sin.\phi)^n = \cos.n\phi + i\sin.n\phi,$$

and thus there will be $x^n = 1 + i\,\text{tang}.n\phi,$ from which it is deduced,

$$\text{tang}.n\phi = \frac{x^n - 1}{i} = i\left(1 - x^n\right):$$

hence on putting in general $\text{tang}.\omega = \text{Z}$, there shall be $\partial\omega = \frac{\partial Z}{1+ZZ}$, for our case there will become :

$$n\partial\phi = \frac{-nix^{n-1}\partial x}{1 + ii - 2iix^n + iix^{2n}},$$

which formula on account of $ii = -1$ is changed into this:

$$\partial\phi = \frac{-ix^{n-1}\partial x}{2x^n - x^{2n}},'$$

and hence this formula

$$\partial p = x\partial\phi = \frac{-i\partial x}{2 - x^n},$$

which since it shall be rational, its integration is subject to no difficulty.

§. 4. But if now, for the other formula, we may put

$$\partial q = \frac{\partial\phi(\cos.\phi - i\sin.\phi)}{\sqrt[n]{\cos.n\phi}} \text{ and } \frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = y,$$

so that by similar operations, there shall be $\partial q = y\partial\phi,$ which differ from the preceding in this alone, so that the letter *i* may be taken to be negative, and this transformation will result : $\partial q = \frac{i\partial y}{2-y^n}$, which since it shall be similar to the former, the same whole matter of integration may be performed, and for the integral sought we shall have :

$$p+q = -i\int \frac{\partial x}{2-x^n} + i\int \frac{\partial y}{2-y^n}.$$

§. 5. Moreover it is agreed that the integration of such formulas from parts of two kinds, evidently to depend on logarithms and circular arcs, thus so that the general form of the former shall be $f\,l(\alpha+\beta x+\gamma xx)$, truly of the latter g Arc.tang.$(\delta+\varepsilon x)$. Whereby if this difference between the two similar formulas may occur, from the individual logarithmic parts the form of such may arise $-i\,f\,l\frac{\alpha+\beta x+\gamma xx}{\alpha+\beta y+\gamma yy}$, where both x as well as y involves imaginary numbers, because of this matter we may put for brevity $x=r+is$ and $y=r-is$, where there shall become:

$$r = \frac{\cos.\phi}{\sqrt[n]{\cos.n\phi}} \text{ and } s = \frac{\sin.\phi}{\sqrt[n]{\cos.n\phi}},$$

therefore with these values substituted, any logarithmic part will become :

$$-i\,f\,l\frac{\alpha+\beta r+\gamma rr-\gamma ss+i(\beta s+2\gamma rs)}{\alpha+\beta r+\gamma rr-\gamma ss-i(\beta s+2\gamma rs)}.$$

§. 6. In place of this more prolix expression for brevity we shall write $-i\,f\,l\frac{t+iu}{t-iu}$, thus, so that there shall become :

$$t=\alpha+\beta r+\gamma rr-\gamma ss \text{ and } u=\beta s+2\gamma rs,$$

and thus these values become known by the angle ϕ also. Therefore because now more often it is required to show, that

$$l\frac{t+u\sqrt{-1}}{t-u\sqrt{-1}} = 2\sqrt{-1}.\text{Arc.tang}\frac{u}{t},$$

that same part of the integral will be $=+2f.\text{Arc.tang}\frac{u}{t}$, which therefore is completely real, while the imaginary parts have mutually cancelled each other out, so that any imaginary logarithmic part will produces a real circular arc.

§. 7. In a similar manner in general we may join together two circular arcs arising from integration, which from the assumed form will become

$$-ig\ \text{Arc.tang.}(\delta+\varepsilon x)+ig\text{Arc.tang.}(\delta+\varepsilon y),$$

which form thus may be contracted into one arc, which will become:

$$-ig\ \text{Arc.tang.}\frac{\varepsilon(x-y)}{1+(\delta+\varepsilon x)(\delta+\varepsilon y)}:$$

which with the assumed values introduced $x = r + is$ and $y = r - is$, will adopt this form

$$-ig \text{ Arc.tang.}\frac{2i\varepsilon s}{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)}.$$

Therefore, since in general there shall be

$$\text{Arc.tang } v\sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+v}{1-v},$$

that same part of the circle will be transformed into the following real logarithm:

$$\frac{g}{2} l \frac{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)+2\varepsilon s}{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)-2\varepsilon s}:$$

therefore in this manner with all the parts of the integrals taken, finally the integral sought will be obtained expressed in real terms by separate logarithms and circular arcs.

Problem. 2.

§. 8. *For the proposed differential formula* $\frac{\partial\varphi \sin.\phi}{\sqrt[n]{\cos.n\phi}}$, *to investigate its integration by logarithms and circular arcs.*

Solution.

Here in place of $\sin.\phi$ this form may be written with the two parts being agreed on

$$\tfrac{1}{2i}(\cos.\phi + i\sin.\phi) - \tfrac{1}{2i}(\cos.\phi - i\sin.\phi),$$

and the formula proposed may be resolved into these parts

$$\partial p = \frac{\partial\phi(\cos.\phi+i\sin.\phi)}{\sqrt[n]{\cos.n\phi}} \text{ and } \partial q = \frac{\partial\phi(\cos.\phi-i\sin.\phi)}{\sqrt[n]{\cos.n\phi}},$$

thus so that the same proposed formula now may become $\frac{\partial p-\partial q}{2i}$, and thus the integral sought itself $\frac{p-q}{2i}$

§. 9. Because if now again as before we may put

$$\frac{\cos.\phi+i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = x \text{ and } \frac{\cos.\phi-i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = y,$$

there will be found as above

$$\partial p = -\frac{i\partial x}{2-x^n} \text{ and } \partial q = \frac{i\partial y}{2-y^n};$$

from which therefore the same integral sought becomes

$$\frac{p-q}{2i} = -\frac{1}{2}\int\frac{\partial x}{2-x^n} - \frac{1}{2}\int\frac{\partial y}{2-y^n},$$

where the real coefficients cancel out.

§. 10. Now we may consider, from the form of each part of the integral, some portion to be logarithmic, which shall be $f\,l\left(\alpha+\beta x+\gamma xx\right)$, and hence from each part there may arise for the integral

$$-\tfrac{1}{2}f\,l\left(\alpha+\beta x+\gamma xx\right)-\tfrac{1}{2}f\,l\left(\alpha+\beta y+\gamma yy\right).$$

But if now as above we may put for the sake of brevity $x=r+is$ and $y=r-is$, then truly

$$t=\alpha+\beta r+\gamma rr-\gamma ss \text{ and } u=\beta s+2\gamma rs,$$

Both these logarithms arrive at

$$=-\tfrac{1}{2}f\,l\left(t+iu\right)-\tfrac{1}{2}f\,l\left(t-iu\right),$$

which contract into $-\tfrac{1}{2}f\,l\left(tt+uu\right)$, which expression is now real, nor needs any further reduction.

§. 11. In the same manner two circular parts arise from the integration :

$$-\tfrac{1}{2}g\text{Arc.tang.}(\delta+\varepsilon x)-\tfrac{1}{2}g\text{Arc.tang.}(\delta+\varepsilon y),$$

which are represented by r and s thus:

$$-\tfrac{1}{2}g\left[\text{Arc.tang.}(\delta+\varepsilon r+i\varepsilon s)+\text{Arc.tang.}(\delta+\varepsilon r-i\varepsilon s)\right],$$

which two arcs may be contracted into one :

$$-\tfrac{1}{2}g\text{Arc.tang.}\frac{2\delta+2\varepsilon r}{1-\left(\delta+\varepsilon r\right)^2-\varepsilon\varepsilon ss},$$

which expression further has now been produced real.

Problem 3.

§.12. *For the proposed differential formula* $\frac{\partial\phi\cos.\lambda\phi}{\sqrt[n]{\cos.n\phi^{\lambda}}}$, *to investigate its integration by logarithms and circular arcs.*

Solution.

Since there shall be

$$\cos.\lambda\phi = \tfrac{1}{2}\left(\cos.\phi + i\sin.\phi\right)^{\lambda} + \tfrac{1}{2}\left(\cos.\phi - i\sin.\phi\right)^{\lambda},$$

the proposed formula may be separated into these two parts :

$$\partial p = \frac{\partial\phi\left(\cos.\phi+i\sin.\phi\right)^{\lambda}}{\sqrt[n]{\cos.n\phi^{\lambda}}} \text{ and } \partial q = \frac{\partial\phi\left(\cos.\phi-i\sin.\phi\right)^{\lambda}}{\sqrt[n]{\cos.n\phi^{\lambda}}}$$

thus so that the integral sought becomes $\frac{p+q}{2}$.

§.13. Now we may put in place, as we have done before :

$$\frac{\cos.\phi+i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = x \text{ and } \frac{\cos.\phi-i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = y,$$

with which done there becomes $\partial p = x^{\lambda}\partial\phi$ and $\partial q = y^{\lambda}\partial\phi.$ But with the calculation effected as above, we will obtain :

$$\partial\phi = -\frac{ix^{n-1}\partial x}{2x^{n}-x^{2n}}, \text{ and hence } \partial p = -\frac{ix^{\lambda-1}\partial x}{2-x^{n}};$$

and in a similar manner there will be $\partial q = -\frac{iy^{\lambda-1}\partial y}{2-y^{n}}$, and thus the whole integral sought will be :

$$= -\frac{i}{2}\int\frac{x^{\lambda-1}\partial x}{2-x^{n}} + \frac{i}{2}\int\frac{y^{\lambda-1}\partial y}{2-y^{n}}.$$

§. 14. Because these two integrals are themselves similar, and thus both the logarithmic as well as the circular similar parts are included, from the logarithmic part, which shall be $f\,l\left(\alpha+\beta x+\gamma xx\right)$, on putting as above $x = r + is$ and $y = r - is$, then truly

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ and } u = \beta s + 2\gamma r\text{s},$$

hence from this first logarithmic part there is deduced $-i\,f\,l\frac{t+iu}{t-iu}$, which since it shall be imaginary is reduced to this real circular arc $= 2f$ Arc.tang.$\frac{u}{t}$: in a similar manner, if the form of the circular arc arising from the integration were $-g$ Arc. tang.$(\delta+\varepsilon x)$, from the circular parts initially the following imaginary arc arises

$$-ig\text{Arc. tang.}\frac{2i\varepsilon s}{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)},$$

which finally is related to that real logarithm

$$\frac{g}{2}l.\frac{1+\delta\delta+2\delta\varepsilon r+\varepsilon\varepsilon(rr+ss)+2\varepsilon s}{1+\delta\delta+2\delta\varepsilon r+\varepsilon\varepsilon(rr+ss)-2\varepsilon s}.$$

Problem 4.

§. 15. *For the proposed differential formula* $\frac{\partial\phi\sin.\lambda\phi}{\sqrt[n]{\cos.n\phi^{\lambda}}}$*, to investigate its integration by logarithms and circular arcs.*

Solution.

Since there shall be

$$\sin.\lambda\phi=\tfrac{1}{2i}(\cos.\phi+i\sin.\phi)^{\lambda}-\tfrac{1}{2i}(\cos.\phi-i\sin.\phi)^{\lambda},$$

at this stage we may put these two parts in place :

$$\partial p=\frac{\partial\phi(\cos.\phi+i\sin.\phi)^{\lambda}}{\sqrt[n]{\cos.n\phi^{\lambda}}}\text{ et }\partial q=\frac{\partial\phi(\cos.\phi-i\sin.\phi)^{\lambda}}{\sqrt[n]{\cos.n\phi^{\lambda}}},$$

thus so that the integral sought shall be $\frac{p-q}{2i}$. Now again we may put in place:

$$\frac{\cos.\phi+i\sin.\phi}{\sqrt[n]{\cos.n\phi}}=x\text{ and }\frac{\cos.\phi-i\sin.\phi}{\sqrt[n]{\cos.n\phi}}=y,$$

so that there becomes $\partial p=x^{\lambda}\partial\phi$ and $\partial q=y^{\lambda}\partial\phi$, and hence with the calculation put in place as above, there will become :

$$\partial p=-\frac{ix^{\lambda-1}\partial x}{2-x^{n}}\text{ and }\partial q=\frac{iy^{\lambda-1}\partial y}{2-y^{n}},$$

and thus the integral sought will be

$$-\frac{1}{2}\int\frac{x^{\lambda-1}\partial x}{2-x^{n}}-\frac{1}{2}\int\frac{y^{\lambda-1}\partial y}{2-y^{n}}.$$

§. 16. But if now, as has been done until now, we may put $x = r + is$ and $y = r - is$, and for the logarithmic parts, the form of which shall be $f\,l\left(\alpha + \beta x + \gamma xx\right)$, we may put

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ and } u = \beta s + 2\gamma r\text{s},$$

the imaginary parts of the two logarithms to be used in the following problem will be contained in one real logarithm, which will be $-\frac{1}{2} f\,l\left(tt + uu\right)$. But if for the circular parts, the form of which shall be g Arc. tang.$\left(\delta + \varepsilon x\right)$, two such imaginary arcs may be taken together, those will come together into one real arc

$$-ig\text{Arc. tang.}\frac{2\delta + 2\varepsilon r}{1 - \left(\delta + \varepsilon r\right)^2 - \varepsilon\varepsilon ss}.$$

Problem in general.

§. 17. *If Φ may denote some rational function itself of tang.$n\phi$, and this formula of the differential were proposed*

$$\frac{\Phi \cdot \partial\phi\left(f\sin.\lambda\phi + g\cos.\lambda\phi\right)}{\sqrt[n]{\left(a\sin.n\phi + b\cos.n\phi\right)^{\lambda}}},$$

to reduce its integration to logarithms and circular arcs.

Solution.

Now from the preceding it is understood easily, the form of the numerator $F\sin.\lambda\phi + G\cos.\lambda\phi$ can always be recalled to such a form

$$F'\left(\cos.\phi + i\sin.\phi\right)^{\lambda} + G'\left(\cos.\phi - i\sin.\phi\right)^{\lambda},$$

and hence that proposed form can be separated into these two parts :

$$\partial p = \frac{\Phi \cdot \partial\phi\left(\cos.\phi + i\sin.\phi\right)^{\lambda}}{\sqrt[n]{\left(a\cos.\phi + b\sin.\phi\right)^{\lambda}}} \text{ and } \partial q = \frac{\Phi \cdot \partial\phi\left(\cos.\phi - i\sin.\phi\right)^{\lambda}}{\sqrt[n]{\left(a\cos.\phi + b\sin.\phi\right)^{\lambda}}}$$

thus so that the integral sought now shall become $\text{F}'p + \text{G}'q$.

§. 18. Now for the former formula ∂p there may be put

$$\frac{\cos.\phi+i\sin.\phi}{\sqrt[n]{(a\cos.n\phi+b\sin.n\phi)}}=x$$

and for the latter

$$\frac{\cos.\phi-i\sin.\phi}{\sqrt[n]{(a\cos.n\phi+b\sin.n\phi)}}=y.$$

thus so that hence there shall become

$$\partial p=\Phi\cdot x^{\lambda}\partial\phi \text{ and } \partial q=\Phi\cdot y^{\lambda}\partial\phi;$$

thence moreover there shall become

$$x^{n}=\frac{cos.n\phi+isin.n\phi}{acos.n\phi+isin.n\phi},$$

from which there is deduced

$$\text{tang.}n\phi=\frac{1-ax^{n}}{bx^{n}-i};$$

whereby since Φ may denote a rational function of tang.$n\phi$, there will be produced also some rational function of x, and thus of x^n, which may be designated by X. Truly in addition the differential $\partial\phi$ will be determined rationally; since there becomes

$$\partial\phi=\frac{(a-b)x^{n-1}\partial x}{(aa+bb)x^{2n}-2(a-ib)x^{"}},$$

therefore in this manner we will have

$$\partial p=\frac{(ia-b)Xx^{\lambda-1}\partial x}{(aa+bb)x^{n}-2(a+ib)},$$

which since it shall be completely rational, it is certain, its integral, however much labour it may have demanded, can be expressed always by logarithms and circular arcs.

§.19. The matter may be resolved in a similar manner in the other formula ∂q, which differs from that only on account of the sign of the letter i, and since here everything expressed will be produced rationally y, with which agreed on, Φ will become Y, and there will be obtained :

$$\partial q=-\frac{(b+ia)Yy^{\lambda-1}\partial y}{(aa+bb)y^{n}-2a+2ib},$$

the integration of which, with everything similar to the preceding, and may be resolved by just the same labour.

§. 20. But it is evident, in a calculation of this kind imaginary quantities are closely mixed together with many real quantities, as usually came about in the preceding problems, since now F' and G' are to be derived at once from the initial conditions now to involve imaginary numbers; then truly also $\text{tang}.n\phi$ is labeled with imaginary numbers on each side, from which also imaginary values will enter into X and Y; on account of which the reduction to real numbers generally will be required to be done with the maximum labour, but according to the calculation the necessary precepts are known well enough now.

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2) A most noteworthy theorem about the integral formula :

$$\int \frac{\partial\phi\cos.\lambda\phi}{(1+aa-2a\cos.\phi)^{n+1}}.$$

M.S. presented to the Academy on the 13th *of August* 1778.

[E672; Opera Omnia Series I, Vol. 19, pp. 141-167.]

§. 21. This formula requires no other restriction other than that the letter λ may designate either a positive or negative whole number. But it is evident negative values do not disagree with positive values, since always there shall be $\cos.-\phi=\cos.+\phi$. With this noted the integral of this formula is extended from the term $\phi=0$ as far as to the term $\phi=180°$ or $\phi=\pi$, the value of which will be expressed always in the following form $\frac{\pi a}{(1-aa)^{2n+1}}\cdot\text{V}$, with there being :

$$\begin{aligned}\text{V} &= \left(\tfrac{n-\lambda}{0}\right)\left(\tfrac{n+\lambda}{\lambda}\right)+\left(\tfrac{n-\lambda}{1}\right)\left(\tfrac{n+\lambda}{\lambda+1}\right)aa\\ &+\left(\tfrac{n-\lambda}{2}\right)\left(\tfrac{n+\lambda}{\lambda+2}\right)a^4+\left(\tfrac{n-\lambda}{3}\right)\left(\tfrac{n+\lambda}{\lambda+3}\right)a^6\\ &+\left(\tfrac{n-\lambda}{4}\right)\left(\tfrac{n+\lambda}{\lambda+4}\right)a^8+\left(\tfrac{n-\lambda}{5}\right)\left(\tfrac{n+\lambda}{\lambda+5}\right)a^{10}\ \text{etc.}\end{aligned}$$

Where the formulas enclosed by the brackets are not fractions, but designate these characters, by which the individual binomial powers are accustomed to be designated, thus so that there shall be

$$\left(\frac{\alpha}{\beta}\right)=\frac{\alpha}{1}\cdot\frac{\alpha-1}{2}\cdot\frac{\alpha-2}{3}\cdot\ \ .\ \ .\ \ .\ \ \ldots\frac{\alpha-\beta+1}{\beta},$$

which expression because in our case β is everywhere an integer, declares the determined value easily in whatever case is being shown, where it will suffice to be noted, whenever there were $\beta=0$ always to become $\left(\frac{\alpha}{0}\right)=1$; but if β were a negative number, the value of these characters will be changed into zero; then truly also it will be agreed to be observed, if there were $\beta=\alpha$ to become $\left(\frac{\alpha}{\alpha}\right)=1$, and if $\beta>\alpha$ the values to vanish equally. Since also there shall be $\left(\frac{\alpha}{\beta}\right)=\left(\frac{\alpha}{\alpha-\beta}\right)$.

[This theorem is proven in Sup. 4*b*.]

§. 22. With these established, we set out the particular cases in which the simpler values 0, 1, 2, 3, 4 etc. are attributed to the exponent n .

Case I.

where $n=0$, and this formula of the integral is proposed

$$\int\frac{\partial\phi\cos.\lambda\phi}{1+aa-2a\cos.\phi}\left[\begin{matrix}from & x=0\\ to & x=\pi\end{matrix}\right].$$

Because here $n=0$, for the first factors [of each of the terms] of the quantity V we will have

$$\left(\frac{0-\lambda}{0}\right)=1;\left(\frac{0-\lambda}{1}\right)=-\lambda;\left(\frac{0-\lambda}{2}\right)=\frac{\lambda}{1}\cdot\frac{\lambda+1}{2};$$
$$\left(\frac{0-\lambda}{3}\right)=-\frac{\lambda}{1}\cdot\frac{\lambda+1}{2}\cdot\frac{\lambda+2}{3};\left(\frac{0-\lambda}{4}\right)=\frac{\lambda}{1}\cdot\frac{\lambda+1}{2}\cdot\frac{\lambda+2}{3}\cdot\frac{\lambda+3}{4};\text{etc.}$$

Truly for the latter factors [of each of the terms] we will have :

$$\left(\frac{0+\lambda}{\lambda}\right)=1;\left(\frac{0+\lambda}{\lambda+1}\right)=0;\left(\frac{0+\lambda}{\lambda+2}\right)=0\text{ etc.}$$

here therefore all these factors except the first vanish; from which the value of the quantity $\mathrm{V}=1$ is deduced, and thus the integral of this outcome sought will be

$$=\frac{\pi a^{\lambda}}{1-aa}.$$

Hence therefore if there were $n=0$, there will be $\int\frac{\partial\phi}{1+aa-2a\cos.\phi}=\frac{\pi}{1-aa}$, which agrees especially well with the well-known integration:

$$\int\frac{\partial\phi}{\alpha+\beta\cos.\phi}=\frac{1}{\sqrt{(\alpha\alpha-\beta\beta)}}\text{Arc.cos.}\frac{\alpha\cos.\phi+\beta}{\alpha+\beta\cos.\phi},$$

which integral now vanishes at once on taking $\phi=0$. Therefore it may be stated, as we have assumed here always, $\phi=180°=\pi$, and on account of $\cos.\phi=-1$, there will be this same integral

$$\frac{1}{\sqrt{(\alpha\alpha-\beta\beta)}}\text{Arc.cos.}-1=\frac{\pi}{\sqrt{(\alpha\alpha-\beta\beta)}}.$$

Now in our case $\alpha=1+aa$ and $\beta=-2a$, from which there becomes $\sqrt{(\alpha\alpha-\beta\beta)}=1-aa.$

Case II.

where $n=1$, and this formula of the integral is proposed :

$$\int\frac{\partial\phi\cos.\lambda\phi}{(1+aa-2a\cos.\phi)^2}\left[\begin{matrix}\textit{from} & \phi=0\\ \textit{to} & \phi=\pi\end{matrix}\right].$$

Because here there is $n=1$, there will be for the first factors of the terms of the magnitude V

$$\left(\tfrac{1-\lambda}{0}\right)=1;\left(\tfrac{1-\lambda}{1}\right)=-(\lambda-1);\ \left(\tfrac{1-\lambda}{2}\right)=\tfrac{\lambda}{1}\cdot\tfrac{(\lambda-1)}{2}.$$

For the latter factors we will have :

$$\left(\tfrac{1+\lambda}{\lambda}\right)=\lambda+1;\left(\tfrac{1+\lambda}{\lambda+1}\right)=1;$$

truly the following formulas vanish, and thus there will be

$$\text{V}=\lambda+1-(\lambda-1)aa\ ;$$

on account of which the value of the proposed integral will be :

$$=\frac{\pi a^{\lambda}}{(1-aa)^3}\left[(\lambda+1)-(\lambda-1)aa\right]:$$

hence therefore it will help to place together the following special cases, where for the sake of brevity in place of the formula $1+aa-2a.\cos.\phi$ we will write the character Δ :

$$\int \frac{\partial\phi}{\Delta^2} = \frac{\pi(1+aa)}{(1-aa)^3},$$

$$\int \frac{\partial\phi\cos.\phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3},$$

$$\int \frac{\partial\phi\cos.2\phi}{\Delta^2} = \frac{\pi a^2(3-aa)}{(1-aa)^3},$$

$$\int \frac{\partial\phi\cos.3\phi}{\Delta^2} = \frac{\pi a^3(4-2aa)}{(1-aa)^3},$$

$$\int \frac{\partial\phi\cos.4\phi}{\Delta^2} = \frac{\pi a^4(5-3aa)}{(1-aa)^3},$$

$$\int \frac{\partial\phi\cos.5\phi}{\Delta^2} = \frac{\pi a^5(6-4aa)}{(1-aa)^3},$$

$$\int \frac{\partial\phi\cos.6\phi}{\Delta^2} = \frac{\pi a^6(7-5aa)}{(1-aa)^3},$$

etc. etc.

Case III.

where $n=2$, and this formula of the integral is proposed :

$$\int \frac{\partial\phi\cos.\lambda\phi}{(1+aa-2a\cos.\phi)^3}\left[\begin{matrix} from & \phi=0 \\ to & \phi=\pi \end{matrix}\right].$$

Here the first factors, which occur in the value of the magnitude V, will be :

$$\left(\tfrac{2-\lambda}{0}\right)=1;\left(\tfrac{2-\lambda}{1}\right)=-(\lambda-2);\ \left(\tfrac{2-\lambda}{2}\right)=\tfrac{(\lambda-2)}{1}\cdot\tfrac{(\lambda-1)}{2};$$
$$\left(\tfrac{2-\lambda}{3}\right)=\tfrac{\lambda-2}{1}\cdot\tfrac{\lambda-1}{2}\cdot\tfrac{\lambda}{3}\text{ etc.}$$

but the latter factors will be :

$$\left(\tfrac{2+\lambda}{\lambda}\right)=\tfrac{\lambda+2}{1}.\tfrac{\lambda+1}{2};\left(\tfrac{2+\lambda}{\lambda+1}\right)=\lambda+2;\left(\tfrac{2+\lambda}{\lambda+2}\right)=1;$$

and all the following vanish ; hence therefore we deduce :

$$V = \tfrac{\lambda+2}{1}.\tfrac{\lambda+1}{2} - (\lambda\lambda-4)aa + \tfrac{(\lambda-2)}{1}.\tfrac{(\lambda-1)}{2}a^4,$$

and with this value found the value of the integral sought will be $\frac{\pi a^\lambda}{(1-aa)^5}.V$, from which we set out the following special cases, by putting in place $1+aa-2a\cos.\phi = \Delta$, as before :

$$\int\frac{\partial\phi}{\Delta^3} = \frac{\pi}{(1-aa)^5}\left(1+4aa+a^4\right),$$

$$\int\frac{\partial\phi\cos.\phi}{\Delta^3} = \frac{3\pi a}{(1-aa)^5}(1+aa),$$

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^3} = \frac{6\pi a^2}{(1-aa)^5},$$

$$\int\frac{\partial\phi\cos.3\phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}\left(10-5aa+a^4\right),$$

$$\int\frac{\partial\phi\cos.4\phi}{\Delta^3} = \frac{3\pi a^4}{(1-aa)^5}\left(5-4aa+a^4\right),$$

$$\int\frac{\partial\phi\cos.5\phi}{\Delta^3} = \frac{3\pi a^5}{(1-aa)^5}\left(7-7aa+2a^4\right),$$

$$\int\frac{\partial\phi\cos.6\phi}{\Delta^3} = \frac{2\pi a^6}{(1-aa)^5}\left(14-16aa+5a^4\right),$$

etc. etc.

Case IV.

where $n=3$, and this formula of the integral is proposed

$$\int\frac{\partial\phi\cos.\lambda\phi}{(1+aa-2a\cos.\phi)^4}\begin{bmatrix} a & \phi=0 \\ ad & \phi=\pi \end{bmatrix}.$$

Here for the first factors of the quantity V we will have

$$\left(\tfrac{3-\lambda}{0}\right)=1;\left(\tfrac{3-\lambda}{1}\right)=-(\lambda-3);\ \left(\tfrac{3-\lambda}{2}\right)=\tfrac{(3-\lambda)}{1}\cdot\tfrac{(2-\lambda)}{2};$$
$$\left(\tfrac{3-\lambda}{3}\right)=\tfrac{3-\lambda}{1}\cdot\tfrac{2-\lambda}{2}\cdot\tfrac{1-\lambda}{3};\ \left(\tfrac{3-\lambda}{4}\right)=\tfrac{3-\lambda}{1}\cdot\tfrac{2-\lambda}{2}\cdot\tfrac{1-\lambda}{3}\cdot\tfrac{-\lambda}{4};$$

but the latter factors will be

$$\left(\tfrac{3+\lambda}{\lambda}\right)=\tfrac{3+\lambda}{1}\cdot\tfrac{2+\lambda}{2}\cdot\tfrac{1+\lambda}{3};\ \left(\tfrac{3+\lambda}{\lambda+1}\right)=\tfrac{3+\lambda}{1}\cdot\tfrac{2+\lambda}{2};\ \left(\tfrac{3+\lambda}{\lambda+2}\right)=3+\lambda;\ \left(\tfrac{3+\lambda}{\lambda+3}\right)=1;$$

and all the following vanish, hence therefore we deduce

$$\mathrm{V}=\frac{(\lambda+1)(\lambda+2)(\lambda+3)}{1.\quad 2.\quad 3}-\frac{(\lambda+2)(\lambda\lambda-9)}{1.\quad 2.}aa+\frac{(\lambda-2)(\lambda\lambda-9)}{1.\quad 2.}a^4-\frac{(\lambda-1)(\lambda-2)(\lambda-3)}{1.\quad 2.\quad 3}a^6.$$

With which value found we deduce the integral sought $=\dfrac{\pi a^{\lambda}}{(1-aa)^7}.\mathrm{V}$,

and hence we may present the following special cases, by putting as usual $1+aa-2a\cos.\phi=\Delta$:

$$\int\frac{\partial\phi}{\Delta^4}=\frac{\pi}{(1-aa)^7}\left(1+9aa+9a^4+a^6\right),$$

$$\int\frac{\partial\phi\cos.\phi}{\Delta^4}=\frac{4\pi a}{(1-aa)^7}\left(1+3aa+a^4\right),$$

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^4}=\frac{10\pi a^2}{(1-aa)^7}(1+aa),$$

$$\int\frac{\partial\phi\cos.3\phi}{\Delta^4}=\frac{20\pi a^3}{(1-aa)^7},$$

$$\int\frac{\partial\phi\cos.4\phi}{\Delta^4}=\frac{\pi a^4}{(1-aa)^7}\left(35-21aa+7a^4-a^6\right).$$

etc. etc.

§. 23. Here to be progressing further would be superfluous, since the whole operation of finding the whole general form for V can be constructed easily; truly it will be helpful, to attribute negative values to the letter *n* also, in which cases the whole integration can be performed by the methods established without any difficulty, from which it will be pleasing to see the pretty agreement of our general forms.

Case I.

when $n=-1$, and this formula of the integral is proposed

$$\int\partial\phi\cos.\lambda\phi\left[\begin{smallmatrix}\textit{from} & \phi=0\\ \textit{to} & \phi=\pi\end{smallmatrix}\right].$$

This formula is completely integrable, since there shall be

$$\int \partial\phi \cos.\lambda\phi = \tfrac{1}{\lambda}\sin.\lambda\phi,$$

which formula now vanishes on putting $\varphi = 0;$ by taking $\varphi = \pi,$ on account of the whole number λ this value will always be $= 0$, only with the case $\lambda = 0$ excepted. Indeed by regarding λ as infinitely small, there will be $\sin.\lambda\pi = \lambda\pi$, and thus in this case the value will be $= \pi$. But now the general form for the quantity V given will become :

$$\begin{aligned} \mathrm{V} = {} & \frac{(-1-\lambda)}{0}\frac{(-1+\lambda)}{\lambda} + \frac{(-1-\lambda)}{1}\frac{(-1+\lambda)}{\lambda+1}a^2 \\ & + \frac{(-1-\lambda)}{2}\frac{(-1+\lambda)}{\lambda+2}a^4 + \frac{(-1-\lambda)}{3}\frac{(-1+\lambda)}{\lambda+3}a^6 \\ & + \frac{(-1-\lambda)}{4}\frac{(-1+\lambda)}{\lambda+4}a^8 + \frac{(-1-\lambda)}{5}\frac{(-1+\lambda)}{\lambda+5}a^{10} \\ & \quad \text{etc.} \qquad\qquad \text{etc.} \end{aligned}$$

All the factors of this latter expression vanish, whenever there was $\lambda = 1$ or $\lambda > 1$, therefore because the lower numbers are greater than the upper ones, each truly will be positive; but which conclusion will not prevail, when the upper number becomes negative, as happens in the case $\lambda = 0$, which therefore only needs to be considered; but in this case the first factors become :

$$\left(\tfrac{-1}{0}\right) = 1;\ \left(\tfrac{-1}{1}\right) = -1;\ \left(\tfrac{-1}{2}\right) = +1;\ \left(\tfrac{-1}{3}\right) = -1;\ \left(\tfrac{-1}{4}\right) = +1;\ \text{etc.}$$

but truly the second factors receive the same determinations; and thus we will have

$$\mathrm{V} = 1 + aa + a^4 + a^6 + a^8 + a^{10} + \text{etc.}$$

which series since it shall be geometric, will become $\mathrm{V} = \frac{1}{1-aa}$ whereby since, on account of $n = -1$ and $\lambda = 0,$ the value sought by our general formula shall become $\pi(1-aa)\mathrm{V}$, this same value now on account of $\mathrm{V} = \frac{1}{1-aa},$ will become π, as above.

Case II.

when $n = -2$, and this formula of the integral is proposed

$$\int \partial\phi \cos.\lambda\phi(1 + aa - 2a\cos.\phi)\left[\begin{smallmatrix} from & \phi=0 \\ to & \phi=\pi \end{smallmatrix}\right].$$

By our general formula the general formula sought will be $= \pi a^{\lambda}(1-aa)^3\,\mathrm{V},$ becoming :

$$\mathrm{V} = \left(\tfrac{-2-\lambda}{0}\right)\left(\tfrac{-2+\lambda}{\lambda}\right) + \left(\tfrac{-2-\lambda}{1}\right)\left(\tfrac{-2+\lambda}{\lambda+1}\right)aa + \left(\tfrac{-2-\lambda}{2}\right)\left(\tfrac{-2+\lambda}{\lambda+2}\right)a^4$$
$$+\left(\tfrac{-2-\lambda}{3}\right)\left(\tfrac{-2+\lambda}{\lambda+3}\right)a^6 + \left(\tfrac{-2-\lambda}{4}\right)\left(\tfrac{-2+\lambda}{\lambda+4}\right)a^8 + \left(\tfrac{-2-\lambda}{5}\right)\left(\tfrac{-2+\lambda}{\lambda+5}\right)a^{10}$$
etc. etc.

Where again it is evident, if there were $\lambda = 2$ or $\lambda > 2$, all the second factors vanish, and thus there becomes $\mathrm{V} = 0$, thus so that the value of the integral sought also will vanish always, that which follows at once from the nature of the formula, clearly its integral in general, on account of

$$\cos.\phi \cos.\lambda\phi = \tfrac{1}{2}\cos.(\lambda-1)\phi + \tfrac{1}{2}\cos.(\lambda+1)\phi,$$

will become

$$\tfrac{1+aa}{\lambda}\sin.\lambda\phi - \tfrac{a}{\lambda-1}\sin.(\lambda-1)\phi = \tfrac{a}{\lambda+1}\sin.(\lambda+1)\phi,$$

which clearly vanishes in the case $\phi = \pi$ because $\lambda > 1$; from which two cases remain to be considered, the one where $\lambda = 0$, and the other where $\lambda = 1$.

1°. Let $\lambda = 0$, and the integral shall be $\pi(1-aa)^3\mathrm{V}$, were the second factors emerge for V:

$$\left(\tfrac{-2}{0}\right) = 1;\ \left(\tfrac{-2}{1}\right) = -2;\ \left(\tfrac{-2}{2}\right) = 3;\ \left(\tfrac{-2}{3}\right) = -4;\ \left(\tfrac{-2}{4}\right) = +5;\ \left(\tfrac{-2}{5}\right) = -6\ ;\text{etc.}$$

in a similar manner the first factors will become :

$$\left(\tfrac{-2}{0}\right) = 1;\ \left(\tfrac{-2}{1}\right) = -2;\ \left(\tfrac{-2}{2}\right) = 3;\ \text{etc.}$$

from which there is deduced to be:

$$\mathrm{V} = 1 + 4aa + 9a^4 + 16a^6 + 25a^8 + 36a^{10} + \text{etc.}$$

From which series requiring to be summed, thence the series V*aa* may be subtracted, and there will remain

$$\mathrm{V}(1-aa) = 1 + 3aa + 5a^4 + 7a^6 + 9a^8 + \text{etc.}$$

Each side again is multiplied by $1-aa$, and there will be produced :

$$\mathrm{V}(1-aa)^2 = 1 + 2aa + 2a^4 + 2a^6 + 2a^8 + \text{ etc.}$$

which further multiplied by $1-aa$ gives

$\mathrm{V}(1-aa)^3 = 1 + aa$, and thus $\mathrm{V} = \frac{1+aa}{(1-aa)^3}$.

Consequently the integral sought will be $=\pi(1+aa)$, that which certainly arises from the actual integration, since there shall be:

$$\int\partial\phi\left(1+aa-2a\cos.\phi\right)=\left(1+aa\right)\phi-2a\sin.\phi,$$

because on making $\phi=\pi$ it becomes $(1+aa)\pi$.

II°. Let $\lambda=1$, and the integral sought shall be $\pi a(1-aa)^3\mathrm{V}$;
where there becomes for the second factors

$$\left(\tfrac{-1}{0}\right)=-1;\ \left(\tfrac{-1}{2}\right)=+1;\ \left(\tfrac{-1}{3}\right)=-1;$$
$$\left(\tfrac{-1}{4}\right)=+1;\ \left(\tfrac{-1}{5}\right)=-1;\ \text{etc.}$$

Truly the first factors arise :

$$\left(\tfrac{-3}{0}\right)=1;\ \left(\tfrac{-3}{1}\right)=-3;\ \left(\tfrac{-3}{2}\right)=6;\ \left(\tfrac{-3}{3}\right)=-10;$$
$$\left(\tfrac{-3}{4}\right)=15;\left(\tfrac{-3}{5}\right)=-21;\ \left(\tfrac{-3}{6}\right)=28;\left(\tfrac{-3}{7}\right)=-36;\ \text{etc.}$$

therefore we will have hence :

$$\mathrm{V}=-1-3aa-6a^4-10a^6-15a^8-21a^{10}-28a^{12}-36a^{14}-\text{etc.}$$

For the summation of which each side may be multiplied by $1-aa$, and there will be produced :

$$\mathrm{V}\left(1-aa\right)=-1-2aa-3a^4-4a^6-5a^8-6a^{10}-7a^{12}-8a^{14}-\text{etc.}$$

on multiplying again by $1-aa$, there will arise :

$$\mathrm{V}\left(1-aa\right)^2=-1-aa-a^4-a^6-a^8-a^{10}-a^{12}-a^{14}-\text{etc.}$$

and on multiplying again by $1-aa$, there will be :

$\mathrm{V}\left(1-aa\right)^3=-1$, and thus $\mathrm{V}=-\frac{1}{(1-aa)^3}$, consequently the integral sought $=-\pi a$. But the same integral itself produced by integration on account of $\cos.^2\phi\ =\frac{1}{2}+\frac{1}{2}\cos.2\phi$:

$$\int\partial\phi\cos.\phi\left(1+aa-2a\cos.\phi\right)=\left(1+aa\right)\sin.\phi-a\phi-\tfrac{1}{2}a\sin.2\phi,$$

from which by substituting $\phi=\pi$, the integral arises $=-a\pi$.

Case III.

where $n=-3$, and this formula of the integral is proposed :

$$\int\partial\phi\cos.\lambda\phi\left(1+aa-2a\cos.\phi\right)^{2}\left[\begin{smallmatrix}from & \phi=0\\ to & \phi=\pi\end{smallmatrix}\right].$$

Therefore in this case from the general form the integral will become $\pi a^{\lambda}(1-aa)^{5}\mathrm{V}$, with V given by :

$$\begin{aligned}\mathrm{V}=&\left(\tfrac{-3-\lambda}{0}\right)\left(\tfrac{-3+\lambda}{\lambda}\right)+\left(\tfrac{-3-\lambda}{1}\right)\left(\tfrac{-3+\lambda}{\lambda+1}\right)a^{2}\\ &+\left(\tfrac{-3-\lambda}{2}\right)\left(\tfrac{-3+\lambda}{\lambda+2}\right)a^{4}+\left(\tfrac{-3-\lambda}{3}\right)\left(\tfrac{-3+\lambda}{\lambda+3}\right)a^{6}\\ &\qquad\text{etc.}\qquad\qquad\text{etc.}\end{aligned}$$

were the second factors clearly all vanish, when $\lambda=3$ or $\lambda>3$, therefore from which cases the whole integral vanishes, as will be apparent from any calculation being put in place : but three cases remain to be considered, for which $\lambda<3$.

I. Let $\lambda=0$, and both the first and second factors will come together, and there will be

$$\left(\tfrac{-3}{0}\right)=1;\ \left(\tfrac{-3}{1}\right)=-3;\ \left(\tfrac{-3}{2}\right)=6;\ \left(\tfrac{-3}{3}\right)=-10;$$
$$\left(\tfrac{-3}{4}\right)=15;\left(\tfrac{-3}{5}\right)=-21;\ \left(\tfrac{-3}{6}\right)=28;\ \text{etc.}$$

from which there is deduced:

$$\mathrm{V}=1+9aa+36a^{4}+100a^{6}+225\,a^{8}+441a^{10}+\text{etc.}$$

which series since finally it shall lead to constant differences, will be able to be summed in the usual similar manner, for first by multiplying by $1-aa$ it provides :

$$\mathrm{V}\left(1-aa\right)=1+8aa+27a^{4}+64a^{6}+125a^{8}+216a^{10}+343a^{12}+\text{etc.}$$

On being multiplied again by $1-aa$ it presents:

$$\mathrm{V}\left(1-aa\right)^{2}=1+7aa+19a^{4}+37a^{6}+61a^{8}+91a^{10}+127a^{12}+\text{etc.}$$

On being multiplied a third time it gives

$$\mathrm{V}\left(1-aa\right)^{3}=1+6aa+12a^{4}+18a^{6}+24a^{8}+30a^{10}+\text{etc.}$$

The fourth multiplication gives

$$\mathrm{V}(1-aa)^4 = 1+5aa+6a^4+6a^6+6a^8+6a^{10}+\text{etc.}$$

and finally

$\mathrm{V}(1-aa)^5 = 1+4aa+a^4$, thus so that there shall be $\mathrm{V}=\frac{1+4aa+a^4}{(1-aa)^5}$;

Consequently the value of the integral sought in this case will be $\pi\left(1+4aa+a^4\right)$, which agrees exactly with the integral found in the usual customary manner.

II.

Let $\lambda = 1$, in which case the first factors of V will become :

$$\left(\tfrac{-4}{0}\right)=1;\ \left(\tfrac{-4}{1}\right)=-4;\ \left(\tfrac{-4}{2}\right)=10;\ \left(\tfrac{-4}{3}\right)=-20;$$
$$\left(\tfrac{-4}{4}\right)=35;\ \left(\tfrac{-4}{5}\right)=-56;\ \left(\tfrac{-4}{6}\right)=84;\ \left(\tfrac{-4}{7}\right)=-120\ \text{etc.}$$

truly the second factors themselves are had thus :

$$\left(\tfrac{-2}{1}\right)=-2;\ \left(\tfrac{-2}{2}\right)=+3;\ \left(\tfrac{-2}{3}\right)=-4;\ \left(\tfrac{-2}{4}\right)=+5;$$
$$\left(\tfrac{-2}{5}\right)=-6;\ \left(\tfrac{-2}{6}\right)=+7;\ \left(\tfrac{-2}{7}\right)=-8;\ \left(\tfrac{-2}{8}\right)=+9;\ \text{etc.}$$

and thus

$$\mathrm{V}=-2-12aa-40a^4-100a^6-210a^8-392a^{10}-672a^{12}-1080a^{14}-\text{etc.}$$

which series since finally it shall lead to constant differences, will be able to be summed in a similar manner as before; indeed the first multiplication by $1-aa$ it gives

$$\mathrm{V}(1-aa)=-2-10a^2-28a^4-60a^6-110a^8-182a^{10}-280a^{12}-\text{etc.}$$

The second multiplication by $1-aa$ presents

$$\mathrm{V}(1-aa)^2=-2-8a^2-18a^4-32a^6-50a^8-72a^{10}-98a^{12}-\text{etc.}$$

The third multiplication gives

$$\mathrm{V}(1-aa)^3=-2-6a^2-10a^4-14a^6-18a^8-22a^{10}-26a^{12}-\text{etc.}$$

The fourth multiplication gives

$$\mathrm{V}(1-aa)^4 = -2-4a^2-4a^4-4a^6-4a^8-4a^{10}-4a^{12}-\text{etc.}$$

and finally the fifth multiplication by $1-aa$ provides

$$\mathrm{V}(1-aa)^5 = -\,2\ -2aa = -2(1+aa)\,;$$

from which there is produced $\mathrm{V} = -\frac{2(1+aa)}{(1-aa)^5}$, and thus the value of the integral sought will be $=2\pi a\left(1-aa\right)$, which agrees precisely with the integral found in the usual manner.

III. Let $\lambda = 1$, and the first factors of this V will be

$$\left(\tfrac{-5}{0}\right)=1;\ \left(\tfrac{-5}{1}\right)=-5;\ \left(\tfrac{-5}{2}\right)=15;\ \left(\tfrac{-5}{3}\right)=-35;$$
$$\left(\tfrac{-5}{4}\right)=70;\ \left(\tfrac{-5}{5}\right)=-126;\ \left(\tfrac{-5}{5}\right)=210;\ \left(\tfrac{-5}{7}\right)=-330;\ \text{etc.}$$

truly the second factors thus will be had :

$$\left(\tfrac{-1}{2}\right)=1;\ \left(\tfrac{-1}{3}\right)=-1;\ \left(\tfrac{-1}{4}\right)=1;\ \left(\tfrac{-1}{5}\right)=-1;$$
$$\left(\tfrac{-1}{6}\right)=1;\ \left(\tfrac{-1}{7}\right)=-1;\ \left(\tfrac{-1}{8}\right)=1;\ \left(\tfrac{-1}{9}\right)=-1;\ \text{etc.}$$

from which there is deduced

$$\mathrm{V} = 1+5a^2+15a^4+35a^6+70a^8+126a^{10}+210a^{12}+330a^{14}+\text{etc.}$$

which series summed in the same manner as before provides $\mathrm{V} = +\frac{1}{(1-aa)^6}$, from which the value of the integral sought is deduced to be $=\pi aa$, which agrees precisely with the integral found in the usual manner.

§. 24. But if we may compare these integrals for which *n* is a negative number with those, for which *n* is a positive number, with the ratio taken between the values of these formulas

$$\int \Delta^n\ \partial\phi\cos.\lambda\phi \text{ and } \int \frac{\partial\phi\cos.\lambda\phi}{\Delta^{n+1}},$$

which relation, if it may be investigated by more cases, supplies us with the following most noteworthy theorem.

Theorem.

§. 25. On putting for brevity $\Delta = 1 + aa - 2a\cos.\phi$, and the integrals may be extended from the limit $\phi = 0$ as far as to the limit $\phi = 180^{\circ}$, the following proportion will be found always :

$$\int \Delta^n \,\partial\phi\cos.\lambda\phi : \int \frac{\partial\phi\cos.\lambda\phi}{\Delta^{n+1}} = \left(\tfrac{n}{\lambda}\right)\left(1-aa\right)^n : \left(\tfrac{-n-1}{\lambda}\right)\left(1-aa\right)^{-n-1},$$

or if we may put

$$\frac{\Delta}{1+aa} = \frac{1+aa-2a\cos.\phi}{1-aa} = \boldsymbol{\Gamma},$$

it will become more simply:

$$\int \boldsymbol{\Gamma}^n \partial\phi\cos.\lambda\phi : \int \frac{\partial\phi\cos.\lambda\phi}{\boldsymbol{\Gamma}^{n+1}} = \left(\tfrac{n}{\lambda}\right) : \left(\tfrac{-n-1}{\lambda}\right).$$

§. 26. Thus for example if we may put $n = 2$, from the first proportion there will become:

$$\int \Delta^2 \,\partial\phi\cos.\lambda\phi : \int \frac{\partial\phi\cos.\lambda\phi}{\Delta^3} = \left(\tfrac{2}{\lambda}\right)\left(1-aa\right)^2 : \left(\tfrac{-3}{\lambda}\right)\left(1-aa\right)^{-3}$$

from which if $\lambda = 0,$ since $\left(\frac{n}{0}\right) = 1$ and $\left(\frac{-3}{0}\right) = 1,$ it will become

$$\int \Delta^2 \,\partial\phi : \int \frac{\partial\phi}{\Delta^3} = \left(1-aa\right)^2 : \frac{1}{(1-aa)^3} = 1 : \frac{1}{(1-aa)^5},$$

and thus there will be :

$$\int \frac{\partial\phi}{\Delta^3} = \frac{1}{(1-aa)^5}\int \Delta^2 \,\partial\phi.$$

Therefore since there shall be :

$$\int \Delta^2 \,\partial\phi = \pi\left(1+4aa+a^4\right), \text{ there will become}$$

$$\int \frac{\partial\phi}{\Delta^3} = \frac{\pi}{(1-aa)^4}\left(1+4aa+a^4\right).$$

§. 27. With $n = 2$ remaining, let $\lambda = 1,$ since $\left(\frac{2}{1}\right) = 2$ and $\left(\frac{-3}{1}\right) = -3,$ there will be

$$\int \Delta^2 \,\partial\phi\cos.\phi : \int \frac{\partial\phi\cos.\phi}{\Delta^3} = 2\left(1-aa\right)^2 : -3\left(1-aa\right)^{-3} = 1 : \frac{-3}{2(1-aa)^5},$$

from which there shall become:

$$\int \frac{\partial\phi\cos.\phi}{\Delta^5} = \frac{-3}{2(1-aa)^5}\int \Delta^2\ \partial\phi\cos.\phi;$$

therefore since there shall be:

$$\int \Delta^2\partial\phi\cos.\phi = -2\pi a(1+aa),\ \text{there will become},$$

$$\int \frac{\partial\phi\cos.\phi}{\Delta^2} = \frac{+3\pi a(1+aa)}{(1-aa)^5}.$$

§. 28. In a similar manner there may be taken
$\lambda = 2$, on account of $\left(\frac{2}{2}\right) = 1$ and $\left(\frac{-3}{2}\right) = 6$, there will become

$$\int \Delta^2\ \partial\phi\cos.2\phi : \int \frac{\partial\phi\cos.2\phi}{\Delta^3} = (1-aa)^2 : 6(1-aa)^{-3} = 1 : \frac{6}{(1-aa)^5},$$

from which there shall become

$$\int \frac{\partial\phi\cos.\phi}{\Delta^3} = \frac{6}{(1-aa)^5}\int \Delta^2\ \partial\phi\cos.2\phi.$$

But there was

$$\varDelta^2\ \partial\phi\cos.2\phi = \pi aa,$$

consequently

$$\int \frac{\partial\phi\cos.2\phi}{\Delta^3} = \frac{6\pi aa}{(1-aa)^5}.$$

§. 29. Since the symbol $\left(\frac{n}{\lambda}\right)$ becomes $=1$ in the case $\lambda = n$, truly in the cases in which $\lambda > n$, $\left(\frac{n}{\lambda}\right) = 0$ always, if indeed λ were a whole number, as we have assumed here always, it is evident in these cases, in which $\lambda > n$, the value of the formula $\int \varDelta^n\partial\phi\cos.\lambda\phi$ to become zero always.

§. 30. The theorem, which we have proposed here, is worthy of attention not only on account of the simplicity of all the reasoning, but also because we have concluded in several cases only by that application, nor at present by any other way may it seem to be apparent, by which its truth may be able to be demonstrated directly ; moreover theorems of this kind shall merit the greatest attention of the geometers. Also at this point we will establish certain other memorable cases of our theorem proposed initially.

Setting out the case where $\lambda = n$, and the proposed formula of the integral

$$\int \frac{\partial\phi \cos.n\phi}{\Delta^{n+1}}.$$

From the general form the integral for this case will become $\frac{\pi a^n}{(1-aa)^{2n+1}}\mathrm{V}$,

with V being given by :

$$\left(\tfrac{0}{0}\right)\left(\tfrac{2n}{n}\right) + \left(\tfrac{0}{1}\right)\left(\tfrac{2n}{n+1}\right)aa + \left(\tfrac{0}{2}\right)\left(\tfrac{2n}{n+2}\right)a^4 + \text{etc.}$$

where evidently all the terms vanish beyond the first, thus so that there shall become $\mathrm{V} = \left(\frac{2n}{n}\right)$, and thus our integral

$$\int \frac{\partial\phi \cos.n\phi}{\Delta^{n+1}} = \frac{\pi a^n}{(1-aa)^{2n+1}} \cdot \left(\frac{2n}{n}\right);$$

where it may be observed, the values of the characters [symbols] $\left(\frac{2n}{n}\right)$ for the various values of the number *n* to be themselves had in the following manner:

n	0, 1, 2, 3, 4, 5, 6, 7 etc.
$\left(\frac{2n}{n}\right)$	1, 2, 6, 20, 70, 252, 924, 3432 etc.

which series may be continued most easily by these factors

$$\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdot\frac{18}{5}\cdot\frac{22}{6}\cdot\frac{26}{7} \text{ etc.}$$

Truly the last theorem found applied to this case will produce this proportion

$$\int \varDelta^n \partial\phi \cos.n\phi : \int \frac{\partial\phi \cos.n\phi}{\Delta^{n+1}} = (1-aa)^n : \left(\frac{-1-n}{n}\right)(1-aa)^{-n-1},$$

from which there becomes

$$\int \varDelta^n \partial\phi \cos.n\phi = \frac{\pi a^n}{\left(\frac{-n-1}{n}\right)} \cdot \left(\frac{2n}{n}\right) = \left(\frac{2n}{n}\right)\pi a^n : \left(\frac{-n-1}{n}\right);$$

where the values of the characters $\left(\frac{-n-1}{n}\right)$ for the various values of n to be:

n	0,	1,	2,	3,	4,	5,	6
$\left(\frac{-n-1}{n}\right)$	−1,	−2,	6,	−20,	70,	−252,	924, etc.

from which it is apparent $\left(\frac{-n-1}{n}\right)=\pm\left(\frac{2n}{n}\right)$, while the upper sign prevails when n is an even number, truly the opposite lower sign, when n is an odd number; hence therefore there will become :

$$\int \varDelta^n \partial\phi \cos.n\phi = \pm\pi a^n.$$

With these noted we may establish the simpler cases for each integral formula :

$n=0$	$\int\frac{\partial\phi}{\varDelta}$	$=\frac{\pi}{1-aa}$	$\int\partial\phi$	$=+\pi$
$n=1$	$\int\frac{\partial\phi\cos.\phi}{\varDelta^2}$	$=\frac{2\pi a}{(1-aa)^3}$	$\int\varDelta\ \partial\phi\cos.\phi$	$=-\pi a$
$n=2$	$\int\frac{\partial\phi\cos.2\phi}{\varDelta^3}$	$=\frac{2\pi a^2}{(1-aa)^5}$	$\int\varDelta^2\partial\phi\cos.2\phi$	$=+\pi a^2$
$n=3$	$\int\frac{\partial\phi\cos.3\phi}{\varDelta^4}$	$=\frac{20\pi a^3}{(1-aa)^7}$	$\int\varDelta^3\partial\phi\cos.3\phi$	$=-\pi a^3$
$n=4$	$\int\frac{\partial\phi\cos.4\phi}{\varDelta^5}$	$=\frac{70\pi a^4}{(1-aa)^9}$	$\int\varDelta^4\partial\phi\cos.4\phi$	$=+\pi a^4$
$n=5$	$\int\frac{\partial\phi\cos.5\phi}{\varDelta^6}$	$=\frac{252\pi a^5}{(1-aa)^{11}}$	$\int\varDelta^5\partial\phi\cos.5\phi$	$=-\pi a^5$
$n=6$	$\int\frac{\partial\phi\cos.6\phi}{\varDelta^7}$	$=\frac{924\pi a^6}{(1-aa)^{13}}$	$\int\varDelta^6\partial\phi\cos.6\phi$	$=+\pi a^6$
etc.	etc.		etc.	

Here it happens to be especially noteworthy, because in these cases $\lambda=n$ as the integrals are expressed more succinctly; but now we are going to consider other cases, in which the letters λ are given successive values 0, 1 , 2, 3, etc.

Establishing the case where $\lambda=0$, and the proposed formula of the integral :

$$\int\frac{\partial\phi}{\varDelta^{n+1}}.$$

§. 31. Since here there shall be $\lambda=0$, the integral sought from our formula will be $\frac{\pi}{(1-aa)^{2n+1}}\mathrm{V}$, with V being

$$=\left(\tfrac{n}{0}\right)+\left(\tfrac{n}{1}\right)^2 aa+\left(\tfrac{n}{2}\right)^2 a^4+\left(\tfrac{n}{3}\right)^2 a^6+\left(\tfrac{n}{4}\right)^2 a^8+\text{etc.}$$

likewise truly hence it will be possible also to assign the value of this formula $\int \varDelta^n \partial\phi,$ since there shall be:

$$\int \varDelta^n \partial\phi : \int \frac{\partial\phi}{\varDelta^{n+1}} = (1-aa)^n : (1-aa)^{-n-1} = (1-aa)^{2n+1} : 1,$$

from which proportion it is deduced : $\int \varDelta^n \partial\phi = \pi.\text{V}.$

Therefore running through the simpler cases for the exponent n, which we add below in the following table :

$$n=0\begin{cases}\int \frac{\partial\phi}{\varDelta}=\frac{\pi}{1-aa}\\ \int \partial\phi=\pi\end{cases}$$

$$n=1\begin{cases}\int \frac{\partial\phi}{\varDelta^2}=\frac{\pi}{(1-aa)^3}(1+aa)\\ \int \varDelta\partial\phi=\pi(1+aa)\end{cases}$$

$$n=2\begin{cases}\int \frac{\partial\phi}{\varDelta^3}=\frac{\pi}{(1-aa)^5}\left(1+2^2aa+a^4\right)\\ \int \varDelta^2\partial\phi=\pi\left(1+2^2aa+a^4\right)\end{cases}$$

$$n=3\begin{cases}\int \frac{\partial\phi}{\varDelta^4}=\frac{\pi}{(1-aa)^7}\left(1+3^2aa+3^2a^4+a^6\right)\\ \int \varDelta^3\partial\phi=\pi\left(1+3^2aa+3^2a^4+a^6\right)\end{cases}$$

$$n=4\begin{cases}\int \frac{\partial\phi}{\varDelta^5}=\frac{\pi}{(1-aa)^9}\left(1+4^2aa+6^2a^4+4^2a^6+a^8\right)\\ \int \varDelta^4\partial\phi=\pi\left(1+4^2aa+6^2a^4+4^2a^6+a^8\right)\end{cases}$$

etc. etc.

Establishment of the cases in which $\lambda=1$, and the proposed formula of the integral

$$\int \frac{\partial\phi\cos.\phi}{\Delta^{n+1}}.$$

§. 32. Therefore in this case the integral sought will be :

$$\frac{\pi a}{(1-aa)^{2n+1}}.\mathrm{V}$$

with V being

$$=\left(\frac{n-1}{0}\right)\left(\frac{n+1}{1}\right)+\left(\frac{n-1}{1}\right)\left(\frac{n+1}{2}\right)aa+\left(\frac{n-1}{2}\right)\left(\frac{n+1}{3}\right)a^4+\left(\frac{n-1}{3}\right)\left(\frac{n+1}{4}\right)a^6+$$
$$\left(\frac{n-1}{4}\right)\left(\frac{n+1}{5}\right)a^8+\left(\frac{n-1}{5}\right)\left(\frac{n+1}{6}\right)a^{10}+\text{etc.}$$

truly since on account of $\lambda=1$ there becomes

$$\int \varDelta^n \partial\phi \cos.\phi : \int \frac{\partial\phi \cos.\phi}{\varDelta^{n+1}} = n\left(1-aa\right)^n : -\left(n+1\right)\left(1-aa\right)^{-n-1}$$

from which there becomes

$$\int \varDelta^n \partial\phi \cos.\phi = -\frac{n}{n+1}\pi a\mathrm{V}.$$

Therefore, for the simpler cases we adjoin the following values of *n* :

$$n=0\begin{cases}\int\frac{\partial\phi\cos.\phi}{\Delta}=\frac{\pi a}{1-aa}\\ \int\partial\phi\cos.\phi=0\end{cases}$$

$$n=1\begin{cases}\int\frac{\partial\phi\cos.\phi}{\Delta^2}=\frac{2\pi a}{(1-aa)^3}\\ \int\Delta\partial\phi\cos.\phi=-\pi a\end{cases}$$

$$n=2\begin{cases}\int\frac{\partial\phi\cos.\phi}{\Delta^3}=\frac{\pi a}{(1-aa)^5}\left(1.3+1.3aa\right)\\ \int\Delta^2\partial\phi\cos.\phi=-\frac{2}{3}\pi a\left(1.3+1.3aa\right)\end{cases}$$

$$n=3\begin{cases}\int\frac{\partial\phi\cos.\phi}{\Delta^4}=\frac{\pi a}{(1-aa)^7}\left(1.4+2.6aa+1.4a^4\right)\\ \int\Delta^3\partial\phi\cos.\phi=-\frac{3}{4}\pi a\left(1.4+2.6aa+1.4a^4\right)\end{cases}$$

$$n=4\begin{cases}\int\frac{\partial\phi}{\Delta^6}\cos.\phi=\frac{\pi a}{(1-aa)^9}\left(1.5+3.10aa+3.10a^4+1.5a^6\right)\\ \int\Delta^4\partial\phi\cos.\phi=-\frac{4}{5}\pi a\left(1.5+3.10aa+3.10a^4+1.5a^6\right)\end{cases}$$

$$n=5\begin{cases}\int\frac{\partial\phi}{\Delta^6}\cos.\phi=\frac{\pi a}{(1-aa)^{11}}\left(1.6+4.15aa+6.20a^4+4.15a^6+1.6a^8\right)\\ \int\Delta^5\partial\phi\cos.\phi=-\frac{5}{6}\pi\left(1.6+\text{etc.}\right)\end{cases}$$

$$n=6\begin{cases}\int\frac{\partial\phi}{\Delta^7}\cos.\phi=\frac{\pi}{(1-aa)^{13}}\left(1.7+5.21aa+10.35a^4+10.35a^6+\text{etc.}\right)\\ \int\Delta^6\partial\phi\cos.\phi=-\frac{6}{7}\pi a\left(1.7+\text{etc.}\right)\end{cases}$$

Establishing the cases in which $\lambda=2$, and the formulas proposed of the integral

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^{n+1}}.$$

§. 33. Therefore in this case the integral sought will be

$$\frac{\pi a^2}{(1-aa)^{2n+1}}.\text{V}$$

with V being equal to

$$\left(\tfrac{n-2}{0}\right)\left(\tfrac{n+2}{2}\right)+\left(\tfrac{n-2}{1}\right)\left(\tfrac{n+2}{3}\right)aa+\left(\tfrac{n-2}{2}\right)\left(\tfrac{n+2}{4}\right)a^4+\left(\tfrac{n-2}{3}\right)\left(\tfrac{n+2}{5}\right)a^6$$
$$+\left(\tfrac{n-2}{4}\right)\left(\tfrac{n+2}{6}\right)a^8+\text{etc.}$$

then truly, the other form will be

$$\int \varDelta^n \partial\phi \cos.2\phi = \frac{n(n-1)}{(n+1)(n+2)}\pi aa\text{V}.$$

Therefore we may run through as far as the simpler cases, and because the integration of the formulas $\int \varDelta^n \partial\phi \cos.2\phi$ is apparent at once from the last formula, it would be superfluous to add proofs to these integrations

$$n=0:\int\frac{\partial\phi\cos.2\phi}{\varDelta}=\frac{\pi aa}{1-aa}$$
$$n=1:\int\frac{\partial\phi\cos.2\phi}{\varDelta^2}=\frac{\pi aa}{(1-aa)^3}(1.3-1.1aa)$$
$$n=2:\int\frac{\partial\phi\cos.2\phi}{\varDelta^3}=\frac{\pi aa}{(1-aa)^5}(1.6)$$
$$n=3:\int\frac{\partial\phi\cos.2\phi}{\varDelta^4}=\frac{\pi aa}{(1-aa)^7}(1.10+1.10aa)$$
$$n=4:\int\frac{\partial\phi}{\varDelta^6}\cos.2\phi=\frac{\pi aa}{(1-aa)^9}\left(1.15+2.20aa+1.15a^4\right)$$
$$n=5:\int\frac{\partial\phi}{\varDelta^6}\cos.2\phi=\frac{\pi aa}{(1-aa)^{11}}\left(1.21+3.35a^2+3.35a^4+1.21a^6\right)$$
$$n=6:\int\frac{\partial\phi}{\varDelta^7}\cos.2\phi=\frac{\pi aa}{(1-aa)^{13}}\left(1.28+4.56aa+6.70a^4+4.56a^6+1.28a^8\right)$$
etc. etc.

The establishment of the cases in which $\lambda=3$ and the formula of the proposed integral

$$\int\frac{\partial\phi\cos.3\phi}{\Delta^{n+1}}.$$

§.34. Therefore in this case the integral will become :

$$\frac{\pi a^3}{(1-aa)^{2n+1}}.\text{V},$$

with V being equal to

$$\left(\tfrac{n-3}{0}\right)\left(\tfrac{n+3}{3}\right)+\left(\tfrac{n-3}{1}\right)\left(\tfrac{n+3}{4}\right)aa+\left(\tfrac{n-3}{2}\right)\left(\tfrac{n+3}{5}\right)a^4+\left(\tfrac{n-3}{3}\right)\left(\tfrac{n+3}{6}\right)a^6+\text{etc.}$$

moreover, we will have for the other formula :

$$\int \Delta^n \partial\phi \cos.3\phi = -\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\pi a^3 \mathrm{V}.$$

Therefore for the following special cases we will have the following table :

$$n=0:\int \frac{\partial\phi\cos.3\phi}{\Delta} = \frac{\pi a^3}{1-aa}$$

$$n=1:\int \frac{\partial\phi\cos.3\phi}{\Delta^2} = \frac{2\pi a^3}{(1-aa)^3}(1.4-2.1aa)$$

$$n=2:\int \frac{\partial\phi\cos.3\phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}(1.10-1.5aa)$$

$$n=3:\int \frac{\partial\phi\cos.3\phi}{\Delta^4} = \frac{\pi a^3}{(1-aa)^7}(1.20)$$

$$n=4:\int \frac{\partial\phi\cos.3\phi}{\Delta^6} = \frac{\pi a^3}{(1-aa)^9}(1.35+1.35aa)$$

$$n=5:\int \frac{\partial\phi\cos.3\phi}{\Delta^6} = \frac{\pi a^3}{(1-aa)^{11}}\left(1.56+2.70aa+1.56a^4\right)$$

$$n=6:\int \frac{\partial\phi\cos.3\phi}{\Delta^7} = \frac{\pi a^3}{(1-aa)^{13}}\left(1.84+3.126aa+3.126a^4+1.84a^6\right)$$

etc. etc.

An observation about the values of λ.

§. 35. Now initially we have advised, it would be required to take positive whole numbers only for the letter λ , from which the general condition of our question is not restricted since always there shall be $\cos.-\lambda\phi = \cos.\lambda\phi$. Yet meanwhile this huge paradox arises, because the solutions found above may emerge false, when negative values are attributed to λ ; so that which may appear clearer, we will consider the case $n=0$; for which we have found above

$$\int \frac{\partial\phi\cos.\lambda\phi}{\Delta} = \frac{\pi a^\lambda}{1-aa},$$

from which it may be seen it must follow, in the case $\lambda=-i$ to become

$$\int \frac{\partial\phi\cos.i\phi}{\Delta} = \frac{\pi}{a^i(1-aa)},$$

but which evidently is false, since truly the integral everywhere shall be $\frac{\pi a^i}{1-aa}$, and likewise if there should be $\lambda = +i$. But truly this same restriction is apparent only, and our general solution is agreed is agreed to be in now manner less true, even if the letters λ may be attributed negative values, provided that they were whole; since we have assumed always, in the case $\phi = \pi$ to be always $\sin.\lambda\phi = 0$; this therefore will be the reward of needing to be shown more clearly .

§. 36. But it will suffice to consider the case where $n = 0$, for which our general solution gives

$$\int \frac{\partial\phi \cos.\lambda\phi}{\Delta} = \frac{\pi a^{\lambda}}{1-aa}\mathrm{V},$$

with V being equal to

$$\left(\tfrac{-\lambda}{0}\right)\left(\tfrac{\lambda}{\lambda}\right)+\left(\tfrac{-\lambda}{1}\right)\left(\tfrac{\lambda}{\lambda+1}\right)aa+\left(\tfrac{-\lambda}{2}\right)\left(\tfrac{\lambda}{\lambda+2}\right)a^4+\left(\tfrac{-\lambda}{3}\right)\left(\tfrac{\lambda}{\lambda+3}\right)a^6+\text{etc.}$$

Only the first part of this expression remains, when λ is a positive integer, since the formulas $\left(\tfrac{\lambda}{\lambda+1}\right),\left(\tfrac{\lambda}{\lambda+2}\right),\left(\tfrac{\lambda}{\lambda+3}\right),$etc. vanish ; but otherwise the it will be longer by far, when a negative number is assumed for λ, just as if we may put $\lambda = -i$ then there will become :

$$\mathrm{V}=\left(\tfrac{i}{0}\right)\left(\tfrac{-i}{-i}\right)+\left(\tfrac{i}{1}\right)\left(\tfrac{-i}{1-i}\right)aa+\left(\tfrac{i}{2}\right)\left(\tfrac{-i}{2-i}\right)a^4+\left(\tfrac{i}{3}\right)\left(\tfrac{-i}{3-i}\right)a^6+\text{etc.}$$

where it may be observed, the values of all the characters vanish as long as the denominator is negative ; truly because the denominators increase continually, at last they will become positive, and thus they will show the determined values. In order to be showing this, we may put initially $\lambda = -1$ or $i = +1$, and there will become $\mathrm{V} = -aa$, where without doubt the first term is $= 0$, truly the second

$$\left(\tfrac{1}{1}\right)\left(\tfrac{+1}{0}\right)aa = aa,$$

Therefore since there shall be $\mathrm{V} = aa$ in the case $\lambda = -1$, our formula provides this integral

$$\int \frac{\partial\phi \cos.-\phi}{\Delta} = \frac{\pi a^{-1}}{1-aa}.aa = \frac{\pi a}{1-aa},$$

in short that is in agreement.

§. 37. Now we may take $\lambda = -2$ or $i = 2$, with $n = 0$ remaining, and there will become

$$\mathrm{V}=\left(\tfrac{2}{0}\right)\left(\tfrac{-2}{-2}\right)+\left(\tfrac{2}{1}\right)\left(\tfrac{-2}{-1}\right)aa+\left(\tfrac{2}{2}\right)\left(\tfrac{-2}{0}\right)a^4,$$

where evidently all the following terms vanish; but on account of the first factors the two first terms vanish also because of the negative denominators ; but the third term on account of $\left(\tfrac{-2}{0}\right)=1$ gives $\mathrm{V}=a^4$, consequently in the case $\lambda=-2$ we will have

$$\int\frac{\partial\phi\cos.-2\phi}{\Delta}=\frac{\pi a^{-2}}{1-aa}\cdot a^4=\frac{\pi aa}{1-aa},$$

and precisely as we have found for $\int\frac{\partial\phi\cos.2\phi}{\Delta}$.

§. 38. It is understood readily in a similar manner, in the case $\lambda=-3$, $\mathrm{V}=a^6$ is going to be produces , and in the same manner for the case $\lambda=-4$, $\mathrm{V}=a^8$ will be found, and thus in general for the case $\lambda=-i$ there will be found $\mathrm{V}=a^{2i}$, and thus the integral of this formula $\int\frac{\partial\phi\cos.-i\phi}{\Delta}$ will become

$$\tfrac{\pi a^{-i}}{1-aa}.a^{2i}=\tfrac{\pi a^{i}}{1-aa},$$

which is itself the integral of the formula $\int\frac{\partial\phi\cos.i\phi}{\Delta}$, as the nature of the integral demands.

§. 39. Moreover such outstanding agreement will be found for all the values of *n*. For example, let $n=2$, and on integrating our equation

$$\int\frac{\partial\phi\cos.\lambda\phi}{\Delta^3}=\frac{\pi a^{\lambda}}{(1-aa)^5}.\mathrm{V}$$

with V being equal to

$$\left(\tfrac{2-\lambda}{0}\right)\left(\tfrac{2+\lambda}{\lambda}\right)+\left(\tfrac{2-\lambda}{1}\right)\left(\tfrac{2+\lambda}{\lambda+1}\right)aa+\left(\tfrac{2-\lambda}{2}\right)\left(\tfrac{2+\lambda}{\lambda+2}\right)a^4+\text{etc.}$$

whereby on taking $\lambda=-3$, so that our form shall become

$$\int\frac{\partial\phi\cos.-3\phi}{\Delta^3}=\frac{\pi a^{-3}}{(1-aa)^5}.\mathrm{V},$$

with V being equal to

$$\left(\tfrac{5}{0}\right)\left(\tfrac{-1}{-3}\right)+\left(\tfrac{5}{1}\right)\left(\tfrac{-1}{-2}\right)aa+\left(\tfrac{5}{2}\right)\left(\tfrac{-1}{-1}\right)a^4+\left(\tfrac{5}{3}\right)\left(\tfrac{-1}{0}\right)a^6$$
$$+\left(\tfrac{5}{4}\right)\left(\tfrac{-1}{1}\right)a^8+\left(\tfrac{5}{5}\right)\left(\tfrac{-1}{2}\right)a^{10},$$

where the first three members vanish, but the following on account of

$$\left(\tfrac{-1}{0}\right)=1,\ \left(\tfrac{-1}{1}\right)=-1,\ \left(\tfrac{-1}{2}\right)=1,$$

will become

$$V=10a^6-5a^8+a^{10}=a^6\left(10-5aa+a^4\right),$$

consequently our integral will become

$$\int\frac{\partial\phi\cos.-3\phi}{\varDelta^3}=\frac{\pi a^3}{(1-aa)^5}\left(10-5aa+a^4\right),$$

just as we have found above for the case $\int\frac{\partial\phi\cos.3\phi}{\varDelta^3}$; but such an agreement will be always understood to be had.

SUPPLEMENTUM IV*a*.

AD TOM. I. CAP.V.
DE
INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM.

1) De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet.

M.S. Academiae exhibit. die 5. *Maii* 1777.

§. 1. Quae jam saepius sum commentatus de formulis differentialibus irrationalibus, quae nulla substitutione ad rationalitatem revocari possunt, nihilo vero minus integrationem per logarithmos et arcus circulares admittunt: etiam transferri possunt ad ejusmodi formulas angulares, quae sinus et cosinus cujuspiam anguli involvunt. Forma autem generalis hujusmodi differentialium, quae hoc modo tractari possunt, sequenti modo repraesentari potest: denotante ϕ angulum quemcunque, designet $\varPhi$ functionem quamcunque rationalem ipsius tang.$n\phi$, atque inveni istam formulam

$$=\boldsymbol{\varPhi}\cdot\frac{\partial\phi(f\sin.\lambda\phi+g\cos.\lambda\phi)}{\sqrt[n]{(a\sin.n\phi+b\cos.n\phi)^\lambda}}$$

semper per logarithmnos et arcus circulares integrari posse, id quod a casibus simplicioribus inchoando in sequentibus problematibus ostendere constitui.

Problema I.

§2. *Proposita formula differentiali* $\frac{\partial\phi\cos.\phi}{\sqrt[n]{\cos.n\phi}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Quoniam mihi quidem alia adhuc via non patet istud praestandi, nisi per imaginaria procedendo, formulam $\sqrt{-1}$ littera *i* in posterum designabo, ita ut sit $ii=-1$, ideoque $\frac{1}{i}=-i.$ Jam ante omnia in numeratore nostrae formulae loco $\cos.\varphi$ has duas partes substituamus

$$\tfrac{1}{2}(\cos.\phi+i\sin.\phi)+\tfrac{1}{2}(\cos.\phi-i\sin.\phi),$$

atque ipsam formulam propositam per duas hujusmodi partes repraesentemus, quae sint

$$\partial p=\frac{\partial\phi(\cos.\phi+i\sin.\phi)}{\sqrt[n]{\cos.n\phi}} \text{ et } \partial q=\frac{\partial\phi(\cos.\phi-i\sin.\phi)}{\sqrt[n]{\cos.n\phi}}$$

ita ut ipsa formula nostra proposita sit $\frac{1}{2}\partial p+\frac{1}{2}\partial q$, ideoque ejus integrale $\frac{p+q}{2}$.

§. 3. Nunc ambas istas partes seorsim sequenti modo tractemus. Pro formula scilicet priore

$$\partial p=\frac{\partial\phi(\cos.\phi+i\sin.\phi)}{\sqrt[n]{\cos.n\phi}} \text{ statuamus } \frac{\cos.\phi+i\sin.\phi}{\sqrt[n]{\cos.n\phi}}=x,$$

ut sit $\partial p=x\partial\phi$, ac sumtis potestatibus exponentis *n* habebimus

$$x^n=\frac{(\cos.\phi+i\sin.\phi)^n}{\cos.n\phi}.$$

Constat autem esse

$$(\cos.\phi+i\sin.\phi)^n=\cos.n\phi+i\sin.n\phi,$$

sicque erit $x^n=1+i\,\text{tang}.n\phi$, unde colligitur

$$\text{tang}.n\phi=\frac{x^n-1}{i}=i\left(1-x^n\right):$$

hinc cum posito in genere $\text{tang}.\omega=\text{Z}$, sit $\partial\omega=\frac{\partial Z}{1+ZZ}$, erit pro nostro casu

$$n\partial\phi=\frac{-nix^{n-1}\partial x}{1+ii-2iix^n+iix^{2n}},$$

quae formula ob $ii=-1$ transmutatur in hanc

$$\partial\phi=\frac{-ix^{n-1}\partial x}{2x^n-x^{2n}},$$

hincque ipsa formula

$$\partial p=x\partial\phi=\frac{-i\partial x}{2-x^n},$$

quae cum sit rationalis, ejus integratio nulli difficultati est subjecta.

§. 4. Quodsi jam simili modo pro altera formula

$$\partial q = \frac{\partial\phi(\cos.\phi - i\sin.\phi)}{\sqrt[n]{\cos.n\phi}} \text{ statuamus } \frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = y,$$

ut sit $\partial q = y\partial\phi$, per similes operationes, quae a praecedentibus in hoc solo discrepabunt, quod littera i negative sit accipienda, resultabit ista transformatio $\partial q = \frac{i\partial y}{2-y^n}$, quae cum priori prorsus sit similis, eadem integratione totum negotium conficietur, et pro ipso integrali quaesito habebimus

$$p + q = -i\int\frac{\partial x}{2-x^n} + i\int\frac{\partial y}{2-y^n}.$$

§. 5. Constat autem integralia talium formularum ex duplicis generis partibus, scilicet logarithmicis et arcubus circularibus constare, ita ut illarum forma generalis sit $f\,l(\alpha + \beta x + \gamma xx)$, harum vera g Arc.tang.$(\delta + \varepsilon x)$. Quare cum hic differentia inter binas formulas integrales similes occurrat, ex singulis partibus logarithmicis orietur talis forma $-i\,f\,l\frac{\alpha+\beta x+\gamma xx}{\alpha+\beta y+\gamma yy}$, ubi tam x quam y imaginaria involvit, hanc ob rem ponamus brevitatis gratia
$x = r + is$ et $y = r - is$, ubi erit

$$r = \frac{\cos.\phi}{\sqrt[n]{\cos.n\phi}} \text{ et } s = \frac{\sin.\phi}{\sqrt[n]{\cos.n\phi}}$$

his igitur valoribus substitutis, quaelibet pars logarithmica erit

$$-i\,f\,l\frac{\alpha+\beta r+\gamma rr-\gamma ss+i(\beta s+2\gamma rs)}{\alpha+\beta r+\gamma rr-\gamma ss-i(\beta s+2\gamma rs)}.$$

§. 6. Loco hujus expressionis prolixioris scribamus brevitatis gratia $-i\,f\,l\frac{t+iu}{t-iu}$, ita ut sit

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta s + 2\gamma rs,$$

sicque etiam hi valores per angulum ϕ innotescunt. Quoniam igitur jam saepius est demonstratum, esse

$$l\frac{t+u\sqrt{-1}}{t-u\sqrt{-1}} = 2\sqrt{-1}.\text{Arc.tang}\frac{u}{t},$$

ista portio integralis erit $= +2f$.Arc.tang$\frac{u}{t}$, quae ergo penitus est realis, dum imaginaria se mutua sustulerunt, ita ut quaelibet portio logarithmica imaginaria producat arcum circularem realem.

§. 7. Simili modo conjungamus in genere binos arcus circulares per integrationem prodeuntes, qui ex forma assumta erunt

$$-ig \text{ Arc.tang.}(\delta + \varepsilon x) + ig\text{Arc.tang.}(\delta + \varepsilon y),$$

quae forma ita in unum arcum contrahetur, qui erit

$$-ig\ \text{Arc.tang.}(\delta+\varepsilon x)+ig\text{Arc.tang.}(\delta+\varepsilon y)$$

quae introductis valoribus assumtis $x=r+is$ et $y=r-is$, induet hanc formam

$$-ig\ \text{Arc.tang.}\frac{2i\varepsilon s}{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)}.$$

Cum igitur in genere sit

$$\text{Arc.tang}\ v\sqrt{-1}=\frac{\sqrt{-1}}{2}l\frac{1+v}{1-v},$$

ista pars circularis transformabitur in sequentem logarithmum realem

$$\frac{g}{2}\ l\frac{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)+2\varepsilon s}{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)-2\varepsilon s}:$$

hoc ergo modo sumendis omnium integralium partibus, tandem obtinebitur integrale quaesitum per meros logarithmos et arcus circulares realiter expressum.

Problema. 2.

§. 8. *Proposita formula differentiali* $\frac{\partial\varphi\sin.\phi}{\sqrt[n]{\cos.n\phi}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Hic loco $\sin.\phi$ scribatur haec forma duabus constans partibus

$$\frac{1}{2i}(\cos.\phi+i\sin.\phi)-\frac{1}{2i}(\cos.\phi-i\sin.\phi),$$

ac formula proposita resolvatur in has partes

$$\partial p=\frac{\partial\phi(\cos.\phi+i\sin.\phi)}{\sqrt[n]{\cos.n\phi}}\ \text{ et }\ \partial q=\frac{\partial\phi(\cos.\phi-i\sin.\phi)}{\sqrt[n]{\cos.n\phi}},$$

ita ut ipsa formula proposita jam fiat $\frac{\partial p-\partial q}{2i}$, ideoque ipsum integrale quaesitum $\frac{p-q}{2i}$

§. 9. Quodsi jam rursus ut ante statuamus

$$\frac{\cos.\phi+i\sin.\phi}{\sqrt[n]{\cos.n\phi}}=x\ \text{ et }\ \frac{\cos.\phi-i\sin.\phi}{\sqrt[n]{\cos.n\phi}}=y,$$

reperietur ut supra

$$\partial p = -\frac{i\partial x}{2-x^n} \text{ et } \partial q = \frac{i\partial y}{2-y^n};$$

unde ergo fiet ipsum integrale quaesitum

$$\frac{p-q}{2i} = -\frac{1}{2}\int\frac{\partial x}{2-x^n} - \frac{1}{2}\int\frac{\partial y}{2-y^n},$$

ubi coefficientes evaserunt reales.

§. 10. Consideremus nunc ex forma integrali utriusque partis quamlibet portionem logarithmicam, quae sit $f\,l(\alpha+\beta x+\gamma xx)$, hincque pro integrali quaesito ex utraque parte orietur

$$-\tfrac{1}{2}f\,l(\alpha+\beta x+\gamma xx)-\tfrac{1}{2}f\,l(\alpha+\beta y+\gamma yy).$$

Quodsi jam ut supra ponamus brevitatis gratia $x = r+is$ et $y=r-is$, tum vero

$$t=\alpha+\beta r+\gamma rr-\gamma ss \text{ et } u=\beta s+2\gamma rs,$$

hi ambo logarithmi evadunt

$$= -\tfrac{1}{2}f\,l(t+iu)-\tfrac{1}{2}f\,l(t-iu),$$

qui contrahuntur in $-\tfrac{1}{2}f\,l(tt+uu)$, quae expressio jam est realis, neque ulla ulteriori reductione indiget.

§. 11. Eadem modo binae partes circulares ex integratione oriundae.

$$-\tfrac{1}{2}g\text{Arc.tang.}(\delta+\varepsilon x)-\tfrac{1}{2}g\text{Arc.tang.}(\delta+\varepsilon y),$$

quae per r et s ita repraesentantur

$$-\tfrac{1}{2}g\left[\text{Arc.tang.}(\delta+\varepsilon r+i\varepsilon s)+\text{Arc.tang.}(\delta+\varepsilon r-i\varepsilon s)\right],$$

qui duo arcus ita in unum contrahuntur

$$-\tfrac{1}{2}g\text{Arc.tang.}\frac{2\delta+2\varepsilon r}{1-(\delta+\varepsilon r)^2-\varepsilon\varepsilon ss},$$

quae expressio jam ultra prodiit realis.

Problema 3.

§.12. *Proposita formula differentiali* $\dfrac{\partial\phi\cos.\lambda\phi}{\sqrt[n]{\cos.n\phi^{\lambda}}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Cum sit

$$\cos.\lambda\phi = \tfrac{1}{2}\left(\cos.\phi + i\sin.\phi\right)^{\lambda} + \tfrac{1}{2}\left(\cos.\phi - i\sin.\phi\right)^{\lambda},$$

formula proposita in has duas partes discerpatur

$$\partial p = \frac{\partial\phi\left(\cos.\phi + i\sin.\phi\right)^{\lambda}}{\sqrt[n]{\cos.n\phi^{\lambda}}} \text{ et } \partial q = \frac{\partial\phi\left(\cos.\phi - i\sin.\phi\right)^{\lambda}}{\sqrt[n]{\cos.n\phi^{\lambda}}}$$

ita ut integrale quaesitum fiat $\frac{p+q}{2}$.

§.13. Jam statuamus, ut ante fecimus,

$$\frac{\cos.\phi + i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = x \text{ et } \frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = y,$$

quo facto fiet $\partial p = x^{\lambda}\partial\phi$ et $\partial q = y^{\lambda}\partial\phi$. Calculo autem ut supra expedite obtinebimus

$$\partial\phi = -\frac{ix^{n-1}\partial x}{2x^{n} - x^{2n}}, \text{ hincque } \partial p = -\frac{ix^{\lambda-1}\partial x}{2 - x^{n}};$$

similique modo erit $\partial q = -\frac{iy^{\lambda-1}\partial y}{2 - y^{n}}$, sicque totum integrale quaesitum erit

$$= -\frac{i}{2}\int \frac{x^{\lambda-1}\partial x}{2 - x^{n}} + \frac{i}{2}\int \frac{y^{\lambda-1}\partial y}{2 - y^{n}}.$$

§. 14.Quoniam haec duo integralia sibi sunt similia, ideoque similes partes tam logarithmicas quam circulares complectuntur , ex parte logarithmica, quae sit $f\,l\left(\alpha + \beta x + \gamma xx\right)$, ponendo ut supra $x = r + is$ et $y = r - is$, tum vero

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta s + 2\gamma rs,$$

hinc primo ista pars logarithmica colligitur $-i\,f\,l\frac{t+iu}{t-iu}$, quae cum sit imaginaria reducitur ad hunc arcum circularem realem $= 2f$ Arc.tang.$\frac{u}{t}$: simili modo si forma arcus circularis ex integratione oriunda fuerit $-g$ Arc. tang.$\left(\delta + \varepsilon x\right)$, ex partibus circularibus primo oritur sequens arcus imaginarius

$$-ig\text{Arc. tang.}\frac{2i\varepsilon s}{1 + \delta\delta + 2\varepsilon\delta r + \varepsilon\varepsilon\left(rr + ss\right)},$$

qui denique ad hunc logarithmum realem revocatur

$$\frac{g}{2}l.\frac{1+\delta\delta+2\delta\varepsilon r+\varepsilon\varepsilon(rr+ss)+2\varepsilon s}{1+\delta\delta+2\delta\varepsilon r+\varepsilon\varepsilon(rr+ss)-2\varepsilon s}.$$

Problema 4.

§. 15. *Proposita formula differentiali* $\frac{\partial\phi\sin.\lambda\phi}{\sqrt[n]{\cos.n\phi^{\lambda}}}$*, ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Cum sit

$$\sin.\lambda\phi=\tfrac{1}{2i}\left(\cos.\phi+i\sin.\phi\right)^{\lambda}-\tfrac{1}{2i}\left(\cos.\phi-i\sin.\phi\right)^{\lambda},$$

constituamus ut hactenus has duas partes

$$\partial p=\frac{\partial\phi\left(\cos.\phi+i\sin.\phi\right)^{\lambda}}{\sqrt[n]{\cos.n\phi^{\lambda}}}\text{ et }\partial q=\frac{\partial\phi\left(\cos.\phi-i\sin.\phi\right)^{\lambda}}{\sqrt[n]{\cos.n\phi^{\lambda}}},$$

ita ut integrale quaesitum sit $\frac{p-q}{2i}$. Statuamus nunc iterum

$$\frac{\cos.\phi+i\sin.\phi}{\sqrt[n]{\cos.n\phi}}=x\text{ et }\frac{\cos.\phi-i\sin.\phi}{\sqrt[n]{\cos.n\phi}}=y,$$

ut fiat $\partial p=x^{\lambda}\partial\phi$ et $\partial q=y^{\lambda}\partial\phi$, hincque calculo ut supra institute, fiet

$$\partial p=-\frac{ix^{\lambda-1}\partial x}{2-x^{n}}\text{ et }\partial q=\frac{iy^{\lambda-1}\partial y}{2-y^{n}},$$

sicque integrale quaesitum erit

$$-\frac{1}{2}\int\frac{x^{\lambda-1}\partial x}{2-x^{n}}-\frac{1}{2}\int\frac{y^{\lambda-1}\partial y}{2-y^{n}}.$$

§. 16. Quodsi jam ut hactenus est factum, ponamus $x=r+is$ et $y=r-is$, et pro partibus logarithmicis, quarurm forma sit $f\,l\left(\alpha+\beta x+\gamma xx\right)$, ponamus

$$t=\alpha+\beta r+\gamma rr-\gamma ss\text{ et }u=\beta s+2\gamma r\mathrm{s},$$

binae partes Iogarithmicae imaginariae uti in problemate secundo in unum logarithmum realem contrahentur, qui erit $-\frac{1}{2}f\,l\left(tt+uu\right)$. At si pro partibus circularibus, quarum forma sit g Arc. tang.$\left(\delta+\varepsilon x\right)$, bini tales arcus imaginarii jungantur, illi coalescent in -unum arcum realem

$$-ig\text{Arc. tang.}\frac{2\delta+2\varepsilon r}{1-(\delta+\varepsilon r)^2-\varepsilon\varepsilon ss}.$$

Problema generale.

§. 17. *Si Φ denotet functionem quamcunque rationalem ipsius tang.$n\phi$, ac proposita fuerit haec formula dijferentialis*

$$\frac{\Phi\partial\phi(F\sin.\lambda\phi+G\cos.\lambda\phi)}{\sqrt[n]{(a\cos.n\phi+b\sin.n\phi)^\lambda}},$$

ejus integrationem ad logarithmos et arcus circulares reducere.

Solutio.

Ex praecedentibus jam facile intelligitur, formulam numeratoris $F\sin.\lambda\phi+G\cos.\lambda\phi$ semper ad talem formam revocari posse

$$F'(\cos.\phi+i\sin.\phi)^\lambda+G'(\cos.\phi-i\sin.\phi)^\lambda,$$

atque hinc ipsa forma proposita discerpatur in has duas partes

$$\partial p=\frac{\Phi\partial\phi(\cos.\phi+i\sin.\phi)^\lambda}{\sqrt[n]{(a\cos.\phi+b\sin.\phi)^\lambda}}\text{ et }\partial q=\frac{\partial\phi(\cos.\phi-i\sin.\phi)^\lambda}{\sqrt[n]{(a\cos.\phi+b\sin.\phi)^\lambda}}$$

ita ut integrale quaesitum jam futurum sit $\mathrm{F}'p+\mathrm{G}'q$.

§. 18. Jam pro formula priori ∂p statuatur

$$\frac{\cos.\phi+i\sin.\phi}{\sqrt[n]{(a\cos.n\phi+b\sin.n\phi)}}=x\text{ et pro posteriori}$$

$$\frac{\cos.\phi-i\sin.\phi}{\sqrt[n]{(a\cos.n\phi+b\sin.n\phi)}}=y.$$

ita ut hinc futurum sit

$$\partial p=\Phi x^\lambda\partial\phi\text{ et }\partial q=\Phi y^\lambda\partial\phi;$$

inde autem fiat

$$x^n=\frac{cos.n\phi+isin.n\phi}{acos.n\phi+isin.n\phi},$$

unde colligitur

$$\text{tang.}n\phi=\frac{1-ax^n}{bx^n-i};$$

quare cum Φ denotet functionem rationalem ipsius tang.$n\phi$, evadet quoque functio rationalis ipsius *x,* atque adeo ipsius x^n, quae designetur per X. Praeterea vero etiam differentiale $\partial\phi$ rationaliter determinabitur; cum fiat

$$\partial\phi = \frac{(a-b)x^{n-1}\partial x}{(aa+bb)x^{2n}-2(a-ib)x^{n}},$$

hoc ergo modo habebimus

$$\partial p = \frac{(ia-b)Xx^{\lambda-1}\partial x}{(aa+bb)x^{n}-2(a+ib)},$$

quae cum sit penitus rationalis, certum est, ejus integrale, quantumcunque etiam laborem postulaverit, semper per logarithmos et arcus circulares expediri posse.

§.19. Simili modo res se habet in altera formula ∂q, quae ab ista tantum ratione signi litterae *i* differet, et quoniam hic omnia rationaliter per *y* prodibunt expressa, quo pacto Φ abeat in Y, atque obtinebitur

$$\partial q = -\frac{(b+ia)Yy^{\lambda-1}\partial y}{(aa+bb)y^{n}-2a+2ib},$$

cujus integratio omnis similis erit praecedenti, et quasi eodem labore absolvetur.

§. 20. Manifestum autem est, in hujusmodi calculo imaginaria cum realibus multo arctius commisceri, quam in praecedentibus problematibus usu venit, quandoquidem jam statim ab initio coefficientes derivati F' et G' jam imaginario involvunt; deinde vero etiam utrinque tang.$n\phi$ imaginariis inquinatur, unde etiam in valores X et Y imaginaria ingredientur; quamobrem reductio ad realitatem plerumque maximum laborem exigere poterit, proque autem negotio praecepta necessaria jam satis sunt cognita.

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2) Theorema maxime memorabile circa formulam integralem

$$\int \frac{\partial\phi \cos.\lambda\phi}{(1+aa-2a\cos.\phi)^{n+1}}.$$

M.S.Academiae exhib. die 13. *Augusti* 1778.

§. 21. Haec formula aliam restrictionem non postulat nisi quod littera λ numeros tantum integros designat sive positivos sive negatives. Evidens autem est valores negatives non discrepare a positivis, cum semper sit $\cos.-\phi = \cos.+\phi$. Hoc notato si istius formulae integrale a termino $\phi = 0$ usque ad terminum $\phi = 180°$ sive $\phi = \pi$ porrigatur, ejus valor semper sequenti formua exprimetur $\frac{\pi a}{(1-aa)^{2n+1}} \cdot \mathrm{V}$ existente

$$\mathrm{V} = \left(\tfrac{n-\lambda}{0}\right)\left(\tfrac{n+\lambda}{\lambda}\right) + \left(\tfrac{n-\lambda}{1}\right)\left(\tfrac{n+\lambda}{\lambda+1}\right)aa$$
$$+\left(\tfrac{n-\lambda}{2}\right)\left(\tfrac{n+\lambda}{\lambda+2}\right)a^4 + \left(\tfrac{n-\lambda}{3}\right)\left(\tfrac{n+\lambda}{\lambda+3}\right)a^6$$
$$+\left(\tfrac{n-\lambda}{4}\right)\left(\tfrac{n+\lambda}{\lambda+4}\right)a^8 + \left(\tfrac{n-\lambda}{5}\right)\left(\tfrac{n+\lambda}{\lambda+5}\right)a^{10} \text{ etc.}$$

Ubi formulae uncinulis inclusae non fractiones, sed eos characteres designant, quibus unciae potestatum Binomii designari solent, ita ut sit

$$\left(\tfrac{\alpha}{\beta}\right) = \tfrac{\alpha}{1}\cdot\tfrac{\alpha-1}{2}\cdot\tfrac{\alpha-2}{3}\cdot\quad.\quad.\quad.\quad..\tfrac{\alpha-\beta+1}{\beta},$$

quae expressio quoniam nostro casu β ubique est numerus integer, determinatum valorem facile quovis casu exhibendam declarat, ubi notasse sufficiet, quoties fuerit $\beta = 0$ semper fore $\left(\tfrac{\alpha}{0}\right) = 1$; sin autem fuerit β numerus negativus, valorem hujus characteris in nihilum abire; tum vero etiam observari convenit, si fuerit $\beta = \alpha$ fore $\left(\tfrac{\alpha}{\alpha}\right) = 1$, et si $\beta > \alpha$ pariter valores evanescere. Cum semper sit $\left(\tfrac{\alpha}{\beta}\right) = \left(\tfrac{\alpha}{\alpha-\beta}\right)$.

§. 22. His explicatis evolvamus praecipuos casus quibus exponenti *n* valores simpliciores 0, 1, 2, 3, 4 etc. tribuuntur.

Casus I.

quo $n = 0$, et formula integralis haec proponitur

$$\int \frac{\partial\phi \cos.\lambda\phi}{1+aa-2a\cos.\phi} \quad \begin{bmatrix} a & x=0 \\ ad & x=\pi \end{bmatrix}.$$

Quia hic $n = 0$, pro prioribus factoribus quantitatis V habemus

$$\left(\tfrac{0-\lambda}{0}\right) = 1; \left(\tfrac{0-\lambda}{1}\right) = -\lambda; \left(\tfrac{0-\lambda}{2}\right) = \tfrac{\lambda}{1}\cdot\tfrac{\lambda+1}{2};$$
$$\left(\tfrac{0-\lambda}{3}\right) = -\tfrac{\lambda}{1}\cdot\tfrac{\lambda+1}{2}\cdot\tfrac{\lambda+2}{3}; \left(\tfrac{0-\lambda}{4}\right) = \tfrac{\lambda}{1}\cdot\tfrac{\lambda+1}{2}\cdot\tfrac{\lambda+2}{3}\cdot\tfrac{\lambda+3}{4}; \text{etc.}$$

Pro posterioribus vero factoribus habebimus

$$\left(\tfrac{0+\lambda}{\lambda}\right) = 1; \left(\tfrac{0+\lambda}{\lambda+1}\right) = 0; \left(\tfrac{0+\lambda}{\lambda+2}\right) = 0 \text{ etc.}$$

hic scilicet omnes isti factores praeter primum evanescunt; unde colligitur valor quantitatis. $\mathrm{V} = 1$, ideoque integrale quaesitum hujus casus erit

$$= \frac{\pi a^{\lambda}}{1-aa}.$$

Hinc ergo si fuerit $n=0$, erit $\int\frac{\partial\phi}{1+aa-2a\cos.\phi}=\frac{\pi}{1-aa}$ quod egregie consentit cum integratione satis cognita

$$\int\frac{\partial\phi}{\alpha+\beta\cos.\phi}=\frac{1}{\sqrt{(\alpha\alpha-\beta\beta)}}\text{Arc.cos.}\frac{\alpha\cos.\phi+\beta}{\alpha+\beta\cos.\phi},$$

quod integrale jam sponte evanescit sumto $\phi=0$. Statuatur igitur, ut hic perpetuo assumimus, $\phi=180°=\pi$, atque ob $\cos.\phi=-1$, erit istud integrale

$$\frac{1}{\sqrt{(\alpha\alpha-\beta\beta)}}\text{Arc.cos.}-1=\frac{\pi}{\sqrt{(\alpha\alpha-\beta\beta)}}.$$

Jam nostro casu est $\alpha=1+aa$ et $\beta=-2a$, unde fit $\sqrt{(\alpha\alpha-\beta\beta)}=1-aa.$

Casus II.

quo $n=1$, et formula integralis haec proponitur

$$\int\frac{\partial\phi\cos.\lambda\phi}{(1+aa-2a\cos.\phi)^2}\begin{bmatrix}a & \phi=0\\ ad & \phi=\pi\end{bmatrix}.$$

Quia hic est $n=1$, erit pro prioribus factoribus quantitatis V

$$\left(\tfrac{1-\lambda}{0}\right)=1;\left(\tfrac{1-\lambda}{1}\right)=-(\lambda-1);\ \left(\tfrac{1-\lambda}{2}\right)=\tfrac{\lambda}{1}\cdot\tfrac{(\lambda-1)}{2}.$$

Pro posterioribus vero factoribus habebimus

$$\left(\tfrac{1+\lambda}{\lambda}\right)=\lambda+1;\left(\tfrac{1+\lambda}{\lambda+1}\right)=1;$$

sequentes vero formulae evanescunt, sicque erit

$$\text{V}=\lambda+1-(\lambda-1)aa\ ;$$

quocirca valor integralis propositi erit

$$=\frac{\pi a^{\lambda}}{(1-aa)^3}\left[(\lambda+1)-(\lambda-1)aa\right]:$$

hinc ergo sequentes casus speciales apposuisse juvabit, ubi brevitatis gratia loco formulae $1+aa-2a.\cos.\phi$ characterem Δ scribamus

$$\int\frac{\partial\phi}{\Delta^2}=\frac{\pi(1+aa)}{(1-aa)^3},$$

$$\int\frac{\partial\phi\cos.\phi}{\Delta^2}=\frac{2\pi a}{(1-aa)^3},$$

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^2}=\frac{\pi a^2(3-aa)}{(1-aa)^3},$$

$$\int\frac{\partial\phi\cos.3\phi}{\Delta^2}=\frac{\pi a^3(4-2aa)}{(1-aa)^3},$$

$$\int\frac{\partial\phi\cos.4\phi}{\Delta^2}=\frac{\pi a^4(5-3aa)}{(1-aa)^3},$$

$$\int\frac{\partial\phi\cos.5\phi}{\Delta^2}=\frac{\pi a^5(6-4aa)}{(1-aa)^3},$$

$$\int\frac{\partial\phi\cos.6\phi}{\Delta^2}=\frac{\pi a^6(7-5aa)}{(1-aa)^3},$$

etc. etc.

Casus III.

quo $n=2$, et formula integralis haec proponitur

$$\int\frac{\partial\phi\cos.\lambda\phi}{(1+aa-2a\cos.\phi)^3}\begin{bmatrix}a & \phi=0\\ ad & \phi=\pi\end{bmatrix}.$$

Hic factores priores, qui in valore quantitatis V occurrunt, erunt

$$\left(\frac{2-\lambda}{0}\right)=1;\left(\frac{2-\lambda}{1}\right)=-(\lambda-2);\ \left(\frac{2-\lambda}{2}\right)=\frac{(\lambda-2)}{1}\cdot\frac{(\lambda-1)}{2};$$
$$\left(\frac{2-\lambda}{3}\right)=\frac{\lambda-2}{1}\cdot\frac{\lambda-1}{2}\cdot\frac{\lambda}{3}\text{ etc.}$$

factores autem posteriores erunt

$$\left(\frac{2+\lambda}{\lambda}\right)=\frac{\lambda+2}{1}\cdot\frac{\lambda+1}{2};\left(\frac{2+\lambda}{\lambda+1}\right)=\lambda+2;\left(\frac{2+\lambda}{\lambda+2}\right)=1;$$

et sequentes omnes evanescunt; hinc ergo colligimus

$$V=\tfrac{\lambda+2}{1}\cdot\tfrac{\lambda+1}{2}-(\lambda\lambda-4)aa+\tfrac{(\lambda-2)}{1}\cdot\tfrac{(\lambda-1)}{2}a^4,$$

hocque valore invento erit integrale quaesitum $\frac{\pi a^{\lambda}}{(1-aa)^5}.V$, unde sequentes casus speciales, statuendo ut ante $1+aa-2a\cos.\phi=\Delta$, evolvamus

$$\int\frac{\partial\phi}{\Delta^3}=\frac{\pi}{(1-aa)^5}\left(1+4aa+a^4\right),$$

$$\int\frac{\partial\phi\cos.\phi}{\Delta^3}=\frac{3\pi a}{(1-aa)^5}(1+aa),$$

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^3}=\frac{6\pi a^2}{(1-aa)^5},$$

$$\int\frac{\partial\phi\cos.3\phi}{\Delta^3}=\frac{\pi a^3}{(1-aa)^5}\left(10-5aa+a^4\right),$$

$$\int\frac{\partial\phi\cos.4\phi}{\Delta^3}=\frac{3\pi a^4}{(1-aa)^5}\left(5-4aa+a^4\right),$$

$$\int\frac{\partial\phi\cos.5\phi}{\Delta^3}=\frac{3\pi a^5}{(1-aa)^5}\left(7-7aa+2a^4\right),$$

$$\int\frac{\partial\phi\cos.6\phi}{\Delta^3}=\frac{2\pi a^6}{(1-aa)^5}\left(14-16aa+5a^4\right),$$

etc. etc.

Casus IV.

quo $n=3$, et formula integralis haec proponitur

$$\int\frac{\partial\phi\cos.\lambda\phi}{(1+aa-2a\cos.\phi)^4}\begin{bmatrix}a & \phi=0\\ ad & \phi=\pi\end{bmatrix}.$$

Hic pro prioribus factoribus quantitatis V habebimus

$$\left(\tfrac{3-\lambda}{0}\right)=1;\left(\tfrac{3-\lambda}{1}\right)=-(\lambda-3);\ \left(\tfrac{3-\lambda}{2}\right)=\tfrac{(3-\lambda)}{1}\cdot\tfrac{(2-\lambda)}{2};$$
$$\left(\tfrac{3-\lambda}{3}\right)=\tfrac{3-\lambda}{1}\cdot\tfrac{2-\lambda}{2}\cdot\tfrac{1-\lambda}{3};\ \left(\tfrac{3-\lambda}{4}\right)=\tfrac{3-\lambda}{1}\cdot\tfrac{2-\lambda}{2}\cdot\tfrac{1-\lambda}{3}\cdot\tfrac{-\lambda}{4};$$

factores autem posteriores erunt

$$\left(\tfrac{3+\lambda}{\lambda}\right)=\tfrac{3+\lambda}{1}\cdot\tfrac{2+\lambda}{2}\cdot\tfrac{1+\lambda}{3};\ \left(\tfrac{3+\lambda}{\lambda+1}\right)=\tfrac{3+\lambda}{1}\cdot\tfrac{2+\lambda}{2};\ \left(\tfrac{3+\lambda}{\lambda+2}\right)=3+\lambda;\ \left(\tfrac{3+\lambda}{\lambda+3}\right)=1;$$

et sequentes onmes evanescunt, hinc ergo colligimus

$$\mathrm{V}=\frac{(\lambda+1)(\lambda+2)(\lambda+3)}{1.\quad 2.\quad 3}-\frac{(\lambda+2)(\lambda\lambda-9)}{1.\quad 2.}aa+\frac{(\lambda-2)(\lambda\lambda-9)}{1.\quad 2.}a^4-\frac{(\lambda-1)(\lambda-2)(\lambda-3)}{1.\quad 2.\quad 3}a^6.$$

Quo valore invento colligimus integrale quaesitum $=\dfrac{\pi a^{\lambda}}{(1-aa)^7}.\mathrm{V}$,

hincque sequentes casus speciales, ponendo ut hactenus $1+aa-2a\cos.\phi=\Delta$, evolvamus

$$\int\frac{\partial\phi}{\Delta^4}=\frac{\pi}{(1-aa)^7}\left(1+9aa+9a^4+a^6\right),$$

$$\int\frac{\partial\phi\cos.\phi}{\Delta^4}=\frac{4\pi a}{(1-aa)^7}\left(1+3aa+a^4\right),$$

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^4}=\frac{10\pi a^2}{(1-aa)^7}(1+aa),$$

$$\int\frac{\partial\phi\cos.3\phi}{\Delta^4}=\frac{20\pi a^3}{(1-aa)^7},$$

$$\int\frac{\partial\phi\cos.4\phi}{\Delta^4}=\frac{\pi a^4}{(1-aa)^7}\left(35-21aa+7a^4-a^6\right).$$

etc. etc.

§. 23. Hic longius progredi superfluum foret, cum forma generalis pro V inventa totum negotium facillime conficiat; verum haud inutile erit, litterae *n* etiam valores negativos tribuere, quibus casibus tota integratio per methodos consuetas haud difficulter expeditur, unde jucundum erit pulcherrimum consensum nostrae formae generalis perspicere.

Casus I.

quo $n=-1$, et formula integralis haec proponitur

$$\int\partial\phi\cos.\lambda\phi\begin{bmatrix}a & \phi=0\\ ad & \phi=\pi\end{bmatrix}.$$

Haec formula absolute est integrabilis, cum sit

$$\int \partial\phi \cos.\lambda\phi = \tfrac{1}{\lambda}\sin.\lambda\phi,$$

quae formula cum jam evanescat posito $\varphi = 0;$ sumendo $\varphi = \pi,$ ob λ numerum integrum iste valor semper erit $= 0$, solo casu excepto $\lambda = 0$. Spectato enim λ tanquam infinite parvo, erit $\sin.\lambda\pi = \lambda\pi$, ideoque hoc casu valor erit $= \pi$. Nunc autem forma generalis pro quantitate V data erit

$$\begin{aligned} \mathrm{V} = {} & \tfrac{(-1-\lambda)}{0}\tfrac{(-1+\lambda)}{\lambda} + \tfrac{(-1-\lambda)}{1}\tfrac{(-1+\lambda)}{\lambda+1}a^2 \\ & + \tfrac{(-1-\lambda)}{2}\tfrac{(-1+\lambda)}{\lambda+2}a^4 + \tfrac{(-1-\lambda)}{3}\tfrac{(-1+\lambda)}{\lambda+3}a^6 \\ & + \tfrac{(-1-\lambda)}{4}\tfrac{(-1+\lambda)}{\lambda+4}a^8 + \tfrac{(-1-\lambda)}{5}\tfrac{(-1+\lambda)}{\lambda+5}a^{10} \\ & \quad \text{etc.} \qquad\qquad \text{etc.} \end{aligned}$$

Cujus expressionis factores posteriores omnes evanescunt, quoties fuerit vel $\lambda = 1$ vel $\lambda > 1$, propterea quod numeri inferiores majores, quam superiores, utrique vero positivi; quae conclusio autem non valet, quando superior numerus evadit negativus, uti evenit casu $\lambda = 0$, quem ergo solum perpendisse necesse est; hoc autem casu factores priores evadent

$$\left(\tfrac{-1}{0}\right) = 1;\ \left(\tfrac{-1}{1}\right) = -1;\ \left(\tfrac{-1}{2}\right) = +1;\ \left(\tfrac{-1}{3}\right) = -1;\ \left(\tfrac{-1}{4}\right) = +1;\ \text{etc.}$$

at vero valores posteriores eosdem determinationes recipiunt; sicque habebimus

$$\mathrm{V} = 1 + aa + a^4 + a^6 + a^8 + a^{10} + \text{etc.}$$

quae series cum sit geometrica, erit $\mathrm{V} = \frac{1}{1-aa}$ quare cum, ob $n = -1$ et $\lambda = 0,$ valor quaesitus per nostram formam generalem sit $\pi(1-aa)\mathrm{V}$, iste valor nunc ob $\mathrm{V} = \frac{1}{1-aa}$, abit in π, uti supra.

Casus II.

quo $n = -2$, et formula integralis haec proponitur

$$\int \partial\phi \cos.\lambda\phi \left(1 + aa - 2a\cos.\phi\right)\left[\begin{smallmatrix} a & \phi=0 \\ ad & \phi=\pi \end{smallmatrix}\right].$$

Per formam nostrum generalem integrale quaesitum erit $= \pi a^{\lambda}\left(1-aa\right)^3 \mathrm{V},$ existente

$$\mathrm{V}=\left(\tfrac{-2-\lambda}{0}\right)\left(\tfrac{-2+\lambda}{\lambda}\right)+\left(\tfrac{-2-\lambda}{1}\right)\left(\tfrac{-2+\lambda}{\lambda+1}\right)aa+\left(\tfrac{-2-\lambda}{2}\right)\left(\tfrac{-2+\lambda}{\lambda+2}\right)a^4$$
$$+\left(\tfrac{-2-\lambda}{3}\right)\left(\tfrac{-2+\lambda}{\lambda+3}\right)a^6+\left(\tfrac{-2-\lambda}{4}\right)\left(\tfrac{-2+\lambda}{\lambda+4}\right)a^8+\left(\tfrac{-2-\lambda}{5}\right)\left(\tfrac{-2+\lambda}{\lambda+5}\right)a^{10}$$
etc. etc.

Ubi iterum evidens est, si fuerit vel $\lambda=2$ vel $\lambda>2$, omnes factores posteriores evanescere, ideoque fieri $\mathrm{V}=0$, ita ut etiam valor integralis quaesitus semper evanescat, id quod ex ipsa natura formulae sponte sequitur, quippe cuius integrale, ob

$$\cos.\phi\cos.\lambda\phi=\tfrac{1}{2}\cos.(\lambda-1)\phi+\tfrac{1}{2}\cos.(\lambda+1)\phi,$$

in genere erit

$$\tfrac{1+aa}{\lambda}\sin.\lambda\phi-\tfrac{a}{\lambda-1}\sin.(\lambda-1)\phi=\tfrac{a}{\lambda+1}\sin.(\lambda+1)\phi,$$

quod quia $\lambda>1$ casu $\phi=\pi$ manifesto evanescit; unde duos casus perpendere superest, alterum quo $\lambda=0$, et alterum quo $\lambda=1$.

1°. Sit $\lambda=0$, et integrale $\pi(1-aa)^3\mathrm{V}$, ubi pro V factores posteriores evadunt

$$\left(\tfrac{-2}{0}\right)=1;\ \left(\tfrac{-2}{1}\right)=-2;\ \left(\tfrac{-2}{2}\right)=3;\ \left(\tfrac{-2}{3}\right)=-4;\ \left(\tfrac{-2}{4}\right)=+5;\ \left(\tfrac{-2}{5}\right)=-6\ ;\text{etc.}$$

simili modo priores factores erunt

$$\left(\tfrac{-2}{0}\right)=1;\ \left(\tfrac{-2}{1}\right)=-2;\ \left(\tfrac{-2}{2}\right)=3;\ \text{etc.}$$

unde colligitur fore

$$\mathrm{V}=1+4aa+9a^4+16a^6+25a^8+36a^{10}+\text{etc.}$$

Pro qua serie summanda, inde subtrahatur series V*aa,* et remanebit

$$\mathrm{V}(1-aa)=1+3aa+5a^4+7a^6+9a^8+\text{etc.}$$

Multipticitur denuo utrinque per $1-aa$, ac prodibit

$$\mathrm{V}(1-aa)^2=1+2aa+2a^4+2a^6+2a^8+\text{ etc.}$$

quae denuo ducta in $1-aa$ praebet

$\mathrm{V}(1-aa)^3=1+aa$, ideoque $\mathrm{V}=\frac{1+aa}{(1-aa)^3}$.

Consequenter integrale quaesitum erit $=\pi(1+aa)$, id quod utique oritur ex integratione actuali, cum sit

$$\int\partial\phi\left(1+aa-2a\cos.\phi\right)=\left(1+aa\right)\phi-2a\sin.\phi,$$

quod facto $\phi=\pi$ abit in $(1+aa)\pi$.

II$^{\text{o}}$. Sit $\lambda=1$, et integrale quaesitum $\pi a(1-aa)^3\text{V}$;
ubi pro factoribus posterioribus est

$$\left(\tfrac{-1}{0}\right)=-1;\ \left(\tfrac{-1}{2}\right)=+1;\ \left(\tfrac{-1}{3}\right)=-1;$$
$$\left(\tfrac{-1}{4}\right)=+1;\ \left(\tfrac{-1}{5}\right)=-1;\ \text{etc.}$$

Factores vero priores evadunt

$$\left(\tfrac{-3}{0}\right)=1;\ \left(\tfrac{-3}{1}\right)=-3;\ \left(\tfrac{-3}{2}\right)=6;\ \left(\tfrac{-3}{3}\right)=-10;$$
$$\left(\tfrac{-3}{4}\right)=15;\left(\tfrac{-3}{5}\right)=-21;\ \left(\tfrac{-3}{6}\right)=28;\left(\tfrac{-3}{7}\right)=-36;\ \text{etc.}$$

hinc igitur habebimus

$$\text{V}=-1-3aa-6a^4-10a^6-15a^8-21a^{10}-28a^{12}-36a^{14}-\text{etc.}$$

Pro cujus summatione multiplicetur utrinque per $1-aa$, et prodibit

$$\text{V}\left(1-aa\right)=-1-2aa-3a^4-4a^6-5a^8-6a^{10}-7a^{12}-8a^{14}-\text{etc.}$$

multiplicando denuo per $1-aa$, prodit

$$\text{V}\left(1-aa\right)^2=-1-aa-a^4-a^6-a^8-a^{10}-a^{12}-a^{14}-\text{etc.}$$

et multiplicando rursus per $1-aa$, erit

$\text{V}\left(1-aa\right)^3=-1$, ideoque $\text{V}=-\frac{1}{(1-aa)^3}$, consequenter integrale quaesitum $=-\pi a$. Ipsa autem integralio ob $\cos.^2\phi=\frac{1}{2}+\frac{1}{2}\cos.2\phi$ praebet

$$\int\partial\phi\cos.\phi\left(1+aa-2a\cos.\phi\right)=\left(1+aa\right)\sin.\phi-a\phi-\tfrac{1}{2}a\sin.2\phi,$$

unde statuendo $\phi=\pi$, oritur integrale $=-a\pi$.

Casus III.

quo $n=-3$, et formula integralis haec proponitur

$$\int \partial\phi \cos.\lambda\phi \left(1+aa-2a\cos.\phi\right)^{2} \begin{bmatrix} a & \phi=0 \\ ad & \phi=\pi \end{bmatrix}.$$

Hoc ergo casu ex forma generali erit integrale $\pi a(1-aa)^{5}\mathrm{V}$, existente

$$\begin{aligned}\mathrm{V} &= \left(\tfrac{-3-\lambda}{0}\right)\left(\tfrac{-3+\lambda}{\lambda}\right)+\left(\tfrac{-3-\lambda}{1}\right)\left(\tfrac{-3+\lambda}{\lambda+1}\right)a^{2} \\ &+\left(\tfrac{-3-\lambda}{2}\right)\left(\tfrac{-3+\lambda}{\lambda+2}\right)a^{4}+\left(\tfrac{-3-\lambda}{3}\right)\left(\tfrac{-3+\lambda}{\lambda+3}\right)a^{6} \\ &\qquad \text{etc.} \qquad\qquad \text{etc.}\end{aligned}$$

ubi factores posteriores manifesto omnes evanescunt, quando fuerit vel $\lambda=3$ vel $\lambda>3$, quibus ergo casibus totum integrale evanescit, ut cuilibet calculum instituenti facile patebit: tres antem casus considerandi restant, quibus $\lambda<3$.

I. Sit $\lambda=0$, atque tam priores quam posteriores factores convenient, eruntque

$$\begin{aligned}&\left(\tfrac{-3}{0}\right)=1;\ \left(\tfrac{-3}{1}\right)=-3;\ \left(\tfrac{-3}{2}\right)=6;\ \left(\tfrac{-3}{3}\right)=-10; \\ &\left(\tfrac{-3}{4}\right)=15;\left(\tfrac{-3}{5}\right)=-21;\ \left(\tfrac{-3}{6}\right)=28;\left(\tfrac{-3}{7}\right)=-36;\ \text{etc.}\end{aligned}$$

unde colligitur

$$\mathrm{V}=1+9aa+36a^{4}+100a^{6}+225\,a^{8}+441a^{10}+\text{etc.}$$

quae series cum tandem perducat ad differentias constantes, simili modo ut hactenus summari poterit, prima enim multiplicatio per $1-aa$ praebet

$$\mathrm{V}\left(1-aa\right)=1+8aa+27a^{4}+64a^{6}+125a^{8}+216a^{10}+343a^{12}+\text{etc.}$$

Secunda multiplicatio per $1-aa$ praebet

$$\mathrm{V}\left(1-aa\right)^{2}=1+7aa+19a^{4}+37a^{6}+61a^{8}+91a^{10}+127a^{12}+\text{etc.}$$

Tertia multiplicatio dat

$$\mathrm{V}\left(1-aa\right)^{3}=1+6aa+12a^{4}+18a^{6}+24a^{8}+30a^{10}+\text{etc.}$$

Quarta multiplicatio dat

$$\mathrm{V}\left(1-aa\right)^{4}=1+5aa+6a^{4}+6a^{6}+6a^{8}+6a^{10}+\text{etc.}$$

ac denique

$V(1-aa)^5 = 1+4aa+a^4$, ita ut sit $V = \frac{1+4aa+a^4}{(1-aa)^5}$;

Consequenter valor integralis quaesitus hoc casu erit $\pi(1+4aa+a^4)$, quod egregie cum integrali more solito invento congruit.

II.

Sit $\lambda = 1$, quo casu priores factores ipsius V erunt

$$\left(\tfrac{-4}{0}\right) = 1;\ \left(\tfrac{-4}{1}\right) = -4;\ \left(\tfrac{-4}{2}\right) = 10;\ \left(\tfrac{-4}{3}\right) = -20;$$
$$\left(\tfrac{-4}{4}\right) = 35;\ \left(\tfrac{-4}{5}\right) = -56;\ \left(\tfrac{-4}{6}\right) = 84;\ \left(\tfrac{-4}{7}\right) = -120 \text{ etc.}$$

posteriores vero ita se habent

$$\left(\tfrac{-2}{1}\right) = -2;\ \left(\tfrac{-2}{2}\right) = +3;\ \left(\tfrac{-2}{3}\right) = -4;\ \left(\tfrac{-2}{4}\right) = +5;$$
$$\left(\tfrac{-2}{5}\right) = -6;\ \left(\tfrac{-2}{6}\right) = +7;\ \left(\tfrac{-2}{7}\right) = -8;\ \left(\tfrac{-2}{8}\right) = +9; \text{ etc.}$$

ideoque

$$V = -2-12aa-40a^4-100a^6-210a^8-392a^{10}-672a^{12}-1080a^{14}-\text{etc.}$$

quae series cum tandem perducat ad differentias constantes, simili modo ut ante summari poterit; prima enim multiplicatio per $1-aa$ dat

$$V(1-aa) = -2-10a^2-28a^4-60a^6-110a^8-182a^{10}-280a^{12}-\text{etc.}$$

Secunda multiplicatio per $1-aa$ praebet

$$V(1-aa)^2 = -2-8a^2-18a^4-32a^6-50a^8-72a^{10}-98a^{12}-\text{etc.}$$

Tertia multiplicatio dat

$$V(1-aa)^3 = -2-6a^2-10a^4-14a^6-18a^8-22a^{10}-26a^{12}-\text{etc.}$$

Quarta multiplicatio dat

$$V(1-aa)^4 = -2-4a^2-4a^4-4a^6-4a^8-4a^{10}-4a^{12}-\text{etc.}$$

ac denique quinta multiplicatio per $1-aa$ praebet

$$V(1-aa)^5 = -2 \; -2aa = -2(1+aa)\,;$$

unde colligitur $V = -\frac{2(1+aa)}{(1-aa)^5}$, ideoque valor integralis quaesitus erit $= 2\pi a\left(1-aa\right)$, qui egregie cum integrali more solito invento congruit.

III. Sit $\lambda = 1$, atque factores priores ipsius V erunt

$$\left(\tfrac{-5}{0}\right)=1;\ \left(\tfrac{-5}{1}\right)=-5;\ \left(\tfrac{-5}{2}\right)=15;\ \left(\tfrac{-5}{3}\right)=-35;$$
$$\left(\tfrac{-5}{4}\right)=70;\ \left(\tfrac{-5}{5}\right)=-126;\ \left(\tfrac{-5}{5}\right)=210;\ \left(\tfrac{-5}{7}\right)=-330;\ \text{etc.}$$

posteriores vero factores ita se habebunt

$$\left(\tfrac{-1}{2}\right)=1;\ \left(\tfrac{-1}{3}\right)=-1;\ \left(\tfrac{-1}{4}\right)=1;\ \left(\tfrac{-1}{5}\right)=-1;$$
$$\left(\tfrac{-1}{6}\right)=1;\ \left(\tfrac{-1}{7}\right)=-1;\ \left(\tfrac{-1}{8}\right)=1;\ \left(\tfrac{-1}{9}\right)=-1;\ \text{etc.}$$

unde colligitur

$$V = 1+5a^2+15a^4+35a^6+70a^8+126a^{10}+210a^{12}+330a^{14}+\text{etc.}$$

quae series eodem modo ut ante summata praebet $V = +\frac{1}{(1-aa)^6}$, unde colligitur valor integralis quaesitus $= \pi aa$, qui cum integrali more solito invento utique egregie congruit.

§. 24. Quodsi haec integralia quibus *n* est numerus negativus cum iis comparemus, quibus *n* est numerus positivus, insignis analogia deprehenditur inter valores harum formularum

$$\int \Delta^n \;\partial\phi \cos.\lambda\phi \ \text{ et } \int \frac{\partial\phi \cos.\lambda\phi}{\Delta^{n+1}},$$

quae affinitas, si per plures casus exploretur, sequens nobis suppeditat theorema maxime notabile.

Theorema.

§. 25. Posito brevitatis gratia $\Delta = 1+aa-2a\cos.\phi$, atque integralia a termino $\phi = 0$ usque ad terminum $\phi = 180^{\circ}$ extendantur, semper locum habebit sequens proportio

$$\int \Delta^n \ \partial\phi\cos.\lambda\phi : \int \frac{\partial\phi\cos.\lambda\phi}{\Delta^{n+1}} = \left(\tfrac{n}{\lambda}\right)\left(1-aa\right)^n : \left(\tfrac{-n-1}{\lambda}\right)\left(1-aa\right)^{-n-1},$$

vel si statuamus

$$\frac{\Delta}{1+aa} = \frac{1+aa-2a\cos.\phi}{1-aa} = \Gamma,$$

simplicius erit

$$\int \Gamma^n \partial\phi\cos.\lambda\phi : \int \frac{\partial\phi\cos.\lambda\phi}{\Gamma^{n+1}} = \left(\tfrac{n}{\lambda}\right) : \left(\tfrac{-n-1}{\lambda}\right).$$

§. 26. Ita exempla gratia si ponamus $n = 2$, erit ex priore proportione

$$\int \Delta^2 \ \partial\phi\cos.\lambda\phi : \int \frac{\partial\phi\cos.\lambda\phi}{\Delta^3} = \left(\tfrac{2}{\lambda}\right)\left(1-aa\right)^2 : \left(\tfrac{-3}{\lambda}\right)\left(1-aa\right)^{-3}$$

unde si $\lambda = 0,$ ob $\left(\tfrac{n}{0}\right) = 1$ et $\left(\tfrac{-3}{0}\right) = 1,$ erit

$$\int \Delta^2 \ \partial\phi : \int \frac{\partial\phi}{\Delta^3} = \left(1-aa\right)^2 : \frac{1}{(1-aa)^3} = 1 : \frac{1}{(1-aa)^5},$$

ideoque erit

$$\int \frac{\partial\phi}{\Delta^3} = \frac{1}{(1-aa)^5} \int \Delta^2 \ \partial\phi.$$

Cum igitur sit

$$\int \Delta^2 \ \partial\phi = \pi\left(1+5aa+a^4\right), \text{ erit}$$

$$\int \frac{\partial\phi}{\Delta^3} = \frac{\pi}{(1-aa)^4}\left(1+4aa+a^4\right).$$

§. 27. Manente $n = 2,$ sit $\lambda = 1,$ ob $\left(\tfrac{2}{1}\right) = 2$ et $\left(\tfrac{-3}{1}\right) = -3,$ erit

$$\int \Delta^2 \ \partial\phi\cos.\phi : \int \frac{\partial\phi\cos.\phi}{\Delta^3} = 2\left(1-aa\right)^2 : -3\left(1-aa\right)^{-3} = 1 : \frac{-3}{2(1-aa)^5},$$

unde fit

$$\int \frac{\partial\phi\cos.\phi}{\Delta^5} = \frac{-3}{2(1-aa)^5} \int \Delta^2 \ \partial\phi\cos.\phi;$$

cum igitur sit

$$\int \Delta^2 \partial\phi\cos.\phi = -2\pi a\left(1+aa\right), \text{ erit},$$

$$\int \frac{\partial\phi\cos.\phi}{\Delta^2} = \frac{+3\pi a(1+aa)}{(1-aa)^5}.$$

§. 28. Simili modo sumatur $\lambda = 2,$ ob $\left(\tfrac{2}{2}\right) = 1$ et $\left(\tfrac{-3}{2}\right) = 6,$ erit

$$\int \Delta^2 \ \partial\phi\cos.2\phi : \int \frac{\partial\phi\cos.2\phi}{\Delta^3} = \left(1-aa\right)^2 : 6\left(1-aa\right)^{-3} = 1 : \frac{6}{(1-aa)^5},$$

unde fit

$$\int \frac{\partial\phi\cos.\phi}{\Delta^3} = \frac{6}{(1-aa)^5}\int \Delta^2 \ \partial\phi\cos.2\phi.$$

Erat autem

$$\varDelta^2 \ \partial\phi\cos.2\phi = \pi aa,$$

consequenter

$$\int \frac{\partial\phi\cos.2\phi}{\Delta^3} = \frac{6\pi aa}{(1-aa)^5}.$$

§. 29. Cum character $\left(\frac{n}{\lambda}\right)$ fiat $=1$ casu $\lambda = n$, casibus vero quibus $\lambda > n$ semper sit $\left(\frac{n}{\lambda}\right) = 0$, siquidem λ fuerit numerus integer, uti hic perpetuo assumimus, .evidens est istis casibus, quibus $\lambda > n$, semper valorem formulae $\int \varDelta^n \partial\phi \cos.\lambda\phi$ in nihilum abire.

§. 30. Theorema, quod hic proposuimus, non solum ob simplicitatem rationis omni attentione est dignum, sed etiam quod id tantum per plures casus sola inductione conclusimus, neque adhuc ulla via patere videtur, qua ejus veritas directe demonstrari queat; hujusmodi autem theoremata summam Geometrarum attentionem merentur. Evolvamus autem adhuc alios quosdam casus memorabiles nostri theorematis initio propositi.

Evolutio casus
quo $\lambda = n$, et formula integralis proposita

$$\int \frac{\partial\phi\cos.n\phi}{\Delta^{n+1}}$$

Ex forma generali hoc casu integrale erit $\frac{\pi a^n}{(1-aa)^{2n+1}}$V,

existente

$$\text{V} = \left(\frac{0}{0}\right)\left(\frac{2n}{n}\right) + \left(\frac{0}{1}\right)\left(\frac{2n}{n+1}\right)aa + \left(\frac{0}{2}\right)\left(\frac{2n}{n+2}\right)a^4 + \text{etc.}$$

ubi manifesto omnes termini praetor primum evanescunt, ita ut sit
$\text{V} = \left(\frac{2n}{n}\right)$, ideoque nostrum integrale

$$\int \frac{\partial\phi\cos.n\phi}{\Delta^{n+1}} = \frac{\pi a^n}{(1-aa)^{2n+1}}.\left(\frac{2n}{n}\right);$$

ubi notetur, valores characteris $\left(\frac{2n}{n}\right)$ pro variis valoribus numeri n sequenti modo se habere

n	0, 1, 2, 3, 4, 5, 6, 7 etc.

$\left(\frac{2n}{n}\right)$	1, 2, 6, 20, 70, 252, 924, 3432 etc.

quae series facillime per hos factores continuatur

$$\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdot\frac{18}{5}\cdot\frac{22}{6}\cdot\frac{26}{7} \text{ etc.}$$

Postremum vera theorema inventum ad hunc casum applicatum praebebit hanc proportionem

$$\int \Delta^n \partial\phi \cos.n\phi : \int \frac{\partial\phi \cos.n\phi}{\Delta^{n+1}} = (1-aa)^n : \left(\frac{-1-n}{n}\right)(1-aa)^{-n-1},$$

unde fit

$$\int \Delta^n \partial\phi \cos.n\phi = \frac{\pi a^n}{\left(\frac{-n-1}{n}\right)} \cdot \left(\frac{2n}{n}\right) = \left(\frac{2n}{n}\right)\pi a^n : \left(\frac{-n-1}{n}\right);$$

ubi notetur valores characteris $\left(\frac{-n-1}{n}\right)$ pro variis valoribus ipsius n esse

n	0, 1, 2, 3, 4, 5, 6
$\left(\frac{-n-1}{n}\right)$	−1, −2, 6, −20, 70, −252, 924, etc.

unde patet esse $\left(\frac{-n-1}{n}\right) = \pm\left(\frac{2n}{n}\right)$, dum signum superius valet, quando n est numerus par, contra vero signum inferius, quando n est numerus impar; hinc ergo erit

$$\int \Delta^n \partial\phi \cos.n\phi = \pm\pi a^n.$$

His notatis evolvamus casus simpliciores pro utraque formula integrali

$n=0$	$\int \frac{\partial\phi}{\Delta}$	$= \frac{\pi}{1-aa}$	$\int \partial\phi$	$= +\pi$
$n=1$	$\int \frac{\partial\phi \cos.\phi}{\Delta^2}$	$= \frac{2\pi a}{(1-aa)^3}$	$\int \Delta\, \partial\phi \cos.\phi$	$= -\pi a$
$n=2$	$\int \frac{\partial\phi \cos.2\phi}{\Delta^3}$	$= \frac{2\pi a^2}{(1-aa)^5}$	$\int \Delta^2 \partial\phi \cos.2\phi$	$= +\pi a^2$
$n=3$	$\int \frac{\partial\phi \cos.3\phi}{\Delta^4}$	$= \frac{20\pi a^3}{(1-aa)^7}$	$\int \Delta^3 \partial\phi \cos.3\phi$	$= -\pi a^3$
$n=4$	$\int \frac{\partial\phi \cos.4\phi}{\Delta^5}$	$= \frac{70\pi a^4}{(1-aa)^9}$	$\int \Delta^4 \partial\phi \cos.4\phi$	$= +\pi a^4$
$n=5$	$\int \frac{\partial\phi \cos.5\phi}{\Delta^6}$	$= \frac{252\pi a^5}{(1-aa)^{11}}$	$\int \Delta^5 \partial\phi \cos.5\phi$	$= -\pi a^5$
$n=6$	$\int \frac{\partial\phi \cos.6\phi}{\Delta^7}$	$= \frac{924\pi a^6}{(1-aa)^{13}}$	$\int \Delta^6 \partial\phi \cos.6\phi$	$= +\pi a^6$

etc.	etc.	etc.

Hic imprimis notatu dignum occurrit, quod his casibus $\lambda = n$ integralia tam succincte exprimuntur; nunc autem alios perpendamus casus, quibus litterae λ successive valores 0, 1 , 2, 3, etc. tribuantur.

Evolutio casus

quo $\lambda = 0$, et formula integralis proposita

$$\int \frac{\partial\phi}{\varDelta^{n+1}}.$$

§. 31. Cum hic sit $\lambda = 0$, integrale quaesitum ex nostra formula erit $\frac{\pi}{(1-aa)^{2n+1}}\mathrm{V}$, existente

$$\mathrm{V} = \left(\frac{n}{0}\right) + \left(\frac{n}{1}\right)^2 aa + \left(\frac{n}{2}\right)^2 a^4 + \left(\frac{n}{3}\right)^2 a^6 + \left(\frac{n}{4}\right)^2 a^8 + \text{etc.}$$

simul vero hinc etiam assignari poterit valor hujus formulae $\int \varDelta^n \partial\phi,$ cum sit

$$\int \varDelta^n \partial\phi : \int \frac{\partial\phi}{\varDelta^{n+1}} = (1-aa)^n : (1-aa)^{-n-1} = (1-aa)^{2n+1} : 1,$$

ex qua proportione colligitur

$$\int \varDelta^n \partial\phi = \pi.\mathrm{V}.$$

Percurramus igitur simpliciores casus pro exponent *n,* quos sequenti tabula subjungamus

$$n=0\begin{cases}\int\frac{\partial\phi}{\Delta}=\frac{\pi}{1-aa}\\ \int\partial\phi=\pi\end{cases}$$

$$n=1\begin{cases}\int\frac{\partial\phi}{\Delta^2}=\frac{\pi}{(1-aa)^3}(1+aa)\\ \int\Delta\partial\phi=\pi(1+aa)\end{cases}$$

$$n=2\begin{cases}\int\frac{\partial\phi}{\Delta^3}=\frac{\pi}{(1-aa)^5}\left(1+2^2aa+a^4\right)\\ \int\Delta^2\partial\phi=\pi\left(1+2^2aa+a^4\right)\end{cases}$$

$$n=3\begin{cases}\int\frac{\partial\phi}{\Delta^4}=\frac{\pi}{(1-aa)^7}\left(1+3^2aa+3^2a^4+a^6\right)\\ \int\Delta^3\partial\phi=\pi\left(1+3^2aa+3^2a^4+a^6\right)\end{cases}$$

$$n=4\begin{cases}\int\frac{\partial\phi}{\Delta^5}=\frac{\pi}{(1-aa)^9}\left(1+4^2aa+6^2a^4+4^2a^6+a^8\right)\\ \int\Delta^4\partial\phi=\pi\left(1+4^2aa+6^2a^4+4^2a^6+a^8\right)\end{cases}$$

etc. etc.

Evolutio casuum
quibus $\lambda=1$, et formula integralis proposita

$$\int\frac{\partial\phi\cos.\phi}{\Delta^{n+1}}.$$

§. 32. Hoc igitur casu integrale quaesitum erit

$$\frac{\pi a}{(1-aa)^{2n+1}}.\mathrm{V}$$

existente

$$\mathrm{V}=\left(\frac{n-1}{0}\right)\left(\frac{n+1}{1}\right)+\left(\frac{n-1}{1}\right)\left(\frac{n+1}{2}\right)aa+\left(\frac{n-1}{2}\right)\left(\frac{n+1}{3}\right)a^4+\left(\frac{n-1}{3}\right)\left(\frac{n+1}{4}\right)a^6+$$
$$\left(\frac{n-1}{4}\right)\left(\frac{n+1}{5}\right)a^8+\left(\frac{n-1}{5}\right)\left(\frac{n+1}{6}\right)a^{10}+\text{etc.}$$

cum vero cum ob $\lambda=1$ fit

$$\int\Delta^n\partial\phi\cos.\phi:\int\frac{\partial\phi\cos.\phi}{\Delta^{n+1}}=n(1-aa)^n:-(n+1)(1-aa)^{-n-1}$$

unde fit

$$\int \Delta^n \partial\phi \cos.\phi = -\tfrac{n}{n+1}\pi a \mathrm{V}.$$

Pro casibus ergo simplicioribus ipsius n sequentem tabulam subjungamus

$$n=0\begin{cases}\int\frac{\partial\phi\cos.\phi}{\Delta}=\frac{\pi a}{1-aa}\\ \int\partial\phi\cos.\phi=0\end{cases}$$

$$n=1\begin{cases}\int\frac{\partial\phi\cos.\phi}{\Delta^2}=\frac{2\pi a}{(1-aa)^3}\\ \int\Delta\,\partial\phi\cos.\phi=-\pi a\end{cases}$$

$$n=2\begin{cases}\int\frac{\partial\phi\cos.\phi}{\Delta^3}=\frac{\pi a}{(1-aa)^5}(1.3+1.3aa)\\ \int\Delta^2\partial\phi\cos.\phi=-\frac{2}{3}\pi a(1.3+1.3aa)\end{cases}$$

$$n=3\begin{cases}\int\frac{\partial\phi\cos.\phi}{\Delta^4}=\frac{\pi a}{(1-aa)^7}\left(1.4+2.6aa+1.4a^4\right)\\ \int\Delta^3\partial\phi\cos.\phi=-\frac{3}{4}\pi a\left(1.4+2.6aa+1.4a^4\right)\end{cases}$$

$$n=4\begin{cases}\int\frac{\partial\phi}{\Delta^6}\cos.\phi=\frac{\pi a}{(1-aa)^9}\left(1.5+3.10aa+3.10a^4+1.5a^6\right)\\ \int\Delta^4\partial\phi\cos.\phi=-\frac{4}{5}\pi a\left(1.5+3.10aa+3.10a^4+1.5a^6\right)\end{cases}$$

$$n=5\begin{cases}\int\frac{\partial\phi}{\Delta^6}\cos.\phi=\frac{\pi a}{(1-aa)^{11}}\left(1.6+4.15aa+6.20a^4+4.15a^6+1.6a^8\right)\\ \int\Delta^5\partial\phi\cos.\phi=-\frac{5}{6}\pi(1.6+\text{etc.})\end{cases}$$

$$n=6\begin{cases}\int\frac{\partial\phi}{\Delta^7}\cos.\phi=\frac{\pi}{(1-aa)^{13}}\left(1.7+5.21aa+10.35a^4+10.35a^6+\text{etc.}\right)\\ \int\Delta^6\partial\phi\cos.\phi=-\frac{6}{7}\pi a(1.7+\text{etc.})\end{cases}$$

Evolutio casuum

quibus $\lambda=2$, et formula integralis proposita

$$\int\frac{\partial\phi\cos.2\phi}{\Delta^{n+1}}.$$

§. 33. Hoc ergo casu integrale quaesitum erit

$$\frac{\pi a^2}{(1-aa)^{2n+1}}.\mathrm{V}$$

existente

$$\mathrm{V} = \left(\tfrac{n-2}{0}\right)\left(\tfrac{n+2}{2}\right) + \left(\tfrac{n-2}{1}\right)\left(\tfrac{n+2}{3}\right)aa + \left(\tfrac{n-2}{2}\right)\left(\tfrac{n+2}{4}\right)a^4 + \left(\tfrac{n-2}{3}\right)\left(\tfrac{n+2}{5}\right)a^6 + \left(\tfrac{n-2}{4}\right)\left(\tfrac{n+2}{6}\right)a^8 + \text{etc.}$$

tum vero, erit altera forma

$$\int \Delta^n \partial\phi \cos.2\phi = \frac{n(n-1)}{(n+1)(n+2)}\pi aa\mathrm{V}.$$

Percurramus ergo ut hactenus casus simpliciores, et quia integratio formulae $\int \Delta^n \partial\phi \cos.2\phi$ sponte patet ex ultima formula, superfluum foret haec integralia allegare

$$n=0: \int \frac{\partial\phi \cos.2\phi}{\Delta} = \frac{\pi aa}{1-aa}$$

$$n=1: \int \frac{\partial\phi \cos.2\phi}{\Delta^2} = \frac{\pi aa}{(1-aa)^3}(1.3-1.1aa)$$

$$n=2: \int \frac{\partial\phi \cos.2\phi}{\Delta^3} = \frac{\pi aa}{(1-aa)^5}(1.6)$$

$$n=3: \int \frac{\partial\phi \cos.2\phi}{\Delta^4} = \frac{\pi aa}{(1-aa)^7}(1.10+1.10aa)$$

$$n=4: \int \frac{\partial\phi}{\Delta^6}\cos.2\phi = \frac{\pi aa}{(1-aa)^9}\left(1.15+2.20aa+1.15a^4\right)$$

$$n=5: \int \frac{\partial\phi}{\Delta^6}\cos.2\phi = \frac{\pi aa}{(1-aa)^{11}}\left(1.21+3.35a^2+3.35a^4+1.21a^6\right)$$

$$n=6: \int \frac{\partial\phi}{\Delta^7}\cos.2\phi = \frac{\pi aa}{(1-aa)^{13}}\left(1.28+4.56aa+6.70a^4+4.56a^6+1.28a^8\right)$$

etc. etc.

Evolutio casuum quibus $\lambda = 3$ et formula integralis proposita

$$\int \frac{\partial\phi \cos.3\phi}{\Delta^{n+1}}.$$

§. 34. Hoc ergo casu integrale erit

$$\frac{\pi a^3}{(1-aa)^{2n+1}}.\mathrm{V},$$

existente

$$\mathrm{V} = \left(\tfrac{n-3}{0}\right)\left(\tfrac{n+3}{3}\right) + \left(\tfrac{n-3}{1}\right)\left(\tfrac{n+3}{4}\right)aa + \left(\tfrac{n-3}{2}\right)\left(\tfrac{n+3}{5}\right)a^4 + \left(\tfrac{n-3}{3}\right)\left(\tfrac{n+3}{6}\right)a^6 + \text{etc.}$$

pro altera autem formula habebimus

$$\int \Delta^n \partial\phi \cos.3\phi = -\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\pi a^3 \mathrm{V}.$$

Pro praecipuis igitur casibus habebimus sequentem tabellam

$$n=0: \int \frac{\partial\phi\cos.3\phi}{\Delta} = \frac{\pi a^3}{1-aa}$$
$$n=1: \int \frac{\partial\phi\cos.3\phi}{\Delta^2} = \frac{2\pi a^3}{(1-aa)^3}(1.4-2.1aa)$$
$$n=2: \int \frac{\partial\phi\cos.3\phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}(1.10-1.5aa)$$
$$n=3: \int \frac{\partial\phi\cos.3\phi}{\Delta^4} = \frac{\pi a^3}{(1-aa)^7}(1.20)$$
$$n=4: \int \frac{\partial\phi\cos.3\phi}{\Delta^6} = \frac{\pi a^3}{(1-aa)^9}(1.35+1.35aa)$$
$$n=5: \int \frac{\partial\phi\cos.3\phi}{\Delta^6} = \frac{\pi a^3}{(1-aa)^{11}}\left(1.56+2.70aa+1.56a^4\right)$$
$$n=6: \int \frac{\partial\phi\cos.3\phi}{\Delta^7} = \frac{\pi a^3}{(1-aa)^{13}}\left(1.84+3.126aa+3.126a^4+1.84a^6\right)$$

etc. etc.

Observatio circa negativos ipsius λ.

§. 35. Jam initio monuimus, pro lettera λ tantum numeros integros positives sumi oportere, qua conditione generalitas nostrae questionis non restringitur cum semper sit $\cos.-\lambda\phi = \cos.\lambda\phi$. Interim tamen hic ingens paradoxon se offert, quod solutiones supra inventae evadant falsae, quando ipsi λ valores negativi tribuuntur; quod quo clarius pateat consideremus casum $n = 0$; pro quo supra invenimus

$$\int \frac{\partial\phi\cos.\lambda\phi}{\Delta} = \frac{\pi a^\lambda}{1-aa},$$

unde videtur sequi debere, casu $\lambda = -i$ fore

$$\int \frac{\partial\phi\cos.i\phi}{\Delta} = \frac{\pi}{a^i(1-aa)},$$

quod autem manifesto est falsum, cum verum integrale utique sit $\frac{\pi a^{i}}{1-aa}$, perinde ac si esset $\lambda=+i$, At vero ista restrictio tantum est apparens, atque solutio nostra generalis nihilo minus veritati est consentanea, etiamsi litterae λ valores negativi tribuantur, dummodo fuerint integri; quandoquidem perpetuo assumimus, casu $\phi=\pi$ semper esse $\sin.\lambda\phi=0$; hoc igitur maxime operae erit pretium clarius ostendisse.

§. 36. Sufficiet autem, casum quo $n=0$ perpendisse, pro quo nostra solutio generalis praebet

$$\int\frac{\partial\phi\cos.\lambda\phi}{\Delta}=\frac{\pi a^{\lambda}}{1-aa}\mathrm{V},$$

existente

$$\mathrm{V}=\left(\tfrac{-\lambda}{0}\right)\left(\tfrac{\lambda}{\lambda}\right)+\left(\tfrac{-\lambda}{1}\right)\left(\tfrac{\lambda}{\lambda+1}\right)aa+\left(\tfrac{-\lambda}{2}\right)\left(\tfrac{\lambda}{\lambda+2}\right)a^{4}+\left(\tfrac{-\lambda}{3}\right)\left(\tfrac{\lambda}{\lambda+3}\right)a^{6}+\text{etc.}$$

Cujus expressionis tantum prima pars remanet, quando λ est numerus positive integer, propterea quod cum formulae $\left(\tfrac{\lambda}{\lambda+1}\right),\left(\tfrac{\lambda}{\lambda+2}\right),\left(\tfrac{\lambda}{\lambda+3}\right)$, etc. evanescunt ; longe secus autem se res habet, quando pro λ assumitur numerus negativus, veluti si ponamus $\lambda=-i$ tum erit

$$\mathrm{V}=\left(\tfrac{i}{0}\right)\left(\tfrac{-i}{-i}\right)+\left(\tfrac{i}{1}\right)\left(\tfrac{-i}{1-i}\right)aa+\left(\tfrac{i}{2}\right)\left(\tfrac{-i}{2-i}\right)a^{4}+\left(\tfrac{i}{3}\right)\left(\tfrac{-i}{3-i}\right)a^{6}+\text{etc.}$$

ubi notetur, omnium horum characterum, quamdiu denominator est negativus, valores evanescere; quoniam vero denominatores continuo crescunt, tandem evadent positivi, atque adeo valores determinatos exhibebunt. Ad hoc ostendendum ponamus primo $\lambda=-1$ sive $i=+1$, eritque $\mathrm{V}=-aa$ ubi primum membrum sine dubio est $=0$, secundum vero

$$\left(\tfrac{1}{1}\right)\left(\tfrac{+1}{0}\right)aa=aa,$$

Cum igitur sit $\mathrm{V}=aa$ casu $\lambda=-1$, nostra formula preabet hoc integrale

$$\int\frac{\partial\phi\cos.-\phi}{\Delta}=\frac{\pi a^{-1}}{1-aa}.aa=\frac{\pi a}{1-aa},$$

id quod prorsus convenit.

§. 37. Sumamus nunc $\lambda=-2$ sive $i=2$, manente $n=0$, eritque

$$\mathrm{V}=\left(\tfrac{2}{0}\right)\left(\tfrac{-2}{-2}\right)+\left(\tfrac{2}{1}\right)\left(\tfrac{-2}{-1}\right)aa+\left(\tfrac{2}{2}\right)\left(\tfrac{-2}{0}\right)a^{4},$$

ubi sequentes termini manifesto evanescunt; ob factores priores autem bini termini initiales etiam evanescunt ob denominatores negativos ; tertius autem terminus ob $\left(\frac{-2}{0}\right)=1$ praebet $\mathrm{V}=a^4$, consequenter casu $\lambda=-2$ habebimus

$$\int\frac{\partial\phi\cos.-2\phi}{\Delta}=\frac{\pi a^{-2}}{1-aa}.a^4=\frac{\pi aa}{1-aa},$$

prorsus atque invenimus pro $\int\frac{\partial\phi\cos.2\phi}{\Delta}$.

§. 38. Simili modo facile intelligitur, casu $\lambda=-3$ proditurum esse $\mathrm{V}=a^6$, eodemque modo casu $\lambda=-4$ reperietur $\mathrm{V}=a^8$, atque adeo in genere casu $\lambda=-i$ obtinebitur $\mathrm{V}=a^{2i}$, sicque hujus formulae $\int\frac{\partial\phi\cos.-i\phi}{\Delta}$ integrale erit

$$\frac{\pi a^{-i}}{1-aa}.a^{2i}=\frac{\pi a^{i}}{1-aa},$$

quod ipsum est integrale formula $\int\frac{\partial\phi\cos.i\phi}{\Delta}$, uti natura rei postulat.

§. 39. Talis autem egregius consensus locum habebit pro omnibus valoribus ipsius n. Sit enim verbi gratia $n=2$, et integratio nostra

$$\int\frac{\partial\phi\cos.\lambda\phi}{\Delta^3}=\frac{\pi a^{\lambda}}{(1-aa)^5}.\mathrm{V}$$

existente

$$\mathrm{V}=\left(\frac{2-\lambda}{0}\right)\left(\frac{2+\lambda}{\lambda}\right)+\left(\frac{2-\lambda}{1}\right)\left(\frac{2+\lambda}{\lambda+1}\right)aa+\left(\frac{2-\lambda}{2}\right)\left(\frac{2+\lambda}{\lambda+2}\right)a^4+\text{etc.}$$

quare sumto $\lambda=-3$, ut forma nostra sit

$$\int\frac{\partial\phi\cos.-3\phi}{\Delta^3}=\frac{\pi a^{-3}}{(1-aa)^5}.\mathrm{V},$$

existente

$$\mathrm{V}=\left(\frac{5}{0}\right)\left(\frac{-1}{-3}\right)+\left(\frac{5}{1}\right)\left(\frac{-1}{-2}\right)aa+\left(\frac{5}{2}\right)\left(\frac{-1}{-1}\right)a^4+\left(\frac{5}{3}\right)\left(\frac{-1}{0}\right)a^6$$
$$+\left(\frac{5}{4}\right)\left(\frac{-1}{1}\right)a^8+\left(\frac{5}{5}\right)\left(\frac{-1}{2}\right)a^{10},$$

ubi tria priora membra evanescunt, sequentia autem ob

$$\left(\frac{-1}{0}\right)=1,\ \left(\frac{-1}{1}\right)=-1,\ \left(\frac{-1}{2}\right)=1,$$

erit

$$\mathrm{V}=10a^6-5a^8+a^{10}=a^6\left(10-5aa+a^4\right),$$

consequenter nostrum integrale fit

$$\int \frac{\partial \phi \cos.-3\phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}\left(10-5aa+a^4\right),$$

prorsus uti supra invenimus pro casu $\int \frac{\partial \phi \cos.3\phi}{\Delta^3};$ talis autem consensus perpetuo deprehendi debebit.
