3) Concerning the integration of the formula $\int \frac{dxlx}{\sqrt{(1-xx)}}$, extended from $x=0$ to $x=1$.

Acta Acad. Imp. Sc. Tom. I. P. II. Pag. 3-28.

§. 117. A most natural method of treating formulas that consist of the kind $\int pdxlx$, shall be that they be reduced to other forms of the kind $\int qdx$, in which the letter q shall be an algebraic function of x; mainly since the rules of integration can be adapted to such formulas. But a reduction of this kind evidently labours with difficulty, thus when no function p has been prepared, so that the integral $\int pdx$ may be able to be shown algebraically. Indeed if there were $\int pdx = \mathrm{P}$, thus so that the proposed formula shall be $\int d\mathrm{P}lx$, that is reduced at once to this expression $\mathrm{P}lx - \int \frac{\mathrm{P}dx}{x}$, and thus now the whole concern has been replaced by the integration of this formula $\int \frac{\mathrm{P}dx}{x}$. Truly when the formula $\int pdx$ does not permit an algebraic integration, such as happens in our proposed formula $\int \frac{dxlx}{\sqrt{(1-xx)}}$, such a reduction cannot be completely successful. Indeed since there shall be $\int \frac{dx}{\sqrt{(1-xx)}} = \mathrm{Asin}.x$, this reduction will give

$$\int \frac{dxlx}{\sqrt{(1-xx)}} = \mathrm{Asin}.x \times lx - \int \frac{dx}{x} \cdot \mathrm{Asin}.x,$$

and thus if after the integral sign the new transcendental function $\mathrm{Asin}.x$ may encountered , whose integration is as equally obscure as that proposed. Whereby since recently I have found by an unusual method, that

$$\int \frac{dxlx}{\sqrt{(1-xx)}} \begin{bmatrix} \textit{from } x=0 \\ \textit{to } x=1 \end{bmatrix} = -\tfrac{1}{2}\pi l2,$$

there the expression of the integral is required to be worthy of consideration with greater attention, because the investigation of that is by no means obvious; from which I have considered it worth the effort to be showing it truth from other sources, before I set out that method itself, which has led me to that.

First demonstration of the proposed integration $\int \frac{dxlx}{\sqrt{(1-xx)}}$:

§.118. Because here mainly there is required to be recourse to infinite series, but lx is denied such a simple resolution formula, instead we may use the substitution $\sqrt{(1-xx)} = y$, from which there shall become $x = \sqrt{(1-yy)}$, and hence again

$$lx = -\frac{yy}{2} - \frac{y^4}{4} - \frac{y^6}{6} - \frac{y^8}{8} - \text{etc.}$$

therefore in this manner, the formula of the proposed integral $\int\frac{dxlx}{\sqrt{(1-xx)}}$ is transformed into the following form

$$\int\frac{dy}{\sqrt{(1-yy)}}\left(\frac{yy}{2}+\frac{y^4}{4}+\frac{y^6}{6}+\frac{y^8}{8}+\text{etc.}\right)$$

where, since there shall be $y=\sqrt{(1-xx)}$, it may be noted the integration must be extended from $y=1$ as far as to $y=0$; whereby if we wish to interchange these terms of the integration, it is required to change the sign of the whole formula.

§. 119. But so that we may be less confused by the change of such signs, we will designate the value sought by the letter S, so that there shall be

$$\mathrm{S}=\int\frac{dxlx}{\sqrt{(1-xx)}}\left[\begin{matrix}\textit{from } x=0\\ \textit{to } x=1\end{matrix}\right]$$

and with the substitution made $y=\sqrt{(1-xx)}$, we will have, as in the manner we have advised :

$$\int\frac{dy}{\sqrt{(1-yy)}}\left(\frac{yy}{2}+\frac{y^4}{4}+\frac{y^6}{6}+\frac{y^8}{8}+\text{etc.}\right)\left[\begin{matrix}\textit{from } y=0\\ \textit{to } y=1\end{matrix}\right].$$

But within the integrations of these terms, evidently from $y=0$ to $y=1$, it is understood well enough, how the individual parts which occur here, are to be reduced to the following values

$$\int\frac{yydy}{\sqrt{(1-yy)}}=\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\int\frac{y^4dy}{\sqrt{(1-yy)}}=\frac{1.3}{2.4}\cdot\frac{\pi}{2}$$

$$\int\frac{y^6dy}{\sqrt{(1-yy)}}=\frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}$$

$$\int\frac{y^8dy}{\sqrt{(1-yy)}}=\frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi}{2}$$

$$\int\frac{y^{10}dy}{\sqrt{(1-yy)}}=\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{\pi}{2}\text{ etc.}$$

where without doubt $\frac{\pi}{2}=\int\frac{dy}{\sqrt{(1-yy)}}$, thus so that $1:\pi$ may express the ratio of the diameter to the periphery of a circle.

§. 120. Therefore, since if we may introduce these individual values, the following infinite series will be obtained for the value S sought :

$$\mathrm{S}=-\frac{\pi}{2}\left(\frac{1}{2^2}+\frac{1.3}{2.4^2}+\frac{1.3.5}{2.4.6^2}+\frac{1.3.5.7}{2.4.6.8^2}+\text{etc.}\right)$$

and thus the whole procedure now has been reduced to this, so that the sum of this infinite series may be investigated ; which labor perhaps can hardly be seen to be made less troublesome, than that which had been proposed by us to be carried out. Yet meanwhile knowledge of the sum of this series can be reached without difficulty in the following manner by us.

§. 121. Since there shall be

$$\frac{1}{\sqrt{(1-zz)}}=1+\frac{1}{2}zz+\frac{1.3}{2.4}z^4+\frac{1.3.5}{2.4.6}z^6+\text{etc.}$$

if we may multiply each side by dz and integrate, we will obtain

$$\int\frac{dz}{z\sqrt{(1-zz)}}=lz+\frac{1}{2^2}zz+\frac{1.3}{2.4^2}z^4+\frac{1.3.5}{2.4.6^2}z^6+\text{etc.}$$

and thus we have been led to our same series, the value of which therefore must be sought from this expression $\int\frac{dz}{z\sqrt{(1-zz)}}-lz$, clearly with the integral taken thus, so that it shall vanish on putting $z=0$, with which done there may be put $z=1$, and that series will be produced:

$$\frac{1}{2^2}+\frac{1.3}{2.4^2}+\frac{1.3.5}{2.4.6^2}+\frac{1.3.5.7}{2.4.6.8^2}+\text{etc.}$$

Therefore in this way the whole procedure has led to the integration of this integral formula $\int\frac{dz}{z\sqrt{(1-zz)}}$, which on putting $\sqrt{(1-zz)}=v$ passes into this form $\frac{-dv}{1-vv}$, the integral of which is agreed to be $-\frac{1}{2}l\frac{1+v}{1-v}=-l\frac{1+v}{\sqrt{1-vv}}$. But if in place of v the value $\sqrt{(1-zz)}$ may be restored, the whole expression, which we need, thus itself will be had :

$$\int\frac{dz}{z\sqrt{(1-zz)}}-lz=-l\frac{\left[1+\sqrt{(1-zz)}\right]}{z}-lz+C$$
$$=C-l\left[1+\sqrt{(1-zz)}\right],$$

where the constant must be taken thus, so that the value may vanish, on putting $z=0$, and thus there will be $\mathrm{C}=l2$. On account of which, on putting $z=1$, the sum of the series sought will be $l2$, and hence the value of the proposed integral formula itself will be :

$$\int\frac{dxlx}{\sqrt{(1-xx)}}=\mathrm{S}=-\frac{\pi}{2}l2:$$

precisely until I had found the answer by another method, from which it is understood well enough now, that same truth to be of a higher investigation, and thus worthy of the greatest attention of the geometers.

Another demonstration of the proposed integration.

§. 122. Since $\frac{dx}{\sqrt{(1-xx)}}$ shall be the element of the arc of a circle of which the sine $= x$, we may put this same angle $= \varphi$, thus so that there shall be

$$x = \sin.\varphi \text{ and } \frac{dx}{\sqrt{(1-xx)}} = d\varphi,$$

and with this substitution made the value of the quantity S, we are looking for, will be represented thus :

$$\mathrm{S} = \int d\varphi l \sin.\varphi \begin{bmatrix} from\ \varphi=0 \\ to\ \varphi=90^0 \end{bmatrix}.$$

Indeed since before the terms were $x = 0$ et $x = 1$, now to these correspond $\varphi = 0,$ and $\varphi = 90^0$ or $\varphi = \frac{\pi}{2}.$ Therefore here the whole investigation reverts to that, so that the formula $l\sin.\varphi$ may be changed conveniently into an infinite series. Finally we may put $l\sin.\varphi = s$, and there will become $ds = \frac{d\varphi\cos.\varphi}{\sin.\varphi}.$ But now we know that

$$\frac{\cos.\varphi}{\sin.\varphi} = 2\sin.2\varphi + 2\sin.4\varphi + 2\sin.6\varphi + 2\sin.8\varphi + \text{etc.}$$

For if we may multiply each side by $\sin.\varphi$; on account of

$$2\sin.n\varphi\sin.\varphi = \cos.(n-1)\varphi - \cos.(n+1)\varphi,,$$

certainly there will become

$$\cos.\varphi = \cos.\varphi + \cos.3\varphi + \cos.5\varphi + \cos.7\varphi + \cos.\,9\varphi + \text{etc.}$$
$$- \cos.3\varphi - \cos.5\varphi - \cos.7\varphi - \cos.\,9\varphi - \text{ etc.}$$

Therefore by calling this series for $\frac{\cos.\varphi}{\sin.\varphi}$ into use, there will become

$$s = \mathrm{C} - \cos.2\varphi - \tfrac{1}{2}\cos.4\varphi - \tfrac{1}{3}\cos.6\varphi - \tfrac{1}{4}\cos.8\varphi - \tfrac{1}{5}\cos.10\varphi - \text{etc.}$$

where since there shall be $s = l\sin.\varphi,$ and thus $s = 0,$ when $\sin.\varphi = 1,$ and thus $\varphi = \frac{\pi}{2},$ thus it is required to define the constant C, so that on putting $\varphi = \frac{\pi}{2} = 90^0,$ there may become $s = 0,$ from which it is deduced that

$$\mathrm{C} = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l2.$$

§.123. Therefore since there shall be :

$$l\sin.\varphi = -l2 - \cos.2\varphi - \tfrac{1}{2}\cos.4\varphi - \tfrac{1}{3}\cos.6\varphi - \tfrac{1}{4}\cos.8\varphi - \text{etc.}$$

the value of the proposed formulae shall become :

$$\int d\varphi l\sin.\varphi = \mathrm{C} - \varphi l2 - \tfrac{1}{2}\sin.2\varphi - \tfrac{1}{8}\sin.4\varphi - \tfrac{1}{18}\sin.6\varphi$$
$$- \tfrac{1}{32}\sin.8\varphi - \tfrac{1}{50}\sin.10\varphi - \text{etc.}$$

which expression since it must vanish on putting $\varphi = 0$, the constant entered here will be $\mathrm{C} = 0$, thus so that now in general there will become

$$\int d\varphi l\sin.\varphi = -\varphi l2 - \frac{2\sin.2\varphi}{2^2} - \frac{2\sin.4\varphi}{4^2} - \frac{2\sin.6\varphi}{6^2} - \frac{2\sin.8\varphi}{8^2}$$
$$- \frac{2\sin.10\varphi}{10^2} - \frac{2\sin.12\varphi}{12^2} - \text{etc.}$$

But if now here there may be taken $\varphi = \frac{\pi}{2} = 90^0$, of all the angles $2\varphi,\ 4\varphi,\ 6\varphi,\ 8\varphi,\ \text{etc.}$, which occur here the sines will vanish, and thus the value sought will be

$$\mathrm{S} = \int d\varphi l\sin.\varphi \left[\begin{smallmatrix} \text{from } \varphi=0 \\ \text{to } \varphi=90^0 \end{smallmatrix}\right] = -\tfrac{\pi}{2} l2;$$

just as also we have shown in the former demonstration.

§. 124. But this same demonstration by the preceding far surpasses not only the value of the formula proposed by us for the case in which it may be shown for $\varphi = 90^0$, but also it shall show its true value, whatever angle may be taken for φ r, that which can be transferred to that same formula $\int \frac{dxlx}{\sqrt{(1-xx)}}$, whose value we will be able to assign precisely for any value of *x* itself. So that if indeed we should require the value of the formula from $x = 0$ as far as to $x = a$, the angle α may be sought the sine of which shall be equal to α itself, and there may be had always :

$$\mathrm{S} = \int d\varphi l\sin.\varphi \left[\begin{smallmatrix} \text{from } \varphi=0 \\ \text{to } \varphi=90^0 \end{smallmatrix}\right] = -\alpha l2 - \frac{2\sin.2\alpha}{2^2} - \frac{2\sin.4\alpha}{4^2} - \frac{2\sin.6\alpha}{6^2} - \frac{2\sin.8\alpha}{8^2} - \text{etc.}$$

From which it is apparent, whenever there were $\alpha = \frac{i\pi}{2}$, with *i* denoting some whole number, since all the sines will vanish, the value of the formula in these cases can be expressed by $-\frac{i\pi}{2}l2$; but truly in all the other cases the value of our formula will be expressed well enough by the infinite series . Thus if $a = \frac{1}{\sqrt{2}}$ may be taken, so that there shall be $\alpha = \frac{\pi}{4}$, the value of our formula will become

$$-\tfrac{\pi}{4}l2 - \frac{2}{2^2} + \frac{2}{6^2} - \frac{2}{10^2} + \frac{2}{14^2} - \frac{2}{18^2} + \frac{2}{22^2} - \text{etc.}$$

which series thus may be expressed more elegantly :

$$-\frac{\pi}{4}l2-\frac{1}{2}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\text{etc.}\right);$$

and thus there occurs here the memorable series

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\frac{1}{121}+\text{etc.},$$

whose sum hitherto in no way can be recalled according to known measures.

§.125. Because such an outstanding series has presented itself here as if beyond expectation, we may establish still other more notable cases, and we may take $a=\frac{1}{2}$, so that there shall become $\alpha=30°=\frac{\pi}{6}$, and in this case the value of our formula will be :

$$-\frac{\pi}{6}l2-\frac{\sqrt{3}}{2^2}-\frac{\sqrt{3}}{4^2}+\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}-\frac{\sqrt{3}}{16^2}+\text{etc.}$$

which expression can be shown thus:

$$-\frac{\pi}{6}l2-\frac{\sqrt{3}}{4}\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right)$$

in which series the squares of multiples of three are missing. In a like manner, we may take $a=\frac{\sqrt{3}}{2}$, so that there shall be $\alpha=60°=\frac{\pi}{3}$, and the value of our formula produced in this case will become

$$-\frac{\pi}{3}l2-\frac{\sqrt{3}}{2^2}+\frac{\sqrt{3}}{4^2}-\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}+\frac{\sqrt{3}}{16^2}-\text{etc.}$$

or it may be expressed in this manner:

$$-\frac{\pi}{3}l2-\frac{\sqrt{3}}{4}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right)$$

Yet another demonstration of the proposed integration.

§. 126. The angle φ may be introduced into our formula , the cosine of which shall be $=x$, or there shall become $x=\cos.\varphi$, and our formula will adopt this form $-\int d\varphi l\cos.\varphi$, [*i.e.* $\frac{dx}{\sqrt{(1-xx)}}=\frac{-\sin\varphi}{\sin\varphi}=-1$] which integral being extended from $\varphi=90°$ as far as to $\varphi=0$. But if we may interchange these limits, the value S, which we seek, may be expressed thus :

$$\mathrm{S}=\int d\varphi l\cos.\varphi\begin{bmatrix}\text{from } \varphi=0\\ \text{to } \varphi=90^0\end{bmatrix}.$$

So that we may convert this $l\cos.\varphi$ into a suitable series, as before we may put $s=l\cos.\varphi$ and here will become $ds=-\frac{d\varphi\sin.\varphi}{\cos.\varphi}$. Moreover it is agreed to be put by the series

$$\frac{\sin.\varphi}{\cos.\varphi}=2\sin.2\varphi-2\sin.4\varphi+2\sin.6\varphi-2\sin.8\varphi+\text{etc.}$$

Since in general there shall become

$$2\sin.n\varphi\ \cos.\varphi=\sin.(n+1)\varphi+\sin.(n-1)\varphi,$$

if we may multiply each side by $\cos.\varphi$, there shall become

$$\begin{aligned}\sin.\varphi&=2\sin.2\varphi\cos.\varphi-2\sin.4\varphi\cos.\varphi+2\sin.6\varphi\cos.\varphi-2\sin.8\varphi\cos.\varphi+\text{etc.}\\&=\sin.3\varphi-\sin.5\varphi+\sin.7\varphi-\sin.9\varphi+\text{etc.}\\&+\sin.\varphi-\sin.3\varphi+\sin.5\varphi-\sin.7\varphi+\sin.9\varphi-\text{etc.}\end{aligned}$$

whereby since there shall be $ds=-\frac{d\varphi\sin.\varphi}{\cos.\varphi}$, now there will become

$$ds=-\frac{d\varphi\sin.\varphi}{\cos.\varphi}=-2\sin.2\varphi d\varphi+2\sin.4\varphi d\varphi-2\sin.6\varphi d\varphi+2\sin.8\varphi d\varphi+\text{etc.}$$

$$s=\mathrm{C}+\frac{\cos.2\varphi}{1}-\frac{\cos.4\varphi}{2}+\frac{\cos.6\varphi}{3}-\frac{\cos.8\varphi}{4}+\frac{\cos.10\varphi}{5}-\text{etc.}$$

Therefore because $s=l\cos.\varphi$, clearly on putting $\varphi=0$, there must become $s=0$, from which it is deduced that :

$$\mathrm{C}=-1+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{5}+\text{etc.}=-l2;$$

and thus there will become

$$l\cos.\varphi=-l2+\frac{\cos.2\varphi}{1}-\frac{\cos.4\varphi}{2}+\frac{\cos.6\varphi}{3}-\frac{\cos.8\varphi}{4}+\text{etc.}$$

which series multiplied by $d\varphi$ and integrated produces

$$\mathrm{S}=\int d\varphi l\cos.\varphi=\mathrm{C}-\varphi l2+\frac{\sin.2\varphi}{2}-\frac{\sin.4\varphi}{8}+\frac{\sin.6\varphi}{18}-\frac{\sin.8\varphi}{32}+\frac{\sin.10\varphi}{50}-\text{etc.}$$

which expression since it vanishes at once on putting $\varphi=0$, thence it is apparent that $\mathrm{C}=0$, and thus we will have :

$$\int d\varphi l\cos.\varphi=-\varphi l2+\tfrac{1}{2}\left(\frac{\sin.2\varphi}{1}-\frac{\sin.4\varphi}{2^2}+\frac{\sin.6\varphi}{3^2}-\frac{\sin.8\varphi}{4^2}+\frac{\sin.10\varphi}{5^2}-\text{etc.}\right)$$

Therefore on taking $\alpha = \frac{\pi}{2} = 90°$, as before there becomes $\mathrm{S} = -\frac{\pi}{2} l2.$ Truly in addition also this integral can be extended to any limit.

§. 127. But if we may subtract the latter formula from the preceding one, we may in general come upon this integration:

$$\int d\varphi l \tan.\varphi = -\sin.2\varphi - \frac{1}{3^2}\sin.6\varphi - \frac{1}{5^2}\sin.10\varphi - \text{etc.}$$

from which it is apparent that this integral vanishes in the cases $\varphi = 0^0$ and in general when $\varphi = \frac{i\pi}{2}$. Therefore after we have demonstrated this same integration in three ways, this same analysis, which had been introduced first by me, I am going to set out here clearly.

Analysis leading to the integration of the formula $\int \frac{dxlx}{\sqrt{(1-xx)}}$ and to others of a similar kind.

§. 128. This whole analysis relies on the following lemmas pointed out by me now some time ago: For the sake of brevity, putting $\left(1-x^n\right)^{\frac{m-n}{n}} = \mathrm{X}$, if hence the two integral formulas may be formed $\int \mathrm{X}x^{p-1}dx$ and $\int \mathrm{X}x^{q-1}dx$, which may be extended from the limit $x=0$ as far as to the limit $x=1$, the ratio of these values can be reduced to a product from an infinite number of factors put together in the following manner:

$$\frac{\int \mathrm{X}x^{p-1}dx}{\int \mathrm{X}x^{q-1}dx} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{etc.}$$

where clearly the factors both of the numerators, as well as of the denominators, may be increased continually by the same amount *n*. But here, it is to be understood properly, the truth of this lemma cannot remain, unless the individual *m*, *n*, *p* and *q* denote positive numbers, which still always must be regarded as whole numbers.

§. 129. Concerning these two integral formulas, extended from the limit $x=0$ as far as to the limit $x=1$, two separate cases are especially noteworthy, in which the integration actually succeeds, and the absolute true value can be assigned. The first special case occurs, if there were $p=n$, thus to that the formula shall become $\int \mathrm{X}x^{n-1}dx$. For on putting $x^n = y$ there will become

$$\mathrm{X} = (1-y)^{\frac{m-n}{n}}, \text{ and } x^{n-1}dx = \frac{1}{n}dy$$

and thus this same formula emerges $\frac{1}{n}\int dy(1-y)^{\frac{m-n}{n}}$, equally from the limit $y=0$ being extended as far as to $y=1$, which again on putting $1-y=z$ will be changed into this formula $-\frac{1}{n}\int z^{\frac{m-n}{n}}dz$, extending from the limit $z=1$ as far as to $z=0$; therefore clearly the integral of this is $-\frac{1}{m}z^{\frac{m}{n}}+\frac{1}{m}$; from which on putting $z=0$ the value will become $\frac{1}{m}$. Consequently, for the case $p=n$, we will have :

$$\int Xx^{n-1}dx\left[\begin{matrix}\text{from } x=0\\ \text{to } x=1\end{matrix}\right]=\frac{1}{m};$$

and thus if there were either $p=n$ or $q=n$; the absolute integral becomes known.

§. 130. The other noteworthy case is when $p=n-m$, thus so that the formula being integrated shall become $\int Xx^{n-m-1}dx$; since then, if there may be put

$$x\left(1-x^n\right)^{\frac{-1}{n}} \text{ or } \frac{x}{\left(1-x^n\right)^{\frac{1}{n}}}=y,$$

so that on putting
$x=0$ there shall become $y=0$, but on putting $x=1$ there shall be $y=\infty$; then moreover there will become

$$y^{n-m}=\frac{x^{n-m}}{\left(1-x^n\right)^{\frac{n-m}{n}}}=Xx^{n-m},$$

from which the formula requiring to be integrated will be $\int y^{n-m}\frac{dx}{x}$. Therefore since there will be

$$\frac{x}{\left(1-x^n\right)^{\frac{1}{n}}}=y, \text{ there will become } \frac{x^n}{1-x^n}=y^n,$$

from which it is deduced that $x^n=\frac{y^n}{1+y^n}$, and thus $nlx=nly-l\left(1+y^n\right)$, the differential of which produces

$$\frac{dx}{x}=\frac{dy}{y\left(1+y^n\right)},$$

with which value substituted our formula being integrated will become

$$\int\frac{y^{n-m-1}dy}{1+y^n},$$

with the limits extending from $y=0$ as far as to $y=\infty$, which formula thus is most noteworthy, because it is freed from all irrationality.

§. 131. Therefore because in this case for the rational formula we have deduced, it is agreed from the elements of the integral calculus, its integration can be resolved always by logarithms and circular arcs, then truly for this case thus I have shown not so long ago, the integration of this formula $\int \frac{x^{m-1}dx}{1+x^n}$, extending from $x=0$ to $x=\infty$, to be reduced to the value $\frac{\pi}{n\sin.\frac{m\pi}{n}}$; therefore with the application made for our case, we will have :

$$\int \frac{y^{n-m-1}dy}{1+y^n} = \frac{\pi}{n\sin.\frac{(n-m)\pi}{n}} = \frac{\pi}{n\sin.\frac{m\pi}{n}};$$

on which account for the case $p=n-m$ the value of the integral can be expressed in the following manner; and there will be:

$$\int \mathrm{X}x^{n-m-1}\partial x \begin{bmatrix} \text{from } x=0 \\ \text{to } x=1 \end{bmatrix} = \frac{\pi}{n\sin.\frac{m\pi}{n}},$$

because the same evidently is understood, if there were $q=n-m$.

§. 132. From these premises, again we may put for the sake of brevity:

$$\int \mathrm{X}x^{p-1}dx \begin{bmatrix} \text{from } x=0 \\ \text{to } x=1 \end{bmatrix} = \mathrm{P}$$

and

$$\int \mathrm{X}x^{q-1}dx \begin{bmatrix} \text{from } x=0 \\ \text{to } x=1 \end{bmatrix} = \mathrm{Q},$$

and the lemma established by us presents this equation :

$$\frac{\mathrm{P}}{\mathrm{Q}} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{etc.}$$

hence therefore with logarithms taken we deduce

$$\begin{aligned} l\mathrm{P} - l\mathrm{Q} &= l(m+p) - lp + l(m+p+n) - l(p+n) + l(m+p+2n) - l(p+2n) + \text{etc.} \\ &+ lq - l(m+q) + l(q+n) - (m+q+n) + l(q+2n) - l(m+q+2n) + \text{etc.} \end{aligned}$$

and this equation will always be found, whatever values may attributed to the letters *m, n, p* and *q* , provided they were positive.

§. 133. Therefore since in general this equality may remain, also there will be agreement about the truth, when some of these letters *m, n, p*, and *q* are made infinitely small, or may be considered as variables. Hence on this account we will consider only the quantity p as variable, thus so that the remaining letters m, n and q will remain constant, and thus also the quantity Q will be constant while the other P is varied; from which by differentiation we may arrive at this equation

$$\frac{d\mathrm{P}}{\mathrm{P}}=\frac{dp}{m+p}-\frac{dp}{p}+\frac{dp}{m+p+n}-\frac{dp}{p+n}+\frac{dp}{m+p+2n}-\frac{dp}{p+2n}$$
$$+\ \frac{dp}{m+p+3n}-\frac{dp}{p+3n}+\text{etc.}$$

where the whole calculation is reduced to that, how the differential of the formula P, which is integral, will be required to be expressed.

§. 134. Therefore since P shall be the integral formula involving the quantity x only as the variable, since in its integration the exponent p must be treated as constant, finally after the integration the quantity P can be considered as a function of the two variables x and p ; from which the question arises from this: it shall be required to investigate how the expression $\left(\frac{d\mathrm{P}}{dp}\right)$ is accustomed to be expressed, which if it may be indicated by the letter Π, the equation found before will adopt this form :

$$\frac{\Pi}{\mathrm{P}}=\frac{1}{m+p}-\frac{1}{p}+\frac{1}{m+p+n}-\frac{1}{p+n}+\frac{1}{m+p+2n}-\frac{1}{p+2n}+\text{etc.}$$

Hence indeed the infinite series can be replaced by a finite expression without difficulty in this manner : There may be put :

$$s=\frac{v^{m+p}}{m+p}-\frac{v^{p}}{p}+\frac{v^{m+p+n}}{m+p+n}-\frac{v^{p+n}}{p+n}+\frac{v^{m+p+2n}}{m+p+2n}-\frac{v^{p+2n}}{p+2n}+\text{etc.}$$

thus so that by setting $v=1$ the letter s may show the value sought $\frac{\Pi}{\mathrm{P}}$ for us ; but truly on differentiation it will give us :

$$\frac{ds}{dv}=v^{m+p-1}-v^{p-1}+v^{m+p+n-1}-v^{p+n-1}+v^{m+p+2n-1}-v^{p+2n-1}+\text{etc.},$$

of which the sum of the infinite series clearly is :

$$\frac{v^{m+p-1}-v^{p-1}}{1-v^{n}}=\frac{v^{p-1}\left(v^{m}-1\right)}{1-v^{n}}.$$

Hence therefore we may conclude to become in turn :

$$s=\int\frac{v^{p-1}\left(v^{m}-1\right)dv}{1-v^{n}},$$

the integration of which formula is required to be extended from $v = 0$ as far as to $v = 1$; and thus we will have

$$\frac{\Pi}{\mathrm{P}} = \int \frac{v^{p-1}\left(v^m - 1\right)dv}{1-v^n}, \left[\begin{array}{l} \textit{from } v = 0 \\ \text{to } v = 1 \end{array}\right].$$

§. 135. Moreover, for the value $\left(\frac{d\mathrm{P}}{dp}\right)$ requiring to be investigated, which we have indicated here by the letter Π, it is now well established from the principles of the calculus applied to functions of two variables , with the differential of the integral formula $\mathrm{P} = \int \mathrm{X}x^{p-1}dx$ to be obtained arising from the variation of the variable *p* itself, if the formula $\mathrm{X}x^{p-1}$ put after the sign of the integral, may be differentiated from the variation of *p* itself, and the element *dp* may be prefixed by the integration sign; but because truly X does not contain *p*, here it must be treated as a constant: indeed with the differential of the power x^{p-1} hence arising will become $x^{p-1}dplx$; which on this account, this differentiation there may arise $d\mathrm{P} = dp\int \mathrm{X}x^{p-1}dxlx$, thus so that finally after the integral sign the factor *lx* will be attached, from which clearly there becomes

$$\Pi = \int \mathrm{X}x^{p-1}dxlx \left[\begin{array}{l} \textit{from } x = 0 \\ \textit{to } x = 1 \end{array}\right],$$

hence, therefore, the following general theorem may be put in place.

General Theorem .

§. 136. On putting $\mathrm{X} = \left(1 - x^n\right)^{\frac{m-n}{n}}$ for the sake of brevity, if all the following integral formulas may be extended from the limit $x = 0$ to the limit $x = 1$, the following equality shall always be agreed to be true :

$$\frac{\int \mathrm{X}x^{p-1}dxlx}{\int \mathrm{X}x^{p-1}dx} = \int \frac{x^{p-1}\left(x^m - 1\right)dx}{1-x^n}$$

for nothing stands in the way, whereby otherwise we may write *x* in place of *v*, since these values depend only on the limits of the integration.

§. 137. Therefore in this manner we have deduced $\int \mathrm{X}x^{p-1}dxlx$ according to the integration of formulas of this kind, in which the logarithmic quantity *lx* after the integral sign belongs as a factor, the value of which may be expressed by the two regular integral formulas, since there shall become :

$$\int Xx^{p-1}dxlx = \int Xx^{p-1}dx \cdot \int \frac{x^{p-1}\left(x^m-1\right)dx}{1-x^n},$$

with the integral clearly extended from $x=0$ to $x=1$, where for the sake of brevity we have put $\left(1-x^n\right)^{\frac{m-n}{n}} = X$. Hence therefore we derive two particular theorems for both the memorable cases established above.

Particular theorem I, where $p=n$.

§. 138. Because above we have seen in the case $p=n$ the integral $\int Xx^{n-1}dx = \frac{1}{m}$, with this value substituted we will have the same elegant equation satisfied:

$$\int Xx^{n-1}dxlx = \frac{1}{m}\int \frac{x^{n-1}\left(x^m-1\right)dx}{1-x^n},$$

while clearly both integrals are extended from $x=0$ to $x=1$.

Particular theorem II, where $p=n-m$.

§. 139. Because for this case, where $p=n-m$, we have shown above that there shall be

$$\int Xx^{n-m-1}dx = \frac{\pi}{n\sin.\frac{m\pi}{n}},$$

now we are led to the following most noteworthy integration:

$$\int Xx^{n-m-1}\partial xlx = \frac{\pi}{n\sin.\frac{m\pi}{n}} \cdot \int \frac{x^{n-m-1}\left(x^m-1\right)\partial x}{1-x^n},$$

if indeed both these integrals may be extended from $x=0$ as far as to $x=1$; where it is required to be remembered that

$$X = \left(1-x^n\right)^{\frac{m-n}{n}}.$$

§. 140. Therefore here it may be noted properly, the general theorem extends the widest, because three indefinite exponents are present in that, clearly *m, n* and *p*, which can be put in place by us quite arbitrarily, which therefore may be defined in an infinite number of ways as it pleases, provided the individual values may be given to be positive, thus so that the value of this formula $\int Xx^{p-1}dxlx$, as it will be required to be regarded as transcendental on account of the factor lx, may be able to be expressed always by the ordinary integral formulas , which since they shall be the most general, it will be worth the effort to present some special cases.

I. Establishing the case where $m=1$ and $n=2$.

§. 141. Therefore in this case there will become $\mathrm{X}=\frac{1}{\sqrt{(1-xx)}}$, so that for this case from the general theorem there will become :

$$\int\frac{x^{p-1}\partial xlx}{\sqrt{(1-xx)}}=-\int\frac{x^{p-1}\partial x}{\sqrt{(1-xx)}}\cdot\int\frac{x^{p-1}\partial x}{1+x},$$

if indeed these individual integrations may be extended from $x=0$ to $x=1$. Therefore since here only the exponent p is left to our choice, hence we will illustrate by the following examples.

Example I., where $p=1$.

§. 142. Therefore in this case the above equation will adopt this form

$$\int\frac{\partial xlx}{\sqrt{(1-xx)}}=-\int\frac{\partial x}{\sqrt{(1-xx)}}\cdot\int\frac{x\partial x}{1+x}$$

where, with the integration extended from $x=0$ to $x=1$, it is noted to become,

$$\int\frac{\partial x}{\sqrt{(1-xx)}}=\frac{\pi}{2}\text{ and }\int\frac{\partial x}{1+x}=l2;$$

thus so that we shall have,

$$\int\frac{\partial xlx}{\sqrt{(1-xx)}}=\begin{bmatrix}\textit{from } x=0\\ \textit{to } x=1\end{bmatrix}=-\frac{\pi}{2}l2,$$

which is that formula itself, that we have treated at the start of this dissertation, and the truth of which we have corroborated now in a three-fold demonstration.

§. 143. The same value may be permitted to be deduced from the second particular theorem, where there was $p=n-m$, if indeed now on account of $n=2$ and $m=1$ there will become $p=1$; for hence since $\mathrm{X}=\frac{1}{\sqrt{(1-xx)}}$, that same theorem becomes

$$\int\frac{\partial xlx}{\sqrt{(1-xx)}}=\frac{\pi}{2\sin.\frac{\pi}{2}}\cdot\int\frac{x\partial x}{1+x}=-\frac{\pi}{2}l2.$$

Example II., where $p = 2$.

§. 144. Therefore in this case the equation will adopt this form :

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} = -\int \frac{x\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x\partial x}{1+x}$$

Now truly with the integrations extended from $x = 0$ to $x = 1$, it is known that

$$\int \frac{x\partial x}{\sqrt{(1-xx)}} = 1 \text{ and } \int \frac{x\partial x}{1+x} = 1 - l2;$$

thus so that we may

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} \left[\begin{matrix} \textit{from } x = 0 \\ \text{to } x = 1 \end{matrix} \right] = l2 - 1.$$

§. 145. Because in this integral formula $\int \frac{x\partial x}{\sqrt{(1-xx)}}$, it can be shown algebraically, since it shall be $= 1 - \sqrt{(1-xx)}$, the value sought can also be determined by the customary reductions, since there shall be

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} = \left[1 - \sqrt{(1-xx)}\right] lx - \int \frac{\partial x}{x} \left[1 - \sqrt{(1-xx)}\right],$$

and on putting $x = 1$, there will be

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} = -\int \frac{\partial x}{x} \left[1 - \sqrt{(1-xx)}\right],$$

the integrand may be made into the form $1 - \sqrt{(1-xx)} = z$, from which it is deduced that $xx = 2z - zz$, therefore $2lx = lz + l(2-z)$, and thus there will become $\frac{\partial x}{x} = \frac{\partial z(1-z)}{z(2-z)}$, from which with the values substituted there will be

$$+\int \frac{\partial x}{x} \left[1 - \sqrt{(1-xx)}\right] = +\int \frac{\partial z(1-z)}{z(2-z)},$$

therefore which value will become $= \mathrm{C} - z - l(2-z)$. Since therefore on putting $x = 0$ there becomes $z = 0$, the constant will be $\mathrm{C} = +\, l2$; therefore on making $x = 1$, because then there becomes fit $z = 1$, this value of the integral will be $l2 - 1$, just as before.

§. 146. The theorem supplies the same value brought forwards above earlier, where there was $p = n = 2$; thence indeed at once there becomes

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} = \int -\frac{x\partial x}{1+x}.$$

But we have seen before that $\int \frac{x\partial x}{1+x} = 1 - l2$; thus so that the value $l2-1$ may also be produced.

Example III., where $p = 3$.

§. 147. Therefore in this case the equation advanced in the general case adopts this form :

$$\int \frac{xx\partial x lx}{\sqrt{(1-xx)}} = -\int \frac{xx\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{xx\partial x}{1+x}.$$

Moreover through the most noteworthy reductions there is agreed to be :

$$\int \frac{xx\partial x}{\sqrt{(1-xx)}} \left[\begin{array}{l} \text{from } x=0 \\ \text{to } x=1 \end{array}\right] = \frac{1}{2} \cdot \frac{\pi}{2},$$

but indeed through a spurious fraction, $\frac{xx}{1+x}$ is resolved into these parts $x - 1 + \frac{1}{1+x}$, from which there becomes

$$\int \frac{xx\partial x}{1+xx} = \frac{1}{2}xx - x + l(1+x),$$

which integral now vanishes on putting $x = 0$; therefore on making $x = 1$ its value will become $= -\frac{1}{2} + l2$; on account of which, the integral we seek will be

$$\int \frac{xx\partial x}{\sqrt{(1-xx)}} \left[\begin{array}{ll} \text{from} & x=0 \\ \text{to} & x=1 \end{array}\right] = -\frac{\pi}{4}\left(l2 - \frac{1}{2}\right).$$

Example IV., where $p = 4$.

§. 148. Therefore in this case the above equation adopts this form :

$$\int \frac{x^3\partial x lx}{\sqrt{(1-xx)}} = -\int \frac{x^3\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^3\partial x}{1+x}.$$

Moreover, by the most noteworthy reductions, there is agreed to become :

$$\int \frac{x^3 \partial x}{\sqrt{(1-xx)}} \begin{bmatrix} \textit{from } x=0 \\ \textit{to} \quad x=1 \end{bmatrix} = \frac{2}{3},$$

then truly the spurious fraction $\frac{x^3}{1+x}$ is resolved into these parts $xx - x + 1 - \frac{1}{1+x}$, from which by integrating there becomes :

$$\int \frac{x^3 \partial x}{1+x} = \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x),$$

from which the value of the formula [when $x = 1$] will become $= \frac{5}{6} - l2$. With these values substituted we arrive at this integration :

$$\int \frac{x^3 \partial x}{\sqrt{(1-xx)}} \begin{bmatrix} \textit{from } x=0 \\ \textit{to} \quad x=1 \end{bmatrix} = -\frac{2}{3}\left(\frac{5}{6} - l2\right).$$

Example V., where $p = 5$.

§. 149. Therefore in this case the above equation adopts this form:

$$\int \frac{x^4 \partial x lx}{\sqrt{(1-xx)}} = -\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^4 \partial x}{1+x}.$$

Moreover there is agreed to be:

$$\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} \begin{bmatrix} \textit{from } x=0 \\ \textit{to} \quad x=1 \end{bmatrix} = \frac{1.3}{2.4} \cdot \frac{\pi}{2},$$

then truly the left over fraction $\frac{x^4}{1+x}$ clearly can be resolved into these parts : $x^3 - xx + x - 1 + \frac{1}{x+1}$, from which on integrating there becomes

$$\int \frac{x^4 \partial x}{1+x} = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}xx - x + l(1+x),$$

from which the value of the formula becomes $= -\frac{7}{12} + l2$. Therefore with these values substituted this integration will be produced :

$$\int \frac{x^4 \partial x lx}{\sqrt{(1-xx)}} \begin{bmatrix} \textit{from } x=0 \\ \textit{to} \quad x=1 \end{bmatrix} = -\frac{1.3}{2.4} \cdot \frac{\pi}{2}\left(l2 - \frac{7}{12}\right).$$

Example VI., where $p=6$.

§. 150. Therefore in this case the above equation adopts this form:

$$\int \frac{x^5 \partial x lx}{\sqrt{(1-xx)}} = -\int \frac{x^5 \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^5 \partial x}{1+x}.$$

But there is agreed by these notable reductions to become

$$\int \frac{x^5 \partial x lx}{\sqrt{(1-xx)}} \left[\begin{array}{l} from\ x=0 \\ to \quad\ \ x=1 \end{array} \right] = \frac{2.4}{3.5} \cdot \frac{\pi}{2},$$

then truly the extra fraction $\frac{x^5}{1+x}$ is resolved into these parts :

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1},$$

from which on integrating we obtain :

$$\int \frac{x^5 \partial x}{1+x} = \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x),$$

from which the value of this formula [when $x=1$] will become $= \frac{47}{60} - l2$; from which with the values substituted , this integration will be produced:

$$\int \frac{x^5 \partial x lx}{\sqrt{(1-xx)}} \left[\begin{array}{l} from\ x=0 \\ to \quad\ \ x=1 \end{array} \right] = \frac{2.4}{3.5} \cdot \frac{\pi}{2}\left(\frac{47}{60} - l2 \right).$$

[In the original text, the factor $\frac{\pi}{2}$ has been omitted.]

II. The case established when $m=3$ and $n=2$.

§. 151. Here therefore there becomes $\mathrm{X} = \sqrt{(1-xx)}$, from which our general theorem will present us with this equation :

$$\int x^{p-1} \partial x lx \cdot \sqrt{(1-xx)} = \int x^{p-1} \partial x \cdot \sqrt{(1-xx)} \cdot \int \frac{x^{p-1}(x^3-1)\partial x}{1-xx},$$

where since there shall be :

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x - \frac{1}{x+1},$$

the last formula of the integral will become :

$$-\int x^p \partial x - \int \tfrac{x^{p-1}\partial x}{1+x};$$

which integrated from $x=0$ to $x=1$ gives:

$$-\tfrac{1}{p+1} - \int \tfrac{x^{p-1}\partial x}{1+x},$$

on account of which we will have :

$$\int x^{p-1}\partial x lx \cdot \sqrt{(1-xx)} = -\int x^{p-1}\partial x \cdot \sqrt{(1-xx)} \cdot \left(\tfrac{1}{p+1} + \int \tfrac{x^{p-1}\partial x}{1+x}\right).$$

Hence therefore it will help to observe the following examples.

Example I., where $p=1$.

§. 152. Therefore in this case the final product emerges, $\frac{1}{2}+l2$, thus so that there shall become:

$$\int \partial x lx\sqrt{(1-xx)} = -\left(\tfrac{1}{2}+l2\right)\cdot \int \partial x\sqrt{(1-xx)}.$$

Moreover, for the formula $\int \partial x\sqrt{(1-xx)}$ we may put in place $\sqrt{(1-xx)} = 1-vx$, and there becomes :

$$x = \tfrac{2v}{1+vv}, \text{and } \sqrt{(1-xx)} = \tfrac{1-vv}{1+vv},$$

and $\partial x = \dfrac{2\partial v(1-vv)}{(1+vv)^2}$, from which here becomes $\partial x\sqrt{(1-xx)} = \dfrac{2\partial v(1-vv)^2}{(1+vv)^3}$,

the integral of which is resolved into these parts :

$$\frac{2v}{(1+vv)^2} - \frac{v}{1+vv} + \text{Arc. tang.} v;$$

which expression, since it must be extended from $x=0$ as far as to $x=1$, the first limit will be $v=0$, and the other truly will be $v=1$; thus so that from that the integral may be extended from $v=0$ as far as to $v=1$. But indeed that expression will vanish at once on putting $v=0$, and moreover on putting $v=1$, and the value of the integral will become $\frac{\pi}{4}$, on account of which we will have

$$\int \partial x lx\sqrt{(1-xx)} \begin{bmatrix} \textit{from } x=0 \\ \textit{to } \quad x=1 \end{bmatrix} = -\tfrac{\pi}{4}\cdot\left(\tfrac{1}{2}+l2\right).$$

§. 153. Indeed here we have presented the calculation in a long winding way, in order that the reduction has led to the rationality of the formula $\sqrt{(1-xx)}$; but truly the only aspect of the formula $\int \partial x\sqrt{(1-xx)}$ is declared at once, that expresses the area of the quadrant of the circle, of which the radius $=1$, which we know to be $-\frac{\pi}{4}$. This reduction may be used henceforth:

$$\int \partial x\sqrt{(1-xx)} = \tfrac{1}{2}x\sqrt{(1-xx)} + \tfrac{1}{2}\int \tfrac{\partial x}{\sqrt{(1-xx)}}$$

the value of which from extending from $x=0$ to $x=1$ manifestly gives $\frac{\pi}{4}$.

Example I., where $p=2$.

§. 154. In this case therefore the final factor becomes

$$\tfrac{1}{3} + \int \tfrac{x\partial x}{1+x} = \tfrac{4}{3} - l2;$$

and thus we will have

$$\int \partial x lx\sqrt{(1-xx)} = -\left(\tfrac{4}{3} - l2\right)\cdot \int x\partial x\sqrt{(1-xx)}:$$

but there can be seen, to be

$$\int x\partial x\sqrt{(1-xx)} = \mathrm{C} - \tfrac{1}{3}(1-xx)^{\frac{3}{2}},$$

which value extending from $x=0$ to $x=1$ gives $\frac{1}{3}$, thus so that we shall have

$$\int \partial x lx\sqrt{(1-xx)}\begin{bmatrix} \textit{from } x=0 \\ \textit{to} \quad x=1 \end{bmatrix} = -\tfrac{1}{3}\left(\tfrac{4}{3} - l2\right).$$

III. Setting out the case where $m=1$ and $n=3$.

§.155. Therefore in this case there will become $\mathrm{X} = \frac{1}{\sqrt[3]{\left(1-x^3\right)^2}}$, from which the general theorem gives us this equation :

$$\int \frac{x^{p-1}\partial x lx}{\sqrt[3]{\left(1-x^3\right)^2}} = \int \frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^2}} \cdot \int \frac{x^{p-1}(x-1)\partial x}{1-x^3},$$

where the final formula is reduced to this :

$$-\int \frac{x^{p-1}\partial x}{xx+x+1},$$

thus so that we may have :

$$\int \frac{x^{p-1}\partial x lx}{\sqrt[3]{\left(1-x^3\right)^2}} = -\int \frac{x^{p-1}\partial x lx}{\sqrt[3]{\left(1-x^3\right)^2}} \cdot \int \frac{x^{p-1}\partial x}{xx+x+1} :$$

thus we may attach the following examples.

Example I., where $p=1$.

§.156. In this case the latter becomes $\frac{\partial x}{xx+x+1}$, the indefinite integral of which is found to be $\frac{2}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2+x}$, which value on putting $x=1$ will go into $\frac{\pi}{3\sqrt{3}}$; on account of which in this case we will have

$$\int \frac{\partial x lx}{\sqrt[3]{\left(1-x^3\right)^2}} = -\frac{\pi}{3\sqrt{3}} \cdot \int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}};$$

but truly the formula of the integral $\int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}$ involves an unusual transcending quantity, which cannot be expressed either in terms of logarithms or by circular arcs.

Example II, where $p=2$.

§.157. Therefore in this case the final factor will be $\int \frac{x\partial x}{1+x+xx}$, which is resolved into these parts

$$\frac{1}{2}\int \frac{2x\partial x+\partial x}{1+x+xx} - \frac{1}{2}\int \frac{\partial x}{1+x+xx},$$

where the integral of the first part is

$$\tfrac{1}{2}l\left(1+x+xx\right) = \tfrac{1}{2}l3 \text{ (clearly on putting } x=1\text{);}$$

and truly the integral of the other part is $\frac{1}{2}\cdot\frac{\pi}{3\sqrt{3}}$, with which value substituted we will have

$$\int \frac{x\partial x lx}{\sqrt[3]{\left(1-x^3\right)^2}} = -\tfrac{1}{2}\left(l3 - \frac{\pi}{3\sqrt{3}}\right) \cdot \int \frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}$$

Now truly the same integral formula can be assigned by the reduction I indicated initially above ; since here there shall be $m=1$ and $n=3$, truly we have assumed $p=2$, then there will become $p=n-m.$ But above §. 131. we have found, in this case the integral to become

$$=\frac{\pi}{n\sin.\frac{m\pi}{n}},$$

which value in our case will become

$$\frac{\pi}{3\sin.\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}.$$

Therefore with this value substituted, we will be able to express our formula in terms of known separate quantities, in this manner :

$$\int\frac{x\partial xlx}{\sqrt[3]{\left(1-x^3\right)^2}}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=1\end{bmatrix}=-\frac{\pi}{3\sqrt{3}}\left(l3-\frac{\pi}{3\sqrt{3}}\right).$$

IV. The evaluation in the case where $m=2$ and $n=3$.

§.158. Therefore in this case there will be $\mathrm{X}=\frac{1}{\sqrt[3]{\left(1-x^3\right)}}$, from which the general theorem presents this same equation

$$\int\frac{x^{p-1}\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}=\int\frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{p-1}(xx-1)\partial x}{1-x^3},$$

where the latter form may be changed into this :

$$-\int\frac{x^{p-1}\partial x(1+x)}{1+x+xx};$$

from which there becomes

$$\int\frac{x^{p-1}\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}=-\int\frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{p-1}\partial x(1+x)}{1+x+xx};$$

from which we may set out the following examples.

Example I., where $p=1$.

§. 159. Therefore in this case, the latter member will become $\int\frac{\partial x(1+x)}{1+x+xx}$, of which the integral may be set out into these parts :

$$\tfrac{1}{2}\int\tfrac{2x\partial x+\partial x}{1+x+xx}+\tfrac{1}{2}\int\tfrac{\partial x}{1+x+xx},$$

from which evidently for the case $x=1$ there is produced $\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)$; on which account our equation will become

$$\int\frac{\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}=-\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)\cdot\int\frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}}.$$

But in this integral formula, on account of $m=2$ and $n=3$, because we have assumed $p=1,$ there will become $p=n-m$; hence for this case by §. 131, the value of this formula can be expressed absolutely, and there will become

$$\int\frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}}=\frac{2\pi}{3\sqrt{3}};$$

consequently also in this case we follow this form through absolute magnitudes :

$$\int\frac{\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}\left[\begin{array}{l}\textit{from } x=0\\ \textit{to} \quad\ \ x=1\end{array}\right]=-\frac{\pi}{3\sqrt{3}}\left(l3+\frac{\pi}{3\sqrt{3}}\right).$$

§. 160. But if we may combine this form with the latter part of the preceding case, which likewise can be expressed absolutely, the sum of these will give in the first place

$$\int\frac{x\partial xlx}{\sqrt[3]{\left(1-x^3\right)^2}}+\int\frac{\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}=-\frac{2\pi l3}{3\sqrt{3}}:$$

but if the latter may be taken from the former, this equation will arise :

$$\int\frac{x\partial xlx}{\sqrt[3]{\left(1-x^3\right)^2}}-\int\frac{\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}=\frac{2\pi\pi}{27}.$$

Since in this way we have been led to simple enough expressions, it will be worth the effort to represent both the equations in other forms, in which both the integral parts may be able to be joined conveniently into one; clearly we may put

$\frac{x}{\sqrt[3]{(1-x^3)}} = z,$ from which there becomes $\frac{xx}{\sqrt[3]{(1-x^3)^2}} = zz$, and thus the first formula adopts this form $\int \frac{zz\partial x lx}{x}$; truly the latter that $\int \frac{z\partial x lx}{x}$; then indeed we will have $\frac{x^3}{1-x^3} = z^3$, from which there shall be $x^3 = \frac{z^3}{1+z^3}$, and thus

$$lx = lz - \tfrac{1}{3} l\left(1+z^3\right) = l\frac{z}{\sqrt[3]{(1+z^3)}},$$

and hence again:

$$\frac{\partial x}{x} = \frac{\partial z}{z} - \frac{zz\partial z}{1+z^3} = \frac{\partial z}{z(1+z^3)};$$

whereby with these values used, the first integral formula will emerge :

$$\int \frac{z\partial z}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}};$$

truly the other will be

$$\int \frac{\partial z}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}}.$$

§. 161. But since the integrals must be extended from $x = 0$ to $x = 1$, it is to be observed, in the case $x = 0$ to become $z = 0$, but truly in the case $x = 1$ to produce $z = \infty$, thus so that it shall be necessary to have extended the same new formulas from $z = 0$ to $z = \infty$. From which consideration the first of these formulas must give

$$\int \frac{z\partial z}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}} \begin{bmatrix} \textit{from } z = 0 \\ \textit{to} \quad z = \infty \end{bmatrix} = -\frac{\pi l3}{3\sqrt{3}} + \frac{\pi\pi}{27},$$

truly the latter

$$\int \frac{\partial z}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}} \begin{bmatrix} \textit{from } z = 0 \\ \textit{to} \quad z = \infty \end{bmatrix} = -\frac{\pi l3}{3\sqrt{3}} - \frac{\pi\pi}{27}.$$

Hence the sum of these formulas therefore will be

$$\int \frac{(1+z)\partial z}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{2\pi l3}{3\sqrt{3}},$$

but truly the difference

$$\int \frac{\partial z(z-1)}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}} = \frac{2\pi\pi}{27}.$$

§. 162. Now here it will be useful to observe, this same logarithm $l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}$ can be conveniently converted into a simple infinite series ; for since there shall become

$$l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}=\frac{1}{3}l\frac{z^3}{1+z^3}=-\frac{1}{3}l\frac{1+z^3}{z^3},$$

truly by setting out this series no use is provided in resolving the integral

$$l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}=-\frac{1}{3}\left(\frac{1}{z^3}-\frac{1}{2z^6}+\frac{1}{3z^9}-\frac{1}{4z^{12}}+\frac{1}{5z^{15}}-\text{etc.}\right),$$

since the powers of z occur in the denominators, and thus the individual parts thus cannot be integrated so that they vanish on putting $z=0$.

[Euler makes the implicit assumption that $xlogx=0$ when $x=0$ is zero throughout this work.]

Example II., where $p=2$.

§. 163. Therefore in this case the latter factor becomes $\int\frac{x\partial x(1+x)}{1+x+xx}$, which can be split into the two parts $\int\partial x-\int\frac{\partial x}{1+x+xx}$, of which the integral extended from $x=0$ to $x=1$ is $=1-\frac{\pi}{3\sqrt{3}}$.
Hence therefore we are led to this equation :

$$\int\frac{x\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}=\left(1-\frac{\pi}{3\sqrt{3}}\right)\cdot\int\frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}}.$$

But here it is to be noted, this same integral formula cannot be shown explicitly in any way, unless it may involve some particular transcending quantity.

V. Evaluating the case, when $m=2$ and $n=4$.

§. 164. Therefore in this case there will be $\mathrm{X}=\frac{1}{\sqrt{\left(1-x^4\right)}}$, from which our general theorem will give us this equation

$$\int\frac{x^{p-1}\partial xlx}{\sqrt{\left(1-x^4\right)}}=-\int\frac{x^{p-1}\partial x}{\sqrt{\left(1-x^4\right)}}\cdot\int\frac{x^{p-1}\partial x}{1+xx};$$

[Recall from §. 137, that $\int \mathrm{X}x^{p-1}dxlx = \int \mathrm{X}x^{p-1}dx \cdot \int \frac{x^{p-1}\left(x^m-1\right)dx}{1-x^n}$, where in this case $\mathrm{X} = \frac{1}{\sqrt{\left(1-x^4\right)}}$.]

but truly the first particular problem for this case gives

$$\int \frac{x^3\partial xlx}{\sqrt[3]{\left(1-x^4\right)}} = -\frac{1}{2}\int \frac{x^3\partial x}{1+xx}.$$

But since there shall be

$$\int \frac{x^3\partial x}{1+xx} = \frac{1}{2} - \frac{1}{2}l2,$$

there shall be absolutely,

$$\int \frac{x^3\partial xlx}{\sqrt{\left(1-x^4\right)}}\left[\begin{matrix} from\ x=0 \\ to\quad x=1\end{matrix}\right] = -\frac{1}{4}\left(1-l2\right),$$

but truly here the case agrees with §. 144. treated above. If indeed here we may put $xx = y$, with which done the terms of the integration remain $y=0$ and $y=1$, there will be $lx = \frac{1}{2}ly$ and $x\partial x = \frac{1}{2}\partial y$; with which values substituted our equation will be changed into this form :

$$\frac{1}{4}\int \frac{y\partial yly}{\sqrt{(1-yy)}} = -\frac{1}{4}\left(1-l2\right),\ \text{or} \int \frac{y\partial y}{\sqrt{(1-yy)}} = l2-1,$$

precisely as above.

§. 165. Truly the other particular theorem adapted to the present case will give :

$$\int \frac{x\partial xlx}{\sqrt{\left(1-x^4\right)}} = -\frac{\pi}{4}\int \frac{x\partial x}{1+xx};$$

thus truly

$$\int \frac{x\partial x}{1+xx} = l\sqrt{(1+xx)} = \frac{1}{2}l2,$$

thus so that we may have:

$$\int \frac{x\partial xlx}{\sqrt{\left(1-x^4\right)}}\left[\begin{matrix} from\ x=0 \\ to\quad x=1\end{matrix}\right] = -\frac{\pi}{8}l2.$$

Indeed if here as before we may put $xx = y$, there will be obtained

$$\int \frac{\partial y l y}{\sqrt{(1-yy)}} = -\frac{\pi}{2} l2,$$

which is the case treated above in §.142. For these two cases the exponent p was an even number, from which it will be convenient to set out the odd cases.

Example I., where $p = 1$.

§. l66. Therefore in this case the latter formula of the integral becomes $\int \frac{\partial x}{1+xx} = \text{Arc.tang.}x,$ thus so that on putting $x = 1$ there may be produced $\text{Arc.tang.}x = \frac{\pi}{4}$; then truly our equation will become

$$\int \frac{\partial x l x}{\sqrt{(1-x^4)}} = -\frac{\pi}{4} \cdot \int \frac{\partial x}{\sqrt{(1-x^4)}},$$

clearly with the integrations extended from $x = 0$ to $x = 1$; where the formula $\int \frac{\partial x}{\sqrt{(1-x^4)}}$ expresses the arc of a rectangular elastic curve, and thus cannot be shown explicitly.

Example II., where $p = 3$.

§. 16 l. In this case therefore the latter formula of the integral will be $\int \frac{xx\partial x}{1+xx} = \int \partial x - \int \frac{\partial x}{1+xx},$ which may be split into these two parts $\int \partial x - \int \frac{\partial x}{1+x+xx},$ of which the integral on putting $x = 1$ becomes $= 1 - \frac{\pi}{4},$ thus so that now our equation may emerge

$$\int \frac{xx\partial x l x}{\sqrt{(1-x^4)}} = -\left(1 - \frac{\pi}{4}\right) \cdot \int \frac{xx\partial x}{\sqrt{(1-x^4)}},$$

of which the formula of the integral equally cannot be shown to be expressed other than by a rectangular elastic curve.

§.168. But nevertheless these two examples for inextricable formulas may be taken together, yet now some time ago I have shown, the product of these two integrals

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} \cdot \int \frac{xx\partial x}{\sqrt{(1-x^4)}}$$

to be equal to the area of a circle, of which the diameter $= 1$, or to be $= \frac{\pi}{4}$; on account of which with both examples taken together, we arrive at this remarkable theorem

$$\int \frac{\partial x l x}{\sqrt{\left(1-x^4\right)}} \cdot \int \frac{x x \partial x l x}{\sqrt{\left(1-x^4\right)}} = \frac{\pi^2}{16}\left(1-\frac{\pi}{4}\right).$$

Moreover it is readily apparent, innumerable other theorems of this kind can be obtained from this source, which considered by themselves, must be considered to be the most remarkable derived.

3) De integratione formulae $\int\frac{dxlx}{\sqrt{(1-xx)}}$, ab $x=0$ ad $x=1$ extensa.

Acta Acad. Imp. Sc. Tom. I. P.II. Pag. 3-28.

§. 117. Methodus maxime naturalis hujusmodi formulas $\int pdxlx$ tractandi in hoc consistit, ut eae ad alias hujusmodi formas $\int qdx$ reducantur, in quibus littera q sit functio algebraica ipsius x; quandoquidem regulae integrandi potissimum ad tales formulas sunt accommodatae. Hujusmodi autem reductio nulla prorsus laborat difficultate, quando functio p ita est comparta, ut integrale $\int pdx$ algebraice exhiberi queat. Si enim fuerit $\int pdx = \mathrm{P}$, ita ut formula proposita sit $\int d\mathrm{P}lx$, ea sponte reducitur ad hanc expressionem $\mathrm{P}lx-\int\frac{\mathrm{P}dx}{x}$, sicque jam totum negotiuin a integrationem hujus formulae $\int\frac{\mathrm{P}dx}{x}$, est perductum. Quando vero formula $\int pdx$ integrationem algebraicam non admittit, quemadmodum evenit in nostra formula proposita $\int\frac{dxlx}{\sqrt{(1-xx)}}$, talis reductio successu penitus caret. Cum enim sit $\int\frac{dx}{\sqrt{(1-xx)}}=\mathrm{A.sin.}x$, ista reductio daret

$$\int\frac{dxlx}{\sqrt{(1-xx)}}=\mathrm{Asin.}x\times lx-\int\frac{dx}{x}\cdot\mathrm{Asin.}x,$$

sicque post signum integrationis nova quantitas transcendens $\mathrm{Asin.}x$ occcurreret, cujus integratio aeque est abscondita ac ipsius propositae. Quare cum nuper singulari methodo invenissem esse

$$\int\frac{dxlx}{\sqrt{(1-xx)}}\begin{bmatrix}ab\ x=0\\ ad\ x=1\end{bmatrix}=-\tfrac{1}{2}\pi l2,$$

expressio integralis eo majori attentione digna est censenda, quod ejus investigatio neutiquam est obvia; unde operae pretium esse duxi ejus veritatem etiam ex aliis fontibus ostendisse, ante quam ipsam methodum, quae me eo perduxit, exponerem.

Prima demonstratio integrationis propositae:

§.118. Quoniam hic potissimum ad series infinitas est recurrendum, formula autem lx talem resolutionem simplicem, respuit, adhibeamus substitutionem, $\sqrt{(1-xx)}=y$, unde fit $x=\sqrt{(1-yy)}$, hincque porro

$$lx=-\frac{yy}{2}-\frac{y^4}{4}-\frac{y^6}{6}-\frac{y^8}{8}-\text{etc.}$$

hoc igitur modo, formula integralis proposita $\int\frac{dxlx}{\sqrt{(1-xx)}}$ transformatur in sequentem formam

$$\int \frac{dy}{\sqrt{(1-yy)}}\left(\frac{yy}{2}+\frac{y^4}{4}+\frac{y^6}{6}+\frac{y^8}{8}+\text{etc.}\right)$$

ubi, cum sit $y=\sqrt{(1-xx)}$, notetur integrationem extendi debere, ab $y=1$ usque ad $y=0$; quare si hos terminas integrationis permutare velimus, signum totius formae mutari oportct.

§. 119. Quo autem minus tali signorum mutatione confundamur, designemus valorem quesitum littera S, ut sit

$$\mathrm{S}=\int \frac{dxlx}{\sqrt{(1-xx)}}\left[\begin{matrix} ab\ x=0 \\ ad\ x=1 \end{matrix}\right]$$

atque facta substitutione $y=\sqrt{(1-xx)}$, habebimus, uti modo monuimus

$$\int \frac{dy}{\sqrt{(1-yy)}}\left(\frac{yy}{2}+\frac{y^4}{4}+\frac{y^6}{6}+\frac{y^8}{8}+\text{etc.}\right)\left[\begin{matrix} ab\ y=0 \\ ad\ y=1 \end{matrix}\right].$$

Sub his autem integrationis terminis, scilicct ab $y=0$ ad $y=1$, jam satis notum est, singulas partes, quae hic occurrunt, ad sequentes valores reduci

$$\int \frac{yydy}{\sqrt{(1-yy)}}=\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\int \frac{y^4dy}{\sqrt{(1-yy)}}=\frac{1.3}{2.4}\cdot\frac{\pi}{2}$$

$$\int \frac{y^6dy}{\sqrt{(1-yy)}}=\frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}$$

$$\int \frac{y^8dy}{\sqrt{(1-yy)}}=\frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi}{2}$$

$$\int \frac{y^{10}dy}{\sqrt{(1-yy)}}=\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{\pi}{2}\ \text{etc.}$$

ubi nimirum est $\frac{\pi}{2}=\int \frac{dy}{\sqrt{(1-yy)}}$, ita ut $1:\pi$ exprimat rationem diametri ad peripheriam circuli.

§. 120. Quodsi ergo singulos istos valores introducamus, pro valore quaesito S impetrabimus sequentem seriem infinitam

$$\mathrm{S}=-\frac{\pi}{2}\left(\frac{1}{2^2}+\frac{1.3}{2.4^2}+\frac{1.3.5}{2.4.6^2}+\frac{1.3.5.7}{2.4.6.8^2}+\text{etc.}\right)$$

sicque nunc totum negotimum eo est reductum, ut istius seriei infinitae summa investigetur ; qui labor fortassc haud minus operosus videri potest, quam id ipsum, quod nobis exsequi est propositum. Interim tamen ad cognitionem summae hujus seriei haud difficulter sequenti modo nobis pertingere licebit.

§. 121. Cum sit

$$\frac{1}{\sqrt{(1-zz)}}=1+\frac{1}{2}zz+\frac{1.3}{2.4}z^4+\frac{1.3.5}{2.4.6}z^6+\text{etc.}$$

si utrinque per dz multiplicemus et integremus, obtinebimus

$$\int\frac{dz}{z\sqrt{(1-zz)}}=lz+\frac{1}{2^2}zz+\frac{1.3}{2.4^2}z^4+\frac{1.3.5}{2.4.6^2}z^6+\text{etc.}$$

sicque ad ipsam seriem nostram sumus perducti, cujus ergo valor quaeri debet ex hac expressione $\int\frac{dz}{z\sqrt{(1-zz)}}-lz$, integrali scilicet ita sumto, ut evanescat posito $z=0$, quo facto statuatur $z=1$, ac prodibit ipsa series

$$\frac{1}{2^2}+\frac{1.3}{2.4^2}+\frac{1.3.5}{2.4.6^2}+\frac{1.3.5.7}{2.4.6.8^2}+\text{etc.}$$

Hoc igitur modo totum negotium perductum est ad istam formulam integralem $\int\frac{dz}{z\sqrt{(1-zz)}}$ quae posito $\sqrt{(1-zz)}=v$ transit in hanc formam $\frac{-dv}{1-vv}$, cujus integrale constat esse $-\frac{1}{2}l\frac{1+v}{1-v}=-l\frac{1+v}{\sqrt{1-vv}}$. Quodsi loco v restituatur valor $\sqrt{(1-zz)}$, tota expressio, qua indigemus, ita se habebit

$$\int\frac{dz}{z\sqrt{(1-zz)}}-lz=-l\frac{\left[1+\sqrt{(1-zz)}\right]}{z}-lz+C$$
$$=C-l\left[1+\sqrt{(1-zz)}\right],$$

ubi constans ita accipi debet, ut valor evanescat, posito $z=0$, ideoque erit $C=l2$. Quamobrem, posito $z=1$, summa seriei quaesita erit $l2$, hincque valor ipsius formulae integralis propositae erit

$$\int\frac{dxlx}{\sqrt{(1-xx)}}=S=-\frac{\pi}{2}l2:$$

prorsus uti longe alia methodo inveneram, ex quo jam satis intelligitur, istam veritatem utique altioris esse indaginis, ideoque attentione Geometrarum maxime dignam.

Alia demonstratio integrationis propositae.

§. 122. Cum sit $\frac{dx}{\sqrt{(1-xx)}}$ elementum arcus circuli cujus sinus $=x$, ponamus istum angulum $=\varphi$, ita ut sit

$$x=\sin.\varphi \text{ et } \frac{dx}{\sqrt{(1-xx)}}=d\varphi,$$

atque facta hac substitutione valor quantitatis S, in quem inquirimus, ita repreasentabitur

$$\mathrm{S} = \int d\varphi l \sin.\varphi \begin{bmatrix} a\ \varphi=0 \\ ad\ \varphi=90^0 \end{bmatrix}.$$

Cum enim ante termini fuissent $x=0$ et $x=1$, iis nunc respondent $\varphi=0$ et $\varphi=90^0$ sive $\varphi=\frac{\pi}{2}$. Hic igitur totum negotium eo redit, ut formula $l\sin.\varphi$ commode in seriem infinitam convertatur. Hunc in finem ponamus $l\sin.\varphi=s$, eritque $ds=\frac{d\varphi\cos.\varphi}{\sin.\varphi}$. Novimus autem esse

$$\frac{\cos.\varphi}{\sin.\varphi} = 2\sin.2\varphi + 2\sin.4\varphi + 2\sin.6\varphi + 2\sin.8\varphi + \text{etc.}$$

Si enim utrinque per $\sin.\varphi$ multiplicemus; ob

$$2\sin.n\varphi\sin.\varphi = \cos.(n-1)\varphi - \cos.(n+1)\varphi,,$$

utique prodit

$$\begin{aligned}\cos.\varphi = \cos.\varphi + \cos.3\varphi + \cos.5\varphi + \cos.7\varphi + \cos.\ 9\varphi + \text{etc.}\\ -\cos.3\varphi - \cos.5\varphi - \cos.7\varphi - \cos.\ 9\varphi - \text{ etc.}\end{aligned}$$

Hac igitur serie pro $\frac{\cos.\varphi}{\sin.\varphi}$ in usum vocata, erit

$$s = \mathrm{C} - \cos.2\varphi - \tfrac{1}{2}\cos.4\varphi - \tfrac{1}{3}\cos.6\varphi - \tfrac{1}{4}\cos.8\varphi - \tfrac{1}{5}\cos.10\varphi - \text{etc.}$$

ubi cum sit $s=l\sin.\varphi,$ ideoque $s=0,$ quando $\sin.\varphi=1,$ ideoque $\varphi=\frac{\pi}{2}$, constantem C ita definire oportet, ut posito $\varphi=\frac{\pi}{2}=90^0,$ evadat $s=0,$ ex quo colligitur fore $\mathrm{C}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\text{etc.}=-l2.$

§.123. Cum igitur sit

$$l\sin.\varphi = -l2 - \cos.2\varphi - \tfrac{1}{2}\cos.4\varphi - \tfrac{1}{3}\cos.6\varphi - \tfrac{1}{4}\cos.8\varphi - \text{etc.}$$

erit valor formulae propositae

$$\begin{aligned}\int d\varphi l\sin.\varphi = \mathrm{C} - \varphi l2 - \tfrac{1}{2}\sin.2\varphi - \tfrac{1}{8}\sin.4\varphi - \tfrac{1}{18}\sin.6\varphi\\ - \tfrac{1}{32}\sin.8\varphi - \tfrac{1}{50}\sin.10\varphi - \text{etc.}\end{aligned}$$

quae expressio cum evanescere debeat posito $\varphi=0$, constans hic ingressa erit $\mathrm{C}=0$, ita ut jam in genere sit

$$\begin{aligned}\int d\varphi l\sin.\varphi = -\varphi l2 - \frac{2\sin.2\varphi}{2^2} - \frac{2\sin.4\varphi}{4^2} - \frac{2\sin.6\varphi}{6^2} - \frac{2\sin.8\varphi}{8^2}\\ - \frac{2\sin.10\varphi}{10^2} - \frac{2\sin.12\varphi}{12^2} - \text{etc.}\end{aligned}$$

Quodsi jam hic capiatur $\varphi = \frac{\pi}{2} = 90^0$, omnium angulorum $2\varphi,\ 4\varphi,\ 6\varphi,\ 8\varphi,$ etc., qui hic occurrunt sinus evanescunt, ideoque valor quaesitus erit

$$S = \int d\varphi l \sin.\varphi \begin{bmatrix} a\ \ \varphi=0 \\ ad\ \ \varphi=90^0 \end{bmatrix} = -\frac{\pi}{2} l2;$$

quemadmodum etiam in priore demonstratione ostendimus.

§. 124. Ista autem demonstratio praecedenti ideo longe antecellit, quod nobis non solum valorem formulae propositae exhibeat casu quo $\varphi = 90^0$, sed etiam verum ejus valorem ostendat, quicunque angulus pro φ accipiatur, id quod ad ipsam formulam propositam $\int \frac{dxlx}{\sqrt{(1-xx)}}$ transferri poterit, cujus adeo valorem pro quolibet valere ipsius x assignare poterimus. Quodsi enim istius formulae valorem desideremus ab $x = 0$ usque ad $x = a$, quaeratur angulus α cujus sinus sit aequalis ipsi α, atque semper hibebitur

$$S = \int d\varphi l \sin.\varphi \begin{bmatrix} a\ \ \varphi=0 \\ ad\ \ \varphi=90^0 \end{bmatrix} = -\alpha l2 - \frac{2\sin.2\alpha}{2^2} - \frac{2\sin.4\alpha}{4^2} - \frac{2\sin.6\alpha}{6^2} - \frac{2\sin.8\alpha}{8^2} - \text{etc.}$$

Unde patet, quoties fuerit $\alpha = \frac{i\pi}{2}$, denotante i numerum integrum quemcunque, quoniam omnes sinus evanescunt, valor formulae his casibus finite exprimi per $-\frac{i\pi}{2} l2$; aliis vero casibus valor nostrae formulae per seriem infinitam satis concinnam exprimetur. Ita si capiatur $a = \frac{1}{\sqrt{2}}$, ut sit $\alpha = \frac{\pi}{4}$, valor nostrae formulae erit

$$-\frac{\pi}{4} l2 - \frac{2}{2^2} + \frac{2}{6^2} - \frac{2}{10^2} + \frac{2}{14^2} - \frac{2}{18^2} + \frac{2}{22^2} - \text{etc.}$$

quae series elegantius ita exprimitur

$$-\frac{\pi}{4} l2 - \frac{1}{2}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \text{etc.}\right);$$

sicque hic occurrit series satis memorabilis

$$1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \frac{1}{121} + \text{etc.}$$

cujus summam nullo adhuc modo ad mensuras cognitas revocare licuit.

§.125. Quoniam tam egregia series hic se quasi praeter exspectationem obtulit, etiam alios casus evolvamus notabiliores, sumamusque $a = \frac{1}{2}$, ut sit $\alpha = 30° = \frac{\pi}{6}$, atque nostrae formulae hoc casu valor erit

$$-\frac{\pi}{6}l2-\frac{\sqrt{3}}{2^2}-\frac{\sqrt{3}}{4^2}+\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}-\frac{\sqrt{3}}{16^2}+\text{etc.}$$

quae expressio ita exhiberi potest

$$-\frac{\pi}{6}l2-\frac{\sqrt{3}}{4}\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right)$$

in qua serie quadrata multiplorum temarii deficiunt. Sumamus nunc simili modo $a=\frac{\sqrt{3}}{2}$, ut sit $\alpha=60°=\frac{\pi}{3}$, ac valor nostrae formulae hoc casu prodibit

$$-\frac{\pi}{3}l2-\frac{\sqrt{3}}{2^2}+\frac{\sqrt{3}}{4^2}-\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}+\frac{\sqrt{3}}{16^2}-\text{etc.}$$

sive hoc modo exprimetur

$$-\frac{\pi}{3}l2-\frac{\sqrt{3}}{4}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right)$$

Adhuc alia demonstratio integrationis propositae.

§. 126. Introducatur in formulam nostram angulus φ, cujus cosinus sit $=x$, sive sit $x=\cos.\varphi$, et formula nostra induet hanc formam $-\int d\varphi l\cos.\varphi$, quod integrale a $\varphi=90°$ usque ad $\varphi=0$ erit extendendum. Quodsi autem hos terminas permutemus, valor S, quem quaerimus, ita exprimetur

$$\mathrm{S}=\int d\varphi l\cos.\varphi\begin{bmatrix}a\ \ \varphi=0\\ ad\ \ \varphi=90^0\end{bmatrix}.$$

Ut hic $l\cos.\varphi$ in seriem idoneam convertamus, statuamus ut ante $s=l\cos.\varphi$ eritque $ds=-\frac{d\varphi\sin.\varphi}{\cos.\varphi}$. Constat autem per seriem esse

$$\frac{\sin.\varphi}{\cos.\varphi}=2\sin.2\varphi-2\sin.4\varphi+2\sin.6\varphi-2\sin.8\varphi+\text{etc.}$$

Cum enim in genere sit

$$2\sin.n\varphi\ \cos.\varphi=\sin.(n+1)\varphi+\sin.(n-1)\varphi,$$

si utrinque per $\cos.\varphi$ multiplicemus, orietur

$$\begin{aligned}\sin.\varphi&=2\sin.2\varphi\cos.\varphi-2\sin.4\varphi\cos.\varphi+2\sin.6\varphi\cos.\varphi-2\sin.8\varphi\cos.\varphi+\text{etc.}\\&=\sin.3\varphi-\sin.5\varphi+\sin.7\varphi-\sin.9\varphi+\text{etc.}\\&+\sin.\varphi-\sin.3\varphi+\sin.5\varphi-\sin.7\varphi+\sin.9\varphi-\text{etc.}\end{aligned}$$

quare cum sit $ds=-\frac{d\varphi\sin.\varphi}{\cos.\varphi}$, erit nunc

$$ds = -\frac{d\varphi \sin.\varphi}{\cos.\varphi} = -2\sin.2\varphi d\varphi + 2\sin.4\varphi d\varphi - 2\sin.6\varphi d\varphi + 2\sin.8\varphi d\varphi + \text{etc.}$$

$$s = \mathrm{C} + \frac{\cos.2\varphi}{1} - \frac{\cos.4\varphi}{2} + \frac{\cos.6\varphi}{3} - \frac{\cos.8\varphi}{4} + \frac{\cos.10\varphi}{5} - \text{etc.}$$

Quia igitur est $s = l\cos.\varphi$, evidens est posito $\varphi = 0$, fieri debere $s = 0$, unde colligitur $\mathrm{C} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \text{etc.} = -l2;$
sicque erit

$$l\cos.\varphi = -l2 + \frac{\cos.2\varphi}{1} - \frac{\cos.4\varphi}{2} + \frac{\cos.6\varphi}{3} - \frac{\cos.8\varphi}{4} + \text{etc.}$$

quae series ducta in $d\varphi$ et integrata praebet

$$\mathrm{S} = \int d\varphi l\cos.\varphi = \mathrm{C} - \varphi l2 + \frac{\sin.2\varphi}{2} - \frac{\sin.4\varphi}{8} + \frac{\sin.6\varphi}{18} - \frac{\sin.8\varphi}{32} + \frac{\sin.10\varphi}{50} - \text{etc.}$$

quae expressio quia sponte evanescit posito $\varphi = 0$, inde patet fore $\mathrm{C} = 0$, sicque habebimus

$$\int d\varphi l\cos.\varphi = -\varphi l2 + \frac{1}{2}\left(\frac{\sin.2\varphi}{1} - \frac{\sin.4\varphi}{2^2} + \frac{\sin.6\varphi}{3^2} - \frac{\sin.8\varphi}{4^2} + \frac{\sin.10\varphi}{5^2} - \text{etc.}\right)$$

Sumto igitur $\alpha = \frac{\pi}{2} = 90°$, oritur ut ante $\mathrm{S} = -\frac{\pi}{2}l2.$ Praeterea vero etiam hinc integrale ad quemvis terminum usque extendere licet.

§. 127. Quodsi formulam posteriorem a praecedente subtrahamus, adipiscemur in genere hanc integrationem

$$\int d\varphi l\tan.\varphi = -\sin.2\varphi - \frac{1}{3^2}\sin.6\varphi - \frac{1}{5^2}\sin.10\varphi - \text{etc.}$$

unde patet hoc integrale evanescere casibus $\varphi = 0^0$ et in genere $\varphi = \frac{i\pi}{2}$. Postquam igitur istam integrationem triplici modo demonstravimus, ipsam Analysin, quae me primum huc perduxit, hic delucide sum expositurus.

Analysis ad integrationem formulae $\int \frac{dxlx}{\sqrt{(1-xx)}}$ aliarumque similium perducens.

§. 128. Tota haec Analysis innititur sequenti lemmati a me jam olim demonstrato: Posito brevitatis gratia $\left(1-x^n\right)^{\frac{m-n}{n}} = \mathrm{X}$, si hinc duae formulae integrales formentur $\int \mathrm{X}x^{p-1}dx$ et $\int \mathrm{X}x^{q-1}dx$, quae a termina $x = 0$ usque ad terminum $x = 1$ extendantur, ratio horum valorum seqenti modo ad productum ex infinitis factoribus conflatum reduci potest

$$\frac{\int \mathrm{X}x^{p-1}dx}{\int \mathrm{X}x^{q-1}dx} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{etc.}$$

ubi scilicet singuli factores tam numeratoris, quam denominatoris continuo eadem quantitate n augentur. Hic autem, probe tenendum est, veritatem istius lemmatis subsistere non posse, nisi singulae m, n, p et q denotent numeros positivos, quos tamen semper tanquam integros spectare licet.

§. 129. Circa has duas formulas integrales, a termina $x=0$ usque ad terminum $x=1$ extensas, duo casus imprimis seorsim notari merentur, quibus integratio actu succedit, verusque valor absolute assignari potest. Prior casus locum habet, si fuerit $p=n$, ita ut formula sit $\int \mathrm{X}x^{n-1}dx$. Posito enim $x^n=y$ fiet

$$\mathrm{X}=(1-y)^{\frac{m-n}{n}}, \text{ et } x^{n-1}dx=\tfrac{1}{n}dy$$

sicque ista formula evadet $\frac{1}{n}\int dy(1-y)^{\frac{m-n}{n}}$, pariter a termino $y=0$ usque ad $y=1$ extendenda, quae porro posito $1-y=z$ abit in hanc formulam $-\frac{1}{n}\int z^{\frac{m-n}{n}}dz$, a termino $z=1$ usque ad $z=0$ extendenam ; ejus ergo integrale manifesto est $-\frac{1}{m}z^{\frac{m}{n}}+\frac{1}{m}$; unde facto $z=0$ valor erit $\frac{1}{m}$. Consequenter pro casu $p=n$ habebimus

$$\int \mathrm{X}x^{n-1}dx\begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix}=\frac{1}{m};$$

sicque si fuerit vel $p=n$ vel $q=n$; integrale absolute innotescit.

§. 130. Alter casus notatu dignus est, quo $p=n-m$, ita ut formula integranda sit $\int \mathrm{X}x^{n-m-1}dx$; tum enim, si ponatur

$$x\left(1-x^n\right)^{\frac{-1}{n}} \text{ sive } \frac{x}{\left(1-x^n\right)^{\frac{1}{n}}}=y,$$

at posito $x=0$ fiet $y=0$, at posito $x=1$ fiet $y=\infty$; tum autem erit

$$y^{n-m}=\frac{x^{n-m}}{\left(1-x^n\right)^{\frac{n-m}{n}}}=\mathrm{X}x^{n-m},$$

unde formula integranda erit $\int y^{n-m}\frac{dx}{x}$. Cum igitur sit

$$\frac{x}{\left(1-x^n\right)^{\frac{1}{n}}}=y, \text{ erit } \frac{x^n}{1-x^n}=y^n,$$

unde colligitur $x^n=\frac{y^n}{1+y^n}$, ideoque $nlx=nly-l\left(1+y^n\right)$, cujus differentiato praebet

$$\frac{dx}{x}=\frac{dy}{y\left(1+y^n\right)},$$

quo valore substituto formula nostra integranda erit

$$\int\frac{y^{n-m-1}dy}{1+y^n},$$

a termina $y=0$ usque ad $y=\infty$ extendenda, quae formula ideo est notatu digna, quod ab omni irrationalitae est liberata.

§. 131. Quoniam igitur hoc casu ad formulam rationalem sumus perducti, ex elementis calculi integralis constat, ejus integrationem semper per logarithmes et arcus circulares absolvi posse, tum vero pro hoc casu non ita pridem ostendi, hujus formulae $\int\frac{x^{m-1}dx}{1+x^n}$ integrale, ab $x=0$ usque ad $x=\infty$ extensum, reduci ad valorem $\frac{\pi}{n\sin.\frac{m\pi}{n}}$; facta igitur applicatione pro nostro casu habebimus

$$\int\frac{y^{n-m-1}dy}{1+y^n}=\frac{\pi}{n\sin.\frac{(n-m)\pi}{n}}=\frac{\pi}{n\sin.\frac{m\pi}{n}};$$

quamobrem pro casu $p=n-m$ valor integralis sequenti modo absolute exprimi potest; eritque .

$$\int Xx^{n-m-1}dx\begin{bmatrix}ab\ x=0\\ ad\ x=1\end{bmatrix}=\frac{\pi}{n\sin.\frac{m\pi}{n}}$$

quod idem manifesto tenendum est, si fuerit $q=n-m$.

§. 132. His praemissis, ponamus porro brevitatis gratia

$$\int Xx^{p-1}dx\begin{bmatrix}ab\ x=0\\ ad\ x=1\end{bmatrix}=\text{P et}$$

$$\int Xx^{q-1}dx\begin{bmatrix}ab\ x=0\\ ad\ x=1\end{bmatrix}=\text{Q},$$

atque lemma allatum nobis praebet hanc aequationem

$$\frac{\text{P}}{\text{Q}}=\frac{(m+p)q}{p(m+q)}\cdot\frac{(m+p+n)(q+n)}{(p+n)(m+q+n)}\cdot\frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)}\text{etc.}$$

hinc igitur sumendis logarithmis deducimus

$$\begin{aligned} lP-lQ = &\, l(m+p)-lp+l(m+p+n)-l(p+n)+l(m+p+2n)-l(p+2n)+\text{etc.} \\ &+lq-l(m+q)+l(q+n)-(m+q+n)+l(q+2n)-l(m+q+2n)+\text{etc.} \end{aligned}$$

haecque aequalitas semper locum habebit, quicunque valores litteris *m, n, p* et *q* tribuantur, dummodo fuerint positivi.

§. 133. Cum igitur haec aequalitas in genere subsistat, etiam veritati erit consentanea, quando quaepiam harum litterarum *m, n, p*, et *q* infinite parum immutantur, sive tanquam variabiles spectantur. Hanc ob rem consideremus solam quantitatem *p* tanquam variabilem, ita ut reliquae litterae *m, n* et *q* maneant constans, ideoque etiam quantitas Q erit constans dum altera P variabitur; ex quo differentiando nanciscemur hanc aequationem

$$\frac{dP}{P}=\frac{dp}{m+p}-\frac{dp}{p}+\frac{dp}{m+p+n}-\frac{dp}{p+n}+\frac{dp}{m+p+2n}-\frac{dp}{p+2n}$$
$$+\ \frac{dp}{m+p+3n}-\frac{dp}{p+3n}+\text{etc.}$$

ubi totum negotium eo redit, quaemadmodum differentiale formulae P, quae est integralis, exprimi oporteat.

§. 134. Cum igitur P sit formula integralis solam quantitatem *x* tanquam variabilem involvens, quandoquidem in ejus integratione exponens *p* ut constans tractari debet, demum post integrationem ipsam quantitatem P tanquam functionem duarum variabilium *x* et *p* spectare licebit; unde quaestio huc redit, quomodo valorem, hoc charactere $\left(\frac{dP}{dp}\right)$ exprimi solitum, investigari oporteat, qui si indicetur littera Π, aequatio ante inventa hanc induet formam

$$\frac{\Pi}{P}=\frac{1}{m+p}-\frac{1}{p}+\frac{1}{m+p+n}-\frac{1}{p+n}+\frac{1}{m+p+2n}-\frac{1}{p+2n}+\text{etc.}$$

Hanc vero seriem infinitam haud difficulter ad expressionem finitam revocare licebit hoc modo: Ponatur

$$s=\frac{v^{m+p}}{m+p}-\frac{v^{p}}{p}+\frac{v^{m+p+n}}{m+p+n}-\frac{v^{p+n}}{p+n}+\frac{v^{m+p+2n}}{m+p+2n}-\frac{v^{p+2n}}{p+2n}+\text{etc.}$$

ita ut facto $v=1$ littera *s* nobis exhibeat valorem quaesitum $\frac{\Pi}{P}$; at vero differentiatio nobis dabit

$$\frac{ds}{dv}=v^{m+p-1}-v^{p-1}+v^{m+p+n-1}-v^{p+n-1}+v^{m+p+2n-1}-v^{p+2n-1}+\text{etc.},$$

cujus serei infinitae summa manifesto est

$$\frac{v^{m+p-1}-v^{p-1}}{1-v^{n}}=\frac{v^{p-1}\left(v^{m}-1\right)}{1-v^{n}}.$$

Hinc igitur vicissim concludimus fore

$$s=\int\frac{v^{p-1}\left(v^{m}-1\right)dv}{1-v^{n}},$$

quae formula integralis a $v=0$ usque ad $v=1$ est extendenda; sicque habebimus

$$\frac{\Pi}{\mathrm{P}}=\int\frac{v^{p-1}\left(v^{m}-1\right)dv}{1-v^{n}},\begin{bmatrix} ab\ v=0\\ ad\ v=1\end{bmatrix}.$$

§. 135. Ad valorem autem $\left(\frac{d\mathrm{P}}{dp}\right)$ quem hic littera Π idicavimus, investigandum, ex principiis calculi integralis ad functiones duarum variabilium applicati jam satis notum est, differentiale formulae integralis $\mathrm{P}=\int \mathrm{X}x^{p-1}dx$ ex sola variabilitate ipsius *p* oriundum obtineri, si formula post signum integrationis posita $\mathrm{X}x^{p-1}$, ex sola variabilitate ipsius *p* differentietur, atque elementum *dp* signa integrationis praefigatur; at vero quia X non continet *p*, hic ut constans tractari debet: potestatis vero x^{p-1} differentiale hinc natum erit $x^{p-1}dplx$; quam ob rem ex hac differentiatione orietur $d\mathrm{P}=dp\int \mathrm{X}x^{p-1}dxlx$, ita ut tantum post signum integrationis factor *lx* accesserit, ex quo manifestum est, fore

$$\Pi=\int \mathrm{X}x^{p-1}dxlx\begin{bmatrix} ab\ x=0\\ ad\ x=1\end{bmatrix},$$

hinc igitur sequens theorema generale constituere licebit.

Theorema generale.

§. 136. Posito brevitatis gratia $\mathrm{X}=\left(1-x^{n}\right)^{\frac{m-n}{n}}$, si sequentes formulae integrales omnes a termina $x=0$ ad termino $x=1$ extendantur, sequens aequalitas semper erit veritati consentanea

$$\frac{\int \mathrm{X}x^{p-1}dxlx}{\int \mathrm{X}x^{p-1}dx}=\int\frac{x^{p-1}\left(x^{m}-1\right)dx}{1-x^{n}}$$

nihil enim obstabat, quo minus loco *v* scriberemus *x*, quandoquidem isti valores tantum a terminis integrationis pendent.

§. 137. Hoc igitur modo deducti sumus ad integrationem hujusmodi formularum $\int \mathrm{X}x^{p-1}dxlx$, in quibus quantitas logarithmica lx post signum integrationis tanquam factor inest, quarum valorem exprimere licuit per binas formulas integrales ordinarias, cum sit

$$\int \mathrm{X}x^{p-1}dxlx = \int \mathrm{X}x^{p-1}dx \int \frac{x^{p-1}\left(x^m-1\right)dx}{1-x^n},$$

integra libus scilicet ab $x=0$ ad $x=1$ extensis, ubi brevitatis gratia posuimus $\left(1-x^n\right)^{\frac{m-n}{n}} = \mathrm{X}$. Hinc igitur pro binis casibus memorabilibus supra expositis bina theoremata particularia derivemus.

Theorema particulare I, quo $p=n$.

§. 138. Quoniam supra vidimus casu $p=n$ fieri $\int \mathrm{X}x^{n-1}dx = \frac{1}{m}$, hoc valore substituto habebimus istam aequationem satis elegantem

$$\int \mathrm{X}x^{n-1}dxlx = \frac{1}{m}\int \frac{x^{n-1}\left(x^m-1\right)dx}{1-x^n},$$

dum scilicet ambo integralia ab $x=0$ ad $x=1$ extenduntur.

Theorema particulare II, quo $p=n-m$.

§. 139. Quoniam pro hoc casu, quo $p=n-m$ supra ostendimus esse

$$\int \mathrm{X}x^{n-m-1}dx = \frac{\pi}{n\sin.\frac{m\pi}{n}},$$

nunc deducimur ad sequentem integrationem maxime notatu dignam

$$\int \mathrm{X}x^{n-m-1}\partial xlx = \frac{\pi}{n\sin.\frac{m\pi}{n}} \cdot \int \frac{x^{n-m-1}\left(x^m-1\right)\partial x}{1-x^n},$$

si quidem haec ambo integralia ab $x=0$ usque ad $x=1$ extendantur; ubi meminisse oportet esse

$$\mathrm{X} = \left(1-x^n\right)^{\frac{m-n}{n}}$$

§. 140. Hic probe notetur, theorema generale latissime patere, propterea quod in eo insunt tres exponentes indefiniti, scilicet *m*, *n* et *p*, qui penitus arbitrio nostro relinquuntur, quos ergo infinitis modis pro lubitu definire licet, dummodo singulis valores positivi tribuantur, ita ut semper valor hujus formulae integralis $\int Xx^{p-1}dxlx$, quam ob factorem lx tanquam transcendentem spectari oportet, per formulas integrales ordinarias exprimi queat, quae cum sint generalissima, operae pretium erit nonnullos casus speciales evolvere.

I. Evolutio casus quo $m=1$ et $n=2$.

§. 141. Hoc igitur casu erit $X=\frac{1}{\sqrt{(1-xx)}}$, unde pro hoc casu theorema generale ita se habebit

$$\int\frac{x^{p-1}\partial xlx}{\sqrt{(1-xx)}}=-\int\frac{x^{p-1}\partial x}{\sqrt{(1-xx)}}\cdot\int\frac{x^{p-1}\partial x}{1+x},$$

siquidem singula haec integralea ab $x=0$ ad $x=1$ extenduntur. Quoniam igitur hic tantum exponens p arbitrio nostro relinquitur, hinc sequentia exempla perlustremus.

Exemplum I. quo $p=1$.

§. 142. Hoc igitur casu aequatio superior hanc induet formam

$$\int\frac{\partial xlx}{\sqrt{(1-xx)}}=-\int\frac{\partial x}{\sqrt{(1-xx)}}\cdot\int\frac{x\partial x}{1+x}$$

ubi, integralibus ab $x=0$ ad $x=1$ extensis, notum est fieri,

$$\int\frac{\partial x}{\sqrt{(1-xx)}}=\frac{\pi}{2}\text{ et }\int\frac{\partial x}{1+x}=l2;$$

ita ut jam habeamus,

$$\int\frac{\partial xlx}{\sqrt{(1-xx)}}=\begin{bmatrix}ab\ x=0\\ ad\ x=1\end{bmatrix}=-\frac{\pi}{2}l2,$$

quae est ea ipsa formula, quam initio hujus dissertationis tractavimus et cujus veritatem jam triplici demonstratione corroboravimus

§. 143. Eundem valorem elicere licet ex theoremate particulari secundo, quo erat $p=n-m$, siquidem nunc ob $n=2$ et $m=1$ erit $p=1$; inde enim ob $X=\frac{1}{\sqrt{(1-xx)}}$, istud theorema praebet

$$\int\frac{\partial xlx}{\sqrt{(1-xx)}}=\frac{\pi}{2\sin.\frac{\pi}{2}}\cdot\int\frac{x\partial x}{1+x}=-\frac{\pi}{2}l2.$$

Exemplum II. quo $p=2$.

§. 144. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} = -\int \frac{x\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x\partial x}{1+x}$$

Jam vero integralibus ab $x=0$ ad $x=1$ extensis, notum est fore

$$\int \frac{x\partial x}{\sqrt{(1-xx)}} = 1 \text{ et } \int \frac{x\partial x}{1+x} = 1-l2;$$

ita ut habeamus

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = l2-1.$$

§. 145. Quoniam in hac formula integrale $\int \frac{x\partial x}{\sqrt{(1-xx)}}$, algebraice exhiberi potest, cum sit $=1-\sqrt{(1-xx)}$, valor quaesitus etiam per reductiones consuetas erui potest, cum sit

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} = \left[1-\sqrt{(1-xx)}\right]lx - \int \frac{\partial x}{x}\left[1-\sqrt{(1-xx)}\right],$$

positoque $x=1$ erit

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} = -\int \frac{\partial x}{x}\left[1-\sqrt{(1-xx)}\right],$$

ad quam formam integrandam fiat $1-\sqrt{(1-xx)}=z$, unde colligitur $xx=2z-zz$, ergo $2lx = lz + l(2-z)$, sicque fiet $\frac{\partial x}{x} = \frac{\partial z(1-z)}{z(2-z)}$, quibus valoribus substitutis erit

$$+\int \frac{\partial x}{x}\left[1-\sqrt{(1-xx)}\right] = +\int \frac{\partial z(1-z)}{z(2-z)},$$

qui ergo valor erit $= \mathrm{C} - z - l(2-z)$. Quia igitur posito $x=0$ fit $z=0$, constans erit $\mathrm{C} = +\,l2$; facto igitur $x=1$, quia tum fit $z=1$, iste valor integralis erit $l2-1$, prorsus ut ante.

§. 146. Eundem valorem suppeditat theorema prius supra allatum, quo erat $p=n=2$; inde enim statim fit

$$\int \frac{x\partial x lx}{\sqrt{(1-xx)}} = \int -\frac{x\partial x}{1+x}.$$

Ante autem vidimus esse $\int \frac{x\partial x}{1+x} = 1-l2$; ita ut etiam hinc prodeat valor quaesitus $l2-1$.

Exemplum III. quo $p=3$.

§. 147. Hoc igitur casu aequatio in theoremate generali allata hanc induet formam

$$\int \frac{xx\partial x lx}{\sqrt{(1-xx)}} = -\int \frac{xx\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{xx\partial x}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{xx\partial x}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = \frac{1}{2} \cdot \frac{\pi}{2},$$

at vero fractio spuria $\frac{xx}{1+x}$ resolvitur in has partes $x-1+\frac{1}{1+x}$, unde erit

$$\int \frac{xx\partial x}{1+xx} = \frac{1}{2}xx - x + l(1+x),$$

quod integrale jam evanescit posito $x=0$; facto ergo $x=1$ ejus valor erit $=-\frac{1}{2}+l2$; quamobrem integrale quod quaerimus, erit

$$\int \frac{xx\partial x}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = -\frac{\pi}{4}\left(l2-\frac{1}{2}\right).$$

Exemplum IV. quo $p=4$.

§. 148. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^3\partial x lx}{\sqrt{(1-xx)}} = -\int \frac{x^3\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^3\partial x}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{x^3\partial x}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = \frac{2}{3},$$

tum vero fractio spuria $\frac{x^3}{1+x}$ resolvitur in has partes $xx-x+1-\frac{1}{1+x}$, unde integrando fit

$$\int \frac{x^3\partial x}{1+x} = \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x),$$

ex quo valor formulae erit $=\frac{5}{6}-l2$. His ergo valoribus substitutis adipiscimur hanc integrationem

$$\int \frac{x^3\partial x}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = -\frac{2}{3}\left(\frac{5}{6}-l2\right).$$

Exemplum V. quo $p = 5$.

§. 149. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^4 \partial x lx}{\sqrt{(1-xx)}} = -\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^4 \partial x}{1+x}.$$

Constat autem esse

$$\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = \frac{1.3}{2.4} \cdot \frac{\pi}{2},$$

tum vero fractio spuria $\frac{x^4}{1+x}$ manifesto resolvitur in has partes $x^3 - xx + x - 1 + \frac{1}{x+1}$ unde integrando fit

$$\int \frac{x^4 \partial x}{1+x} = \frac{1}{4} x^4 - \frac{1}{3} x^3 + \frac{1}{2} xx - x + l(1+x),$$

ex quo valor formulae erit $= -\frac{7}{12} + l2$. His igitur valoribus substitutis prodibit ista integratio

$$\int \frac{x^4 \partial x lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = -\frac{1.3}{2.4} \cdot \frac{\pi}{2} \left(l2 - \frac{7}{12} \right).$$

Exemplum VI. quo $p = 6$.

§. 150. Hoc igitur casu aequatio superior induet hanc formam

$$\int \frac{x^5 \partial x lx}{\sqrt{(1-xx)}} = -\int \frac{x^5 \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^5 \partial x}{1+x}.$$

Constat autem per reductiones notas esse

$$\int \frac{x^5 \partial x lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = \frac{2.4}{3.5},$$

tum vero fractio spuria $\frac{x^5}{1+x}$, resolvitur in has partes

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1},$$

unde integrando nanciscimur

$$\int \frac{x^5 \partial x}{1+x} = \frac{1}{5} x^5 - \frac{1}{4} x^4 + \frac{1}{3} x^3 - \frac{1}{2} xx + x - l(1+x),$$

ex quo valor hujus formulae erit $= \frac{47}{60} - l2$; quibus valoribus substitutis prodibit ista integratio

$$\int \frac{x^5 \partial x lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = \frac{2.4}{3.5} \cdot \left(\frac{47}{60} - l2 \right).$$

II. Evolutio casus quo $m=3$ et $n=2$.

§. 151. Hic ergo erit $\mathrm{X} = \sqrt{(1-xx)}$, unde theorema nostrum generale nobis praebebit hanc aequationem

$$\int x^{p-1} \partial x lx \cdot \sqrt{(1-xx)} = \int x^{p-1} \partial x \cdot \sqrt{(1-xx)} \cdot \int \frac{x^{p-1}(x^3-1)\partial x}{1-xx},$$

ubi cum sit

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x - \frac{1}{x+1},$$

erit postrema formula integralis

$$-\int x^p \partial x - \int \frac{x^{p-1}\partial x}{1+x};$$

quae integrata ab $x=0$ ad $x=1$ dat

$$-\frac{1}{p+1} - \int \frac{x^{p-1}\partial x}{1+x},$$

quamobrem habebimus

$$\int x^{p-1} \partial x lx \cdot \sqrt{(1-xx)} = -\int x^{p-1} \partial x \cdot \sqrt{(1-xx)} \cdot \left(\frac{1}{p+1} + \int \frac{x^{p-1}\partial x}{1+x} \right).$$

Hinc igitur sequentia exempla notasse juvabit.

Exemplum I. quo $p=1$.

§. 152. Pro hoc igitur casu postremis factor evadet, $\frac{1}{2} + l2$, ita ut sit

$$\int \partial x lx \sqrt{(1-xx)} = -\left(\tfrac{1}{2} + l2 \right) \cdot \int \partial x \sqrt{(1-xx)}.$$

Pro formula autem $\int \partial x \sqrt{(1-xx)}$ statuatur $\sqrt{(1-xx)} = 1 - vx$, fietque

$$x = \frac{2v}{1+vv}, \text{ et } \sqrt{(1-xx)} = \frac{1-vv}{1+vv},$$

atque $\partial x = \frac{2\partial v(1-vv)}{(1+vv)^2}$, unde fiet $\partial x \sqrt{(1-xx)} = \frac{2\partial v(1-vv)^2}{(1+vv)^3}$,

cujus integrale resolvitur in has partes

$$\frac{2v}{(1+vv)^2} - \frac{v}{1+vv} + \text{Arc.tang.} v;$$

quae expressio, cum extendi debeat ab $x=0$ usque ad $x=1$, prior terminus erit $v=0$, alter vero terminus est $v=1$; ita ut integrale illud a $v=0$ usque ad $v=1$ extendi debeat. At vero illa expressio sponte evanescit posito $v=0$, facto autem $v=1$, valor integralis erit $\frac{\pi}{4}$, quamobrem habebimus

$$\int \partial x l x \sqrt{(1-xx)} \begin{bmatrix} ab \ x=0 \\ ad \ x=1 \end{bmatrix} = -\frac{\pi}{4} \cdot \left(\frac{1}{2}+l2\right).$$

§. 153. Hic quidem calculum per longas ambages evolvimus, prouti reductio ad rationalitatem formulae $\sqrt{(1-xx)}$ manuduxit; at vero solus aspectus formulae $\int \partial x \sqrt{(1-xx)}$ statim declarat, eam exprimere aream quadrantis circuli, cujus radius $=1$, quem novimus esse $-\frac{\pi}{4}$. Caeterum adhiberi potuisset ista reductio

$$\int \partial x \sqrt{(1-xx)} = \frac{1}{2} x \sqrt{(1-xx)} + \frac{1}{2} \int \frac{\partial x}{\sqrt{(1-xx)}}$$

cujus valor ab $x=0$ ad $x=1$ extensus manifesta dat $\frac{\pi}{4}$.

Exemplum I. quo $p=2$.

§. 154. Hoc ergo casu postremus factor fit

$$\frac{1}{3} + \int \frac{x\partial x}{1+x} = \frac{4}{3} - l2;$$

sicque habebimus

$$\int \partial x l x \sqrt{(1-xx)} = -\left(\frac{4}{3} - l2\right) \cdot \int x \partial x \sqrt{(1-xx)} :$$

perspicuum autem est, esse

$$\int x \partial x \sqrt{(1-xx)} = \mathrm{C} - \frac{1}{3}(1-xx)^{\frac{3}{2}},$$

qui valor ab $x=0$ ad $x=1$ extensus praebet $\frac{1}{3}$, ita ut habeamus

$$\int \partial x l x \sqrt{(1-xx)} \begin{bmatrix} ab \ x=0 \\ ad \ x=1 \end{bmatrix} = -\frac{1}{3}\left(\frac{4}{3} - l2\right).$$

III. Evolutio casus quo $m=1$ et $n=3$.

§.155. Hoc igitur casu erit $\mathrm{X} = \frac{1}{\sqrt[3]{(1-x^3)^2}}$, unde theorema generale nobis praebet hanc aequationem

$$\int \frac{x^{p-1}\partial x l x}{\sqrt[3]{\left(1-x^3\right)^2}} = \int \frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^2}} \cdot \int \frac{x^{p-1}(x-1)\partial x}{1-x^3},$$

ubi postrema formula reducitur ad hanc

$$-\int \frac{x^{p-1}\partial x}{xx+x+1},$$

ita ut habeamus

$$\int \frac{x^{p-1}\partial x l x}{\sqrt[3]{\left(1-x^3\right)^2}} = -\int \frac{x^{p-1}\partial x l x}{\sqrt[3]{\left(1-x^3\right)^2}} \cdot \int \frac{x^{p-1}\partial x}{xx+x+1}:$$

sequentia igitur exempla adiungamus.

Exemplum I. quo $p=1$.

§.156. Hoc igitur casu postremus factor evadit $\frac{\partial x}{xx+x+1}$, cujus integrale indefinitum reperitur $\frac{2}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2+x}$, qui valor posito $x=1$ abit in $\frac{\pi}{3\sqrt{3}}$; quocirca hoc casu habebimus

$$\int \frac{\partial x l x}{\sqrt[3]{\left(1-x^3\right)^2}} = -\frac{\pi}{3\sqrt{3}} \cdot \int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}};$$

at vero formula integralis $\int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}$ peculiarem quantitatem transcendentem involvit, quam neque per logarithmos, neque per arcus circulares explicare licet.

Exemplum II quo $p=2$.

§.157. Hoc igitur casu postremus factor erit $\int \frac{x\partial x}{1+x+xx}$, qui in has partes resolvatur

$$\frac{1}{2}\int \frac{2x\partial x+\partial x}{1+x+xx} - \frac{1}{2}\int \frac{\partial x}{1+x+xx},$$

ubi partis prioris integrale est

$$\tfrac{1}{2} l\left(1+x+xx\right) = \tfrac{1}{2} l3 \text{ (posito scilicet } x=1\text{);}$$

alterius vero partis integrale est $\frac{1}{2}\cdot\frac{\pi}{3\sqrt{3}}$, quo valore substituto habebimus

$$\int \frac{x\partial x l x}{\sqrt[3]{\left(1-x^3\right)^2}} = -\tfrac{1}{2}\left(l3 - \frac{\pi}{3\sqrt{3}}\right) \cdot \int \frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}$$

Nunc vero istam formulam integralem commode assignare licet per reductionem supra initio indicatam; cum enim hic sit $m=1$ et $n=3$, tum vero sumserimus $p=2$, erit $p=n-m.$ Supra autem §. 131. invenimus, hoc casu integrale fore

$$=\frac{\pi}{n\sin.\frac{m\pi}{n}},$$

qui valor nostro casu abit in

$$\frac{\pi}{3\sin.\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}.$$

Hoc igitur valore substituto, nostram formulam per meras quantitales cognitas exprimere poterimus, hoc modo

$$\int\frac{x\partial xlx}{\sqrt[3]{\left(1-x^3\right)^2}}\begin{bmatrix}ab\ x=0\\ ad\ x=1\end{bmatrix}=-\frac{\pi}{3\sqrt{3}}\left(l3-\frac{\pi}{3\sqrt{3}}\right).$$

IV. Evolutio casus quo $m=2$ et $n=3$.

§.158. Hoc igitur casu erit $\mathrm{X}=\frac{1}{\sqrt[3]{\left(1-x^3\right)}}$, unde theorema generale praebet istam aequationem

$$\int\frac{x^{p-1}\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}=\int\frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{p-1}(xx-1)\partial x}{1-x^3},$$

ubi forma postrema transmutatur in hanc

$$-\int\frac{x^{p-1}\partial x(1+x)}{1+x+xx};$$

unde fiet

$$\int\frac{x^{p-1}\partial xlx}{\sqrt[3]{\left(1-x^3\right)}}=-\int\frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{p-1}\partial x(1+x)}{1+x+xx};$$

unde sequentia exempla expediamus.

Exemplum I. quo $p=1$.

§. 159. Hoc ergo casu membrum postremum erit $\int\frac{\partial x(1+x)}{1+x+xx}$, cujus integrale in has partes distribuatur

$$\frac{1}{2}\int\frac{2x\partial x+\partial x}{1+x+xx}+\frac{1}{2}\int\frac{\partial x}{1+x+xx},$$

unde manifesta pro casu $x=1$ prodit $\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)$; quamobrem nostra aequatio erit

$$\int\frac{\partial x lx}{\sqrt[3]{\left(1-x^3\right)}}=-\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)\cdot\int\frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}}.$$

In hac autem formula integrali, ob $m=2$ et $n=3$, quia sumsimus $p=1,$ erit $p=n-m$; pro hoc ergo casu per §. 131. valor istius formulae absolute exprimi poterit, eritque

$$\int\frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}}=\frac{2\pi}{3\sqrt{3}};$$

consequenter etiam hoc casu per quantitates absolutas consequimur
hanc formam

$$\int\frac{\partial x lx}{\sqrt[3]{\left(1-x^3\right)}}\begin{bmatrix}ab\ x=0\\ ad\ x=1\end{bmatrix}=-\frac{\pi}{3\sqrt{3}}\left(l3+\frac{\pi}{3\sqrt{3}}\right).$$

§. 160. Quodsi hanc formam cum postrema casus praecedentis, quae itidem absolute prodiit expressa, combinemus, earum summa primo dabit

$$\int\frac{x\partial x lx}{\sqrt[3]{\left(1-x^3\right)^2}}+\int\frac{\partial x lx}{\sqrt[3]{\left(1-x^3\right)}}=-\frac{2\pi l3}{3\sqrt{3}}:$$

sin autem posterior a priore subtrahatur orietur ista aequatio

$$\int\frac{x\partial x lx}{\sqrt[3]{\left(1-x^3\right)^2}}-\int\frac{\partial x lx}{\sqrt[3]{\left(1-x^3\right)}}=\frac{2\pi\pi}{27}.$$

Quoniam hoc modo ad expressiones satis simplices sumus perducti, operae pretium erit ambas aequationes sub alia forma repraesentare, qua binae partes integrales commode in unam conjungi queant; statuamus
scilicet $\frac{x}{\sqrt[3]{\left(1-x^3\right)}}=z,$ unde fit $\frac{xx}{\sqrt[3]{\left(1-x^3\right)^2}}=zz$, sicque prior formula induet hanc speciem $\int\frac{zz\partial x lx}{x}$; posterior vero istam $\int\frac{z\partial x lx}{x}$; tum vero habebimus $\frac{x^3}{1-x^3}=z^3$, unde sit $x^3=\frac{z^3}{1+z^3}$, ideoque

$$lx=lz-\frac{1}{3}l\left(1+z^3\right)=l\frac{z}{\sqrt[3]{\left(1+z^3\right)}},$$

hincque porro

$$\frac{\partial x}{x}=\frac{\partial z}{z}-\frac{zz\partial z}{1+z^3}=\frac{\partial z}{z\left(1+z^3\right)};$$

quare his valoribus adhibitis, prior formula integralis evadit

$$\int\frac{z\partial z}{1+z^3}\cdot l\frac{z}{\sqrt[3]{\left(1+z^3\right)}};$$

altera vero formula erit

$$\int\frac{\partial z}{1+z^3}\cdot l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}.$$

§. 161. Quoniam autem integralia ab $x=0$ ad $x=1$ extendi debent, notandum est, casu $x=0$ fieri $z=0,$ at vero casu $x=1$ prodire $z=\infty,$ ita ut novas istas formas a $z=0$ ad $z=\infty$ extendi oporteat. Quo observato prior harum formularum dabit

$$\int\frac{z\partial z}{1+z^3}\cdot l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}\begin{bmatrix}a\ x=0\\ ad\ x=\infty\end{bmatrix}=-\frac{\pi l3}{3\sqrt{3}}+\frac{\pi\pi}{27},$$

posterior vero

$$\int\frac{\partial z}{1+z^3}\cdot l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}\begin{bmatrix}a\ z=0\\ ad\ z=\infty\end{bmatrix}=-\frac{\pi l3}{3\sqrt{3}}-\frac{\pi\pi}{27}.$$

Hinc igitur summa harum formularum erit

$$\int\frac{(1+z)\partial z}{1+z^3}\cdot l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}=-\frac{2\pi l3}{3\sqrt{3}},$$

at vero differentia

$$\int\frac{\partial z(z-1)}{1+z^3}\cdot l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}=\frac{2\pi\pi}{27}.$$

§. 162. Hic non inutile erit observasse, istum logarithmum $l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}$ commode in seriem infinitam satis simplicem converti posse; cum enim sit

$$l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}=\frac{1}{3}l\frac{z^3}{1+z^3}=-\frac{1}{3}l\frac{1+z^3}{z^3},$$

erit per seriem

$$l\frac{z}{\sqrt[3]{\left(1+z^3\right)}}=-\frac{1}{3}\left(\frac{1}{z^3}-\frac{1}{2z^6}+\frac{1}{3z^9}-\frac{1}{4z^{12}}+\frac{1}{5z^{15}}-\text{etc.}\right)$$

verum ista resolutio nullum usum praestare potest ad integralia haec per series evolvenda, propterea quod potestates ipsius z in denominatoribus occurrunt, ideoque singulae partes non ita integrari possunt ut evanescant posito $z = 0$.

Exemplum II. quo $p = 2$.

§. 163. Hoc igitur casu factor postremus evadit $\int \frac{x\partial x(1+x)}{1+x+xx}$, qui in has duas partes discerpitur $\int \partial x - \int \frac{\partial x}{1+x+xx}$, cujus ergo integrale ab $x = 0$ ad $x = 1$ extensum est $= 1 - \frac{\pi}{3\sqrt{3}}$. Hinc igitur deducimur ad hanc aequationem

$$\int \frac{x\partial x lx}{\sqrt[3]{(1-x^3)}} = \left(1 - \frac{\pi}{3\sqrt{3}}\right) \cdot \int \frac{\partial x}{\sqrt[3]{(1-x^3)}}.$$

Hic autem notandum, istam formulam integralem nullo modo absolute exhiberi posse, sed peculiarem quandam quantitatem transcendentem involvere.

V. Evolutio casus, quo $m = 2$ et $n = 4$.

§. 164. Hoc igitur casu erit $\mathrm{X} = \frac{1}{\sqrt{(1-x^4)}}$, unde theorema nostrum generale nobis dabit hanc aequationem

$$\int \frac{x^{p-1}\partial x lx}{\sqrt{(1-x^4)}} = -\int \frac{x^{p-1}\partial x}{\sqrt{(1-x^4)}} \cdot \int \frac{x^{p-1}\partial x}{1+xx};$$

at vero problema particulare prius pro hoc casu praebet

$$\int \frac{x^3\partial x lx}{\sqrt[3]{(1-x^4)}} = -\frac{1}{2}\int \frac{x^3\partial x}{1+xx}.$$

Cum autem sit

$$\int \frac{x^3\partial x}{1+xx} = \frac{1}{2} - \frac{1}{2}l2,$$

erit absolute

$$\int \frac{x^3\partial x lx}{\sqrt{(1-x^4)}} \begin{bmatrix} ab\ x = 0 \\ ad\ x = 1 \end{bmatrix} = -\frac{1}{4}(1 - l2),$$

at vero hic casus congruit cum supra §. 144. tractato. Si enim hic ponamus $xx = y$, quo facto termini integrationis manent $y = 0$ et $y = 1$, erit $lx = \frac{1}{2}ly$ et $x\partial x = \frac{1}{2}\partial y$; quibus valoribus substitutis nostra aequatio abibit in hanc formam

$$\frac{1}{4}\int \frac{y\partial y ly}{\sqrt{(1-yy)}} = -\frac{1}{4}(1 - l2), \text{ sive } \int \frac{y\partial y}{\sqrt{(1-yy)}} = l2 - 1,$$

prorsus ut supra.

§. 165. Alterum vero theorema particulare ad praesentem casum accommodatum dabit

$$\int \frac{x\partial x lx}{\sqrt{\left(1-x^4\right)}} = -\frac{\pi}{4}\int \frac{x\partial x}{1+xx};$$

est vero

$$\int \frac{x\partial x}{1+xx} = l\sqrt{(1+xx)} = \frac{1}{2}l2,$$

ita ut habeamus

$$\int \frac{x\partial x lx}{\sqrt{\left(1-x^4\right)}} \begin{bmatrix} ab\ x=0 \\ ad\ \ x=1 \end{bmatrix} = -\frac{\pi}{8}l2.$$

Quodsi vero hic ut ante statuamus $xx = y$, obtinebitur

$$\int \frac{\partial y ly}{\sqrt{(1-yy)}} = -\frac{\pi}{2}l2,$$

qui est casus supra §. 142. tractatus. His duobus casibus exponens *p* erat numerus par, unde casus impares evolvi conveniet.

Exemplum I. quo $p = 1$.

§. l66. Hoc igitur casu formula integralis postrema fiet $\int \frac{\partial x}{1+xx} = \text{Arc.tang.}x$, ita ut posito $x=1$ prodeat $\text{Arc.tang.}x = \frac{\pi}{4}$; tum vero aequatio nostra erit

$$\int \frac{\partial x lx}{\sqrt{\left(1-x^4\right)}} = -\frac{\pi}{4}\cdot\int \frac{\partial x}{\sqrt{\left(1-x^4\right)}},$$

integralibus scilicet ab $x=0$ ad $x=1$ extensis; ubi formula $\int \frac{\partial x}{\sqrt{\left(1-x^4\right)}}$ arcum curvae elasticae rectangulae exprimit, ideoque absolute exhiberi nequit.

Exemplum II. quo $p = 3$.

§. 16 l. Hoc ergo casu formula integralis postrema erit $\int \frac{xx\partial x}{1+xx} = \int \partial x - \int \frac{\partial x}{1+xx}$, qui in has duas partes discerpitur $\int \partial x - \int \frac{\partial x}{1+x+xx}$, cujus integrale posito $x=1$ fit $= 1-\frac{\pi}{4}$, ita ut nunc aequatio nostra evadat

$$\int \frac{xx\partial xlx}{\sqrt{\left(1-x^4\right)}} = -\left(1-\tfrac{\pi}{4}\right)\cdot\int \frac{xx\partial x}{\sqrt{\left(1-x^4\right)}},$$

quae formula integralis pariter absolute exhiberi nequit exprimit enim applicatam curvae elasticae rectangulae.

§.168. Quanquam autem haec duo exempla ad formulas inextricabiles perduxerunt, tamen jam pridem demonstravi, productum horum duorum integralium

$$\int \frac{\partial x}{\sqrt{\left(1-x^4\right)}}\cdot\int \frac{xx\partial x}{\sqrt{\left(1-x^4\right)}}$$

aequari areae circuli, cujus diameter $=1$, sive esse $=\frac{\pi}{4}$; quamobrem binis exemplis conjungendis, hoc insigne theorema adipiscimur

$$\int \frac{\partial xlx}{\sqrt{\left(1-x^4\right)}}\cdot\int \frac{xx\partial xlx}{\sqrt{\left(1-x^4\right)}} = \frac{\pi^2}{16}\left(1-\tfrac{\pi}{4}\right).$$

Facile autem patet, innumera alia hujusmodi theoremata ex hoc fonte hauriri posse, quae, per se spectata, profundissimae indaginis sunt censenda.