SUPPLEMENT III*a*. TO BOOK I. CHAP. IV.

CONCERNING THE INTEGRATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

1). The evolution of the integral formula $\int x^{f-1}\partial x\left(lx\right)^{\frac{m}{n}}$, with the integration extending from $x=0$ to $x=1$.

Nov. Commentarii Acad. Imp. Sc. Petrapolitanae. Tom. XVL. Pag. 91−139.

Theorem 1.

§. 1. If *n* may denote some positive whole number, and if the integration of the formula $\int x^{f-1}\partial x\left(1-x^{g}\right)^{n}$ may be extended from $x=0$ as far as $x=1$, the value of which will be :

$$=\frac{g^{n}}{f}\cdot\frac{1.\quad 2.\quad 3.\quad ....n}{(f+g)(f+2g)(f+3g)...........(f+ng)}.$$

Demonstration.

It is observed in general, the integration of the formula

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{m}$$

can be reduced to the integration of this $\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}$, because it is possible to define constant quantities A and B thus, so that

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{m}=\mathrm{A}\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}+\mathrm{B}x^{f}\left(1-x^{g}\right)^{m};$$

indeed with the differentials taken this equation is produced

$$x^{f-1}\partial x\left(1-x^{g}\right)^{m}=\mathrm{A}x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}+\mathrm{B}fx^{f-1}\partial x\left(1-x^{g}\right)^{m}-\mathrm{B}mgx^{f+g-1}\partial x\left(1-x^{g}\right)^{m-1},$$

which divided by $x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}$ gives

$$1-x^{g}=\mathrm{A}+\mathrm{B}f\left(1-x^{g}\right)-\mathrm{B}mgx^{g}\partial x\left(1-x^{g}\right),\text{or}$$
$$1-x^{g}=\mathrm{A}-\mathrm{B}mg+\mathrm{B}\left(f+mg\right)\left(1-x^{g}\right),$$

which in order that the equation shall be consistent, it is necessary that

$$1 = \mathrm{B}\,(f + mg) \text{ and } \mathrm{A} = \mathrm{B}mg\ ;$$

from which we gather

$$\mathrm{B} = \tfrac{1}{f+mg} \text{ and } \mathrm{A} = \tfrac{mg}{f+mg}.$$

On account of which we will have the following general reduction :

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{m} = \tfrac{mg}{f+mg}\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1} + \tfrac{1}{f+mg}\cdot x^{f}\left(1-x^{g}\right)^{m}$$

which since it may vanish on putting $x=0$, if indeed there shall be $f>0$, there is no need for a constant. Whereby with the extension of the integral as far as to $x=1$, the latter part of the integral vanishes at once, and for the case $x=1$ there will be

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{m} = \tfrac{mg}{f+mg}\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}.$$

Therefore since on taking $m=1$ there shall be $\int x^{f-1}\partial x\left(1-x^{g}\right)^{0} = \tfrac{1}{f}x^{f} = \tfrac{1}{f}$, on putting $x=1$, we arrive at the following values for the same case $x=1$:

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{1} = \tfrac{g}{f}\cdot\tfrac{1}{f+g}$$

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{2} = \tfrac{g^{2}}{f}\cdot\tfrac{1}{f+g}\cdot\tfrac{2}{f+2g}$$

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{3} = \tfrac{g^{3}}{f}\cdot\tfrac{1}{f+g}\cdot\tfrac{2}{f+2g}\cdot\tfrac{3}{f+3g}$$

etc.

and hence for any positive whole number *n* we conclude there shall become

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{n} = \tfrac{g^{n}}{f}\cdot\tfrac{1}{f+g}\cdot\tfrac{2}{f+2g}\cdot\tfrac{3}{f+3g}\cdots\cdots\cdot\tfrac{n}{f+ng}$$

But only if the numbers f and g shall be positive.

Corollary I.

§. 2. Hence in turn therefore the value of any product of this kind formed from any number of factors, can be expressed by the integral formula, thus so that there shall be

$$\frac{1.\quad 2.\quad 3.\quad \cdots\cdots n}{(f+g)(f+2g)(f+3g)\cdots\cdots(f+ng)} = \frac{f}{g^{n}}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n},$$

here with the value of the integral extended from $x=0$ as far as to $x=1$.

Corollary 2.

§. 3. Whereas therefore a progression of this kind may be had,

$$\frac{1}{f+g};\frac{1.\quad 2}{(f+g)(f+2g)};\frac{1.\quad 2.\quad 3}{(f+g)(f+2g)(f+3g)};\frac{1.\quad 2.\quad 3.\quad 4}{(f+g)(f+2g)(f+3g)(f+4g)};\text{etc.},$$

the general term of which is agreed to be indicated by the index *n*, which may be represented conveniently by this form of integral $\frac{f}{g^n}\int x^{f-1}\partial x\left(1-x^g\right)^n$, with the aid of which that progression can be interpolated, and its corresponding terms can be indicated by fractional indices.

Corollary 3.

§. 4. If we shall write $n-1$ in place of n, we will have

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\ldots(n-1)}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots\ldots(f+(n-1)g)}=\frac{f}{g^{n-1}}\int x^{f-1}\partial x\left(1-x^g\right)^{n-1},$$

which multiplied by $\frac{n}{f+ng}$ gives

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots\ldots(f+ng)}=\frac{f.ng}{g^n(f+ng)}\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}.$$

Scholium 1.

§5. This latter form may be derived at once from the preceding, in the same manner as we have shown, to be

$$\int x^{f-1}\partial x\left(1-x^g\right)^n=\frac{ng}{f+ng}\int x^{f-1}\partial x\left(1-x^g\right)^{n-1},$$

if indeed each integral may be extended from the value $x=0$ as far as to $x=1$; as will be required to understand everywhere in the following determination of the integrals. Then it is required also to be kept in mind always, that the quantities *f* and *g* shall be positive, clearly as the condition demonstrated demands absolutely to be put in place. But concerning the number *n*, in so far as each value is the index of its term, and the terms of which progression may be designated as in (§. 3.), nothing prevents any number there which may be designated either positive or negative, since all the terms of this progression, also those with corresponding negative indices, may be agreed to be shown by the integral formula. Yet meanwhile it is required to be kept in mind properly, that this reduction

$$\int x^{f-1}\partial x\left(1-x^g\right)^n=\frac{mg}{f+mg}\int x^{f-1}\partial x\left(1-x^g\right)^{m-1}$$

cannot be agreed to be true, unless there shall be $m>0$; because otherwise the algebraic part $\frac{1}{f+mg}x^f\left(1-x^g\right)^m$ will not disappear on putting $x=1$.

Scholium 2.

§ 6. Series of this kind, which it is appropriate to call transcending, because the terms corresponding to fractional indices are transcending quantities, originally found in the *Comment. Petrop*. Vol. V. (*Institut. Calc. Integralis* Vol. I. Sect. I. Chap. IV.) I now pursue further; so that here I am going to examine carefully not only these progressions, but also the comparisons with the extraordinary integral formulas, which may be derived from these. Clearly since I had shown, how the value of this indefinite product 1. 2. 3…..n to be expressed by the integral formula $\int \partial x\left(l\frac{1}{x}\right)^{n}$ extended from $x=0$ to $x=1$, which matter is evident by integration whenever n is a whole positive number, these cases I have subjected to examination, in which fractions are accepted for n ; where indeed by no means is it apparent from these integral formulas, to what kind of transcending quantities these terms must be referred. But with the aid of a single artifice these same terms are able to be reduced to a well known quadrature, which therefore is seen to be of the greatest worth, so that it may be considered with the greatest interest.

Problem 1.

§. 7. *Since*

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots\ldots(f+ng)}=\frac{f}{g^{n}}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n}$$

it may be demonstrated, with the integral extended from $x=0$ *to* $x=1$*, to assign the value of this same product in the same case where* $g=0$*, from the integral formula.*

Solution.

On putting $g=0$ into the formula of the integral the term $\left(1-x^{g}\right)^{n}$ vanishes, likewise also the denominator g^{n}, from which the question here is reduced to a form, so that the value of the fraction $\frac{\left(1-x^{g}\right)^{n}}{x^{g}}$ may be defined in the case $g=0$, where both the numerator as well as the denominator vanish. This g in the end may considered as an infinitely small quantity, and since there shall be $x^{g}=e^{glx}$, there becomes $x^{g}=1+glx$, and thus

$$\left(1-x^{g}\right)^{n}=g^{n}\left(-lx\right)^{n}=g^{n}\left(l\frac{1}{x}\right)^{n};$$

from which for this case our formula of the integral will be changed into

$$f\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n};$$

thus so that now there may be had

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\ldots n}{f^{n}}=f\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n},\ \text{seu}$$

$$1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\ldots n=f^{f+1}\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n}.$$

Corollary 1.

§. 8. Whenever *n* is a whole positive number, the integration of the formula $\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^n$ succeeds, and that extended from $x=0$ to $x=1$ actually produces that product, that we have found equal to the same formula. But if a fractional number may be taken for *n*, the same formula of the integral does not arise for this hyper geometric interpolation of the progression,

$$\begin{array}{llllll} 1; & 1.\,2; & 1.\,2.\,3; & 1.\,2.\,3.\,4; & 1.\,2.\,3.\,4.\,5; & 1.\,2.\,5.\,4.\,5.\,6;\ \text{etc. or,} \\ 1; & 2; & 6; & 24; & 120; & 720;\ \text{etc.} \end{array}$$

Corollary 2.

§. 9. But if the expression may be divided by the principal, a product arises, whose factors shall be progressing in some arithmetical progression

$$(f+g)(f+2g)(f+3g)........(f+ng)=f^n g^n.\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^n}{\int x^{f-1}\partial x\left(1-x^g\right)^n},$$

therefore the values of this also, if *n* shall be a fractional number, hence will be allowed to assign.

Corollary 3.

§.10. Since there shall be

$$\int x^{f-1}\partial x\left(1-x^g\right)^n=\frac{ng}{f+ng}\int x^{f-1}\partial x\left(1-x^g\right)^{n-1},$$

also there will be in a manner similar for the case $g=0$

$$\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^n=\frac{n}{f}\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1},$$

and hence by these other integral formulas

$$1.\,2.\,3...............n=nf^n\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1},$$

and

$$(f+g)(f+2g)......(f+ng)=f^{n-1}g^{n-1}.\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}}.$$

Scholium.

§.11. Since we have found

$$1.\ 2.\ 3.......n = f^{n+1}\int x^{f-1}\partial x\left(l\tfrac{1}{x}\right)^n,$$

it is apparent this integral formula does not depend on the value of the quantity *f*, which is easily seen also by putting $x^f = y$, from which there shall be $fx^{f-1}\partial x = \partial y$, and $l\frac{1}{x} = -lx = -\frac{1}{f}ly = \frac{1}{f}l\frac{1}{y}$,
and thus $f^n\left(l\frac{1}{x}\right)^n = \left(l\frac{1}{y}\right)^n$, thus so that there becomes

$$1.\ 2.\ 3.......n = \int \partial y\left(l\tfrac{1}{y}\right)^n,$$

which formula arises from the first by putting $f = 1$. Therefore for the interpolation of formulas of this kind the whole matter is reduced to this, so that the values of these integral formulas $\int \partial x\left(l\frac{1}{x}\right)^n$ may be defined, when the exponent *n* is a fractional number. So that if *n* shall be $\frac{1}{2}$, it is required to assign the value of this formula $\int \partial x\sqrt{l\frac{1}{x}}$, which was shown some time ago to be $= \frac{1}{2}\sqrt{\pi}$, with *π* denoting the periphery of a circle whose diameter $= 1$: but for the value of these other fractional numbers, I have shown to be generated from the quadratures of algebraic curves of higher orders. Which reduction shall be performed with little difficulty, and that occurring only when the integration of the formula $\int \partial x\left(l\frac{1}{x}\right)^n$ is extended from the value $x = 0$ to $x = 1$, shall be worthy of our particular attention. But now I have examined this very argument some time ago also, yet because I was led astray by several long-winded arguments, I have decided to resume and to set out the same here with more care.

Theorem 2.

§. 12. If the integral formulas may be extended from the value $x = 0$ as far as to $x = 1$, and *n* shall denote a positive integer, there will be

$$\frac{1.\quad 2.\quad 3.\quad ....n}{(n+1)(n+2)(n+3)............2n} = \frac{1}{2}ng\int x^{f+ng-1}\partial x\left(1-x^g\right)^{n-1} \times \frac{\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}}$$

whatever the positive numbers may the taken in place of *f* and *g*.

Demonstration.

Since above (§. 4.) we have shown to have,

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots(f+ng)}=\frac{f.ng}{g^{n}(f+ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1},$$

if in place of *n* we may write 2*n,*

$$\frac{1.\quad 2.\quad 3.\ldots\ldots\ldots 2n}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots(f+2ng)}=\frac{f.2ng}{g^{2n}(f+2ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}.$$

Now the first equation may be divided by the second, and this third will be produced:

$$\frac{(f+(n+1)g)(f+(n+2)g)\ldots\ldots\ldots(f+2ng)}{(n+1)(n+2)\ldots\ldots 2n}=\frac{g^{n}(f+2ng)}{2(f+ng)}\cdot\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}.$$

But if in the first equation in place of *f* there may be written $f+ng$, this fourth equation will arise :

$$\frac{1.\quad 2.\quad 3.\ldots\ldots\ldots n}{(f+(n+1)g)(f+(n+2)g)\ldots\ldots\ldots(f+2ng)}=\frac{(f+ng)ng}{g^{n}(f+2ng)}\int x^{f+ng-1}\partial x\left(1-x^{g}\right)^{n-1}.$$

This fourth equation may be multiplied by that third, and the equation requiring to be shown will be found,

$$\frac{1.\quad 2.\quad 3.\ldots\ldots\ldots n}{(n+1)(n+2)(n+3)\ldots\ldots\ldots 2n}=\frac{1}{2}ng\int x^{f+ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}$$

Corollary 1.

§. 13. If $f=n$ and $g=1$ may be put into the first equation, there will arise also

$$\frac{1.\quad 2\ \ldots\ldots\ldots n}{(n+1)(n+2)\ldots\ldots\ldots 2n}=\frac{1}{2}n\int x^{f-1}\partial x\left(1-x\right)^{n-1},$$

which collated with that equation we arrive at :

$$\frac{\int x^{n-1}\partial x(1-x)^{n-1}}{g\int x^{f+ng-1}\partial x\left(1-x^{g}\right)^{n-1}}=\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}.$$

Corollary 2.

§. 14. If in that equation we may write x^{g} in place of *x*, there becomes

$$\frac{1.\quad 2\ \ldots\ldots\ldots n}{(n+1)(n+2)\ldots\ldots\ldots 2n}=\frac{1}{2}ng\int x^{ng-1}\partial x\left(1-x^{g}\right)^{n-1};$$

thus so that now we may put together this comparison between the following integral formulas:

$$\int x^{ng-1}\partial x\left(1-x^{g}\right)^{n-1}=\int x^{f+ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}.$$

Corollary 3.

§. 15. If we may put $g=0$ into the equation of the theorem, on account of $\left(1-x^{g}\right)^{m}=g^{m}\left(l\frac{1}{x}\right)^{m}$, the powers of g will cancel out, and this equation shall arise

$$\frac{1.\quad 2.\quad 3.\ldots\ldots n}{(n+1)(n+2)(n+3)\ldots\ldots 2n}=\frac{1}{2}n\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}};$$

from which we deduce

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^{2}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}}=g\int x^{ng-1}\partial x\left(1-x^{g}\right)^{n-1},$$

or on account of

$$\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}=\frac{f}{n}\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n},\ \text{this :}$$

$$\frac{2f}{n}\cdot\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n}\right]^{2}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n}}=g\int x^{ng-1}\partial x\left(1-x^{g}\right)^{n-1}.$$

Corollary 4.

§.16. We may put here $f=1$, $g=2$ and $n=\frac{m}{2}$, so that m shall be a positive integer; and because $\int\partial x\left(l\frac{1}{x}\right)^{m}=1.\ 2.\ 3.\ \ldots..m$, there will become

$$\frac{4}{m}\cdot\frac{\left[\int\partial x\left(l\frac{1}{x}\right)^{\frac{m}{2}}\right]^{2}}{1.\ 2.\ 3.\ \ldots..m}=2\int x^{m-1}\partial x\left(1-x^{2}\right)^{\frac{m}{2}-1},$$

and hence

$$\int\partial x\left(l\frac{1}{x}\right)^{\frac{m}{2}}=\sqrt{}1.\ 2.\ 3.\ \ldots..m.\frac{m}{2}\int x^{m-1}\partial x\left(1-x^{2}\right)^{\frac{m}{2}-1},$$

[***Here, and unless stated otherwise in the sequence, the root*** $\sqrt{}$ ***of whatever size applies to the whole expression to the right of the sign***.]

and by taking $m=1$, on account of $\int\frac{\partial x}{\sqrt{(1-xx)}}=\frac{\pi}{2}$, there will be had

$$\int \partial x\sqrt{\left(l\tfrac{1}{x}\right)} = \sqrt{\ }\tfrac{1}{2}\int \frac{\partial x}{\sqrt{1-x^2}} = \tfrac{1}{2}\sqrt{\pi}.$$

Scholium.

§.17. At one time a more succinct demonstration of the theorem was advanced by me, so that there shall be $\int \partial x\sqrt{\left(l\frac{1}{x}\right)} = \frac{1}{2}\sqrt{\pi}$, and that on account of interpolation, which then I was free to use. Clearly with this deduced here from that theorem, where I had found to be

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}} = g\int x^{ng-1}\partial x\left(1-x^g\right)^{n-1}.$$

But the principal theorem, from which that is deduced, may be had thus :

$$g\cdot\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}\times\int x^{f+ng-1}\partial x\left(1-x^g\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}} = \int x^{n-1}\partial x\left(1-x\right)^{n-1};$$

indeed each part may be worked out by integrating from $x=0$ to $x=1$, to be established by this numerical product

$$\frac{1.\quad 2.\quad 3.\dots\dots\dots\dots (n-1)}{(n+1)\ (n+2)\ \dots\dots\ (2n-1)}.$$

And if we would like to attribute a broader view set out for the other part, the theorem can be proposed so that it becomes

$$g\cdot\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}\times\int x^{f+ng-1}\partial x\left(1-x^g\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}} = k\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1},$$

and hence if there may be taken $g=0$, it becomes

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}} = k\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1}.$$

Therefore in the first place it is to be noted, because that equality may remain, whatever numbers may be taken in place of f and g : indeed in the case $f=g$, that is evident, since there shall be

$$\int x^{g-1}\partial x\left(1-x^g\right)^{n-1} = \frac{1-\left(1-x^g\right)^n}{ng} = \frac{1}{ng},$$

indeed there becomes

$$2g\int x^{ng+g-1}\partial x\left(1-x^g\right)^{n-1} = k\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1},$$

and because

$$\int x^{ng+g-1}\partial x\left(1-x^{g}\right)^{n-1}=\tfrac{1}{2}\int x^{ng-1}\partial x\left(1-x^{g}\right)^{n-1},$$

equality is evident, because *k* can be taken arbitrarily. Moreover in the same manner as this theorem has been reached, thus it may be extended to other similar theorems.

Theorem 3.

§.18. If the following integral formulas may be extended from the value $x=0$ to $x=1$, and *n* may denote some positive integer, there will be

$$\frac{1.\quad 2.\quad 3.\ldots\ldots\ldots n}{(2n+1)(2n+2)(2n+3)\ldots.3n}=\tfrac{2}{3}ng\int x^{f+2ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{3n-1}},$$

whatever positive numbers may be taken for f and g.

Demonstration.

Now we have seen in the preceding theorem

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots 2n}{(f+g)(f+2g)\ldots\ldots\ldots(f+2ng)g}=\frac{f.2ng}{g^{2n}(f+2ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}.$$

however in a similar manner, if in place of *n* in the principal form we may write 3*n*, we will have

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots 3n}{(f+g)(f+2g)\ldots\ldots\ldots(f+3ng)}=\frac{f.3ng}{g^{3n}(f+3ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{3n-1},$$

from which with that equation divided by this, there is produced

$$\frac{(f+(2n+1)g)(f+(2n+2)g)\ldots\ldots\ldots(f+3ng)}{(2n+1)(2n+2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3n}=\frac{2g^{n}(f+3ng)}{3(f+2ng)}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{3n-1}}.$$

Truly if in the principal equation (§. 4.) we may write $f+2ng$ in place of f, we obtain this equation

$$\frac{1.\quad 2.\quad 3.\ldots\ldots\ldots\ldots\ldots\ldots\ldots n}{(f+(2n+1)g)(f+(2n+2)g)\ldots\ldots\ldots(f+3ng)}=\frac{(f+2ng).ng}{g^{n}(f+3ng)}\times\int x^{f+2ng-1}\partial x\left(1-x^{g}\right)^{n-1}.$$

Now this equation is multiplied by the preceding, and that equation may arise, which it is required to be shown

$$\frac{1.\quad 2.\quad 3............n}{(2n+1)(2n+2)(2n+3)....3n}=\tfrac{2}{3}ng\int x^{f+2ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{3n-1}},$$

Corollary 1.

§. 19. We obtain the same value from the principal equation, by putting $f=2n$ and $g=1$, thus so that there shall be

$$\frac{1.\quad 2.\quad 3............n}{(2n+1)(2n+2)(2n+3)....3n}=\tfrac{2}{3}n\int x^{2n-1}\partial x\left(1-x\right)^{n-1}$$

which integral formula, by writing x^{k} in place of x is changed into this :

$$\tfrac{2}{3}n\int x^{2nk-1}\partial x\left(1-x^{k}\right)^{n-1}$$

thus so that there shall be

$$g\int x^{f+2ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{3n-1}}=k\int x^{2nk-1}\partial x\left(1-x^{k}\right)^{n-1}.$$

Corollary 2.

§. 20. If here we may put $g=0,$ because $1-x^{g}=gl\frac{1}{x}$ we will have this equation

$$\int x^{f-1}\partial x\left(l\tfrac{1}{x}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}\partial x\left(1-x^{k}\right)^{n-1}:$$

therefore since we have found before

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^{2}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}\partial x\left(1-x^{k}\right)^{n-1},$$

we will have these equations multiplying each other

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^{3}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{3n-1}}=k^{2}\int x^{nk-1}\partial x\left(1-x^{k}\right)^{n-1}\times\int x^{2nk-1}\partial x\left(1-x^{k}\right)^{n-1}.$$

Corollary 3.

§. 21. Without any restriction it is allowed here to put $f=1$; therefore on assuming then $n=\frac{1}{3}$ and $k=3$, there will become

$$\frac{\left[\int \partial x\left(l\frac{1}{x}\right)^{-\frac{2}{3}}\right]^3}{\int \partial x\left(l\frac{1}{x}\right)^0}=9\int \partial x\left(1-x^3\right)^{-\frac{2}{3}}\times\int x\partial x\left(1-x^3\right)^{-\frac{2}{3}},$$

and on account of

$$\int \partial x\left(l\frac{1}{x}\right)^{-\frac{2}{3}}=3\int \partial x\left(l\frac{1}{x}\right)^{\frac{1}{3}} \text{ and } \int \partial x\left(l\frac{1}{x}\right)^0=1,$$

we will obtain

$$\int \partial x\left(l\frac{1}{x}\right)^{-\frac{2}{3}}=3\int \partial x\left(l\frac{1}{x}\right)^{\frac{1}{3}} \text{ and } \int \partial x\left(l\frac{1}{x}\right)^0=1,$$

then truly by taking $n=\frac{2}{3}$ and $k=3$, there will be

$$\frac{\left[\int \partial x\left(l\frac{1}{x}\right)^{-\frac{1}{3}}\right]^3}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)}=9\int x\partial x\left(1-x^3\right)^{-\frac{1}{3}}\times\int x^3\partial x\left(1-x^3\right)^{-\frac{1}{3}}.$$

or

$$\left[\int \partial x\left(l\frac{1}{x}\right)^{\frac{2}{3}}\right]^3=4\int x\partial x\left(1-x^3\right)^{-\frac{1}{3}}\times\int x^3\partial x\left(1-x^3\right)^{-\frac{1}{3}}.$$

General Theorem .

§. 22. If the following integral formulas may be extended from the value $x=0$ as far as to $x=1$, and n may denote any positive integer, there will be

$$\frac{1.\quad 2.\quad 3............n}{(\lambda n+1)(\lambda n+2)....(\lambda+1)n}=\frac{n}{\lambda+1}ng\int x^{f+\lambda ng-1}\partial x\left(1-x^g\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{\lambda n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{(\lambda+1)n-1}},$$

whatever positive numbers may be taken for the letters f and g .

Demonstration.

Since there may be used as we have shown above

$$\frac{1.\quad 2.\quad 3.\quad ....3n}{(f+g)(f+2g)...........(f+3ng)}=\frac{f.ng}{g^n(f+ng)}\int x^{f-1}\partial x\left(1-x^g\right)^{n-1},$$

if here in the first place we may write λn in place of n , then truly $(\lambda+1)n$, we will come upon these two equations :

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\lambda n}{(f+g)(f+2g)\ldots\ldots\ldots(f+\lambda ng)}=\frac{f\cdot\lambda ng}{g^{\lambda n}(f+\lambda ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{\lambda n-1},$$

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots(\lambda+1)n}{(f+g)(f+2g)\ldots\ldots\ldots(f+(\lambda+1)ng)}=\frac{f\cdot(\lambda+1)g}{g^{(\lambda+1)n}(f+(\lambda+1)ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{(\lambda+1)n-1},$$

of which that divided by this gives

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\ldots\ldots\ldots(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\ldots\ldots\ldots(\lambda n+n)}=g^{n}\,\frac{\lambda\cdot(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)}\cdot\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{\lambda n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{(\lambda+1)n-1}},$$

But if in the first equation we may write $f+\lambda ng$ in place of f, we shall obtain

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots n}{(f+\lambda ng+g)(f+\lambda ng+2g)\ldots\ldots\ldots(f+\lambda ng+ng)}=\frac{(f+\lambda ng)\cdot ng}{g^{n}(f+\lambda ng+ng)}\cdot\int x^{f+\lambda ng-1}\partial x\left(1-x^{g}\right)^{n-1},$$

which two equations multiplied together produce that equation required to be shown

$$\frac{1.\quad 2.\quad 3.\ldots\ldots\ldots n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda n+n)}=\frac{\lambda ng}{\lambda+1}\int x^{f+\lambda ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{\lambda n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{(\lambda+1)n-1}}.$$

Corollary 1.

§. 23. If we may put in the principal equation $f=\lambda n$ and $g=1$ we shall find

$$\frac{1.\quad 2.\quad 3.\ldots\ldots\ldots n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda n+n)}=\frac{\lambda n}{\lambda+1}\int x^{\lambda n-1}\partial x(1-x)^{n-1},$$

which form, by writing x^{k} in place of x, will be changed into this

$$\frac{\lambda nk}{\lambda+1}\int x^{\lambda nk-1}\partial x\left(1-x^{k}\right)^{n-1};$$

Thus so that we may have this theorem extended as widely as possible

$$g\int x^{f+\lambda ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{\lambda n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{(\lambda+1)n-1}}=k\int x^{\lambda nk-1}\partial x\left(1-x^{k}\right)^{n-1}.$$

Corollary 2.

§. 24. Now this theorem has a place, even if n shall not be a whole number ; indeed, there is no reason why we cannot also write *m* in place of *λn* for the number *λ*, which can be taken as it pleases, and we will arrive at this equation

$$\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{m+n-1}}=\frac{k\int x^{mk-1}\partial x\left(1-x^{k}\right)^{n-1}}{g\int x^{f+mg-1}\partial x\left(1-x^{g}\right)^{n-1}}$$

Corollary 3.

§. 25. If we may put $g=0$, because $1-x^{g}=gl\frac{1}{x}$, this theorem itself assumes the form :

$$\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{m-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{m+n-1}}=\frac{k\int x^{mk-1}\partial x\left(1-x^{k}\right)^{n-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}},$$

which thus may be represented more conveniently

$$\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\times\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{m-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{m+n-1}}=k\int x^{mk-1}\partial x\left(1-x^{k}\right)^{n-1},$$

where it is clear that the numbers *m* and *n* can be interchanged between themselves.

Scholium.

§. 26. Therefore we have found a two-fold source, from which innumerable combinations of integral formulas can be extracted; the one source shown in §. 24, includes integral formulas of this kind

$$\int x^{p-1}\partial x\left(1-x^{g}\right)^{q-1},$$

which I had treated now some time before the observations about the integral formulas [referred to by Lagrange] (in the *Miscellanea Tuarinensia*. Book. III.)

$$\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q}{n}-1}$$

extended from the value $x=0$ as far as to $x=1$, where I had shown initially how the letters *p* and *q* may be interchanged between themselves, so that there shall be

$$\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q}{n}-1}=\int x^{q-1}\partial x\left(1-x^{n}\right)^{\frac{p}{n}-1},$$

then truly there shall be also

$$\int \frac{x^{p-1}\partial x}{\left(1-x^{n}\right)^{\frac{p}{m}}}=\frac{\pi}{n\sin.\frac{p\pi}{n}}:$$

but I have shown especially to be true that

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}\times\int \frac{x^{p+q-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-r}}}=\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-r}}}\times\int \frac{x^{p+r-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}},$$

in which equation a comparison may be made with that now found in §. 24 ; thus hence so that there is nothing new, which I have not already set out, which might be able to be deduced.

Therefore the other source indicated in §. 25, I undertake to be investigating here mainly, or if with some restriction it may be assumed that $f=1$, our first equation will become

$$\frac{\int\partial x\left(l\frac{1}{x}\right)^{n-1}\times\int\partial x\left(l\frac{1}{x}\right)^{m-1}}{\int\partial x\left(l\frac{1}{x}\right)^{m+n-1}}=k\int x^{mk-1}\partial x\left(1-x^{k}\right)^{n-1},$$

with the benefit of which the values of the integral formula $\int\partial x\left(l\frac{1}{x}\right)^{\lambda}$, when λ is not a whole number, will be able to be related to the quadratures of algebraic curves ; since whenever λ is a whole number, a complete integration will be had since there occurs

$$\int\partial x\left(l\frac{1}{x}\right)^{\lambda}=1.2.3\ldots\ldots\ldots\ldots\ldots\lambda.$$

But the question of the greatest interest is concerned with these cases, in which λ is a fractional number, which therefore I am going to define here successively for rational denominators.

Problem 2.

§. 27. *With i denoting a whole positive number, to define the value of the integral formula* $\int\partial x\left(l\frac{1}{x}\right)^{\frac{i}{2}}$, *with the integration extended from* $x=0$ *as far as to* $x=1$.

Solution.

In our general equation we may put $m=n$, and there will be

$$\frac{\left[\int \partial x\left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int \partial x\left(l\frac{1}{x}\right)^{2n-1}} = k\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1}.$$

Now let $n-1=\frac{1}{2}$, and because $2n-1=i+1$, there will be

$$\int \partial x\left(l\frac{1}{x}\right)^{2n-1} = 1.2.3.......................(i+1):$$

again there may be taken k =2, so that there shall be $nk-1=i+1$, and there becomes

$$\frac{\left[\int \partial x\sqrt{\left(l\frac{1}{x}\right)^i}\right]^2}{1.2.3.......................(i+1)} = 2\int x^{i+1}\partial x\left(1-x^2\right)^{\frac{i}{2}},$$

and thus

$$\frac{\int \partial x\sqrt{\left(l\frac{1}{x}\right)^i}}{\sqrt{\left[1.2.3.......................(i+1)\right]}} = \sqrt{2}\int x^{i+1}\partial x\sqrt{\left(1-x^2\right)^i},$$

where it is clear, to be agreed only odd numbers to be taken for i ; because for even numbers the same calculation is evident between themselves.

Corollary 1.

§. 28. But all cases are reduced easily for $i=1$, or thus for $i=-1$; indeed a reduction can be found, provided that $i+1$ shall not be a negative number. Therefore regarding this case, there will be

$$\int \frac{\partial x}{\sqrt{\left(l\frac{1}{x}\right)}} = \sqrt{2}\int \frac{\partial x}{\sqrt{(1-xx)}} = \sqrt{\pi},\text{ because } \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}.$$

Corollary 2.

§. 29. But in this case it is helped by a rule, because

$$\int \partial x\left(l\frac{1}{x}\right)^n = n\int \partial x\left(l\frac{1}{x}\right)^{n-1},$$

we will have

$$\int \partial x\sqrt{l\frac{1}{x}} = \frac{1}{2}\sqrt{\pi};\int\left(l\frac{1}{x}\right)^{\frac{3}{2}} = \frac{1\cdot 3}{2\cdot 2}\sqrt{\pi},$$

and in general

$$\int \partial x \left(l\tfrac{1}{x} \right)^{\frac{2n+1}{2}} = \tfrac{1}{2} \cdot \tfrac{3}{2} \cdot \tfrac{5}{2} \cdot \tfrac{7}{2} \cdots\cdots \tfrac{(2n+1)}{2} \sqrt{\pi}.$$

Problem 3.

§. 30. *With i denoting some positive whole number, to define the value of the integral formula* $\int \partial x \left(l\tfrac{1}{x} \right)^{\frac{i}{3}-1}$, *with the integration extending from* $x = 0$ *to* $x = 1$.

Solution.

We may begin from the equation of the preceding problem

$$\frac{\left[\int \partial x \left(l\tfrac{1}{x} \right)^{n-1} \right]^2}{\int \partial x \left(l\tfrac{1}{x} \right)^{2n-1}} = k \int x^{nk-1} \partial x \left(1 - x^k \right)^{n-1},$$

and in the general form we may put $m = 2n$, so that there may be had

$$\frac{\int \partial x \left(l\tfrac{1}{x} \right)^{n-1} \times \int \partial x \left(l\tfrac{1}{x} \right)^{3n-1}}{\int \partial x \left(l\tfrac{1}{x} \right)^{3n-1}} = k \int x^{2nk-1} \partial x \left(1 - x^k \right)^{n-1},$$

and by multiplying these two equations we arrive at

$$\frac{\left| \int \partial x \left(l\tfrac{1}{x} \right)^{n-1} \right|^3}{\int \partial x \left(l\tfrac{1}{x} \right)^{3n-1}} = kk \int x^{nk-1} \partial x \left(1 - x^k \right)^{n-1} \times \int x^{2nk-1} \partial x \left(1 - x^k \right)^{n-1}.$$

Now here there may be put $n = \frac{i}{3}$, so that there shall be

$$\int \partial x \left(l\tfrac{1}{x} \right)^{i-1} = 1 \cdot 2 \cdot 3 \cdots\cdots (i-1),$$

and there may be assumed $k = 3$, and there will be produced

$$\frac{\left[\int \partial x \sqrt[3]{\left(l\tfrac{1}{x} \right)^{i-3}} \right]^3}{1 \cdot 2 \cdot 3 \cdots\cdots (i-1)} = 9 \int x^{i-1} \partial x \sqrt[3]{\left(1 - x^3 \right)^{i-3}} \times \int x^{2i-1} \partial x \sqrt[3]{\left(1 - x^3 \right)^{i-3}};$$

from which we conclude

$$\frac{\int \partial x \sqrt[3]{\left(l\frac{1}{x}\right)^{i-3}}}{\sqrt[3]{1\cdot 2\cdot 3 \cdots\cdots (i-1)}} = \sqrt[3]{}9\int \frac{x^{i-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-i}}}.$$

Corollary 1.

§. 31. Here two principal cases occur, on which all the rest will depend; clearly by putting either $i=1$ or $i=2$, which are :

$$\text{I.}\int \frac{\partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)^2}} = \sqrt[3]{}9\int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}} \times \int \frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}$$

$$\text{II.}\int \frac{\partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)}} = \sqrt[3]{}9\int \frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)}} \times \int \frac{x^3\partial x}{\sqrt[3]{\left(1-x^3\right)}},$$

of which the latter form on account of

$$\int \frac{x^3\partial x}{\sqrt[3]{\left(1-x^3\right)}} = \frac{1}{3}\int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}}$$

will become

$$\frac{\int \partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)}} = \sqrt[3]{}3\int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}} \times \int \frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)}}.$$

Corollary 2.

§. 32. In order to make use of my observations mentioned before, for brevity we may put

$$\int \frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-q}}} = \left(\frac{p}{q}\right),$$

and so that for this case

$$\left(\frac{2}{1}\right) = \frac{\pi}{3\sin.\frac{\pi}{2}} = \alpha,$$

then truly,

$$\left(\frac{1}{1}\right) = \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}} = \text{A}, \text{ there will become}$$

$$\text{I.}\int \frac{\partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)^2}} = \sqrt[3]{9\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)} = \sqrt[3]{9\alpha\text{A}},$$

$$\text{II.}\int \frac{\partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)^1}} = \sqrt[3]{3\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)} = \sqrt[3]{\frac{3\alpha\alpha}{\text{A}}}.$$

Corollary 3.

§. 33. Therefore for the first case we will have

$$\int \partial x \sqrt[3]{\left(l\tfrac{1}{x}\right)^{-2}} = \sqrt[3]{9\alpha\mathrm{A}},\ \int \partial x \sqrt[3]{\left(l\tfrac{1}{x}\right)} = \tfrac{1}{3}\sqrt[3]{9\alpha\mathrm{A}},\ \text{and}$$
$$\int \partial x \sqrt[3]{\left(l\tfrac{1}{x}\right)^{3n+1}} = \tfrac{1}{3}\cdot\tfrac{4}{3}\cdot\tfrac{7}{3}\cdots\cdots\tfrac{3n+1}{3}\sqrt[3]{9\alpha\mathrm{A}}\ :$$

truly for the other case,

$$\int \partial x \sqrt[3]{\left(l\tfrac{1}{x}\right)^{-1}} = \sqrt[3]{\tfrac{3\alpha\alpha}{\mathrm{A}}},\ \int \partial x \sqrt[3]{\left(l\tfrac{1}{x}\right)^{2}} = \tfrac{2}{3}\sqrt[3]{\tfrac{3\alpha\alpha}{\mathrm{A}}},\ \text{and}$$
$$\int \partial x \sqrt[3]{\left(l\tfrac{1}{x}\right)^{3n-1}} = \tfrac{2}{3}\cdot\tfrac{5}{3}\cdot\tfrac{8}{3}\cdots\cdots\tfrac{3n-1}{3}\cdot\sqrt[3]{\tfrac{3\alpha\alpha}{\mathrm{A}}}.$$

Problem 4.

§. 34. *With i denoting a positive whole number, to define the value of the integral formula* $\int \partial x\left(l\tfrac{1}{x}\right)^{\frac{i}{4}-1}$, *with the integration extending from* $x=0$ *to* $x=1$.

Solution.

In the solution of the preceding problem we have been led to this equation

$$\frac{\left|\int \partial x\left(l\frac{1}{x}\right)^{n-1}\right|^{3}}{\int \partial x\left(l\frac{1}{x}\right)^{3n-1}} = kk\int \frac{x^{nk-1}\partial x}{\left(1-x^{k}\right)^{1-n}} \times \int \frac{x^{2nk-1}\partial x}{\left(1-x^{k}\right)^{1-n}};$$

but the general form [*cf* § 26: $\dfrac{\int \partial x\left(l\frac{1}{x}\right)^{n-1} \times \int \partial x\left(l\frac{1}{x}\right)^{m-1}}{\int \partial x\left(l\frac{1}{x}\right)^{m+n-1}} = k\int x^{mk-1}\partial x\left(1-x^{k}\right)^{n-1},$]

by taking $m=3n$ produces

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{n-1} \times \int \partial x\left(l\frac{1}{x}\right)^{3n-1}}{\int \partial x\left(l\frac{1}{x}\right)^{4n-1}} = k\int \frac{x^{3nk-1}\partial x}{\left(1-x^{k}\right)^{1-n}},$$

with which taken together we obtain:

$$\frac{\left|\int \partial x\left(l\frac{1}{x}\right)^{n-1}\right|^4}{\int \partial x\left(l\frac{1}{x}\right)^{4n-1}} = k^3\int\frac{x^{nk-1}\partial x}{\left(1-x^k\right)^{1-n}}\times\int\frac{x^{2nk-1}\partial x}{\left(1-x^k\right)^{1-n}}\times\int\frac{x^{3nk-1}\partial x}{\left(1-x^k\right)^{1-n}}.$$

Now let $n=\frac{i}{4}$, and $k=4$ may be taken, and there becomes :

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{\frac{i}{4}-1}}{\sqrt[4]{1.2.3....(i-1)}} = \sqrt[4]{}4^3\int\frac{x^{i-1}\partial x}{\sqrt[4]{\left(1-x^4\right)^{4-i}}}\times\int\frac{x^{2i-1}\partial x}{\sqrt[4]{\left(1-x^4\right)^{4-i}}}\times\int\frac{x^{3i-1}\partial x}{\sqrt[4]{\left(1-x^4\right)^{4-i}}}.$$

Corollary 1.

§. 35. Therefore if there shall be $i=1$, we will have

$$\int \partial x\sqrt[4]{}\left(l\frac{1}{x}\right)^{-3} = \sqrt[4]{}4^3\int\frac{\partial x}{\sqrt[4]{\left(1-x^4\right)^3}}\times\int\frac{x\partial x}{\sqrt[4]{\left(1-x^4\right)^3}}\times\int\frac{xx\partial x}{\sqrt[4]{\left(1-x^4\right)^3}}.$$

If which expression may be designated by the letter P, in general there will be

$$\int \partial x\sqrt[4]{}\left(l\frac{1}{x}\right)^{4n-3} = \frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\cdots\cdots\frac{4n-3}{4}\cdot \mathrm{P}$$

Corollary 2.

§. 36. For the other principal case we may take $i=3$, and there will be

$$\int \partial x\sqrt[4]{}\left(l\frac{1}{x}\right)^{-1} = \sqrt[4]{}2.4^3\int\frac{x^2\partial x}{\sqrt[4]{\left(1-x^4\right)}}\times\int\frac{x^5\partial x}{\sqrt[4]{\left(1-x^4\right)}}\times\int\frac{x^8\partial x}{\sqrt[4]{\left(1-x^4\right)}},$$

or with the product reduced to simpler forms :

$$\int \partial x\sqrt[4]{}\left(l\frac{1}{x}\right)^{-1} = \sqrt[4]{}8\int\frac{xx\partial x}{\sqrt[4]{\left(1-x^4\right)}}\times\int\frac{x\partial x}{\sqrt[4]{\left(1-x^4\right)}}\times\int\frac{\partial x}{\sqrt[4]{\left(1-x^4\right)}},$$

which expression if it may be designated by the letter Q, will be generally

$$\int \partial x\sqrt[4]{\left(l\frac{1}{x}\right)^{4n-1}} = \frac{3}{4}\cdot\frac{7}{4}\cdot\frac{11}{4}\cdots\cdots\frac{4n-1}{4}\cdot \mathrm{Q}.$$

Scholium.

§. 37. If we may indicate the integral formula

$$\int \frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^4\right)^{4-q}}}$$

by this sign $\left(\frac{p}{q}\right)$, the solution of the problem thus will be had

$$\int \partial x \sqrt[4]{\left(l\frac{1}{x}\right)^{i-4}} = \sqrt[4]{\ }1\cdot 2\cdot 3\cdots\cdots(i-1)\cdot 4^3\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right),$$

and for both cases presented there becomes

$$\mathrm{P} = \sqrt[4]{\ }4^3\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right) \text{ and } \mathrm{Q} = \sqrt[4]{\ }8\left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right).$$

We may now put these formulas in place which depend on the circle:

$$\left(\frac{3}{1}\right) = \frac{\pi}{4\sin.\frac{\pi}{4}} = \alpha \text{ and } \left(\frac{2}{2}\right) = \frac{\pi}{4\sin.\frac{2\pi}{4}} = \beta,$$

for transcendental functions, but of higher orders,

$$\left(\frac{2}{1}\right) = \int \frac{x\partial x}{\sqrt[4]{\left(1-x^4\right)^3}} = \int \frac{\partial x}{\sqrt[4]{\left(1-x^4\right)}} = \mathrm{A},$$

certainly on which all the remaining will depend, and we will find

$$\mathrm{P} = \sqrt[4]{4^3\cdot\frac{\alpha\alpha}{\beta}\cdot\mathrm{AA}} \text{ and } \mathrm{Q} = \sqrt[4]{4\cdot\alpha\alpha\beta\cdot\frac{1}{\mathrm{AA}}};$$

from which it is evident,

$$\mathrm{PQ} = 4\alpha = \frac{\pi}{\sin.\frac{\pi}{4}}.$$

But since there shall be

$$\alpha = \frac{\pi}{2\sqrt{2}} \text{ and } \gamma = \frac{\pi}{4}, \text{ there will be}$$

$$\mathrm{P} = \sqrt[4]{\ }32\pi\mathrm{AA},\ \mathrm{Q} = \sqrt[4]{\ }\frac{\pi^3}{8\mathrm{AA}} \text{ and } \frac{\mathrm{P}}{\mathrm{Q}} = \frac{4\mathrm{A}}{\sqrt{\pi}}.$$

Problem 5.

§. 38. *With i denoting a positive integer, to define the value of the integral formula* $\int \partial x \sqrt[4]{\ }\left(l\frac{1}{x}\right)^{i-5}$, *with the integration extended from* $x = 0$ *to* $x = 1$.

Solution.

From the preceding solutions it is seen now well enough for in this case going to arise according to this form :

$$\frac{\int \partial x\sqrt[5]{\left(l\frac{1}{x}\right)^{i-5}}}{\sqrt[5]{1.2.3....(i-1)}}=\sqrt[5]{}5^4\int\frac{x^{i-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-i}}}\times\int\frac{x^{2i-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-i}}}\times\int\frac{x^{3i-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-i}}}\times\int\frac{x^{4i-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-i}}},$$

Which integral formulas have been referred to the fifth class of my dissertation mentioned above. Whereby only if the sign $\left(\frac{p}{q}\right)$ may be used again here, this may denote formula

$$\int\frac{x^{p-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-q}}},$$

and the value sought thus will be allowed to be expressed more conveniently, so that there becomes

$$\int\partial x\sqrt[5]{\left(l\frac{1}{x}\right)^{i-5}}=\sqrt[5]{}1.2.3....(i-1)5^4\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\left(\frac{4i}{i}\right),$$

where indeed the five values of i less than five may suffice to be attributed, but when the numbers of the five are greater, there shall be had

$$\left(\frac{5+m}{i}\right)=\frac{m}{m+i}\left(\frac{m}{i}\right);$$

truly again

$$\left(\frac{10+m}{i}\right)=\frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\left(\frac{m}{i}\right)$$

$$\left(\frac{15+m}{i}\right)=\frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\cdot\frac{m+10}{m+i+10}\left(\frac{m}{i}\right).$$

Then truly for this class two formulas involve the quadratures of circles, which shall be

$$\left(\frac{4}{1}\right)=\frac{\pi}{5\sin.\frac{\pi}{5}}=\alpha \text{ and } \left(\frac{3}{2}\right)=\frac{\pi}{5\sin.\frac{2\pi}{5}}=\beta,$$

but two contain higher quadratures, which may be put

$$\left(\frac{3}{1}\right)=\int\frac{xx\partial x}{\sqrt[5]{\left(1-x^5\right)^4}}=\int\frac{\partial x}{\sqrt[5]{\left(1-x^5\right)^2}}=\mathrm{A} \text{ and}$$

$$\left(\frac{2}{2}\right)=\int\frac{x\partial x}{\sqrt[5]{\left(1-x^5\right)^3}}=\mathrm{B};$$

and from these values I have assigned all the remaining formulas of this class : clearly,

$$\left(\tfrac{2}{2}\right)=1;\left(\tfrac{5}{2}\right)=\tfrac{1}{2};\left(\tfrac{5}{4}\right)=\tfrac{1}{4};\left(\tfrac{5}{5}\right)=\tfrac{1}{5}$$

$$\left(\tfrac{4}{1}\right)=\alpha;\left(\tfrac{4}{2}\right)=\tfrac{\beta}{\mathrm{A}};\left(\tfrac{4}{3}\right)=\tfrac{\beta}{2\mathrm{B}};\left(\tfrac{4}{4}\right)=\tfrac{\alpha}{3\mathrm{A}}$$

$$\left(\tfrac{3}{1}\right)=\mathrm{A};\left(\tfrac{3}{2}\right)=\beta;\left(\tfrac{3}{3}\right)=\tfrac{\beta\beta}{\alpha\mathrm{B}}$$

$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha\mathrm{B}}{\beta};\left(\tfrac{2}{2}\right)=\mathrm{B}$$

$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha\mathrm{A}}{\beta}.$$

Corollary 1.

§. 39. With the exponent $i=1$ taken, there will be

$$\int\partial x\sqrt[5]{\left(l\tfrac{1}{x}\right)^{-4}}=\sqrt[5]{5^4}\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)=\sqrt[5]{}5^4\cdot\tfrac{\alpha^3}{\beta^2}\cdot\mathrm{A}^2\mathrm{B};$$

from which we may conclude to become in general, with n denoting some whole number

$$\int\partial x\sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-4}}=\tfrac{1}{5}\cdot\tfrac{6}{5}\cdot\tfrac{11}{5}\cdot\ldots\ldots\tfrac{5n-4}{5}\sqrt[5]{}5^4\cdot\tfrac{\alpha^3}{\beta^2}\cdot\mathrm{A}^2\mathrm{B}.$$

Corollary 2.

§. 40. Now let $i=2$, and

$$\int\partial x\sqrt[5]{\left(l\tfrac{1}{x}\right)^{-3}}=\sqrt[5]{}1.5^4\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{8}{2}\right),$$ may arise, because

$\left(\tfrac{6}{2}\right)=\tfrac{1}{3}\left(\tfrac{1}{2}\right)=\tfrac{1}{3}\left(\tfrac{2}{1}\right)$ and $\left(\tfrac{8}{2}\right)=\tfrac{3}{5}\left(\tfrac{3}{2}\right)$,

this expression arises :

$$\sqrt[5]{}5^3\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{2}\right)\sqrt[5]{}5^3\cdot\alpha\beta\cdot\tfrac{\mathrm{BB}}{\mathrm{A}}$$

and in general

$$\int\partial x\sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-3}}=\tfrac{2}{5}\cdot\tfrac{7}{5}\cdot\tfrac{12}{5}\cdot\ldots\ldots\tfrac{5n-3}{5}\sqrt[5]{}5^3\cdot\alpha\beta\cdot\tfrac{\mathrm{BB}}{\mathrm{A}}.$$

Corollary 3.

§. 41. Let $i=3$, and the form found

$\int \partial x \sqrt[5]{\left(l\frac{1}{x}\right)^{-2}} = \sqrt[5]{\ } 2.5^4 \left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{9}{3}\right)\left(\frac{12}{3}\right)$, on account of

$\left(\frac{6}{3}\right) = \frac{1}{4}\left(\frac{3}{1}\right);\ \left(\frac{9}{3}\right) = \frac{4}{7}\left(\frac{4}{5}\right);\ \left(\frac{12}{3}\right) = \frac{2}{5}\cdot\frac{7}{10}\left(\frac{3}{2}\right)$, will change into

$\sqrt[5]{\ } 2.5^2 \left(\frac{3}{3}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{3}{2}\right) = \sqrt[5]{\ } 5^2 \frac{\beta^4}{\alpha}\cdot\frac{\mathrm{A}}{\mathrm{BB}};$

from which in general it is deduced

$$\int \partial x \sqrt[5]{\left(l\frac{1}{x}\right)^{5n-2}} = \frac{3}{5}\cdot\frac{8}{5}\cdot\frac{13}{5}\cdots\cdots\frac{5n-2}{5}\sqrt[5]{\ } 5^2 \cdot\frac{\beta^4}{\alpha}\cdot\frac{\mathrm{A}}{\mathrm{BB}}.$$

Corollary 4.

§. 42. And then by putting $i = 4$, our form

$$\int \partial x \sqrt[5]{\left(l\frac{1}{x}\right)^{-1}} = \sqrt[5]{\ } 6.5^4 \left(\frac{4}{4}\right)\left(\frac{8}{4}\right)\left(\frac{12}{4}\right)\left(\frac{16}{4}\right), \text{because}$$

$$\left(\frac{8}{4}\right) = \frac{3}{7}\left(\frac{4}{1}\right);\ \left(\frac{12}{4}\right) = \frac{2}{6}\cdot\frac{7}{11}\left(\frac{4}{2}\right);\ \left(\frac{16}{4}\right) = \frac{1}{5}\cdot\frac{6}{10}\cdot\frac{11}{15}\left(\frac{4}{1}\right),$$

it will be transformed into this form

$$\sqrt[5]{\ } 6.5^4 \left(\frac{4}{4}\right)\left(\frac{4}{3}\right)\left(\frac{4}{2}\right)\left(\frac{4}{1}\right) = \sqrt[5]{\ } \frac{\alpha\alpha\beta\beta}{\mathrm{AAB}};$$

thus so that in general there shall be

$$\int \partial x \sqrt[5]{\left(l\frac{1}{x}\right)^{5n-1}} = \frac{4}{5}\cdot\frac{9}{5}\cdot\frac{14}{5}\cdots\cdots\frac{5n-1}{5}\sqrt[5]{\ } 5.\alpha\alpha\beta\beta\cdot\frac{1}{\mathrm{AAB}}.$$

Scholium.

§. 43. If we may represent the value of the integral formula $\int \partial x \left(l\frac{1}{x}\right)^{\lambda}$ by this sign [λ], the present cases transformed become

$$\left[-\frac{4}{5}\right] = \sqrt[3]{\ } 5^4.\frac{\alpha^3}{\beta^2}.\mathrm{A}^2\mathrm{B}; \qquad \left[+\frac{1}{5}\right] = \frac{1}{5}\sqrt[3]{\ } 5^4.\frac{\alpha^3}{\beta^2}.\mathrm{A}^2\mathrm{B}$$

$$\left[-\frac{3}{5}\right] = \sqrt[5]{\ } 5^3.\alpha\beta.\frac{\mathrm{BB}}{\mathrm{A}}; \qquad \left[+\frac{2}{5}\right] = \frac{1}{5}\sqrt[3]{\ } 5^3.\alpha\beta.\frac{\mathrm{BB}}{\mathrm{A}}$$

$$\left[-\frac{2}{5}\right] = \sqrt[5]{\ } 5^2.\frac{\beta^4}{\alpha}.\frac{\mathrm{A}}{\mathrm{BB}}; \qquad \left[+\frac{3}{5}\right] = \frac{3}{5}\sqrt[3]{\ } 5^2.\frac{\beta}{\alpha}.\frac{\mathrm{A}}{\mathrm{BB}}$$

$$\left[-\frac{1}{5}\right] = \sqrt[5]{\ } 5.\alpha^2\beta^2.\frac{1}{\mathrm{AAB}}; \qquad \left[+\frac{4}{5}\right] = \frac{4}{5}\sqrt[3]{\ } 5.\alpha^2\beta^2.\frac{1}{\mathrm{BBA}};$$

from which each two, the indices of which taken together become $=0$, we deduce by being multiplied :

$$\left[+\tfrac{1}{5}\right]\cdot\left[-\tfrac{1}{5}\right]=\alpha=\frac{\pi}{5\sin.\frac{\pi}{5}}$$

$$\left[+\tfrac{2}{5}\right]\cdot\left[-\tfrac{2}{5}\right]=2\beta=\frac{2\pi}{5\sin.\frac{2\pi}{5}}$$

$$\left[+\tfrac{3}{5}\right]\cdot\left[-\tfrac{3}{5}\right]=3\beta=\frac{3\pi}{5\sin.\frac{3\pi}{5}}$$

$$\left[+\tfrac{4}{5}\right]\cdot\left[-\tfrac{4}{5}\right]=4\alpha=\frac{2\pi}{5\sin.\frac{4\pi}{5}}$$

Moreover from the preceding problem in a similar manner, we deduce

$$\left[-\tfrac{3}{4}\right]=\mathrm{P}=\sqrt[4]{4^3.\tfrac{\alpha\alpha}{\beta}}.\mathrm{AA};\qquad \left[+\tfrac{1}{4}\right]=\tfrac{1}{4}\sqrt[4]{4^3.\tfrac{\alpha\alpha}{\beta}}.\mathrm{AA}$$

$$\left[-\tfrac{1}{4}\right]=\mathrm{Q}=\sqrt[4]{4.\alpha\alpha\beta.\tfrac{1}{\mathrm{AA}}};\qquad \left[+\tfrac{3}{4}\right]=\tfrac{3}{4}\sqrt[4]{4.\alpha\alpha\beta.\tfrac{1}{\mathrm{AA}}}$$

and hence

$$\left[+\tfrac{1}{4}\right]\cdot\left[-\tfrac{1}{4}\right]=\alpha=\frac{\pi}{4\sin.\frac{\pi}{4}}$$

$$\left[+\tfrac{3}{4}\right]\cdot\left[-\tfrac{3}{4}\right]=3\alpha=\frac{3\pi}{4\sin.\frac{3\pi}{4}}$$

from which in general we arrive at this theorem, which shall be

$$[+\lambda]\cdot[-\lambda]=\frac{\lambda\pi}{\sin.\lambda\pi},$$

whose ratio can thus be determined from the method of interpolation established some time ago, since there shall be

$$[\lambda]=\frac{1^{1-\lambda}.2^{\lambda}}{1+\lambda}\cdot\frac{2^{1-\lambda}.3^{\lambda}}{2+\lambda}\cdot\frac{3^{1-\lambda}.4^{\lambda}}{3+\lambda}\cdot\text{etc.}$$

there will be

$$[-\lambda]=\frac{1^{1+\lambda}.2^{-\lambda}}{1-\lambda}\cdot\frac{2^{1+\lambda}.3^{-\lambda}}{2-\lambda}\cdot\frac{3^{1+\lambda}.4^{-\lambda}}{3-\lambda}\cdot\text{etc.}$$

and hence,

$$[\lambda].[-\lambda]=\frac{1.1}{1-\lambda\lambda}\cdot\frac{2.2}{4-\lambda\lambda}\cdot\frac{3.3}{9-\lambda\lambda}\cdot\text{etc.}=\frac{\lambda\pi}{\sin.\lambda\pi};$$

as I have shown elsewhere.

General Problem 6 .

§. 44. *If the letters i and n shall denote positive integers, to define the value of the integral formula*

$$\int\partial x\left(l\tfrac{1}{x}\right)^{\frac{i-n}{n}}\text{ , or }\int\partial x\sqrt[n]{\left(l\tfrac{1}{x}\right)^{i-n}},$$

with the integration extended from $x = 0$ *to* $x = 1$.

Solution.

The present method will show the value sought by the usual method, expressed by the quadrature of algebraic curves

$$\frac{\int \partial x \sqrt[n]{\left(l\frac{1}{x}\right)^{i-n}}}{\sqrt[n]{1\cdot 2\cdot 3\cdots\cdots(i-1)}} = \sqrt[n]{\quad} n^{n-1}\int \frac{x^{i-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-i}}} \times \ldots\ldots \int \frac{x^{(n-1)i-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-i}}}.$$

Because now for the sake of brevity, the integral formula

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} \text{ may be expressed by this character } \left(\frac{p}{q}\right),$$

truly we will designate $\int \partial x \sqrt[n]{\left(l\frac{1}{x}\right)^m}$ in a like manner by $\left[\frac{m}{n}\right]$, thus so that $\left[\frac{m}{n}\right]$ will denote the value of this indefinite product $1.\ 2.\ \ldots .\ z$, with $z = \frac{m}{n}$, a more succinct value will be produced expressed in this manner :

$$\left[\frac{i-n}{n}\right] = \sqrt[n]{\quad} 1\cdot 2\cdot 3\cdots\cdots(i-1)\, n^{n-1}.\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\cdot\left(\frac{ni-i}{i}\right);$$

from which also it may be deduced :

$$\left[\frac{i}{n}\right] = \frac{i}{n}\sqrt[n]{\quad} 1\cdot 2\cdot 3\cdots\cdots(i-1)\, n^{n-1}.\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\cdot\left(\frac{ni-i}{i}\right).$$

Here it will suffice to have taken the number *i* always to be less than the number *n* ; because for greater values there is known to be,

$$\left[\frac{i+n}{n}\right] = \frac{i+n}{n}\left[\frac{i}{n}\right], \text{likewise } \left[\frac{i+2n}{n}\right] = \frac{i+n}{n}\cdot\frac{i+2n}{n}\left[\frac{i}{n}\right] \text{etc.}$$

and in this way the whole investigation is reduced to these cases only, in which the numerator *i* of which fraction $\frac{i}{n}$ is less than the denominator *n* . Truly in addition, with regard to the integral formulas

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} = \left(\frac{q}{p}\right),$$

it will help to have observed the following:

I. The letters *p* and *q* to be interchangeable between themselves, so that there shall be

$$\left(\frac{p}{p}\right)=\left(\frac{q}{p}\right).$$

II. If either of the numbers p or q may be put equal to the exponent n , the value of the integral formula becomes algebraic, clearly

$$\left(\frac{n}{p}\right)=\left(\frac{p}{n}\right)=\frac{1}{p},\ \text{or}\ \left(\frac{n}{q}\right)=\left(\frac{q}{n}\right)=\frac{1}{q}.$$

III. If the sum of the numbers $p+q$ may be put equal to the exponent n itself, the value of the integral formula $\left(\frac{p}{q}\right)$ can be shown by the circle, since there shall be

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}},\ \text{and}\ \left(\frac{q}{n-q}\right)=\left(\frac{n-q}{q}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

IV. If either of the numerators p or q shall be greater than the exponent n, the integral formula $\left(\frac{p}{q}\right)$ can be restored to another form, of which the terms shall be less than n, which shall be done with the aid of this reduction

$$\left(\frac{p+n}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

V. Between several integral formulas of this kind, such a relation can be put in place, so that there shall be

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

with the aid of which all the reductions are found, which I have presented in the observations concerning these formulas.

Corollary 1.

§. 45. If in this manner with the aid of reduction N°. IV. indicated we may accommodate the form found for individual cases, we will be able to show these by the following most simple account. And indeed at first for the case $n=2$, where we will have no help in the reduction

$$\left[\frac{1}{2}\right]=\frac{1}{2}\sqrt[2]{2\left(\frac{1}{1}\right)}=\frac{1}{2}\sqrt[2]{\frac{\pi}{\sin.\frac{\pi}{2}}}=\frac{1}{2}\sqrt{\pi}.$$

Corollary 2.

§. 46. For the case $n=3$ we will have these reductions :

$$\left[\tfrac{1}{3}\right]=\tfrac{1}{3}\sqrt[3]{3^2.\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)}$$
$$\left[\tfrac{2}{3}\right]=\tfrac{2}{3}\sqrt[3]{3.1.\left(\tfrac{2}{2}\right)\left(\tfrac{1}{2}\right)}.$$

Corollary 3.

§. 47. For the case $n=4$ we will obtain these three reductions

$$\left[\tfrac{3}{4}\right]=\tfrac{1}{4}\sqrt[4]{4^3\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)}$$
$$\left[\tfrac{2}{4}\right]=\tfrac{2}{4}\sqrt[4]{4^2.2.\left(\tfrac{4}{2}\right)}=\tfrac{1}{2}\sqrt[2]{4\left(\tfrac{2}{2}\right)}, \text{because } \left(\tfrac{4}{2}\right)=\tfrac{1}{2}$$
$$\left[\tfrac{3}{4}\right]=\tfrac{3}{4}\sqrt[4]{4.1.2.\left(\tfrac{3}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)};$$

since in the middle, there shall be $\left(\tfrac{2}{2}\right)=\left(\tfrac{2}{4-2}\right)=\tfrac{\pi}{4}$, there will be everywhere as before
$$\left[\tfrac{2}{4}\right]=\left[\tfrac{1}{2}\right]=\tfrac{1}{2}\sqrt{\pi}.$$

Corollary 4.

§. 48. Now there shall be $n=5$, and these four reductions arise :

$$\left[\tfrac{1}{5}\right]=\tfrac{1}{5}\sqrt[5]{5^4\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)}$$
$$\left[\tfrac{2}{5}\right]=\tfrac{2}{5}\sqrt[5]{5^3.1\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)}$$
$$\left[\tfrac{3}{5}\right]=\tfrac{3}{5}\sqrt[5]{5^2.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{2}{3}\right)}$$
$$\left[\tfrac{4}{5}\right]=\tfrac{4}{5}\sqrt[5]{5.1.2.3\left(\tfrac{4}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{4}\right)}.$$

Corollary 5.

§. 49. Let $n=6$, and we will have these reductions:

$$\left[\tfrac{1}{6}\right]=\tfrac{1}{6}\sqrt[6]{6^5\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)}$$

$$\left[\tfrac{2}{6}\right]=\tfrac{2}{6}\sqrt[6]{6^4.2\left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2\left(\tfrac{6}{2}\right)}=\tfrac{1}{2}\sqrt[3]{6^2\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)}$$

$$\left[\tfrac{3}{6}\right]=\tfrac{3}{6}\sqrt[6]{6^3.3.3\left(\tfrac{3}{3}\right)^3\left(\tfrac{6}{3}\right)^2}=\tfrac{1}{2}\sqrt[2]{6\left(\tfrac{3}{3}\right)}$$

$$\left[\tfrac{4}{6}\right]=\tfrac{4}{6}\sqrt[6]{6^2.2.4.2\left(\tfrac{4}{4}\right)^2\left(\tfrac{2}{4}\right)^2\left(\tfrac{6}{4}\right)}=\tfrac{2}{3}\sqrt[3]{6^2.2\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right)}$$

$$\left[\tfrac{5}{6}\right]=\tfrac{5}{6}\sqrt[6]{6.1.2.3.4\left(\tfrac{5}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{1}{5}\right)}.$$

Corollary 6.

§. 50. By putting $n=7$, the following six equations arise:

$$\left[\tfrac{1}{7}\right]=\tfrac{1}{7}\sqrt[7]{7^6\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)}$$

$$\left[\tfrac{2}{7}\right]=\tfrac{2}{7}\sqrt[7]{7^5.1\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)}$$

$$\left[\tfrac{3}{7}\right]=\tfrac{3}{7}\sqrt[7]{7^4.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)}$$

$$\left[\tfrac{4}{7}\right]=\tfrac{4}{7}\sqrt[7]{7^3.1.2.3\left(\tfrac{4}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{3}{4}\right)}$$

$$\left[\tfrac{5}{7}\right]=\tfrac{5}{7}\sqrt[7]{7^2.1.2.3.4\left(\tfrac{5}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{2}{5}\right)}$$

$$\left[\tfrac{6}{7}\right]=\tfrac{6}{7}\sqrt[7]{7.1.2.3.4.5\left(\tfrac{6}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)}.$$

Corollary 7.

§. 51. Let $n=8$, and these seven reductions will be obtained :

$$\left[\tfrac{1}{8}\right]=\tfrac{1}{8}\sqrt[8]{8^7\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)\left(\tfrac{7}{1}\right)}$$

$$\left[\tfrac{2}{8}\right]=\tfrac{2}{8}\sqrt[8]{8^6.2\left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2\left(\tfrac{6}{2}\right)^2\left(\tfrac{8}{2}\right)}=\tfrac{1}{4}\sqrt[4]{8^3\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)}$$

$$\left[\tfrac{3}{8}\right]=\tfrac{3}{8}\sqrt[8]{8^5.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{7}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)}$$

$$\left[\tfrac{4}{8}\right]=\tfrac{4}{8}\sqrt[8]{8^4.4.4.4\left(\tfrac{4}{4}\right)^4\left(\tfrac{8}{4}\right)^3}=\tfrac{1}{2}\sqrt[2]{8\left(\tfrac{4}{4}\right)}$$

$$\left[\tfrac{5}{8}\right]=\tfrac{5}{8}\sqrt[8]{8^3.1.2.3.4\left(\tfrac{5}{5}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{7}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{3}{5}\right)}$$

$$\left[\tfrac{6}{8}\right]=\tfrac{6}{8}\sqrt[8]{8^2.4.2.6.4.2\left(\tfrac{6}{6}\right)^2\left(\tfrac{4}{6}\right)^2\left(\tfrac{2}{6}\right)^2\left(\tfrac{3}{6}\right)}=\tfrac{3}{4}\sqrt[4]{8.2.4\left(\tfrac{4}{6}\right)\left(\tfrac{2}{6}\right)}$$

$$\left[\tfrac{7}{8}\right]=\tfrac{7}{8}\sqrt[8]{8.1.2.3.4.5.6\left(\tfrac{7}{7}\right)\left(\tfrac{6}{7}\right)\left(\tfrac{5}{7}\right)\left(\tfrac{4}{7}\right)\left(\tfrac{3}{7}\right)\left(\tfrac{2}{7}\right)\left(\tfrac{1}{7}\right)}.$$

Scholium.

§. 52. It may be superfluous to set out these cases further, since the arrangement of these formulas may be evident from these presented. Indeed if in the proposed formula $\left[\frac{m}{n}\right]$, the numbers *m* and *n* shall be prime between themselves, the law is evident, since there becomes

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}1.2....\ (m-1).\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\ \left(\frac{n-1}{m}\right)},$$

but if these numbers *m* and *n* may have a common divisor, a certain fraction $\frac{m}{n}$ may be extricated to be reduced to the minimum form, and from which preceding cases the value desired sought ; yet meanwhile also an operation will be able to be put in place in this way. Since the expression sought certainly shall have this form

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}\mathrm{P.Q}},$$

where Q is the product from $n-1$ integrable formulas, P truly a product from some absolute numbers; initially Q to be found for that product, this series of formulas shall be present $\left(\frac{m}{m}\right)\ \left(\frac{2m}{m}\right)\ \left(\frac{3m}{m}\right)$, while the numerator may exceed the exponent *n*, and in place of this the excess above *n* may be written, which if it may be put $=\alpha$, so that now the formula shall be $\left(\frac{\alpha}{m}\right)$, here the numerator α itself will give a factor of the product P, then hence again the series of the formula may be put in place again $\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+m}{m}\right)\left(\frac{\alpha+2m}{m}\right)$, etc. which again it may arrive at a numerator with a greater exponent *n* , and the formula $\left(\frac{n+\beta}{m}\right)$ may arise, and in place of this it is required to write $\left(\frac{\beta}{m}\right)$, and likewise hence the factor β may be introduced into the product P ; and it may be agreed to progress thus ; while for Q , $n-1$ formulas will have emerged. So that which operations may be understood more easily, we may set out in this manner the case of the formula $\left[\frac{9}{12}\right]=\frac{9}{12}\sqrt[12]{12^3\mathrm{P.Q}}$, where a search of the letters Q and P thus may be put in place,

for Q ..... $\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)\left(\frac{12}{9}\right)\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)\left(\frac{12}{9}\right)\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)$,
for P 6. 3. 9. 6. 3 9. 6. 3,

and thus there will be found

$$\mathrm{Q}=\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3\left(\frac{12}{9}\right)^2 \text{ and}$$
$$\mathrm{P}=6^3.\,3^3.\,9^2.$$

Therefore since there shall be $\left(\frac{12}{9}\right)=\frac{1}{9}$, $\mathrm{P.Q}=6^3.\,3^3.\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3$, and thus

$$\left[\frac{9}{12}\right]=\frac{3}{4}\sqrt[4]{12.6.3.}\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right).$$

Theorem.

§. 53. Any positive whole numbers may be indicated by the letters *m* and *n*, now truly by the theorem there will indicated in the manner set out previously

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}1.2....\ (m-1).}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\ \left(\frac{n-1}{m}\right).$$

Demonstration.

For the case, in which *m* and *n* are relatively prime, the truth of the theorem has prevailed in the antecedents ; but because a case may occur also, if these numbers *m* and *n* may share a common divisor, then indeed it may not be apparent : truly for the cases in which *m* and *n* are prime numbers, it may be agreed to be true without risk, from which it may be permitted to conclude that the theorem in general shall be true. Certainly it concerns me little that numbers of this kind shall be the only case, and most others must be considered to be suspect. Whereby so that no doubt may remain, for the cases, in which the numbers *m* and *n* are composite between themselves, we have obtained two expressions, the agreement of each for the cases being set out before will help [to show the validity of the theorem]. Initially the conspicuous case $m = n$ supplies the basis, where our formula clearly produces unity.

Corollary 1.

§. 54. The first case agreed to require a demonstration is where $m = 2$ and $n = 4$, for which we have found above in §. 47 :

$$\left[\frac{2}{4}\right]=\frac{2}{4}\sqrt[4]{4^2.}\left(\frac{2}{2}\right)^2,$$

but now by the strength of the theorem, there is

$$[i.e.\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}1.2....\ (m-1).}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\ \left(\frac{n-1}{m}\right).\ ]:$$

$$\left[\frac{2}{4}\right]=\frac{2}{4}\sqrt[4]{4^2.1.}\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right),$$

from which by comparison there shall be put in place $\left(\frac{2}{2}\right)=\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)$, the truth of which is confirmed in the observations advanced above.

Corollary 2.

§. 55. If $m = 2$ and $n = 6$, from §. 49 above, there is :

$$\left[\frac{2}{6}\right]=\frac{2}{6}\sqrt[6]{6^4.}\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2.$$

$$\left[\tfrac{2}{6}\right]=\tfrac{2}{6}\sqrt[6]{6^4}.1\left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{5}{2}\right),$$

and thus it is necessary that

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)=\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right),$$

the truth of which likewise is apparent.

Corollary 3.

§56. If $m=3$ and $n=6$, this equation is arrived at :

$$\left(\tfrac{3}{6}\right)^2=1.2.\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{3}\right),$$

but if $m=4$ and $n=6$, in a similar manner there shall be

$$2^2\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right)=1.2.3\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right),$$

or

$$\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right)=\tfrac{3}{2}\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right),$$

which also it taken to be true

Corollary 4.

§. 57. In the case $m=2$ and $n=8$ this equation arises :

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)=\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)\left(\tfrac{7}{2}\right);$$

but the case $m=4$ and $n=8$ this :

$$\left(\tfrac{4}{4}\right)^3=1.2.3\left(\tfrac{1}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{7}{4}\right);$$

and finally in the case $m=6$ and $n=8$ this one :

$$2.4\left(\tfrac{6}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{2}{4}\right)=1.3.5\left(\tfrac{1}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{7}{6}\right),$$

which also are true.

Scholium.

§.58. But in general if the numbers *m* and *n* may have a common factor 2, and the proposed formula shall be $\left[\tfrac{2m}{2n}\right]=\left[\tfrac{m}{n}\right]$, because there is

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{\quad n^{n-m} 1.2.3...\ (m-1).\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\ \left(\frac{n-1}{m}\right)},$$

the same will be returned for the exponent $2n$:

$$\frac{m}{n}\sqrt[2n]{\quad 2n^{2n-2m}.2^2.4^2.6^2....(2m-2)^2.\left(\frac{2}{2m}\right)^2\left(\frac{4}{2m}\right)^2\left(\frac{6}{2m}\right)^2\cdots\ \left(\frac{2n-2}{2m}\right)^2}.$$

Truly by the theorem the same expression shall be :

$$\frac{m}{n}\sqrt[2n]{\quad 2n^{2n-2m} 1.3.5....(2m-1)\left(\frac{1}{2m}\right)\left(\frac{3}{2m}\right)\left(\frac{5}{2m}\right)\cdots\ \left(\frac{2n-1}{2m}\right)},$$

from which for the exponent $2n$ there will be

$$\begin{aligned}&2.4.6....(2m-2)\left(\tfrac{2}{2m}\right)\left(\tfrac{4}{2m}\right)\left(\tfrac{6}{2m}\right)\cdots\cdots\left(\tfrac{2n-2}{2m}\right)\\&=1.3.5....(2m-1)\left(\tfrac{1}{2m}\right)\left(\tfrac{3}{2m}\right)\left(\tfrac{5}{2m}\right)\cdots\ \left(\tfrac{2n-1}{2m}\right).\end{aligned}$$

In a similar manner if the common divisor shall be 3, there shall be found for the exponent $3n$

$$\begin{aligned}&3^2.6^2.9^2....(3m-3)^2\left(\tfrac{3}{3m}\right)^2\left(\tfrac{6}{3m}\right)^2\left(\tfrac{9}{3m}\right)^2\cdots\cdots\left(\tfrac{3n-3}{3m}\right)^2\\&=1.2.4.5....(3m-2)(3m-1)\left(\tfrac{1}{3m}\right)\left(\tfrac{2}{3m}\right)\left(\tfrac{4}{3m}\right)\left(\tfrac{5}{3m}\right)\cdots\ \left(\tfrac{3n-1}{3m}\right).\end{aligned}$$

which equation thus contracted shows :

$$\frac{1.2.4.5.7.8.10...(3m-2)(3m-1)}{3^2.6^2.9^2....(3m-3)^2} = \frac{\left(\frac{3}{3m}\right)^2\cdot\left(\frac{6}{3m}\right)^2......\left(\frac{3n-3}{3m}\right)^2}{\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\left(\frac{7}{3m}\right)....\ \left(\frac{3n-2}{3m}\right)\left(\frac{3n-1}{3m}\right)}.$$

Moreover in general if the common divisor shall be d and the exponent dn, there will be had :

$$\begin{aligned}&\left[d.2d.3d....(dm-d)\left(\tfrac{d}{dm}\right)\left(\tfrac{2d}{dm}\right)\left(\tfrac{3d}{dm}\right)\cdots\cdots\left(\tfrac{dn-d}{dm}\right)\right]^d\\&=1.2.4.5....(dm-1)\left(\tfrac{1}{dm}\right)\left(\tfrac{2}{dm}\right)\left(\tfrac{3}{dm}\right)\cdots\cdots\ \left(\tfrac{dn-1}{dm}\right),\end{aligned}$$

which equation can be adapted to whatever case it pleases, from which the following theorem deserves to be noted.

Theorem.

§. 59. If α were a common divisor of the numbers *m* and *n*, and this formula $\left(\frac{p}{q}\right)$ may denote the value of the integral

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$$

extended from $x=0$ as far as to $x=1$, there will be

$$\left[\alpha.2\alpha.3\alpha........(m-\alpha)\left(\tfrac{\alpha}{m}\right)\left(\tfrac{2\alpha}{m}\right)\left(\tfrac{3\alpha}{m}\right)\cdots\cdot\left(\tfrac{n-\alpha}{m}\right)\right]^{\alpha}$$
$$=1.2.3........(m-1)\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\cdot\left(\tfrac{n-1}{m}\right).$$

Demonstration.

The truth of this theorem may be seen from the preceding scholium, for since there the common divisor shall be $=d$, and the two numbers proposed *dm* and *dn*, here in place of these I have written *m* and *n*, moreover in place of their divisor *d* I have written the letter α as the equal ratio of the divisors included set out thus, as in an arithmetical progression α, 2α, 3α, etc. the numbers *m* and *n* may be assumed to occur continually, and thus also the numbers $m-\alpha$ and $n-\alpha$. The remainder I confess to be inferred, as this demonstration depends mainly on induction, by no means to be found most rigorously: but since nevertheless we may be convinced about the truth of this, this theorem therefore may be considered worthy of great attention, yet meanwhile there is no doubt, why a superior demonstration of the integral formulas of this kind may not be provided in a perfect demonstration, but since for now before this truth it may be permitted by us to consider, how this conspicuous example of an analytical investigation may be elucidated.

Corollary 1.

§. 60. If in place of the symbols used we may substitute these integral formulas themselves, our theorem thus may be obtained so that there shall be :

$$\alpha.2\alpha.3\alpha........(m-\alpha)\int\frac{x^{\alpha-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-m}}}\cdot\int\frac{x^{2\alpha-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-m}}}\cdots\int\frac{x^{n-\alpha-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-m}}}$$
$$=\sqrt{}1.2.3........(m-1)\int\frac{\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-m}}}\cdot\int\frac{x\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-m}}}\cdots\int\frac{x^{n-2}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-m}}}.$$

Corollary 2.

§. 61. Or, if for an abbreviation, we may put

$$\sqrt[n]{\left(1-x^{n}\right)^{n-m}} = X,$$

$$\alpha.2\alpha.3\alpha........(m-\alpha)\int\frac{x^{\alpha-1}\partial x}{X}\cdot\int\frac{x^{2\alpha-1}\partial x}{X}\cdots\int\frac{x^{n-\alpha-1}\partial x}{X}$$

$$=\sqrt[\alpha]{1.2.3........(m-1)\int\frac{\partial x}{X}\cdot\int\frac{x\partial x}{X}\cdots\int\frac{x^{n-2}\partial x}{X}}.$$

The Theorem in General.

§. 62. If the common divisors of the two numbers *m* and *n* shall be $\alpha, \beta, \gamma,$ etc. and the formula $\left(\frac{p}{q}\right)$ shall denote the value of the integral

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}$$

extended from $x=0$ to $x=1$; the following expressions of the forms from integral formulas of this kind will be equal to each other

$$\left[\alpha.2\alpha.3\alpha........(m-\alpha)\left(\tfrac{\alpha}{m}\right)\left(\tfrac{2\alpha}{m}\right)\left(\tfrac{3\alpha}{m}\right)\cdots\cdot\left(\tfrac{n-\alpha}{m}\right)\right]^{\alpha}=$$

$$\left[\beta.2\beta.3\beta........(m-\beta)\left(\tfrac{\beta}{m}\right)\left(\tfrac{2\beta}{m}\right)\left(\tfrac{3\beta}{m}\right)\cdots\cdot\left(\tfrac{n-\beta}{m}\right)\right]^{\beta}=$$

$$\left[\gamma.2\gamma.3\gamma........(m-\gamma)\left(\tfrac{\gamma}{m}\right)\left(\tfrac{2\gamma}{m}\right)\left(\tfrac{3\gamma}{m}\right)\cdots\cdot\left(\tfrac{n-\gamma}{m}\right)\right]^{\gamma}=\text{etc.}$$

$$=1.2.3........(m-1)\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\cdot\left(\tfrac{n-1}{m}\right).$$

Demonstration.

The truth of this evidently follows from the preceding theorem, since any of these expressions themselves shall be equal to this one

$$1.2.3........(m-1)\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\cdot\left(\tfrac{n-1}{m}\right),$$

which, as with the minimum of unity, agrees with the number of common divisors of the numbers *m* and *n* . Therefore just as many expressions of this kind can be shown to be equal to each other, as there shall be common divisors of the two numbers *m* and *n*.

Corollary 1.

§. 63. Since this formula $\left(\frac{n}{m}\right)=\frac{1}{m}$ and thus there shall be $m\left(\frac{n}{m}\right)=1$, our expressions can be represented more neatly in this manner :

$$\left[\alpha.2\alpha.3\alpha........m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdot\left(\frac{n}{m}\right)\right]^{\alpha}=$$

$$\left[\beta.2\beta.3\beta........m\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\cdot\left(\frac{n}{m}\right)\right]^{\beta}=$$

$$\left[\gamma.2\gamma.3\gamma........m\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\cdot\left(\frac{n}{m}\right)\right]^{\gamma}=\text{etc.}$$

And indeed here if the number of factors is increased, yet the ratio of the composition can be seen more easily.

Corollary 2.

§. 64. Therefore if there shall be $m=6$ and $n=12$, on account of the number of common divisor of these 6, 3, 2, 1, the four following forms will be obtained equal to each other :

$$\left[6\left(\tfrac{6}{6}\right)\left(\tfrac{12}{6}\right)\right]^{6}=\left[3.6\left(\tfrac{3}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{9}{6}\right)\left(\tfrac{12}{6}\right)\right]^{3}=$$

$$\left[2.4.6\left(\tfrac{2}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{8}{6}\right)\left(\tfrac{10}{6}\right)\left(\tfrac{12}{6}\right)\right]^{2}=$$

$$1.2.3.4.5.6\left(\tfrac{1}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{3}{6}\right)........\left(\tfrac{12}{6}\right).$$

Corollary 3.

§. 65. If the last may be combined with the second last, this equation arises :

$$\frac{1.3.5}{2.4.6}=\frac{\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{6}\right)\left(\frac{11}{6}\right)},$$

but the last may be compared with the last but two provides

$$\frac{1.2.4.5}{3.3.5.5}=\frac{\left(\frac{3}{6}\right)\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{11}{6}\right)}.$$

Scholium.

§. 66. Therefore infinitely many relations follow between these integral formulas

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}=\left(\frac{p}{q}\right),$$

which therefore are more noteworthy, because they have been produced by a single straightforward method according to these we have produced here. And if anyone may doubt the truth about these at some point, he may consult my observations about these formulas, and thus for any case presented he may be convinced easily about the truth. But even if that discussion provided being a confirmation of this, yet these relations elicited are of greater value, because within these a certain order is discerned, and that for all the kinds, however great the exponent *n* may be wished to be taken, may be continued in an easy manner ; truly from earlier handling, the calculation for higher order classes may become continually more laborious and complicated.

Supplement containing a demonstration of the proposed
Theorem §. 53.

§. 67. This demonstration aims at providing further agreement; clearly equation §. 25 may be assumed given, which on putting $f=1$ and with the letters changed is

$$\frac{\int\partial x\left(l\frac{1}{x}\right)^{\nu-1}\times\int\partial x\left(l\frac{1}{x}\right)^{\mu-1}}{\int\partial x\left(l\frac{1}{x}\right)^{\mu+\nu-1}}=\chi\int\frac{x^{\chi\mu-1}\partial x}{\left(1-x^{\chi}\right)^{1-\nu}},$$

and that by the observed reductions may be represented in this form :

$$\frac{\int\partial x\left(l\frac{1}{x}\right)^{\nu}\times\int\partial x\left(l\frac{1}{x}\right)^{\mu}}{\int\partial x\left(l\frac{1}{x}\right)^{\mu+\nu}}=\frac{\chi\mu\nu}{\mu+\nu}\int\frac{x^{\chi\mu-1}\partial x}{\left(1-x^{\chi}\right)^{1-\nu}}.$$

Now there may be put in place $\nu=\frac{m}{n}$ and $\mu=\frac{\lambda}{n}$, then truly $\chi=n$, so that we shall have

$$\frac{\int\partial x\left(l\frac{1}{x}\right)^{\frac{m}{n}}\times\int\partial x\left(l\frac{1}{x}\right)^{\frac{\lambda}{n}}}{\int\partial x\left(l\frac{1}{x}\right)^{\frac{\lambda+m}{n}}}=\frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}\partial x}{\sqrt[\chi]{\left(1-x^{n}\right)^{n-m}}}$$

which for the sake of brevity, in the above usual manner, thus may be restored neatly

$$\frac{\left[\frac{m}{n}\right]\cdot\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{m}\right]}=\frac{\lambda m}{\lambda+m}\cdot\left(\frac{\lambda}{m}\right).$$

Now in place of λ the numbers 1,2, 3, 4 …. *n* may be written successively, and all these equations, the number of which is $=n$, may be multiplied together in turn, and the resulting equation will be

$$\left[\frac{m}{n}\right]^n \cdot \frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{m+1}{m}\right]\left[\frac{m+2}{m}\right]\left[\frac{m+3}{m}\right]\cdots\cdots\left[\frac{m+n}{m}\right]} =$$
$$m^n \cdot \frac{1}{m+1} \cdot \frac{2}{m+2} \cdot \frac{3}{m+3} \cdot\cdots\cdots \frac{n}{m+n} \cdot \left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right) =$$
$$m^n \cdot \frac{1.2.3\ldots\ldots\ldots\ldots m}{(n+1)(n+2)(n+3)\ldots\ldots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Moreover in a similar manner the first part may be transformed so that it shall become

$$\left[\frac{m}{n}\right]^n \cdot \frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+1}{n}\right]\left[\frac{n+2}{n}\right]\left[\frac{n+3}{n}\right]\cdots\cdots\left[\frac{n+m}{n}\right]},$$

of which the agreement being cross-multiplied by the preceding form, as they say, is itself produced at once. Truly since from the nature of these formulas there shall be :

$$\left[\frac{n+1}{n}\right] = \frac{n+1}{n}\left[\frac{1}{n}\right], \left[\frac{n+2}{n}\right] = \frac{n+2}{n}\left[\frac{2}{n}\right], \left[\frac{n+3}{n}\right] = \frac{n+3}{n}\left[\frac{3}{n}\right], \text{etc.}$$

on account of the number of these formulas $= m$, this former part will emerge

$$\left[\frac{m}{n}\right]^n \cdot \frac{n^m}{(n+1)(n+2)(n+3)\ldots\ldots(n+m)},$$

which since it shall be equal to the other part shown before

$$m^n \cdot \frac{1.2.3\ldots\ldots\ldots\ldots m}{(n+1)(n+2)(n+3)\ldots\ldots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)$$

we arrive at this equation

$$\left[\frac{m}{n}\right]^n = \frac{m^n}{n^m}.1.2.3\ldots\ldots\ldots\ldots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

so that thus there shall be

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1.2.3\ldots\ldots\ldots\ldots m}{n^m}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)},$$

which agrees entirely as proposed in (§. 53.) on account of $\left(\frac{n}{m}\right) = \frac{1}{m}$, from which its truth now is vindicated from the most certain principles.

Demonstration of the theorem proposed in §.59.

§. 68. This theorem needs to be established more firmly also, as from the equation made firm before : thus I set up

$$\frac{\left[\frac{m}{n}\right]\cdot\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{m}\right]}=\frac{\lambda m}{\lambda+m}\cdot\left(\frac{\lambda}{m}\right).$$

With the common divisor α of the numbers present m and n, in place of λ, the numbers α, 2α, 3α; etc. may be written successively as far as to n, the number of which is $=\frac{n}{\alpha}$, and all the equations resulting in this manner may be multiplied together, so that this equation may be produced :

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}}\cdot\frac{\left[\frac{\alpha}{n}\right]\cdot\left[\frac{2\alpha}{n}\right]\cdot\left[\frac{3\alpha}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\cdots\left[\frac{m+n}{n}\right]}=$$

$$m^{\frac{n}{\alpha}}\cdot\frac{1\alpha}{m+\alpha}\cdot\frac{2\alpha}{m+2\alpha}\cdot\frac{3\alpha}{m+3\alpha}\cdots\cdots\frac{n}{m+n}\cdot\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Now the first part may be transformed into this equal form :

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}}\cdot\frac{\left[\frac{\alpha}{n}\right]\cdot\left[\frac{2\alpha}{n}\right]\cdot\left[\frac{3\alpha}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\cdots\left[\frac{m+n}{n}\right]},$$

which on account of $\left[\frac{n+\alpha}{n}\right]=\frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$, and thus with the others, is reduced to this :

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}}\cdot\frac{n}{n+\alpha}\cdot\frac{n}{n+2\alpha}\cdot\frac{n}{n+3\alpha}\cdots\cdots\frac{n}{n+m}.$$

Truly the latter part of the equation may be transformed in a similar manner into

$$m^{\frac{n}{\alpha}}\cdot\frac{\alpha}{n+\alpha}\cdot\frac{2\alpha}{n+2\alpha}\cdot\frac{3\alpha}{n+3\alpha}\cdots\cdots\frac{m}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdot\left(\frac{n}{m}\right),$$

from which this equation arises :

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}}m^{\frac{m}{\alpha}}=m^{\frac{n}{\alpha}}\alpha\cdot 2\alpha\cdot 3\alpha\cdots\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdot\left(\frac{n}{m}\right),$$

and hence

$$\left[\frac{m}{n}\right]=m\sqrt[n]{\frac{1}{m^n}\left[\alpha\cdot 2\alpha\cdot 3\alpha\cdots\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdot\left(\frac{n}{m}\right)\right]^{\alpha}},$$

which expression coupled with the preceding provides this equation

$$\left[\alpha\cdot 2\alpha\cdot 3\alpha\cdots\cdots m\left(\tfrac{\alpha}{m}\right)\left(\tfrac{2\alpha}{m}\right)\left(\tfrac{3\alpha}{m}\right)\cdots\left(\tfrac{n}{m}\right)\right]^{\alpha}=$$
$$1\cdot 2\cdot 3\cdots\cdots m\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\left(\tfrac{n}{m}\right),$$

which is understood to be formed from all the common divisors of the two numbers m and n .

SUPPLEMENTUM III*a*. AD TOM.I. CAP. IV.

DE INTEGRATIONE FORMULARUM LOGARITHMICARUM ET EXPONENTIALIUM .

1). Evolutio formulae integralis $\int x^{f-1}\partial x(lx)^{\frac{m}{n}}$, integratione a valore $x=0$ ad $x=1$ extensa.

Nov. Commentarii Acad. Imp. Sc. Petrapolitanae. Tom. XVL. Pag. 91−139.

Theorema 1.

§. 1. Si *n* denotat numerum integrum positivum quemcunque, et formulae $\int x^{f-1}\partial x\left(1-x^{g}\right)^{n}$ integratio a valore $x=0$ usque ad $x=1$ extendatur, erit ejus valor :

$$=\frac{g^{n}}{f}\cdot\frac{1.\quad 2.\quad 3.\quad ....n}{(f+g)(f+2g)(f+3g)............(f+ng)}.$$

Demonstratio.

Notum est in genere, integrationem formulae

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{m}$$

reduci posse ad integrationem hujus $\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}$, quoniam quantitates constantes A et B ita definire licet, ut fiat

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{m}=\mathrm{A}\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}+\mathrm{B}x^{f}\left(1-x^{g}\right)^{m};$$

sumtis enim differentialibus prodit haec aequatio

$$x^{f-1}\partial x\left(1-x^{g}\right)^{m}=\mathrm{A}x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}+\mathrm{B}fx^{f-1}\partial x\left(1-x^{g}\right)^{m}-\mathrm{B}mgx^{f+g-1}\partial x\left(1-x^{g}\right)^{m-1},$$

quae per $x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}$ divisa dat

$$1-x^{g}=\mathrm{A}+\mathrm{B}f\left(1-x^{g}\right)-\mathrm{B}mgx^{g}\partial x\left(1-x^{g}\right),\text{seu}$$

$$1-x^{g}=\mathrm{A}-\mathrm{B}mg+\mathrm{B}(f+mg)\left(1-x^{g}\right),$$

quae aequatio ut consistere possit, necesse est sit

$$1=\mathrm{B}\ (f+mg)\ \text{et}\ \ \mathrm{A}=\mathrm{B}mg\ ;$$

unde colligimus

$$\mathrm{B}=\tfrac{1}{f+mg}\ \text{ et }\ \mathrm{A}=\tfrac{mg}{f+mg}.$$

Quocirca habebimus sequentem reductionem generalem

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{m}=\tfrac{mg}{f+mg}\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}+\tfrac{1}{f+mg}\cdot x^{f}\left(1-x^{g}\right)^{m}$$

quae cum evanescat posito $x=0$, siquidem sit $f>0$, constantis additione haud est opus. Quare extenso utroque integrali usque ad $x=1$, pars integralis postrema sponte evanescit, eritque pro casu $x=1$

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{m}=\tfrac{mg}{f+mg}\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}.$$

Cum igitur sumto $m=1$ sit $\int x^{f-1}\partial x\left(1-x^{g}\right)^{0}=\tfrac{1}{f}x^{f}=\tfrac{1}{f}$, posito $x=1$, nanciscimur pro eodem casu $x=1$ sequentes valores

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{1}=\tfrac{g}{f}\cdot\tfrac{1}{f+g}$$
$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{2}=\tfrac{g^{2}}{f}\cdot\tfrac{1}{f+g}\cdot\tfrac{2}{f+2g}$$
$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{3}=\tfrac{g^{3}}{f}\cdot\tfrac{1}{f+g}\cdot\tfrac{2}{f+2g}\cdot\tfrac{3}{f+3g}$$
etc.

hincque pro numero quocunque integra positivo *n* concludimus fore

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{n}=\tfrac{g^{n}}{f}\cdot\tfrac{1}{f+g}\cdot\tfrac{2}{f+2g}\cdot\tfrac{3}{f+3g}\cdots\cdots\cdot\tfrac{n}{f+ng}$$

Si modo numeri *f* et *g* sint positivi.

Corollarium I.

§. 2. Hinc ergo vicissim valor hujusmodi producti ex quotcunque factoribus formati, per formulam integralem exprimi potest, ita ut sit

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots(f+ng)}=\frac{f}{g^{n}}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n},$$

integrali hoc a valore $x=0$ usque ad $x=1$ extenso.

Corollarium 2.

§. 3. Quodsi ergo hujusmodi habeatur progressio

$$\frac{1}{f+g};\frac{1.\quad 2}{(f+g)(f+2g)};\frac{1.\quad 2.\quad 3}{(f+g)(f+2g)(f+3g)};\frac{1.\quad 2.\quad 3.\quad 4}{(f+g)(f+2g)(f+3g)(f+4g)};\text{etc.}$$

ejus terminus generalis qui indici indefinito *n* convenit, commode hac forma integrali $\frac{f}{g^{n}}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n}$ repraesentatur, cujus ope ea progressio interpolari , ejusque termini indicibus fractis respondentes exhiberi poterunt.

Corollarium 3.

§. 4. Si loco n scribamus $n-1$, habebimus

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots(n-1)}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots(f+(n-1)g)}=\frac{f}{g^{n-1}}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1},$$

quae per $\frac{n}{f+ng}$ multiplicata praebet

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots(f+ng)}=\frac{f.ng}{g^{n}(f+ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1}.$$

Scholion 1.

§5. Hanc posteriorem formam immediate ex praecedente derivare licuisset, cum modo demonstraverimus esse

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{n}=\frac{ng}{f+ng}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1},$$

siquidem utrumque integrale a valore $x=0$ usque ad $x=1$ extendatur; quam integralium determinationem in sequentibus ubique subintelligi oportet. Deinde etiam perpetuo est tenendum, quantitates f et g esse positivas, quippe quam conditionem demonstratio allata absolute postulat. Quod autem ad numerum n attinet, quatenus eo index cujusque termini, progressionis (§. 3.) designatur, nihil impedit, quominus eo numeri quicunque sive positivi sive negativi denotentur, quandoquidem ejus progressionis omnes termini etiam indicibus negativis respondentes per formulam integralem datam exhiberi censentur. Interim tamen probe tenendum est, hanc reductionem

$$\int x^{f-1}\partial x\left(1-x^{g}\right)^{n}=\frac{mg}{f+mg}\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}$$

non esse veritati consentaneam, nisi sit $m>0$; quia alioquin pars algebraica $\frac{1}{f+mg}x^{f}\left(1-x^{g}\right)^{m}$ non evanesceret posito $x=1$.

Scholion 2.

§ 6. Hujusmodi series, quas transcendentes appellare licet, quia termini indicibus fractis respondentes sunt quantitates transcendentes, jam olim in Comment. Petrop. Tomo V. (Institut. Calc. integralis Tom. I. Sect. I. Cap. IV.) fusius sum prosecutus; unde hoc loco non tam istas progressiones, quam eximias formlarum integralium comparationes, quae inde derivantur, diligentius sum scrutaturus. Cum scilicet ostendissem, hujus producti indefiniti 1. 2. 3…..n valorem hac formula integrali $\int\partial x\left(l\frac{1}{x}\right)^{n}$ ab $x=0$ ad $x=1$ extensa exprimi, quae res quoties n est numerus integer positivus per ipsam integrationem est manifesta, eos casus examini subjeci, quibus pro n numeri fracti accipiuntur; ubi quidem ex ipsa formula integrali neutiquam patet, ad quodnam genus quantitatum transcendentium hi termini referri debeant. Singulari autem artificio eosdem terminos ad quadraturas magis cognitas, reduxi, quod propterea maxime dignum videtur, ut majori studio perpendatur.

Problema 1.

§. 7. *Cum demonstratum sit esse*

$$\frac{1. \quad 2. \quad 3. \quad \ldots\ldots\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots(f+ng)} = \frac{f}{g^n}\int x^{f-1}\partial x\left(1-x^g\right)^n$$

integrali ab $x=0$ *ad* $x=1$ *extenso, ejusdem producti casu quo* $g=0$ *valorem per formulam integralem assignare.*

Solutio.

Posito $g=0$ in formula integrali membrum $\left(1-x^g\right)^n$ evanescit, simul vero etiam denominator g^n, unde quaestio huc redit, ut fractionis $\frac{\left(1-x^g\right)^n}{x^g}$ valor definiatur casu $g=0$, quo tam numerator quam denominator evanescit. Hunc in finem spectetur g ut quantitas infinite parva, et cum sit $x^g = e^{glx}$, fiet $x^g = 1+glx$, ideoque

$$\left(1-x^g\right)^n = g^n\left(-lx\right)^n = g^n\left(l\tfrac{1}{x}\right)^n;$$

ex quo pro hoc casu formula nostra integralis abit in

$$f\int x^{f-1}\partial x\left(l\tfrac{1}{x}\right)^n;$$

ita ut jam habeatur

$$\frac{1. \quad 2. \quad 3. \quad \ldots\ldots\ldots\ldots n}{f^n} = f\int x^{f-1}\partial x\left(l\tfrac{1}{x}\right)^n, \text{ seu}$$

$$1. \quad 2. \quad 3. \quad \ldots\ldots\ldots\ldots n = f^{f+1}\int x^{f-1}\partial x\left(l\tfrac{1}{x}\right)^n.$$

Corollarium 1.

§. 8. Quoties n est, numerus integer positivus, integratio formulae $\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^n$ succedit, eaque ab $x=0$ ad $x=1$ extensa revera prodit id productum, cui istam formulam aequalem invenimus. Sin autem pro n capiantur numeri fracti, eadem formula integralis inserviet, huic progressioni hypergeometricae interpolandae

$$1;\ 1.\ 2;\ 1.\ 2.\ 3;\ 1.\ 2.\ 3.\ 4;\ 1.\ 2.\ 3.\ 4.\ 5;\ 1.\ 2.\ 5.\ 4.\ 5.\ 6;\ \text{etc. seu}$$

$$1;\quad 2;\quad 6;\quad 24;\quad 120;\quad 720;\ \text{etc.}$$

Corollarium 2.

§. 9. Si expressio modo inventa per principalem dividatur, orietur productum, cujus factores in progressione arithmetica quacunque progrediuntur

$$(f+g)(f+2g)(f+3g)........(f+ng)=f^n g^n.\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^n}{\int x^{f-1}\partial x\left(1-x^g\right)^n},$$

cujus ergo etiam valores, si n sit numerus fractus, hinc assignare licebit.

Corollarium 3.

§.10. Cum sit

$$\int x^{f-1}\partial x\left(1-x^g\right)^n=\frac{ng}{f+ng}\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}$$

erit etiam simili modo pro casu $g=0$

$$\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^n=\frac{n}{f}\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1},$$

hincque per istas alteros formulas integrales

$$1.\,2.\,3...............n=nf^n\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}$$

et

$$(f+g)(f+2g)......(f+ng)=f^{n-1}g^{n-1}.\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}}.$$

Scholion.

§.11. Cum invenerimus esse

$$1.\,2.\,3.......n=f^{n+1}\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^n,$$

patet hanc formulam integralem non a valore quantitatis f pendere, quod etiam facile perspicitur ponendo $x^f=y$, unde fit $fx^{f-1}\partial x=\partial y$, et $l\frac{1}{x}=-lx=-\frac{1}{f}ly=\frac{1}{f}l\frac{1}{y}$, ideoque $f^n\left(l\frac{1}{x}\right)^n=\left(l\frac{1}{y}\right)^n$, ita ut sit

$$1.\,2.\,3.......n=\int\partial y\left(l\frac{1}{y}\right)^n,$$

quae formula ex priori nascitur ponendo $f=1$. Pro interpolatione ergo hujusmodi formarum totum negotium huc reducitur, ut istius formulae integralis $\int\partial x\left(l\frac{1}{x}\right)^n$ valores definiantur, quando exponens n est numerus fractus. Veluti si n sit $\frac{1}{2}$, assignari oportet valorem hujus formulae $\int\partial x\sqrt{l\frac{1}{x}}$, quem olim jam ostendi esse $=\frac{1}{2}\sqrt{\pi}$, denotante π circuli peripheriam cujus diameter $=1$: pro aliis autem numeris fractis cujus valorem, ad quadraturas curvarum algebraicarum altioris ordinis revocare docui. Quae reductio cum minime sit obvia, atque tum solum locum habeat, quando formulae $\int \partial x \left(l \frac{1}{x} \right)^n$ integratio a valore $x = 0$ ad $x = 1$ extenditur, singulari attentione digna videtur. Etsi autem jam olim hoc argumentum tractavi, tamen quia per plures ambages eo sum perductus, idem hic resumere et concinnius evolvere constitui.

Theorema 2.

§. 12. Si formulae integrales a valore $x = 0$ usque ad $x = 1$ extendantur, et *n* denotet numerum integrum positivum, erit

$$\frac{1. \quad 2. \quad 3. \quad \ldots\ldots\ldots\ldots n}{(n+1)(n+2)(n+3)\ldots\ldots\ldots\ldots 2n} = \frac{1}{2} ng \int x^{f+ng-1} \partial x \left(1 - x^g\right)^{n-1} \times \frac{\int x^{f-1} \partial x \left(1 - x^g\right)^{n-1}}{\int x^{f-1} \partial x \left(1 - x^g\right)^{2n-1}}$$

quicunque numeri positivi loco *f* et *g* accipiantur.

Demonstratio.

Cum supra (§. 4.) ostenderimus esse

$$\frac{1. \quad 2. \quad 3. \quad \ldots\ldots\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots\ldots(f+ng)} = \frac{f.ng}{g^n (f+ng)} \int x^{f-1} \partial x \left(1 - x^g\right)^{n-1},$$

habebimus, si loco *n* scribamus 2*n,*

$$\frac{1. \quad 2. \quad 3. \quad \ldots\ldots\ldots\ldots 2n}{(f+g)(f+2g)(f+3g)\ldots\ldots\ldots\ldots(f+2ng)} = \frac{f.2ng}{g^{2n} (f+2ng)} \int x^{f-1} \partial x \left(1 - x^g\right)^{2n-1}.$$

Dividatur nunc prima aequatio per secondam, ac prodibit ista tertia

$$\frac{\left(f+(n+1)g\right)\left(f+(n+2)g\right)\ldots\ldots\ldots\ldots(f+2ng)}{(n+1)(n+2)\ldots\ldots\ldots 2n} = \frac{g^n (f+2ng)}{2(f+ng)} \cdot \frac{\int x^{f-1} \partial x \left(1 - x^g\right)^{n-1}}{\int x^{f-1} \partial x \left(1 - x^g\right)^{2n-1}}.$$

At si in prima aequatione loco *f* scribatur $f + ng$, orietur haec aequatio quarta

$$\frac{1. \quad 2. \quad 3. \quad \ldots\ldots\ldots\ldots n}{\left(f+(n+1)g\right)\left(f+(n+2)g\right)\ldots\ldots\ldots\ldots(f+2ng)} = \frac{(f+ng)ng}{g^n (f+2ng)} \int x^{f+ng-1} \partial x \left(1 - x^g\right)^{n-1}.$$

Multiplicetur haec quarta aequatio per illam tertiam, ac reperietur ipsa aequatio demonstranda,

$$\frac{1. \quad 2. \quad 3. \quad \ldots\ldots\ldots\ldots n}{(n+1)(n+2)(n+3)\ldots\ldots\ldots\ldots 2n} = \frac{1}{2} ng \int x^{f+ng-1} \partial x \left(1 - x^g\right)^{n-1} \times \frac{\int x^{f-1} \partial x \left(1 - x^g\right)^{n-1}}{\int x^{f-1} \partial x \left(1 - x^g\right)^{2n-1}}$$

Corollarium 1.

§. 13. Si in prima aequatione statuatur $f = n$ et $g = 1$, orietur idem

$$\frac{1.\quad 2\ ....n}{(n+1)(n+2)...........2n} = \frac{1}{2}n\int x^{f-1}\partial x(1-x)^{n-1},$$

qua aequatione cum illa collata adipiscimur

$$\frac{\int x^{n-1}\partial x(1-x)^{n-1}}{g\int x^{f+ng-1}\partial x\left(1-x^g\right)^{n-1}} = \frac{\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}}.$$

Corollarium 2.

§. 14. Si in illa aequatione loco x scribamus x^g, fiet

$$\frac{1.\quad 2\ ....n}{(n+1)(n+2)...........2n} = \frac{1}{2}ng\int x^{ng-1}\partial x\left(1-x^g\right)^{n-1};$$

ita ut jam consequamur istam comparationem inter sequentes formulas integrales

$$\int x^{ng-1}\partial x\left(1-x^g\right)^{n-1} = \int x^{f+ng-1}\partial x\left(1-x^g\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}}.$$

Corollarium 3.

§. 15. Si in aequatione theorematis ponamus $g = 0$, ob $\left(1-x^g\right)^m = g^m\left(l\frac{1}{x}\right)^m$, potestates ipsius g se destruent, orieturque haec aequatio

$$\frac{1.\quad 2.\quad 3..................n}{(n+1)(n+2)(n+3)............2n} = \frac{1}{2}n\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}};$$

unde colligimus

$$\frac{4}{m}\cdot\frac{\left[\int\partial x\left(l\frac{1}{x}\right)^{\frac{m}{2}}\right]^2}{1.\ 2.\ 3.\ .....m} = 2\int x^{m-1}\partial x\left(1-x^2\right)^{\frac{m}{2}-1},$$

seu ob

$$\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1} = \frac{f}{n}\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n},\ \text{hanc}$$

$$\frac{2f}{n}\cdot\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n}}=g\int x^{ng-1}\partial x\left(1-x^g\right)^{n-1}.$$

Corollarium 4.

§.16. Ponamus hic $f=1,\ g=2$ et $n=\frac{m}{2}$, ut m sit numerus integer positivus; et ob $\int\partial x\left(l\frac{1}{x}\right)^m=1.\ 2.\ 3.\ .....m,$ erit

$$\frac{4}{m}\cdot\frac{\left[\int\partial x\left(l\frac{1}{x}\right)^{\frac{m}{2}}\right]^2}{1.\ 2.\ 3.\ .....m}=2\int x^{m-1}\partial x\left(1-x^2\right)^{\frac{m}{2}-1},$$

hincque

$$\int\partial x\left(l\frac{1}{x}\right)^{\frac{m}{2}}=\sqrt{\ }1.\ 2.\ 3.\ .....m.\frac{m}{2}\int x^{m-1}\partial x\left(1-x^2\right)^{\frac{m}{2}-1},$$

et sumendo $m=1$, ob $\int\frac{\partial x}{\sqrt{(1-xx)}}=\frac{\pi}{2}$ habebitur

$$\int\partial x\sqrt{\left(l\frac{1}{x}\right)}=\sqrt{\ }\frac{1}{2}\int\frac{\partial x}{\sqrt{1-x^2}}=\frac{1}{2}\sqrt{\pi}.$$

Scholion.

§.17. En ergo succinctam demonstrationem theorematis olim a me prolati, quod sit $\int\partial x\sqrt{\left(l\frac{1}{x}\right)}=\frac{1}{2}\sqrt{\pi}$, eamque ab interpolationis ratione, qua tum usus fueram, libera. Deducta scilicet hic ea ex hoc theoremate, quo inveni esse

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}}=g\int x^{ng-1}\partial x\left(1-x^g\right)^{n-1}.$$

Principale autem theorema, unde hoc est deductum ita se habet

$$g\cdot\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{n-1}\times\int x^{f+ng-1}\partial x\left(1-x^g\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}}=\int x^{n-1}\partial x\left(1-x\right)^{n-1};$$

utrumque enim membrum per integrationem ab $x=0$ ad $x=1$ extensam evolvitur in hoc productum numericum

$$\frac{1.\ \ 2.\ \ 3............\ (n-1)}{(n+1)\ (n+2)\ ....\ (2n-1)}.$$

Ac si alteri membro speciem latius patentem tribuere velimus, theorema ita proponi poterit ut sit

$$g\cdot\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1}\times\int x^{f+ng-1}\partial x\left(1-x^{g}\right)^{n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}=k\int x^{nk-1}\partial x\left(1-x^{k}\right)^{n-1},$$

hicque si capiatur $g=0$, fit

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^{2}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}\partial x\left(1-x^{k}\right)^{n-1}.$$

Imprimis igitur notandum est, quod illa aequalitas subsistat, quicunque numeri loco *f* et *g* accipiantur: casu quidem $f=g$, ea est manifesta, cum sit

$$\int x^{g-1}\partial x\left(1-x^{g}\right)^{n-1}=\frac{1-\left(1-x^{g}\right)^{n}}{ng}=\frac{1}{ng},$$

fiet enim

$$2g\int x^{ng+g-1}\partial x\left(1-x^{g}\right)^{n-1}=k\int x^{nk-1}\partial x\left(1-x^{k}\right)^{n-1},$$

et quia

$$\int x^{ng+g-1}\partial x\left(1-x^{g}\right)^{n-1}=\frac{1}{2}\int x^{ng-1}\partial x\left(1-x^{g}\right)^{n-1},$$

aequalitas est perspicua, quia *k* pro lubitu accipere licet. Eodem autem modo, quo ad hoc theorema perveni, ad alia similia pertingere licet.

Theorema 3.

§.18. Si sequentes formulae integrales a valore $x=0$ ad $x=1$ extendantur, et *n* denotet numerum integrum positivum quemcunque, erit

$$\frac{1.\quad 2.\quad 3.\ldots\ldots\ldots n}{(2n+1)(2n+2)(2n+3)\ldots.3n}=\frac{2}{3}ng\int x^{f+2ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{3n-1}},$$

quicumque numeri positivi pro *f* et *g* accipiantur.

Demonstratio.

In praecedente theoremate jam vidimus esse

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\ldots 2n}{(f+g)(f+2g)\ldots\ldots\ldots(f+2ng)g}=\frac{f.2ng}{g^{2n}(f+2ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{2n-1}.$$

simili autem modo, si in forma principali loco *n* scribamus 3*n* habebimus

$$\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\ldots 3n}{(f+g)(f+2g)\ldots\ldots\ldots(f+3ng)}=\frac{f.3ng}{g^{3n}(f+3ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{3n-1},$$

ex quo illa aequatio per hanc divisa producit

$$\frac{(f+(2n+1)g)(f+(2n+2)g)..........(f+3ng)}{(2n+1)(2n+2)..3n}=\frac{2g^n(f+3ng)}{3(f+2ng)}\times\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{3n-1}}.$$

Verum si in aequatione principali (§. 4.) loco f scribamus $f+2ng$, adipiscimur hanc aequationem

$$\frac{1.\quad 2.\quad 3..n}{(f+(2n+1)g)(f+(2n+2)g)..........(f+3ng)}=\frac{(f+2ng).ng}{g^n(f+3ng)}\times\int x^{f+2ng-1}\partial x\left(1-x^g\right)^{n-1}.$$

Multiplicetur nunc haec aequatio per praecedentem, et orietur ipsa aequatio, quam demonstrari oportet

$$\frac{1.\quad 2.\quad 3...........n}{(2n+1)(2n+2)(2n+3)....3n}=\tfrac{2}{3}ng\int x^{f+2ng-1}\partial x\left(1-x^g\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{3n-1}},$$

Corollarium 1.

§. 19. Eundem valorem ex aequatione principali nanciscimur, ponendo $f=2n$ et $g=1$, ita ut sit

$$\frac{1.\quad 2.\quad 3...........n}{(2n+1)(2n+2)(2n+3)....3n}=\tfrac{2}{3}n\int x^{2n-1}\partial x\left(1-x\right)^{n-1}$$

quae formula integralis, loco x scribendo x^k transformatur in hanc

$$\tfrac{2}{3}n\int x^{2nk-1}\partial x\left(1-x^k\right)^{n-1}$$

ita ut sit

$$g\int x^{f+2ng-1}\partial x\left(1-x^g\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{2n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{3n-1}}=k\int x^{2nk-1}\partial x\left(1-x^k\right)^{n-1}.$$

Corollarium 2.

§. 20. Si hic statuamus $g=0$, ob $1-x^g=gl\frac{1}{x}$ habebimus hanc aequationem

$$\int x^{f-1}\partial x\left(l\tfrac{1}{x}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}\partial x\left(1-x^k\right)^{n-1}:$$

cum igitur ante invenissemus

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1},$$

habebimus has aequationes in se multiplicando

$$\frac{\left[\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\right]^3}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{3n-1}}=k^2\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1}\times\int x^{2nk-1}\partial x\left(1-x^k\right)^{n-1}.$$

Corollarium 3.

§. 21. Sine ulla restrictione hic ponere licet $f=1$; tum ergo sumto $n=\frac{1}{3}$ et $k=3$, erit

$$\frac{\left[\int \partial x\left(l\frac{1}{x}\right)^{-\frac{2}{3}}\right]^3}{\int \partial x\left(l\frac{1}{x}\right)^{0}}=9\int \partial x\left(1-x^3\right)^{-\frac{2}{3}}\times\int x\partial x\left(1-x^3\right)^{-\frac{2}{3}},$$

et ob

$$\int \partial x\left(l\frac{1}{x}\right)^{-\frac{2}{3}}=3\int \partial x\left(l\frac{1}{x}\right)^{\frac{1}{3}} \text{ et } \int \partial x\left(l\frac{1}{x}\right)^{0}=1,$$

obtinebimus

$$\int \partial x\left(l\frac{1}{x}\right)^{-\frac{2}{3}}=3\int \partial x\left(l\frac{1}{x}\right)^{\frac{1}{3}} \text{ et } \int \partial x\left(l\frac{1}{x}\right)^{0}=1,$$

tum vero sumto $n=\frac{2}{3}$ et $k=3$, erit

$$\frac{\left[\int \partial x\left(l\frac{1}{x}\right)^{-\frac{1}{3}}\right]^3}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)}=9\int x\partial x\left(1-x^3\right)^{-\frac{1}{3}}\times\int x^3\partial x\left(1-x^3\right)^{-\frac{1}{3}}.$$

seu

$$\left[\int \partial x\left(l\frac{1}{x}\right)^{\frac{2}{3}}\right]^3=4\int x\partial x\left(1-x^3\right)^{-\frac{1}{3}}\times\int x^3\partial x\left(1-x^3\right)^{-\frac{1}{3}}.$$

Theorema generale.

§. 22. Si sequentes formulae integrales a valore $x=0$ usque ad $x=1$ extendantur, et n denotet numerum integrum positivum quemcunque, erit

$$\frac{1.\quad 2.\quad 3............n}{(\lambda n+1)(\lambda n+2)....(\lambda+1)n}=\frac{n}{\lambda+1}ng\int x^{f+\lambda ng-1}\partial x\left(1-x^g\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^g\right)^{\lambda n-1}}{\int x^{f-1}\partial x\left(1-x^g\right)^{(\lambda+1)n-1}},$$

quicunque numeri positivi pro litteris *f* et *g* accipiantur.

Demonstatio.

Cum sit uti supra ostendimus

$$\frac{1.\quad 2.\quad 3.\quad ....3n}{(f+g)(f+2g)...........(f+3ng)}=\frac{f.ng}{g^{n}(f+ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{n-1},$$

si hic loca *n* scribamus primo *λn,* tum vero $(\lambda+1)n$, nanciscemur has duas aequationes

$$\frac{1.\quad 2.\quad 3.\quad ....\lambda n}{(f+g)(f+2g)...........(f+\lambda ng)}=\frac{f\cdot\lambda ng}{g^{\lambda n}(f+\lambda ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{\lambda n-1},$$

$$\frac{1.\quad 2.\quad 3.\quad ....(\lambda+1)n}{(f+g)(f+2g)...........(f+(\lambda+1)ng)}=\frac{f\cdot(\lambda+1)g}{g^{(\lambda+1)n}(f+(\lambda+1)ng)}\int x^{f-1}\partial x\left(1-x^{g}\right)^{(\lambda+1)n-1},$$

quarum illa per hanc divisa praebet

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)...........(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)...........(\lambda n+n)}=g^{n}\,\frac{\lambda\cdot(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)}\cdot\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{\lambda n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{(\lambda+1)n-1}},$$

At si in aequatione prima loco *f* scribamus $f+\lambda ng$, obtinebus

$$\frac{1.\quad 2.\quad 3.\quad ....n}{(f+\lambda ng+g)(f+\lambda ng+2g)...........(f+\lambda ng+ng)}=\frac{(f+\lambda ng)\cdot ng}{g^{n}(f+\lambda ng+ng)}\cdot\int x^{f+\lambda ng-1}\partial x\left(1-x^{g}\right)^{n-1},$$

quae duae aequationes in se ductae producunt ipsam aequalitatem demonstrandam

$$\frac{1.\quad 2.\quad 3............n}{(\lambda n+1)(\lambda n+2)....(\lambda n+n)}=\frac{\lambda ng}{\lambda+1}\int x^{f+\lambda ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{\lambda n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{(\lambda+1)n-1}}.$$

Corollarium1.

§. 23. Si in aequatione principali statuamus $f=\lambda n$ et $g=1$ reperiemus etiam

$$\frac{1.\quad 2.\quad 3............n}{(\lambda n+1)(\lambda n+2)....(\lambda n+n)}=\frac{\lambda n}{\lambda+1}\int x^{\lambda n-1}\partial x(1-x)^{n-1},$$

quae forma loco *x* scribendo x^{k} abit in hanc

$$\frac{\lambda nk}{\lambda+1}\int x^{\lambda nk-1}\partial x\left(1-x^{k}\right)^{n-1};$$

ita ut habeamus hoc theorema latissime patens

$$g\int x^{f+\lambda ng-1}\partial x\left(1-x^{g}\right)^{n-1}\times\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{\lambda n-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{(\lambda+1)n-1}}=k\int x^{\lambda nk-1}\partial x\left(1-x^{k}\right)^{n-1}.$$

Corollarium 2.

§. 24. Hoc jam theorema locum habet, etiamsi n non fit numerus integer; quin etiam cum numerum λ pro lubitu accipere liceat, loco λn scribamus m, et perveniemus ad hoc theorema

$$\frac{\int x^{f-1}\partial x\left(1-x^{g}\right)^{m-1}}{\int x^{f-1}\partial x\left(1-x^{g}\right)^{m+n-1}}=\frac{k\int x^{mk-1}\partial x\left(1-x^{k}\right)^{n-1}}{g\int x^{f+mg-1}\partial x\left(1-x^{g}\right)^{n-1}}$$

Corollarium 3.

§. 25. Si ponamus $g=0$, ob $1-x^{g}=gl\frac{1}{x}$, hoc theorema istam induet formam

$$\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{m-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{m+n-1}}=\frac{k\int x^{mk-1}\partial x\left(1-x^{k}\right)^{n-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}},$$

quae commodius ita repraesentatur

$$\frac{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{n-1}\times\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{m-1}}{\int x^{f-1}\partial x\left(l\frac{1}{x}\right)^{m+n-1}}=k\int x^{mk-1}\partial x\left(1-x^{k}\right)^{n-1},$$

ubi evidens est numeros m et n inter se permutari posse.

Scholion.

§. 26. Duplicem ergo deteximus fontem, unde innumerabiles formularum integralium comparationes haurire licet; alter fons §. 24, patefactus complectitur hujusmodi formulas integrales

$$\int x^{p-1}\partial x\left(1-x^{g}\right)^{q-1},$$

quas jam ante aliquod tempus pertractavi in observationibus circa integralia formularum (Miscellanea Tuarinensia. Tom. III.)

$$\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q}{n}-1}$$

a valere $x=0$ usque ad $x=1$ extensa, ubi ostendi primo litteras p et q inter se permutari posse, ut sit

$$\int x^{p-1}\partial x\left(1-x^{n}\right)^{\frac{q}{n}-1}=\int x^{q-1}\partial x\left(1-x^{n}\right)^{\frac{p}{n}-1},$$

tum vero etiam esse

$$\int \frac{x^{p-1}\partial x}{\left(1-x^n\right)^{\frac{p}{m}}} = \frac{\pi}{n \sin . \frac{p\pi}{n}} :$$

imprimis autem demonstravi esse

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} \times \int \frac{x^{p+q-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-r}}} = \int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-r}}} \times \int \frac{x^{p+r-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}},$$

in qua aequatione comparatio in §. 24. inventa jam continetur; ita ut hinc nihil novi, quod non jam evolvi, deduci queat. Alterum igitur fontem §. 25. indicatum hic potissimum investigandum suscipio, ubi cum sive ulla restrictione sumi queat $f=1$, aequatio nostra primaria erit

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{n-1} \times \int \partial x\left(l\frac{1}{x}\right)^{m-1}}{\int \partial x\left(l\frac{1}{x}\right)^{m+n-1}} = k\int x^{mk-1}\partial x\left(1-x^k\right)^{n-1},$$

cujus beneficio valores formulae integralis $\int \partial x\left(l\frac{1}{x}\right)^{\lambda}$, quando λ non est numerus integer, ad quadraturas curvarum algebraicarum revocare licebit; quandoquidem quoties λ est numerus integer, integratio habetur absoluta quoniam est

$$\int \partial x\left(l\frac{1}{x}\right)^{\lambda} = 1.2.3.......................\lambda.$$

Maximi autem momenti quaestio versatur circa eos casus, quibus λ est numerus fractus, quos ergo pro ratione denominationis hic successive sum definiturus.

Problema 2.

§. 27. *Denotante i numerian integrum positivum, definire valorem formulae integralis* $\int \partial x\left(l\frac{1}{x}\right)^{\frac{i}{2}}$, *integratione ab* $x=0$ *usque ad* $x=1$ *extensa.*

Solutio.

In aequatione nostra generali faciamus $m=n$, eritque

$$\frac{\left[\int \partial x\left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int \partial x\left(l\frac{1}{x}\right)^{2n-1}} = k\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1}.$$

Sit jam $n-1=\frac{1}{2}$, et ob $2n-1=i+1$, erit

$$\int \partial x\left(l\tfrac{1}{x}\right)^{2n-1} = 1.2.3.......................(i+1):$$

sumatur porro k =2, ut sit $nk-1=i+1$, fietque

$$\frac{\left[\int \partial x\sqrt{\left(l\frac{1}{x}\right)^{i}}\right]^{2}}{1.2.3.......................(i+1)} = 2\int x^{i+1}\partial x\left(1-x^{2}\right)^{\frac{i}{2}},$$

ideoque

$$\frac{\int \partial x\sqrt{\left(l\frac{1}{x}\right)^{i}}}{\sqrt{\left[1.2.3.......................(i+1)\right]}} = \sqrt{2}\int x^{i+1}\partial x\sqrt{\left(1-x^{2}\right)^{i}},$$

ubi evidens est, pro i numeros tantum impares sumi convenire; quoniam pro paribus evoluto per se est manifesta.

Corollarium 1.

§. 28. Omnes autem casus facile reducuntur ad $i=1$, vel adeo ad $i=-1$; dummodo enim $i+1$ non sit numerus negativus, reductio inventa locum habet. Pro hoc ergo casu erit

$$\int \frac{\partial x}{\sqrt{\left(l\frac{1}{x}\right)}} = \sqrt{2}\int \frac{\partial x}{\sqrt{(1-xx)}} = \sqrt{\pi}, \text{ ob } \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}.$$

Corollarium 2.

§. 29. Hoc autem casu principali expedita, ob

$$\int \partial x\left(\tfrac{1}{x}\right)^{n} = n\int \partial x\left(l\tfrac{1}{x}\right)^{n-1},$$

habebimus

$$\int \partial x\sqrt{l\tfrac{1}{x}} = \tfrac{1}{2}\sqrt{\pi}; \int \left(l\tfrac{1}{x}\right)^{\frac{3}{2}} = \tfrac{1\cdot 3}{2\cdot 2}\sqrt{\pi},$$

atque in genere.

$$\int \partial x\left(l\tfrac{1}{x}\right)^{\frac{2n+1}{2}} = \tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\tfrac{7}{2}\cdots\cdots\tfrac{(2n+1)}{2}\sqrt{\pi}.$$

Problema 3.

§. 30. *Denotante i numerum integrum positivum, definire valorem formulae integralis* $\int \partial x\left(l\frac{1}{x}\right)^{\frac{i}{3}-1}$, *integratione ab* $x=0$ *ad* $x=1$ *extensa.*

Solutio.

Inchoemus ab aequatione praecedentis problematis

$$\frac{\left[\int \partial x\left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int \partial x\left(l\frac{1}{x}\right)^{2n-1}} = k\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1},$$

atque in forma generali statuamus $m=2n$, ut habeatur

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{n-1}\times\int \partial x\left(l\frac{1}{x}\right)^{3n-1}}{\int \partial x\left(l\frac{1}{x}\right)^{3n-1}} = k\int x^{2nk-1}\partial x\left(1-x^k\right)^{n-1},$$

ac multiplicando has duas aequalitates adipiscimur

$$\frac{\left|\int \partial x\left(l\frac{1}{x}\right)^{n-1}\right|^3}{\int \partial x\left(l\frac{1}{x}\right)^{3n-1}} = kk\int x^{nk-1}\partial x\left(1-x^k\right)^{n-1}\times\int x^{2nk-1}\partial x\left(1-x^k\right)^{n-1}.$$

Hic jam ponatur $n=\frac{i}{3}$ ut sit

$$\int \partial x\left(l\frac{1}{x}\right)^{i-1} = 1\cdot 2\cdot 3\cdots\cdots(i-1),$$

sumaturque $k=3$, ac prodibit

$$\frac{\left[\int \partial x\sqrt[3]{\left(l\frac{1}{x}\right)^{i-3}}\right]^3}{1\cdot 2\cdot 3\cdots\cdots(i-1)} = 9\int x^{i-1}\partial x\sqrt[3]{\left(1-x^3\right)^{i-3}}\times\int x^{2i-1}\partial x\sqrt[3]{\left(1-x^3\right)^{i-3}};$$

unde concludimus

$$\frac{\int \partial x\sqrt[3]{\left(l\frac{1}{x}\right)^{i-3}}}{\sqrt[3]{1\cdot 2\cdot 3\cdots\cdots(i-1)}} = \sqrt[3]{9}\int\frac{x^{i-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-i}}}\times\int\frac{x^{2i-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-i}}}.$$

Corollarium 1.

§. 31. Bini hic occurrunt casus principales, a quibus reliqui omnes pendent; ponendo scilicet vel $i=1$ vel $i=2$, qui sunt

$$\text{I.}\int\frac{\partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)^2}} = \sqrt[3]{9}\int\frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}\times\int\frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)^2}}$$

$$\text{II.}\int\frac{\partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)}} = \sqrt[3]{9}\int\frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)}}\times\int\frac{x^3\partial x}{\sqrt[3]{\left(1-x^3\right)}},$$

quae posterior forma ob

$$\int\frac{x^3\partial x}{\sqrt[3]{\left(1-x^3\right)}} = \frac{1}{3}\int\frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}}$$

abit in

$$\frac{\int \partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)}} = \sqrt[3]{3}\int \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)}} \times \int \frac{x\partial x}{\sqrt[3]{\left(1-x^3\right)}}.$$

Corollarium 2.

§. 32. Si uti in observationibus meis ante allegatis brevitatis gratia ponamus

$$\int \frac{x^{p-1}\partial x}{\sqrt[3]{\left(1-x^3\right)^{3-q}}} = \left(\frac{p}{q}\right),$$

atque ut ibi pro hac classe

$$\left(\frac{2}{1}\right) = \frac{\pi}{3\sin.\frac{\pi}{2}} = \alpha,$$

tum vero

$$\left(\frac{1}{1}\right) = \frac{\partial x}{\sqrt[3]{\left(1-x^3\right)^2}} = \mathrm{A},\ \text{erit}$$

$$\text{I.}\int \frac{\partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)^2}} = \sqrt[3]{9}\left(\frac{1}{1}\right)\left(\frac{2}{1}\right) = \sqrt[3]{9}\alpha\mathrm{A},$$

$$\text{II.}\int \frac{\partial x}{\sqrt[3]{\left(l\frac{1}{x}\right)^1}} = \sqrt[3]{3}\left(\frac{1}{2}\right)\left(\frac{2}{2}\right) = \sqrt[3]{\frac{3\alpha\alpha}{\mathrm{A}}}.$$

Corollarium 3.

§. 33. Pro casu ergo priori habebimus

$$\int \partial x\sqrt[3]{\left(l\frac{1}{x}\right)^{-2}} = \sqrt[3]{9\alpha\mathrm{A}},\ \int \partial x\sqrt[3]{\left(l\frac{1}{x}\right)} = \frac{1}{3}\sqrt[3]{9\alpha\mathrm{A}},\ \text{et}$$

$$\int \partial x\sqrt[3]{\left(l\frac{1}{x}\right)^{3n+1}} = \frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdots\cdots\frac{3n+1}{3}\sqrt[3]{9\alpha\mathrm{A}}\ :$$

pro altero vero casu

$$\int \partial x\sqrt[3]{\left(l\frac{1}{x}\right)^{-1}} = \sqrt[3]{\frac{3\alpha\alpha}{\mathrm{A}}},\ \int \partial x\sqrt[3]{\left(l\frac{1}{x}\right)^{2}} = \frac{2}{3}\sqrt[3]{\frac{3\alpha\alpha}{\mathrm{A}}},\ \text{et}$$

$$\int \partial x\sqrt[3]{\left(l\frac{1}{x}\right)^{3n-1}} = \frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdots\cdots\frac{3n-1}{3}\cdot\sqrt[3]{\frac{3\alpha\alpha}{\mathrm{A}}}.$$

Problema 4.

§. 34. De*notante i numerum integrum positivum, definire valorem formulae integralis* $\int \partial x\left(l\frac{1}{x}\right)^{\frac{i}{4}-1}$, *integratione ab* $x=0$ *ad* $x=1$ *extensa.*

Solutio.

In solutione problematis praecedentis perducti sumus ad hanc aequationem

$$\frac{\left|\int \partial x\left(l\frac{1}{x}\right)^{n-1}\right|^3}{\int \partial x\left(l\frac{1}{x}\right)^{3n-1}} = kk\int \frac{x^{nk-1}\partial x}{\left(1-x^k\right)^{1-n}} \times \int \frac{x^{2nk-1}\partial x}{\left(1-x^k\right)^{1-n}};$$

forma generalis autem sumendo $m = 3n$ praebet

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{n-1} \times \int \partial x\left(l\frac{1}{x}\right)^{3n-1}}{\int \partial x\left(l\frac{1}{x}\right)^{4n-1}} = k\int \frac{x^{3nk-1}\partial x}{\left(1-x^k\right)^{1-n}},$$

quibus conjungendis adipiscimur

$$\frac{\left|\int \partial x\left(l\frac{1}{x}\right)^{n-1}\right|^4}{\int \partial x\left(l\frac{1}{x}\right)^{4n-1}} = k^3\int \frac{x^{nk-1}\partial x}{\left(1-x^k\right)^{1-n}} \times \int \frac{x^{2nk-1}\partial x}{\left(1-x^k\right)^{1-n}} \times \int \frac{x^{3nk-1}\partial x}{\left(1-x^k\right)^{1-n}};$$

Sit nunc $n = \frac{i}{4}$, et sumatur $k = 4$, fietque

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{\frac{i}{4}-1}}{\sqrt[4]{1.2.3....(i-1)}} = \sqrt[4]{}\, 4^3\int \frac{x^{i-1}\partial x}{\sqrt[4]{\left(1-x^4\right)^{4-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[4]{\left(1-x^4\right)^{4-i}}} \times \int \frac{x^{3i-1}\partial x}{\sqrt[4]{\left(1-x^4\right)^{4-i}}}.$$

Corollarium 1.

§. 35. Si igitur sit $i = 1$, habebimus

$$\int \partial x \sqrt[4]{}\left(l\frac{1}{x}\right)^{-3} = \sqrt[4]{}\, 4^3\int \frac{\partial x}{\sqrt[4]{\left(1-x^4\right)^3}} \times \int \frac{x\partial x}{\sqrt[4]{\left(1-x^4\right)^3}} \times \int \frac{xx\partial x}{\sqrt[4]{\left(1-x^4\right)^3}}.$$

quae expressio si littera P designetur, erit in genere

$$\int \partial x\sqrt[4]{\left(l\frac{1}{x}\right)^{4n-3}} = \frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\cdots\cdots\frac{4n-3}{4}\cdot \mathrm{P}$$

Corollarium 2.

§. 36. Pro altera casu principali sumamus $i = 3$, eritque

$$\int \partial x \sqrt[4]{\left(l\tfrac{1}{x}\right)^{-1}} = \sqrt[4]{2.4^3} \int \frac{x^2\partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x^5\partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x^8\partial x}{\sqrt[4]{(1-x^4)}},$$

seu facta reductione ad simpliciores formas

$$\int \partial x \sqrt[4]{\left(l\tfrac{1}{x}\right)^{-1}} = \sqrt[4]{8} \int \frac{xx\partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x\partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{\partial x}{\sqrt[4]{(1-x^4)}},$$

quae expressio si littera Q designetur, erit generatim

$$\int \partial x \sqrt[4]{\left(l\tfrac{1}{x}\right)^{4n-1}} = \tfrac{3}{4} \cdot \tfrac{7}{4} \cdot \tfrac{11}{4} \cdots\cdots \tfrac{4n-1}{4} \cdot \mathrm{Q}.$$

Scholion.

§. 37. Si formulam integralem

$$\int \frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^4)^{4-q}}}$$

hoc signo $\left(\frac{p}{q}\right)$ indicemus, solutio problematis ita se habebit

$$\int \partial x \sqrt[4]{\left(l\tfrac{1}{x}\right)^{i-4}} = \sqrt[4]{1 \cdot 2 \cdot 3 \cdots\cdots (i-1) \cdot 4^3} \left(\tfrac{i}{i}\right)\left(\tfrac{2i}{i}\right)\left(\tfrac{3i}{i}\right),$$

et pro binis casibus evolutis fit

$$\mathrm{P} = \sqrt[4]{4^3} \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right) \text{ et } \mathrm{Q} = \sqrt[4]{8} \left(\tfrac{3}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right).$$

Statuamus nunc pro iis formulis quae a circulo pendent:

$$\left(\tfrac{3}{1}\right) = \frac{\pi}{4\sin.\frac{\pi}{4}} = \alpha \text{ et } \left(\tfrac{2}{2}\right) = \frac{\pi}{4\sin.\frac{2\pi}{4}} = \beta,$$

pro transcendentibus autem altioris ordinis

$$\left(\tfrac{2}{1}\right) = \int \frac{x\partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = \mathrm{A},$$

quippe a qua omnes reliquae pendent ac reperiemus

$$\mathrm{P} = \sqrt[4]{4^3} \cdot \frac{\alpha\alpha}{\beta} \cdot \mathrm{AA} \text{ et } \mathrm{Q} = \sqrt[4]{4} \cdot \alpha\alpha\beta \cdot \frac{1}{\mathrm{AA}};$$

unde pate esse

$$\mathrm{PQ} = 4\alpha = \frac{\pi}{\sin.\frac{\pi}{4}}.$$

Cum autem sit

$\alpha = \frac{\pi}{2\sqrt{2}}$ et $\gamma = \frac{\pi}{4}$, erit

$$\mathrm{P} = \sqrt[4]{32\pi\mathrm{AA}},\ \mathrm{Q} = \sqrt[4]{\frac{\pi^3}{8\mathrm{AA}}} \text{ et } \frac{\mathrm{P}}{\mathrm{Q}} = \frac{4\mathrm{A}}{\sqrt{\pi}}.$$

Problemata 5.

§. 38. *Denotante i numerum integrum positivum, definïre valorem formulae integralis* $\int \partial x \sqrt[4]{\left(l\frac{1}{x}\right)^{i-5}}$, *integratione ab* $x=0$ *ad* $x=1$ *extensa.*

Solutio.

Ex praecentibus solutionis jam satis est perspicuum pro hoc casu perventum iri ad hanc formam

$$\frac{\int \partial x \sqrt[5]{\left(l\frac{1}{x}\right)^{i-5}}}{\sqrt[5]{1.2.3....(i-1)}} = \sqrt[5]{5^4}\int \frac{x^{i-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-i}}} \times \int \frac{x^{3i-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-i}}} \times \int \frac{x^{4i-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-i}}},$$

quae formulae integrales ad classem quintam dissertationis meae supra allegatae sunt referendae. Quare si modo ibi recepto signum $\left(\frac{p}{q}\right)$ denotet hanc formulam

$$\int \frac{x^{p-1}\partial x}{\sqrt[5]{\left(1-x^5\right)^{5-q}}},$$

valorem quaesitum ita commodius exprimere licebit, ut sit

$$\int \partial x \sqrt[5]{\left(l\frac{1}{x}\right)^{i-5}} = \sqrt[5]{1.2.3....(i-1)5^4}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\left(\frac{4i}{i}\right),$$

ubi quidem sufficit ipsi *i* valores quinario minores, tribuisse, quando autem numeratores quinarium superant, tenendum est esse

$$\left(\frac{5+m}{i}\right) = \frac{m}{m+i}\left(\frac{m}{i}\right);$$

tum vero porro

$$\left(\frac{10+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\left(\frac{m}{i}\right)$$
$$\left(\frac{15+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\cdot\frac{m+10}{m+i+10}\left(\frac{m}{i}\right).$$

Deinde vero pro hac classe binae formulae quadraturam circuli involvunt, quae sint

$$\left(\frac{4}{1}\right) = \frac{\pi}{5\sin.\frac{\pi}{5}} = \alpha \text{ et } \left(\frac{3}{2}\right) = \frac{\pi}{5\sin.\frac{2\pi}{5}} = \beta,$$

duae autem quadraturas altiores continent, quae ponantur

$$\left(\tfrac{3}{1}\right)=\int\frac{xx\partial x}{\sqrt[5]{\left(1-x^5\right)^4}}=\int\frac{\partial x}{\sqrt[5]{\left(1-x^5\right)^2}}=\mathrm{A}\ \text{et}$$

$$\left(\tfrac{2}{2}\right)=\int\frac{x\partial x}{\sqrt[5]{\left(1-x^5\right)^3}}=\mathrm{B};$$

atque ex his valores omnium reliquarum formularum hujus classis assignavi, scilicet :

$$\left(\tfrac{2}{2}\right)=1;\left(\tfrac{5}{2}\right)=\tfrac{1}{2};\left(\tfrac{5}{4}\right)=\tfrac{1}{4};\left(\tfrac{5}{5}\right)=\tfrac{1}{5}$$

$$\left(\tfrac{4}{1}\right)=\alpha;\left(\tfrac{4}{2}\right)=\tfrac{\beta}{\mathrm{A}};\left(\tfrac{4}{3}\right)=\tfrac{\beta}{2\mathrm{B}};\left(\tfrac{4}{4}\right)=\tfrac{\alpha}{3\mathrm{A}}$$

$$\left(\tfrac{3}{1}\right)=\mathrm{A};\left(\tfrac{3}{2}\right)=\beta;\left(\tfrac{3}{3}\right)=\tfrac{\beta\beta}{\alpha\mathrm{B}}$$

$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha\mathrm{B}}{\beta};\left(\tfrac{2}{2}\right)=\mathrm{B}$$

$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha\mathrm{A}}{\beta}.$$

Corollarium 1.

§. 39. Sumto exponente $i=1$, erit

$$\int\partial x\sqrt[5]{\left(l\tfrac{1}{x}\right)^{-4}}=\sqrt[5]{5^4}\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)=\sqrt[5]{}\,5^4\cdot\tfrac{\alpha^3}{\beta^2}\cdot\mathrm{A}^2\mathrm{B};$$

unde in genere concludimus fore, denotante *n* numerum integrum quemcunque

$$\int\partial x\sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-4}}=\tfrac{1}{5}\cdot\tfrac{6}{5}\cdot\tfrac{11}{5}\cdots\cdots\tfrac{5n-4}{5}\sqrt[5]{}\,5^4\cdot\tfrac{\alpha^3}{\beta^2}\cdot\mathrm{A}^2\mathrm{B}.$$

Corollarium 2.

§. 40. Sit nunc $i=2$, et cum prodeat

$$\int\partial x\sqrt[5]{\left(l\tfrac{1}{x}\right)^{-3}}=\sqrt[5]{}\,1.5^4\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{8}{2}\right),\ \text{ob}$$

$$\left(\tfrac{6}{2}\right)=\tfrac{1}{3}\left(\tfrac{1}{2}\right)=\tfrac{1}{3}\left(\tfrac{2}{1}\right)\ \text{et}\ \left(\tfrac{8}{2}\right)=\tfrac{3}{5}\left(\tfrac{3}{2}\right),$$

erit haec expressio

$$\sqrt[5]{}\,5^3\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{2}\right)\sqrt[5]{}\,5^3\cdot\alpha\beta\cdot\tfrac{\mathrm{BB}}{\mathrm{A}}$$

et in genere

$$\int\partial x\sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-3}}=\tfrac{2}{5}\cdot\tfrac{7}{5}\cdot\tfrac{12}{5}\cdots\cdots\tfrac{5n-3}{5}\sqrt[5]{}\,5^3\cdot\alpha\beta\cdot\tfrac{\mathrm{BB}}{\mathrm{A}}.$$

Corollarium 3.

§. 41. Sit $i = 3$, et forma inventa

$$\int \partial x \sqrt[5]{\left(l\tfrac{1}{x}\right)^{-2}} = \sqrt[5]{\ } 2.5^4 \left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{9}{3}\right)\left(\tfrac{12}{3}\right), \text{ob}$$

$$\left(\tfrac{6}{3}\right) = \tfrac{1}{4}\left(\tfrac{3}{1}\right);\ \left(\tfrac{9}{3}\right) = \tfrac{4}{7}\left(\tfrac{4}{5}\right);\ \left(\tfrac{12}{3}\right) = \tfrac{2}{5}\cdot\tfrac{7}{10}\left(\tfrac{3}{2}\right), \text{ abit in}$$

$$\sqrt[5]{\ } 2.5^2 \left(\tfrac{3}{3}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{3}{2}\right) = \sqrt[5]{\ } 5^2\, \tfrac{\beta^4}{\alpha}\cdot\tfrac{\mathrm{A}}{\mathrm{BB}};$$

unde in genere colligitur

$$\int \partial x \sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-2}} = \tfrac{3}{5}\cdot\tfrac{8}{5}\cdot\tfrac{13}{5}\cdots\cdots\tfrac{5n-2}{5}\sqrt[5]{\ } 5^2\cdot\tfrac{\beta^4}{\alpha}\cdot\tfrac{\mathrm{A}}{\mathrm{BB}}.$$

Corollarium 4.

§. 42. Posito denique $i = 4$, forma nostra

$$\int \partial x \sqrt[5]{\left(l\tfrac{1}{x}\right)^{-1}} = \sqrt[5]{\ } 6.5^4 \left(\tfrac{4}{4}\right)\left(\tfrac{8}{4}\right)\left(\tfrac{12}{4}\right)\left(\tfrac{16}{4}\right), \text{ob}$$

$$\left(\tfrac{8}{4}\right) = \tfrac{3}{7}\left(\tfrac{4}{1}\right);\ \left(\tfrac{12}{4}\right) = \tfrac{2}{6}\cdot\tfrac{7}{11}\left(\tfrac{4}{2}\right);\ \left(\tfrac{16}{4}\right) = \tfrac{1}{5}\cdot\tfrac{6}{10}\cdot\tfrac{11}{15}\left(\tfrac{4}{1}\right),$$

transformabitur in hanc

$$\sqrt[5]{\ } 6.5^4 \left(\tfrac{4}{4}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{4}{1}\right) = \sqrt[5]{\ } \tfrac{\alpha\alpha\beta\beta}{\mathrm{AAB}};$$

ita ut sit in genere

$$\int \partial x \sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-1}} = \tfrac{4}{5}\cdot\tfrac{9}{5}\cdot\tfrac{14}{5}\cdots\cdots\tfrac{5n-1}{5}\sqrt[5]{\ } 5.\alpha\alpha\beta\beta\cdot\tfrac{1}{\mathrm{AAB}}.$$

Scholion.

§. 43. Si valorem formulae integralis $\int \partial x \left(l\tfrac{1}{x}\right)^{\lambda}$ hoc signo [λ] repraesentemus, casus hactenus evoluti praebent

$$\left[-\tfrac{4}{5}\right] = \sqrt[3]{\ } 5^4.\tfrac{\alpha^3}{\beta^2}.\mathrm{A}^2\mathrm{B}; \qquad \left[+\tfrac{1}{5}\right] = \tfrac{1}{5}\sqrt[3]{\ } 5^4.\tfrac{\alpha^3}{\beta^2}.\mathrm{A}^2\mathrm{B}$$

$$\left[-\tfrac{3}{5}\right] = \sqrt[5]{\ } 5^3.\alpha\beta.\tfrac{\mathrm{BB}}{\mathrm{A}}; \qquad \left[+\tfrac{2}{5}\right] = \tfrac{1}{5}\sqrt[3]{\ } 5^3.\alpha\beta.\tfrac{\mathrm{BB}}{\mathrm{A}}$$

$$\left[-\tfrac{2}{5}\right] = \sqrt[5]{\ } 5^2.\tfrac{\beta^4}{\alpha}\cdot\tfrac{\mathrm{A}}{\mathrm{BB}}; \qquad \left[+\tfrac{3}{5}\right] = \tfrac{3}{5}\sqrt[3]{\ } 5^2.\tfrac{\beta}{\alpha}\cdot\tfrac{\mathrm{A}}{\mathrm{BB}}$$

$$\left[-\tfrac{1}{5}\right] = \sqrt[5]{\ } 5.\alpha^2\beta^2.\tfrac{1}{\mathrm{AAB}}; \qquad \left[+\tfrac{4}{5}\right] = \tfrac{4}{5}\sqrt[3]{\ } 5.\alpha^2\beta^2.\tfrac{1}{\mathrm{BBA}};$$

unde binis, quarum indices simul sumti fiunt $= 0$, conjungendis colligimus

$$\left[+\tfrac{1}{5}\right]\cdot\left[-\tfrac{1}{5}\right]=\alpha=\frac{\pi}{5\sin.\frac{\pi}{5}}$$

$$\left[+\tfrac{2}{5}\right]\cdot\left[-\tfrac{2}{5}\right]=2\beta=\frac{2\pi}{5\sin.\frac{2\pi}{5}}$$

$$\left[+\tfrac{3}{5}\right]\cdot\left[-\tfrac{3}{5}\right]=3\beta=\frac{3\pi}{5\sin.\frac{3\pi}{5}}$$

$$\left[+\tfrac{4}{5}\right]\cdot\left[-\tfrac{4}{5}\right]=4\alpha=\frac{2\pi}{5\sin.\frac{4\pi}{5}}$$

Ex antecedente autem problemate simili modo deducimus

$$\left[-\tfrac{3}{4}\right]=\mathrm{P}=\sqrt[4]{4^3.\tfrac{\alpha\alpha}{\beta}.\mathrm{AA}};\qquad \left[+\tfrac{1}{4}\right]=\tfrac{1}{4}\sqrt[4]{4^3.\tfrac{\alpha\alpha}{\beta}.\mathrm{AA}}$$

$$\left[-\tfrac{1}{4}\right]=\mathrm{Q}=\sqrt[4]{4.\alpha\alpha\beta.\tfrac{1}{\mathrm{AA}}};\qquad \left[+\tfrac{3}{4}\right]=\tfrac{3}{4}\sqrt[4]{4.\alpha\alpha\beta.\tfrac{1}{\mathrm{AA}}}$$

hincque

$$\left[+\tfrac{1}{4}\right]\cdot\left[-\tfrac{1}{4}\right]=\alpha=\frac{\pi}{4\sin.\frac{\pi}{4}}$$

$$\left[+\tfrac{3}{4}\right]\cdot\left[-\tfrac{3}{4}\right]=3\alpha=\frac{3\pi}{4\sin.\frac{3\pi}{4}}$$

unde in genere hoc Theorema adipiscimur, quod sit

$$[+\lambda]\cdot[-\lambda]=\frac{\lambda\pi}{\sin.\lambda\pi},$$

cujus ratio ex methodo interpolandi olim exposita ita reddi potest cum sit

$$[\lambda]=\frac{1^{1-\lambda}.2^{\lambda}}{1+\lambda}\cdot\frac{2^{1-\lambda}.3^{\lambda}}{2+\lambda}\cdot\frac{3^{1-\lambda}.4^{\lambda}}{3+\lambda}\cdot\text{etc.}$$

erit

$$[-\lambda]=\frac{1^{1+\lambda}.2^{-\lambda}}{1-\lambda}\cdot\frac{2^{1+\lambda}.3^{-\lambda}}{2-\lambda}\cdot\frac{3^{1+\lambda}.4^{-\lambda}}{3-\lambda}\cdot\text{etc.}$$

hincque ,

$$[\lambda].[-\lambda]=\frac{1.1}{1-\lambda\lambda}\cdot\frac{2.2}{4-\lambda\lambda}\cdot\frac{3.3}{9-\lambda\lambda}\cdot\text{etc.}=\frac{\lambda\pi}{\sin.\lambda\pi};$$

uti alibi demonstravi.

Problema 6 generale.

§. 44. *Si litterae i et n denotent numeros integras positivos, definire valorem formulae integralis*

$$\int\partial x\left(l\tfrac{1}{x}\right)^{\frac{i-n}{n}},\ seu\ \int\partial x\sqrt[n]{\left(l\tfrac{1}{x}\right)^{i-n}},$$

integratione $x=0$ *ad* $x=1$ *ab extensa.*

Solutio.

Methodus hactenus usitata quaesitum valorem sequenti modo per quadraturas curvarum algebraicarum expressum exhibebit

$$\frac{\int \partial x \sqrt[n]{\left(l\frac{1}{x}\right)^{i-n}}}{\sqrt[n]{1\cdot 2\cdot 3\cdots\cdots(i-1)}} = \sqrt[n]{} n^{n-1}\int \frac{x^{i-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-i}}} \times \ldots\ldots \int \frac{x^{(n-1)i-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-i}}}.$$

Quod si jam brevitatis gratia formulam integralem

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} \text{ hoc charactere } \left(\frac{p}{q}\right),$$

formulam vero $\int \partial x \sqrt[n]{\left(l\frac{1}{x}\right)^{m}}$ isthoc $\left[\frac{m}{n}\right]$ designemus, ita ut $\left[\frac{m}{n}\right]$ valorem hujus producti indefiniti $1.\ 2.\ \ldots.\ z$ denotet, existente $z=\frac{m}{n}$, succinctius valor quaesitus hoc modo expressus prodibit

$$\left[\frac{i-n}{n}\right] = \sqrt[n]{} 1\cdot 2\cdot 3\cdots\cdots(i-1)\, n^{n-1}.\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\cdots\left(\frac{ni-i}{i}\right);$$

unde etiam colligitur

$$\left[\frac{i}{n}\right] = \frac{i}{n}\sqrt[n]{} 1\cdot 2\cdot 3\cdots\cdots(i-1)\, n^{n-1}.\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\cdots\left(\frac{ni-i}{i}\right).$$

Hic semper numerum i ipso n minorem accepisse sufficiet; quoniam pro majoribus notum est esse,

$$\left[\frac{i+n}{n}\right] = \frac{i+n}{n}\left[\frac{i}{n}\right], \text{item } \left[\frac{i+2n}{n}\right] = \frac{i+n}{n}\cdot\frac{i+2n}{n}\left[\frac{i}{n}\right] \text{etc.}$$

hocque modo tota investigatio ad eos tantum casus reducitur, quibus fractionis $\frac{i}{n}$ numerator i denominatore n est minor. Praeterea vero de formulis integralibus

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} = \left(\frac{q}{p}\right),$$

sequentia notasse juvabit

I. Litteras p et q inter se esse permutabiles ut sit

$$\left(\frac{p}{p}\right) = \left(\frac{q}{p}\right).$$

II. Si alteruter numerorum p vel q ipsi exponenti n aequetur, valorem formulae integralis fore algebraicum, scilicet

$$\left(\frac{n}{p}\right) = \left(\frac{p}{n}\right) = \frac{1}{p}, \text{ seu } \left(\frac{n}{q}\right) = \left(\frac{q}{n}\right) = \frac{1}{q}.$$

III. Si summa numerorum $p+q$ ipsi exponenti *n* aequatur, formulae integralis $\left(\frac{p}{q}\right)$ valorem per circulum exhiberi posse, cum sit

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}},\ \text{et}\ \left(\frac{q}{n-q}\right)=\left(\frac{n-q}{q}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

IV. Si alteruter numerorum *p* vel *q* major sit exponente *n,* formulam integralem $\left(\frac{p}{q}\right)$ ad aliam revocari posse, cujus termini sint ipso *n* minores, quod sit ope hujus reductionis

$$\left(\frac{p+n}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

V. Inter plures hujusmodi formulas integrales talem relationem intercedere, ut sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

cujus ope omnes reductiones reperiuntur, quas in observationibus circa has formulas exposui.

Corollarium 1.

§. 45. Si hoc modo ope reductionis N°. IV. indîcatae formam inventam ad singulos casus accommodemus, eos sequenti ratione simplicissime exhibere poterimus. Ac primo quidem pro casu $n=2$, quo nulla opus est reductione habebimus

$$\left[\frac{1}{2}\right]=\frac{1}{2}\sqrt[2]{2\left(\frac{1}{1}\right)}=\frac{1}{2}\sqrt[2]{\frac{\pi}{\sin.\frac{\pi}{2}}}=\frac{1}{2}\sqrt{\pi}.$$

Corollarium 2.

§. 46. Pro casu $n=3$ habebimus has reductiones

$$\left[\frac{1}{3}\right]=\frac{1}{3}\sqrt[3]{3^2.\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)}$$
$$\left[\frac{2}{3}\right]=\frac{2}{3}\sqrt[3]{3.1.\left(\frac{2}{2}\right)\left(\frac{1}{2}\right)}.$$

Corollarium 3.

§. 47. Pro casu $n=4$ hae tres reductiones obtinentur

$$\left[\tfrac{3}{4}\right]=\tfrac{1}{4}\sqrt[4]{4^3\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)}$$
$$\left[\tfrac{2}{4}\right]=\tfrac{2}{4}\sqrt[4]{4^2.2.\left(\tfrac{4}{2}\right)}=\tfrac{1}{2}\sqrt[2]{4\left(\tfrac{2}{2}\right)},\text{ob }\left(\tfrac{4}{2}\right)=\tfrac{1}{2}$$
$$\left[\tfrac{3}{4}\right]=\tfrac{3}{4}\sqrt[4]{4.1.2.\left(\tfrac{3}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)};$$

cum in media, sit $\left[\tfrac{2}{4}\right]=\left[\tfrac{1}{2}\right]=\tfrac{1}{2}\sqrt{\pi}$, erit utique ut ante

$$\left(\tfrac{2}{4}\right)=\left(\tfrac{1}{2}\right)=\tfrac{1}{2}\sqrt{\pi}.$$

Corollarium 4.

§. 48. Sit nunc $n=5$, et prodeunt hae quatuor reductiones

$$\left[\tfrac{1}{5}\right]=\tfrac{1}{5}\sqrt[5]{5^4\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)}$$
$$\left[\tfrac{2}{5}\right]=\tfrac{2}{5}\sqrt[5]{5^3.1\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)}$$
$$\left[\tfrac{3}{5}\right]=\tfrac{3}{5}\sqrt[5]{5^2.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{2}{3}\right)}$$
$$\left[\tfrac{4}{5}\right]=\tfrac{4}{5}\sqrt[5]{5.1.2.3\left(\tfrac{4}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{4}\right)}.$$

Corollarium 5.

§. 49. Sit $n = 6$, et habebimus has reductiones

$$\left[\tfrac{1}{7}\right]=\tfrac{1}{7}\sqrt[7]{7^6\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)}$$
$$\left[\tfrac{2}{7}\right]=\tfrac{2}{7}\sqrt[7]{7^5.1\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)}$$
$$\left[\tfrac{3}{7}\right]=\tfrac{3}{7}\sqrt[7]{7^4.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)}$$
$$\left[\tfrac{4}{7}\right]=\tfrac{4}{7}\sqrt[7]{7^3.1.2.3\left(\tfrac{4}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{3}{4}\right)}$$
$$\left[\tfrac{5}{7}\right]=\tfrac{5}{7}\sqrt[7]{7^2.1.2.3.4\left(\tfrac{5}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{2}{5}\right)}$$
$$\left[\tfrac{6}{7}\right]=\tfrac{6}{7}\sqrt[7]{7.1.2.3.4.5\left(\tfrac{6}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)}.$$

Corollarium 6.

§. 50. Posito $n=7$, sequentes sex prodeunt aequationes .

$$\left[\tfrac{1}{7}\right]=\tfrac{1}{7}\sqrt[7]{7^6}\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)$$
$$\left[\tfrac{2}{7}\right]=\tfrac{2}{7}\sqrt[7]{7^5}.1\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)$$
$$\left[\tfrac{3}{7}\right]=\tfrac{3}{7}\sqrt[7]{7^4}.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)$$
$$\left[\tfrac{4}{7}\right]=\tfrac{4}{7}\sqrt[7]{7^3}.1.2.3\left(\tfrac{4}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{3}{4}\right)$$
$$\left[\tfrac{5}{7}\right]=\tfrac{5}{7}\sqrt[7]{7^2}.1.2.3.4\left(\tfrac{5}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{2}{5}\right)$$
$$\left[\tfrac{6}{7}\right]=\tfrac{6}{7}\sqrt[7]{7}.1.2.3.4.5\left(\tfrac{6}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right).$$

Corollarium 7.

§. 51. Sit $n=8$, et septem hae reductiones impetrabuntur

$$\left[\tfrac{1}{8}\right]=\tfrac{1}{8}\sqrt[8]{8^7}\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)\left(\tfrac{7}{1}\right)$$
$$\left[\tfrac{2}{8}\right]=\tfrac{2}{8}\sqrt[8]{8^6}.2\left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2\left(\tfrac{6}{2}\right)^2\left(\tfrac{8}{2}\right)=\tfrac{1}{4}\sqrt[4]{8^3}\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)$$
$$\left[\tfrac{3}{8}\right]=\tfrac{3}{8}\sqrt[8]{8^5}.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{7}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)$$
$$\left[\tfrac{4}{8}\right]=\tfrac{4}{8}\sqrt[8]{8^4}.4.4.4\left(\tfrac{4}{4}\right)^4\left(\tfrac{8}{4}\right)^3=\tfrac{1}{2}\sqrt[2]{8}\left(\tfrac{4}{4}\right)$$
$$\left[\tfrac{5}{8}\right]=\tfrac{5}{8}\sqrt[8]{8^3}.1.2.3.4\left(\tfrac{5}{5}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{7}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{3}{5}\right)$$
$$\left[\tfrac{6}{8}\right]=\tfrac{6}{8}\sqrt[8]{8^2}.4.2.6.4.2\left(\tfrac{6}{6}\right)^2\left(\tfrac{4}{6}\right)^2\left(\tfrac{2}{6}\right)^2\left(\tfrac{3}{6}\right)=\tfrac{3}{4}\sqrt[4]{8}.2.4\left(\tfrac{4}{6}\right)\left(\tfrac{2}{6}\right)$$
$$\left[\tfrac{7}{8}\right]=\tfrac{7}{8}\sqrt[8]{8}.1.2.3.4.5.6\left(\tfrac{7}{7}\right)\left(\tfrac{6}{7}\right)\left(\tfrac{5}{7}\right)\left(\tfrac{4}{7}\right)\left(\tfrac{3}{7}\right)\left(\tfrac{2}{7}\right)\left(\tfrac{1}{7}\right).$$

Scholion.

§. 52. Superfluum foret hos casus ulterius evolvere, cum ex allatis ordo istarum formularum satis perspiciatur. Si enim in formula proposita $\left[\tfrac{m}{n}\right]$ numeri *m* et *n* sint inter se primi lex est manifesta, cum fiat

$$\left[\tfrac{m}{n}\right]=\tfrac{m}{n}\sqrt[n]{n^{n-m}}1.2....\,(m-1).\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\left(\tfrac{n-1}{m}\right),$$

sin autem hi numeri *m* et *n* communem habeant divisorem, expediet quidem fractionem $\frac{m}{n}$ ad minimam formam reduci, et ex casibus praecedentibus quaesitum valorem peti ; interim tamen etiam operatio hoc modo institui poterit. Cum expressio quaesita certe hanc habeat formam

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}}\text{P.Q},$$

ubi Q est productum ex $n-1$ formulis integralibus, P vero productum ex aliquot numeris absolutis; primum pro illo producto Q inveniendo, continuetur haec formularum series $\left(\frac{m}{m}\right)\ \left(\frac{2m}{m}\right)\ \left(\frac{3m}{m}\right)$, donec numerator superet exponentem n, ejusque loco excessus supra n scribatur, qui si ponatur $=\alpha$, ut jam formula sit $\left(\frac{\alpha}{m}\right)$, hic ipse numerator α dabit factorem producti P, tum hinc formularum series porro statuatur $\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+m}{m}\right)\left(\frac{\alpha+2m}{m}\right)$, etc. donec iterum ad numeratorem exponente n majorem perveniatur, formulaque prodeat $\left(\frac{n+\beta}{m}\right)$, cujus loco scibi oportet $\left(\frac{\beta}{m}\right)$, simulque hinc factor β in productum P inferatur; sicque progredi conveniet; donec pro Q prodierint $n-1$ formulae. Quae operationes quo faciliüs intelligantur, casum formulae $\left[\frac{9}{12}\right]=\frac{9}{12}\sqrt[12]{12^3}\text{P.Q}$ hoc modo evolvamus, ubi investigatio litterarum Q et P ita instituetur,

$$\text{pro Q}\ \ldots\ \left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right)\left(\tfrac{12}{9}\right)\left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right)\left(\tfrac{12}{9}\right)\left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right),$$
$$\text{pro P}\ \ldots\ \quad 6.\ \ 3.\qquad 9.\ \ 6.\ \ 3\qquad 9.\ \ 6.\ \ 3,$$

sicque reperitur

$$\text{Q}=\left(\tfrac{9}{9}\right)^3\left(\tfrac{6}{9}\right)^3\left(\tfrac{3}{9}\right)^3\left(\tfrac{12}{9}\right)^2\ \text{et}$$
$$\text{P}=6^3.\,3^3.\,9^2.$$

Cum igitur sit $\left(\frac{12}{9}\right)=\frac{1}{9}$, $\text{PQ}=6^3.\,3^3.\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3$, ideoque

$$\left[\frac{9}{12}\right]=\frac{3}{4}\sqrt[4]{12.6.3.}\left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right).$$

Theorema.

§. 53. Quinque numeri integri positivi litteris m et n indicentur, erit semper signandi modo ante exposito

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}1.2\ldots.\ (m-1)}.\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\left(\tfrac{n-1}{m}\right).$$

Demonstratio.

Pro casu, quo m et n sunt numeri inter se primi,veritas theorematis in antecedentibus est evicta; quod autem etiam locum habeat, si illi numeri m et n commune divisore gaudeant, inde quidem non liquet: verum ex hoc ipso, quod pro casibus, quibus m et n sunt numeri primi, veritas constat, tuto concludere licet, theorema in genere esse verum. Minime quidem diffiteor hoc concludendi genus prorsus esse singulare, ac plerisque suspectum videri debere. Quare quo nullum dubium relinquatur, quoniam pro casibus, quibus numeri m et n inter se sunt compositi, geminam expressionem sumus nacti, utriusque consensum pro casibus ante evolutis ostendisse juvabit. Insigne autem jam suppeditat firmamentum, casus $m=n$, quo forma nostra manifesto unitatem producit.

Corollarium 1.

§. 54. Primus casus consensus demonstrationem postulans est quo $m=2$ et $n=4$, pro quo supra §. 47 invenimus

$$\left[\tfrac{2}{4}\right]=\tfrac{2}{4}\sqrt[4]{4^2.\left(\tfrac{2}{2}\right)^2},$$

nunc autem vi theorematis est

$$\left[\tfrac{2}{4}\right]=\tfrac{2}{4}\sqrt[4]{4^2.1.\left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)},$$

unde comparatione instituta fit $\left(\frac{2}{2}\right)=\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)$, cujus veritas in observationibus supra allegatis est confirmata .

Corollarium 2.

§. 55. Si $m=2$ et $n=6$, ex superioribus §. 49 est

$$\left[\tfrac{2}{6}\right]=\tfrac{2}{6}\sqrt[6]{6^4.\left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2}$$

nunc vero per theorema

$$\left[\tfrac{2}{6}\right]=\tfrac{2}{6}\sqrt[6]{6^4.1\left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{5}{2}\right)},$$

ideoque necesse est sit

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)=\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right),$$

cujus veritas indidem patet.

Corollarium 3.

§56. Si $m=3$ et $n=6$, pervenitur ad hanc aequationem

$$\left(\tfrac{3}{6}\right)^2=1.2.\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{3}\right),$$

at si $m=4$ et $n=6$, fit simili modo

$$2^2\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right)=1.2.3\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right),\text{seu}$$
$$\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right)=\tfrac{3}{2}\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right),$$

quod etiam, verum deprehenditur

Corollarium 4.

§. 57. Casus $m=2$ et $n=8$ praebet hanc aequalitatem

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)=\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)\left(\tfrac{7}{2}\right);$$

at casus $m=4$ et $n=8$ hanc

$$\left(\tfrac{4}{4}\right)^3=1.2.3\left(\tfrac{1}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{7}{4}\right);$$

casu denique $m=6$ et $n=8$ istam

$$2.4\left(\tfrac{6}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{2}{4}\right)=1.3.5\left(\tfrac{1}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{7}{6}\right),$$

quae etiam veritati sunt consentaneae.

Scholion.

§.58. In genere autem si numeri m et n communem habeant factorem 2, et formula proposita sit $\left[\tfrac{2m}{2n}\right]=\left[\tfrac{m}{n}\right]$, quia est

$$\left[\tfrac{m}{n}\right]=\tfrac{m}{n}\sqrt[n]{n^{n-m}1.2....(m-1).\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\left(\tfrac{n-1}{m}\right)},$$

erit eadem ad exponentem $2n$ reducta

$$\tfrac{m}{n}\sqrt[2n]{2n^{2n-2m}}.2^2.4^2.6^2....(2m-2)^2.\left(\tfrac{2}{2m}\right)^2\left(\tfrac{4}{2m}\right)^2\left(\tfrac{6}{2m}\right)^2\cdots\left(\tfrac{2n-2}{2m}\right)^2.$$

Per theorema vero eadem expresso fit

$$\tfrac{m}{n}\sqrt[2n]{2n^{2n-2m}1.3.5....(2m-1)\left(\tfrac{1}{2m}\right)\left(\tfrac{3}{2m}\right)\left(\tfrac{5}{2m}\right)\cdots\left(\tfrac{2n-1}{2m}\right)},$$

unde pro exponente $2n$ erit

$$2.4.6....(2m-2)\left(\tfrac{2}{2m}\right)\left(\tfrac{4}{2m}\right)\left(\tfrac{6}{2m}\right)\cdots\cdots\left(\tfrac{2n-2}{2m}\right)$$
$$=1.3.5....(2m-1)\left(\tfrac{1}{2m}\right)\left(\tfrac{3}{2m}\right)\left(\tfrac{5}{2m}\right)\cdots\left(\tfrac{2n-1}{2m}\right).$$

Simili modo si communis divisor sit 3, pro exponente $3n$ reperietur

$$3^2.6^2.9^2....(3m-3)^2\left(\tfrac{3}{3m}\right)^2\left(\tfrac{6}{3m}\right)^2\left(\tfrac{9}{3m}\right)^2\cdots\cdots\left(\tfrac{3n-3}{3m}\right)^2$$
$$=1.2.4.5....(3m-2)(3m-1)\left(\tfrac{1}{3m}\right)\left(\tfrac{2}{3m}\right)\left(\tfrac{4}{3m}\right)\left(\tfrac{5}{3m}\right)\cdots\left(\tfrac{3n-1}{3m}\right).$$

quae aequatio concinnius ita exhiberi potest

$$\frac{1.2.4.5.7.8.10...(3m-2)(3m-1)}{3^2.6^2.9^2....(3m-3)^2}=\frac{\left(\tfrac{3}{3m}\right)^2\cdot\left(\tfrac{6}{3m}\right)^2......\left(\tfrac{3n-3}{3m}\right)^2}{\left(\tfrac{1}{3m}\right)\left(\tfrac{2}{3m}\right)\left(\tfrac{4}{3m}\right)\left(\tfrac{5}{3m}\right)\left(\tfrac{7}{3m}\right)....\left(\tfrac{3n-2}{3m}\right)\left(\tfrac{3n-1}{3m}\right)}.$$

In genere autem si communis divisor sit *d* et exponens *dn*, habitur

$$\left[d.2d.3d....(dm-d)\left(\tfrac{d}{dm}\right)\left(\tfrac{2d}{dm}\right)\left(\tfrac{3d}{dm}\right)\cdots\cdots\left(\tfrac{dn-d}{dm}\right)\right]^{d}$$
$$=1.2.4.5....(dm-1)\left(\tfrac{1}{dm}\right)\left(\tfrac{2}{dm}\right)\left(\tfrac{3}{dm}\right)\cdots\cdots\ \left(\tfrac{dn-1}{dm}\right),$$

quae aequatio facile ad quosvis casus accommodari potest, unde sequens Theorema notari meretur.

Theorema.

§. 59. Si α fuerit divisor commums numerorum *m* et *n*, haecque formula $\left(\frac{p}{q}\right)$ denotet valorem integralis

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}$$

ab $x=0$ usque ad $x=1$ extensi, erit

$$\left[\alpha.2\alpha.3\alpha........(m-\alpha)\left(\tfrac{\alpha}{m}\right)\left(\tfrac{2\alpha}{m}\right)\left(\tfrac{3\alpha}{m}\right)\cdots\cdot\left(\tfrac{n-\alpha}{m}\right)\right]^{\alpha}$$
$$=1.2.3........(m-1)\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\cdot\left(\tfrac{n-1}{m}\right).$$

Demonstratio.

Ex praecedente scholio veritas hujus theorematis perspicitur, cum enim ibi divisor communis esset $=d$, binique numeri propositi *dm* et *dn*, horum loco hic scripsi *m* et *n*, loco divisoris eorum autem *d* litteram α quam divisoris rationem aequalitas enunciata ita complectitur, ut in progressione arithmetica α, 2α, 3α, etc. continuata occurrere assumantur ipsi numeri *m* et *n* ideoque etiam $m-\alpha$ et $n-\alpha$. Caeterum faleri cogor; hanc demonstrationem utpote inductioni potissimum innixam, neutiquam pro rigorosa haberi posse: cum autem nihilominus de ejus veritate simus convicti, hoc theorema eo majori attentione dignum videtur, interim tamen nullum est dubium, quin uberior hujusmodi formularum integralium evolutio tandem perfectam demonstrationem sit largitura, quod autem jam ante hanc veritatem nobis perspicere licuerit, insigne hinc specimen analyticae investigationis elucet.

Corollarium1.

§. 60. Si loco signorum adhibitorum ipsas formulas integrales substituamus, theorema nostrum ita se habebit ut sit

$$\alpha.2\alpha.3\alpha........(m-\alpha)\int\frac{x^{\alpha-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-m}}}\cdot\int\frac{x^{2\alpha-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-m}}}\cdots\int\frac{x^{n-\alpha-1}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-m}}}$$
$$=\sqrt{\ }1.2.3........(m-1)\int\frac{\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-m}}}\cdot\int\frac{x\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-m}}}\cdots\int\frac{x^{n-2}\partial x}{\sqrt[n]{\left(1-x^{n}\right)^{n-m}}}.$$

Corollarium 2.

§. 61. Vel si ad abbreviandum statuamus

$$\sqrt[n]{\left(1-x^n\right)^{n-m}} = X,$$

$$\alpha.2\alpha.3\alpha........(m-\alpha)\int\frac{x^{\alpha-1}\partial x}{X}\cdot\int\frac{x^{2\alpha-1}\partial x}{X}\cdots\int\frac{x^{n-\alpha-1}\partial x}{X}$$

$$=\sqrt[\alpha]{}1.2.3........(m-1)\int\frac{\partial x}{X}\cdot\int\frac{x\partial x}{X}\cdots\int\frac{x^{n-2}\partial x}{X}.$$

Theorema generale.

§. 62. Si binorum numerorum *m* et *n* divisores communes sint α, β, γ, etc. formulaque $\left(\frac{p}{q}\right)$ denotet valorem integralis

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$$

$x=0$ ad $x=1$ extensi; sequentes expressiones ex hujusmodi formulis integralibus formatae inter se erunt aequales

$$\left[\alpha.2\alpha.3\alpha........(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdots\left(\frac{n-\alpha}{m}\right)\right]^{\alpha}=$$

$$\left[\beta.2\beta.3\beta........(m-\beta)\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\cdots\left(\frac{n-\beta}{m}\right)\right]^{\beta}=$$

$$\left[\gamma.2\gamma.3\gamma........(m-\gamma)\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\cdots\left(\frac{n-\gamma}{m}\right)\right]^{\gamma}=\text{etc.}$$

$$=1.2.3........(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\cdots\left(\frac{n-1}{m}\right).$$

Demonstratio.

Ex praecedente Theoremate hujus veritas manifesta sequitur, cum quaelibet harum expressionum seorsim aequetur huic

$$1.2.3........(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\cdots\left(\frac{n-1}{m}\right),$$

quae unitati utpote minima communi divisori numerorum *m* et *n* convenit. Tot igitur hujusmodi expressiones inter se aequales exhiberi possunt, quot fuerint divisores communes binorum numerorum *m* et *n*.

Corollarium 1.

§. 63. Cum sit haec formula $\left(\frac{n}{m}\right)=\frac{1}{m}$ ideoque $m\left(\frac{n}{m}\right)=1$, expressiones nostrae aequales succinctius hoc modo repraesentari possunt

$$\left[\alpha.2\alpha.3\alpha........m\left(\tfrac{\alpha}{m}\right)\left(\tfrac{2\alpha}{m}\right)\left(\tfrac{3\alpha}{m}\right)\cdots\left(\tfrac{n}{m}\right)\right]^{\alpha}=$$
$$\left[\beta.2\beta.3\beta........m\left(\tfrac{\beta}{m}\right)\left(\tfrac{2\beta}{m}\right)\left(\tfrac{3\beta}{m}\right)\cdots\left(\tfrac{n}{m}\right)\right]^{\beta}=$$
$$\left[\gamma.2\gamma.3\gamma........m\left(\tfrac{\gamma}{m}\right)\left(\tfrac{2\gamma}{m}\right)\left(\tfrac{3\gamma}{m}\right)\cdots\left(\tfrac{n}{m}\right)\right]^{\gamma}=\text{etc.}$$

Etsi enim hic factorum numerus est auctos, tamen ratio compositionis facilius in oculos incurrit.

Corollarium 2.

§. 64. Si ergo sit $m=6$ et $n=12$, ob horum numerorum divisores communes 6, 3, 2, 1, quatuor sequentes formae inter se aequales habebuntur

$$\left[6\left(\tfrac{6}{6}\right)\left(\tfrac{12}{6}\right)\right]^{6}=\left[3.6\left(\tfrac{3}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{9}{6}\right)\left(\tfrac{12}{6}\right)\right]^{3}=$$
$$\left[2.4.6\left(\tfrac{2}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{8}{6}\right)\left(\tfrac{10}{6}\right)\left(\tfrac{12}{6}\right)\right]^{2}=$$
$$1.2.3.4.5.6\left(\tfrac{1}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{3}{6}\right)........\left(\tfrac{12}{6}\right).$$

Corollarium 3.

§. 65. Si ultima cum penultima combinetur, nascetur haec aequatio

$$\frac{1.3.5}{2.4.6}=\frac{\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{6}\right)\left(\frac{11}{6}\right)},$$

ultima autem cum antepenultima comparata praebet

$$\frac{1.2.4.5}{3.3.5.5}=\frac{\left(\frac{3}{6}\right)\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{11}{6}\right)}.$$

Scholion.

§. 66. Infinitae igitur hinc consequuntur relationes inter formulas integrales formae

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} = \left(\frac{p}{q}\right),$$

quae eo magis sunt notatu dignae, quod singulari prorsus methodo ad eas hic sumus perducti. Ac si quis de earum veritate adhuc dubitet, observationes meas circa has formulas integrales consulat, indeque pro quovis casu oblato de veritate facile convincetur. Etsi autem illa tractatio huic confirmandae inservit, tamen relationes hic erutae eo majoris sunt momenti, quod in iis certus ordo cernitur, eaeque per omnes classes, quantumvis exponentem n accipere lubeat, facili negotio continuentur ; in priori vero tractatione calculus pro classibus altioribus continuo fiat operosior et intricatior.

Supplementum continens demonstrationem
Theorematis §. 53. propositi.

§. 67. Demonstrationem hanc altius peti convenit; sumatur scilicet aequatio §. 25, data, quae posito $f=1$ et mutatis litteris est

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{\nu-1} \times \int \partial x\left(l\frac{1}{x}\right)^{\mu-1}}{\int \partial x\left(l\frac{1}{x}\right)^{\mu+\nu-1}} = \chi \int \frac{x^{\chi\mu-1}\partial x}{\left(1-x^{\chi}\right)^{1-\nu}},$$

eaque per reductiones notas hac forma repraesentetur

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{\nu} \times \int \partial x\left(l\frac{1}{x}\right)^{\mu}}{\int \partial x\left(l\frac{1}{x}\right)^{\mu+\nu}} = \frac{\chi\mu\nu}{\mu+\nu} \int \frac{x^{\chi\mu-1}\partial x}{\left(1-x^{\chi}\right)^{1-\nu}}.$$

Statuatur nunc $\nu=\frac{m}{n}$ et $\mu=\frac{\lambda}{n}$, tum vero $\chi=n$, ut habeamus,

$$\frac{\int \partial x\left(l\frac{1}{x}\right)^{\frac{m}{n}} \times \int \partial x\left(l\frac{1}{x}\right)^{\frac{\lambda}{n}}}{\int \partial x\left(l\frac{1}{x}\right)^{\frac{\lambda+m}{n}}} = \frac{\lambda m}{\lambda+m} \int \frac{x^{\lambda-1}\partial x}{\sqrt[\chi]{\left(1-x^n\right)^{n-m}}}$$

quae brevitatis gratia, more supra usitato, ita concinne referatur

$$\frac{\left[\frac{m}{n}\right]\cdot\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{m}\right]} = \frac{\lambda m}{\lambda+m}\cdot\left(\frac{\lambda}{m}\right).$$

Jam loco λ successive scribantur numeri 1,2, 3, 4 …. n omnesque hae aequationes, quarum numerus est $=n$, in se invicem ducantur, et aequatio resultans erit

$$\left[\frac{m}{n}\right]^n \cdot \frac{\left[\frac{1}{n}\right]\cdot\left[\frac{2}{n}\right]\cdot\left[\frac{3}{n}\right]\ldots\ldots\left[\frac{n}{n}\right]}{\left[\frac{m+1}{m}\right]\left[\frac{m+2}{m}\right]\left[\frac{m+3}{m}\right]\ldots\ldots\left[\frac{m+n}{m}\right]} =$$

$$m^n \cdot \frac{1}{m+1} \cdot \frac{2}{m+2} \cdot \frac{3}{m+3} \cdots\cdots \frac{n}{m+n} \cdot \left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right) =$$

$$m^n \cdot \frac{1.2.3\ldots\ldots\ldots\ldots m}{(n+1)(n+2)(n+3)\ldots\ldots.(n+m)} \left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Simili autem modo pars prior transformetur ut sit

$$\left[\frac{m}{n}\right]^n \cdot \frac{\left[\frac{1}{n}\right]\cdot\left[\frac{2}{n}\right]\cdot\left[\frac{3}{n}\right]\ldots\ldots\left[\frac{m}{n}\right]}{\left[\frac{n+1}{n}\right]\left[\frac{n+2}{n}\right]\left[\frac{n+3}{n}\right]\ldots\ldots\left[\frac{n+m}{n}\right]},$$

cujus convenientia cum forma praecedente multiplicando per crucem, ut ajunt, sponte se prodit. Cum vero ex natura harum formularum sit

$$\left[\frac{n+1}{n}\right] = \frac{n+1}{n}\left[\frac{1}{n}\right], \left[\frac{n+2}{n}\right] = \frac{n+2}{n}\left[\frac{2}{n}\right], \left[\frac{n+3}{n}\right] = \frac{n+3}{n}\left[\frac{3}{n}\right], \text{etc.}$$

ob harum formularum numerum $= m$, evadet haec prior pars

$$\left[\frac{m}{n}\right]^n \cdot \frac{n^m}{(n+1)(n+2)(n+3)\ldots\ldots.(n+m)},$$

quae cum aequalis sit parti alteri ante exhibitae

$$m^n \cdot \frac{1.2.3\ldots\ldots\ldots\ldots m}{(n+1)(n+2)(n+3)\ldots\ldots.(n+m)} \left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)$$

adipiscimur hanc aequationem

$$\left[\frac{m}{n}\right]^n = \frac{m^n}{n^m}.1.2.3\ldots\ldots\ldots\ldots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

ita ut sit

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1.2.3\ldots\ldots\ldots\ldots m}{n^m}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)},$$

quae cum proposita in (§. 53.) ob $\left(\frac{n}{m}\right) = \frac{1}{m}$ omnino congruit, ex quo ejus veritas nunc quidem ex principiis certissimis est victa.

Demonstratio Theorematis §.59. propositi.

§. 68. Etiam hoc Theorema firmiori demonstratione indiget, quam ex aequalitate ante stabilita

$$\frac{\left[\frac{m}{n}\right]\cdot\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{m}\right]}=\frac{\lambda m}{\lambda+m}\cdot\left(\frac{\lambda}{m}\right)$$

îta adorno. Existente α communi divisore numerorum m et n, 1oco λ, successive scribantur numeri α, 2α, 3α; etc. usque ad n, quorum multitudo est $=\frac{n}{\alpha}$, atque omnes aequalitates hoc modo resultantes in se invicem ducantur, ut prodeat haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}}\cdot\frac{\left[\frac{\alpha}{n}\right]\cdot\left[\frac{2\alpha}{n}\right]\cdot\left[\frac{3\alpha}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\cdots\left[\frac{m+n}{n}\right]}=$$

$$m^{\frac{n}{\alpha}}\cdot\frac{1\alpha}{m+\alpha}\cdot\frac{2\alpha}{m+2\alpha}\cdot\frac{3\alpha}{m+3\alpha}\cdots\cdots\frac{n}{m+n}\cdot\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Jam prior pars in hanc formam ipsi aequalem transmutetur

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}}\cdot\frac{\left[\frac{\alpha}{n}\right]\cdot\left[\frac{2\alpha}{n}\right]\cdot\left[\frac{3\alpha}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\cdots\left[\frac{m+n}{n}\right]},$$

quae ob $\left[\frac{n+\alpha}{n}\right]=\frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$, sicque de caeteris, reducitur ad hanc

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}}\cdot\frac{n}{n+\alpha}\cdot\frac{n}{n+2\alpha}\cdot\frac{n}{n+3\alpha}\cdots\cdots\frac{n}{n+m}.$$

Posterior vero aequationis pars simili modo transformatur in

$$m^{\frac{n}{\alpha}}\cdot\frac{\alpha}{n+\alpha}\cdot\frac{2\alpha}{n+2\alpha}\cdot\frac{3\alpha}{n+3\alpha}\cdots\cdots\frac{m}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdot\left(\frac{n}{m}\right),$$

unde enascitur haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}}m^{\frac{m}{\alpha}}=m^{\frac{n}{\alpha}}\alpha\cdot 2\alpha\cdot 3\alpha\cdots\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdot\left(\frac{n}{m}\right),$$

hincque

$$\left[\frac{m}{n}\right]=m\sqrt[n]{\frac{1}{m^n}}\left[\alpha\cdot 2\alpha\cdot 3\alpha\cdots\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdot\left(\frac{n}{m}\right)\right]^{\alpha},$$

quae expressio cum praecedente comparata praebet hanc aequationem

$$\left[\alpha\cdot 2\alpha\cdot 3\alpha\cdots\cdots m\left(\tfrac{\alpha}{m}\right)\left(\tfrac{2\alpha}{m}\right)\left(\tfrac{3\alpha}{m}\right)\cdots\cdot\left(\tfrac{n}{m}\right)\right]^{\alpha} =$$
$$1\cdot 2\cdot 3\cdots\cdots m\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\cdot\left(\tfrac{n}{m}\right),$$

quod de omnibus divisoribus communibus binorum numerorum *m* et *n* est intelligendum .