# SUPPLEMENT II. TO BOOK I, CH. III.

## THE INTEGRATION OF DIFFERENTIAL FORMULAS BY INFINITE SERIES.

The series $\int x^{m-1}\partial x\left(\Delta + x^{m-1}\right)^{\lambda}$ must always be converging for the resolution of integral formulas. Where likewise many conspicuous artifices about the summation of series are set forth. *M. S. of the Academy, presented on the* 12th *day of Aug.* 1779.

§.1. Recently this integral formula presented itself to me : $\int \partial x\sqrt{\left(\Delta + x^4\right)}$; since the value of which, in the case when $\Delta = 0$, shall become $\frac{1}{3}x^3$, it came to mind, to investigate these of its values, which it adopts, when $\Delta$ is an extremely small quantity. But I saw soon, this could by no means be set out in the usual manner. Since indeed there shall be

$$\sqrt{\left(\Delta + x^4\right)} = \sqrt{\Delta} \times \left(1 + \tfrac{x^4}{\Delta}\right)^{\frac{1}{2}},$$

and thus expanded by the series

$$\sqrt{\left(\Delta + x^4\right)} = \sqrt{\Delta}\left(1 + \tfrac{1}{2}\cdot\tfrac{x^4}{\Delta} - \tfrac{1\cdot 1}{2\cdot 4}\cdot\tfrac{x^8}{\Delta^2} + \tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\tfrac{x^{12}}{\Delta^3} - \text{etc.}\right).$$

the value of the formula of this integral will be

$$\int \partial x\sqrt{\left(\Delta + x^4\right)} = x\sqrt{\Delta} \times \left(1 + \frac{1}{2}\cdot\frac{x^4}{5\Delta} - \frac{1\cdot 1}{2\cdot 4}\cdot\frac{x^8}{9\Delta^2} + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{x^{12}}{13\Delta^3} - \text{etc.}\right).$$

which series clearly diverges greatly, whenever $\Delta$ was an extremely small quantity, and thus, and just as often as the fraction $\frac{x^n}{\Delta}$ were greater than one.

§. 2. Therefore in order that I might reach the aim of the proposition, I have considered this same question in this form : *To express the value of the integral formula* $\int x^{m-1}\partial x\sqrt{\left(\Delta + x^4\right)}$ *set out by always converging series, from the end* $x = 0$ *as far as to the end* $x = a$, *whatever value may be attributed to the letter* $\Delta$. I represent the formula $\Delta + x^4$ in the end under this kind:

$$\Delta + a^4 - \left(a^4 - x^4\right),$$

or this:

$$\left(\Delta + a^4\right)\left(1 - \tfrac{a^4 - x^4}{\Delta + a^4}\right).$$

Hence therefore there will be

$$\sqrt{\left(\Delta+x^4\right)}=\sqrt{\left(\Delta+a^4\right)}\times\left(1+\tfrac{1}{2}\cdot\tfrac{a^4-x^4}{\Delta+a^4}-\tfrac{1\cdot1}{2\cdot4}\cdot\left(\tfrac{a^4-x^4}{\Delta+a^4}\right)^2-\text{etc.}\right).$$

And thus the whole concern is returned here, so that the values of these integral formulas may be investigated, extending from $x=0$ as far as to $x=a$
$\int\partial x\left(a^4-x^4\right)$, $\int\partial x\left(a^4-x^4\right)^2$, $\int\partial x\left(a^4-x^4\right)^3$, etc., from which the first term $\int\partial x$ will give $a$.

§.3. For the second term there will be had on integrating
$\int\partial x\left(a^4-x^4\right)=a^4x-\tfrac{1}{5}x^5$, the value of which with $x=a$ will become $\tfrac{4}{5}a^5$. For the third term there will be
$\int\partial x\left(a^4-x^4\right)^2=a^8x-\tfrac{2}{5}a^4x^5+\tfrac{1}{9}x^9$, which expression on putting $x=a$ will become $\tfrac{4\cdot8}{5\cdot9}a^9$. We will have in a similar manner for the fourth term

$$\int\partial x\left(a^4-x^4\right)^3=a^{13}\left(1-\tfrac{3}{5}+\tfrac{3}{9}-\tfrac{1}{13}\right)^3=\tfrac{4\cdot8\cdot12}{5\cdot9\cdot13}a^{13}.$$

In the same manner there may be found :

$$\int\partial x\left(a^4-x^4\right)^4=\tfrac{4\cdot8\cdot12\cdot16}{5\cdot9\cdot13\cdot17}a^{17},$$

and thus so on. Moreover, I am about to show the rule of this elegant progression below.

§. 4. Therefore with these values substituted , the whole value of the integral sought may be found to be

$$a\sqrt{\left(\Delta+a^4\right)}\times\left[1-\tfrac{1}{2}\cdot\tfrac{4}{5}\tfrac{a^4}{\Delta+a^4}-\tfrac{1\cdot1}{2\cdot4}\cdot\tfrac{4\cdot8}{5\cdot9}\cdot\left(\tfrac{a^4}{\Delta+a^4}\right)^2-\tfrac{1\cdot1\cdot3}{2\cdot4\cdot6}\cdot\tfrac{4\cdot8\cdot12}{5\cdot9\cdot13}\cdot\left(\tfrac{a^4}{\Delta+a^4}\right)^3-\text{etc.}\right].$$

Since here duplicate coefficients are present both above and below, if we may double the individual factors of the first, thus this series may be contracted into the following :

$$a\sqrt{\left(\Delta+a^4\right)}\times\left[1-\tfrac{2}{5}\tfrac{a^4}{\Delta+a^4}-\tfrac{2\cdot2}{5\cdot9}\cdot\left(\tfrac{a^4}{\Delta+a^4}\right)^2-\tfrac{2\cdot2\cdot6}{5\cdot9\cdot13}\cdot\left(\tfrac{a^4}{\Delta+a^4}\right)^3-\text{etc.}\right],$$

which series evidently converges always, therefore so that not only do the coefficients decrease rapidly, but also the formula $\tfrac{a^4}{\Delta+a^4}$ is less than one.

§. 5. Now nothing prevents us whereby in place of the smaller $a$ we may restore the variable quantity $x$ itself, and thus the value of this integral formula $\int \partial x \sqrt{(\Delta + x^4)}$ may be expressed always by a part of always converging series :

$$x\sqrt{(\Delta + x^4)} \times \left[1 - \frac{2}{5}\frac{x^4}{\Delta+x^4} - \frac{2\cdot2}{5\cdot9}\cdot\left(\frac{x^4}{\Delta+x^4}\right)^2 - \frac{2\cdot2\cdot6}{5\cdot9\cdot13}\cdot\left(\frac{x^4}{\Delta+x^4}\right)^3 - \text{etc.}\right].$$

Here the case in which the series converges minimally, is that one itself, which we have commented on initially, where $\Delta = 0$, and its integral $= \frac{1}{3}x^3$. Therefore on putting $\Delta = 0$ we come upon the following most noteworthy series

$$x^3\left(1 - \frac{2}{5} - \frac{2\cdot2}{5\cdot9} - \frac{2\cdot2\cdot6}{5\cdot9\cdot13} - \frac{2\cdot2\cdot6\cdot10}{5\cdot9\cdot13\cdot17} - \text{etc.}\right),$$

whose sum therefore we know to be $\frac{1}{3}x^3$, thus so that we shall have this summation

$$\frac{1}{3} = 1 - \frac{2}{5} - \frac{2}{5}\cdot\frac{2}{9} - \frac{2}{5}\cdot\frac{2}{9}\cdot\frac{6}{13} - \frac{2}{5}\cdot\frac{2}{9}\cdot\frac{6}{13}\cdot\frac{10}{17} - \text{etc.}$$

whose demonstration may be seen to be more difficult to find. Yet meanwhile since its sum is known, the truth can be shown in the following manner.
Hence indeed there will be

$$\frac{2}{5} + \frac{2\cdot2}{5\cdot9} + \frac{2\cdot2\cdot6}{5\cdot9\cdot13} + \text{etc.} = \frac{2}{3}.$$

which equation multiplied by $\frac{5}{2}$ gives

$$1 + \frac{2}{9} + \frac{2\cdot6}{9\cdot13} + \frac{2\cdot6\cdot10}{9\cdot13\cdot17} + \text{etc.} = \frac{5}{3}.$$

Here the first term may be moved to the opposite side, and on multiplying by $\frac{9}{2}$ there will be produced

$$1 + \frac{6}{13} + \frac{6\cdot10}{13\cdot17} + \frac{6\cdot10\cdot14}{13\cdot17\cdot21} + \text{etc.} = \frac{2}{3}.$$

Again the first term is moved to the other side and with the multiplication made by $\frac{13}{2}$ , there is deduced:

$$1 + \frac{10}{17} + \frac{10\cdot14}{17\cdot21} + \frac{10\cdot14\cdot18}{17\cdot21\cdot25} + \text{etc.} = \frac{13}{3}.$$

By progressing in a similar manner there will be produced

$$1 + \frac{14}{21} + \frac{14\cdot18}{21\cdot25} + \frac{14\cdot18\cdot22}{21\cdot25\cdot29} + \text{etc.} = \frac{17}{3}.$$

$$1 + \frac{18}{25} + \frac{18\cdot22}{25\cdot29} + \frac{18\cdot22\cdot26}{25\cdot29\cdot33} + \text{etc.} = \frac{21}{3}.$$

And thus we have obtained innumerable series, the sum of which is known, and because by the same law they can be progressed further, this proof is certain, just the first sum is to be given. But we will show this remarkable truth more accurately, when the matter is viewed below generally.

## Problem general.

*To express the value of the integral formula* $\int x^{m-1}\partial x(\Delta+x^n)^\lambda$ *between* $x=0$ *and* $x=a$ *by a series which is always convergent.*

## Solution.

§.6. We may represent the formula $\Delta+x^n$ in this form $\Delta+x^n-(a^n-x^n)$ , which is reduced to this

$$\left(\Delta+x^n\right)\left(1-\tfrac{a^n-x^n}{\Delta+x^n}\right),$$

and thus the proposed integral formula will be

$$(\Delta+a^n)^\lambda\int x^{m-1}\partial x\left(1-\tfrac{a^n-x^n}{\Delta+x^n}\right)^\lambda .$$

But with the expansion made there becomes

$$\left(1-\tfrac{a^n-x^n}{\Delta+x^n}\right)^\lambda=1-\tfrac{\lambda}{1}\left(\tfrac{a^n-x^n}{\Delta+x^n}\right)+\tfrac{\lambda(\lambda-1)}{2}\left(\tfrac{a^n-x^n}{\Delta+x^n}\right)^2-\text{etc.}$$

which series therefore multiplied by $x^{m-1}\partial x$ thus must be integrated, so that the integral may be extended from $x=0$ as far as to $x=a$ . Hence it will be apparent the whole question to be reduced to integrations of the form $\int x^{m-1}\partial x(a^n-x^n)^\theta$ , the value of which for the case where $\theta=0$ evidently is $\tfrac{x^m}{m}=\tfrac{a^m}{m}$. Truly in the case where $\theta=1$ it will become

$$\int x^{m-1}\partial x(a^n-x^n)=\tfrac{a^nx^m}{m}-\tfrac{x^{m+n}}{m+n}\cdot$$

which value, on putting $x=a$, becomes $\tfrac{n}{m(m+n)}a^{m+n}$. And in the case when $\theta=2$ it will become

$$\int x^{m-1}\partial x(a^n-x^n)^2=a^{2n}\tfrac{x^m}{m}-2a^n\tfrac{x^{m+2n}}{m+2n}+\tfrac{x^{m+2n}}{m+2n},$$

which expression on putting $x=a$ will go into this form $\tfrac{n\cdot 2n}{m(m+n)(m+2n)}a^{m+2n}$.

In a similar manner with the calculation undertaken there will be found

$$\int x^{m-1}\partial x(a^n-x^n)^3=\tfrac{n\cdot 2n\cdot 3n}{m(m+n)(m+2n)(m+3n)}a^{m+3n}$$

But here, least we may attribute too much to induction, we will demonstrate this progression more carefully in the following manner.

§.7. Now we may put the value of the formula found, $\int x^{m-1}\partial x(a^n-x^n)^\theta$ to be $=\mathrm{V}$, and hence we seek the value of the following formula [in the series] $\int x^{m-1}\partial x(a^n-x^n)^{\theta+1}$.

To this end, we may put

$$\int x^{m-1}\partial x(a^n-x^n)^{\theta+1}=\mathrm{A}\int x^{m-1}\partial x(a^n-x^n)^{\theta}+\mathrm{B}x^m(a^n-x^n)^{\theta+1},$$

which formula, differentiated and divided by $x^{m-1}\partial x(a^n-x^n)^{\theta}$ , will give

$$a^n-x^n=\mathrm{A}+m\mathrm{B}(a^n-x^n)-(\theta+1)n\mathrm{B}x^n;$$

from which these two determinations arise

$$\mathrm{A}+m\mathrm{B}a^n=a^n \text{ and } mB+(\theta+1)nB=1,$$

which provide

$$\mathrm{A}=\frac{(\theta+1)na^n}{m+(\theta+1)n} \text{ and } \mathrm{B}=\frac{1}{m+(\theta+1)n}.$$

§.8. Therefore, since after the integration there must be $x=a$, the part influenced by the letter B vanishes, and there will be

$$\int x^{m-1}\partial x(a^n-x^n)^{\theta+1}=\frac{(\theta+1)na^n}{m+(\theta+1)n}.\mathrm{V}.$$

Therefore since in the case $\theta=0$ there shall be $\mathrm{V}=\frac{a^m}{m}$, there will be

$$\int x^{m-1}\partial x(a^n-x^n)=\frac{n}{m(m+n)}a^{m+n},$$

$$\int x^{m-1}\partial x(a^n-x^n)^2=\frac{n\cdot 2n}{m(m+n)(m+2n)}a^{m+2n},$$

$$\int x^{m-1}\partial x(a^n-x^n)^3=\frac{n\cdot 2n\cdot 3n}{m(m+n)(m+2n)(m+3n)}a^{m+3n}.$$

From which it is apparent the above order is observed to be established in the nature of the calculations.

§. 9. Because here the integrals must be taken thus, so that they vanish on putting $x=0$ , in the general reduction, which we have used where the latter part was $\mathrm{B}x^m(a^n-x^n)^{\theta+1}$, it is clear, this part does not vanish, unless $m>0$ ; on account of which, if perhaps formulas of this kind occur, where the exponent *m* were either 0 or indeed negative, the reductions found here could not be used.

§.10. These individual terms involve the common factor $\frac{a^m}{m}$, which as since they may be connected generally by multiplication, the series will arise by integration :

$$\frac{a^m}{m}\left(\Delta+a^n\right)^{\lambda}\left\{1-\frac{\lambda}{1}\cdot\frac{n}{m+n}\left(\frac{a^n}{\Delta+x^n}\right)+\frac{\lambda(\lambda-1)}{1.2}\cdot\frac{m.n}{(m+n)(m+2n)}\left(\frac{a^n}{\Delta+x^n}\right)^2-\text{etc}\right\}$$

where the coefficients will be able to be contracted in the following manner:

$$\frac{a^m}{m}\left(\Delta+a^n\right)^{\lambda}\left\{1-\frac{\lambda n}{m+n}\left(\frac{a^n}{\Delta+a^n}\right)+\frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\left(\frac{a^n}{\Delta+a^n}\right)^2-\frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\frac{(\lambda-2)n}{m+3n}\cdot\left(\frac{a^n}{\Delta+a^n}\right)^3+\text{etc.}\right\}.$$

Because if now, here in place of *a,* we may substitute that same variable quantity $x$ , this series

$$\frac{x^m}{m}\left(\Delta+x^n\right)^{\lambda}\left\{1-\frac{\lambda n}{m+n}\left(\frac{x^n}{\Delta+x^n}\right)+\frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\left(\frac{x^n}{\Delta+x^n}\right)^2-\frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\frac{(\lambda-2)n}{m+3n}\cdot\left(\frac{x^n}{\Delta+x^n}\right)^3+\text{etc.}\right\}.$$

expresses the value of the integral formula $\int x^{m-1}\partial x\left(\Delta+x^{m-1}\right)^{\lambda}$ obtained from the initial term $x=0$.

§.11. For the cases in which the exponent $\lambda$ is a whole positive number, the truth of the series found is apparent at once ; as in these cases :

1°) If $\lambda=1,$ there will be

$$\int x^{m-1}\partial x\left(\Delta+x^n\right)=\frac{x^m}{m}\left(\Delta+x^n\right)\left(1-\frac{n}{m+n}\frac{x^n}{\Delta+x^n}\right),$$

which expression is reduced to this $\frac{x^m}{m}\left(\Delta+x^n-\frac{n}{m+n}x^n\right)$ truly with the integral taken in the ordinary manner there will be $\frac{\Delta x^m}{m}+\frac{x^{m+n}}{m+n},$ which agrees with the preceding.

2°) If there were $\lambda=2$ , there will be

$$\int x^{m-1}\partial x\left(\Delta+x^n\right)^2=\frac{x^m}{m}\left(\Delta+x^n\right)^2\left[1-\frac{2n}{m+n}\left(\frac{x^n}{\Delta+x^n}\right)+\frac{2n}{m+n}\cdot\frac{2}{m+n}\left(\frac{x^n}{\Delta+x^n}\right)^2\right]$$

which expression is reduced to this :

$$\frac{x^m}{m}\left\{\Delta\Delta+2\Delta x^n+x^{2n}-\frac{2n}{m+n}\Delta x^n-\frac{2n}{m+n}x^{2n}+\frac{n\cdot 2n}{(m+n)(m+2n)}x^{2n}\right\},$$

or contracted to this:

$$\frac{x^m}{m}\left(\Delta\Delta+\frac{2n}{m+n}\Delta x^n+\frac{m}{m+2n}x^{2n}\right),$$

which agrees exactly with the integral taken in the usual manner. For the rest it helps to remember, this integration cannot be used, unless *m* were greater than zero, because otherwise will not be able to be taken, as it may vanish in the case $x=0$.

§.12. But if the exponent $\lambda$ were not a whole number, the series found is progressing to infinity, and its truth can no longer be seen at once. But in these cases the form of our integral may be returned simpler and more condensed, if we should put $\lambda=-\frac{\mu}{n}$ ; indeed then the integral of this formula $\int x^{m-1}\partial x\left(\Delta+x^n\right)^{-\frac{\mu}{n}}$ will become

$$\frac{x^m}{m\left(\Delta+x^n\right)^{\frac{\mu}{n}}}\left\{\begin{array}{l}1+\frac{\mu}{m+n}\left(\frac{x^n}{\Delta+x^n}\right)+\frac{\mu}{m+n}\cdot\frac{\mu}{m+2n}\left(\frac{x^n}{\Delta+x^n}\right)^2\\ +\frac{\mu}{m+n}\cdot\frac{\mu}{m+2n}\cdot\frac{\mu}{m+3n}\left(\frac{x^n}{\Delta+x^n}\right)^3+\text{etc.}\end{array}\right\}.$$

§.13. Hence the sum of this general series will now be able to be assigned

$$1+\tfrac{a}{b}\chi+\tfrac{a}{b}\cdot\tfrac{a+n}{b+n}\chi^2+\tfrac{a}{b}\cdot\tfrac{a+n}{b+n}\cdot\tfrac{a+2n}{b+2n}\chi^3+\text{etc.}$$

Indeed if we may compare this series with that found, there will be $\mu=a$, and $m+n=b$, and thus $m=b-n$ ; then indeed there becomes $\chi=\frac{x^n}{\Delta+x^n}$ , from which the relation between $\chi$ and $x$ is known. Then, therefore the sum of this series will be equal to this integral formula

$$\int\frac{x^{b-n-1}\partial x}{\left(\Delta+x^n\right)^{\frac{a}{n}}}$$

divided by this quantity $\frac{x^{b-n}}{(b-n)\left(\Delta+x^n\right)^{\frac{a}{n}}}$; and thus this sum will be

$$\frac{(b-n)\left(\Delta+x^n\right)^{\frac{a}{n}}}{x^{b-n}}\cdot\int\frac{x^{b-n-i}\partial x}{\left(\Delta+x^n\right)^{\frac{a}{n}}},$$

but this sum is unable to be present, unless $b>n$. For the rest it is evident, this series is convergent always, since not only the fraction $\frac{x^n}{\Delta+x^n}$ shall be smaller than one, but also all the coefficients shall be less than one.

§.14. But the most memorable case, which happens here, is when $\Delta=0$ ; then indeed our formula of the integral will be

$$\int x^{m-\mu-1}\partial x=\frac{x^{m-\mu}}{m-\mu}.$$

to which quantity the following series therefore will be equal always

$$\frac{x^{m-\mu}}{m}\left(1+\frac{\mu}{m+n}+\frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}+\frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\cdot\frac{\mu+2n}{m+3n}+\text{etc.}\right)$$

but only if *m* were a positive number, as now is noted a number of times. Consequently the sum of this series

$$1+\frac{\mu}{m+n}+\frac{\mu(\mu+n)}{(m+n)(m+2n)}+\frac{\mu(\mu+n)(\mu+2n)}{(m+n)(m+2n)(m+3n)}+\text{etc.}$$

is $\frac{m}{m-\mu}$, which summation is there the most noteworthy, because hardly any way is apparent, how its truth is to be investigated.

§.15. But it is at once apparent, that this sum cannot exist, unless both $n$ as well as $m-\mu$ were a positive number. Since indeed our integral formula in the case $\Delta=0$ shall be $\int x^{m-1-\mu}\partial x$ as it is required to start from $x=0$, it is evident that this cannot happen, unless $m-\mu$ were a positive number ; besides also it is required to be noted; the exponent $n$ by necessity must be positive . For since in the analysis set out above this integral may occur $\int x^{m-1}\partial x\left(a^{n}-x^{n}\right)^{\theta}$, it is evident , if $n$ were a negative number, the integration thus cannot be performed, so that it may vanish in the case $x=0$ . With the same noted I am going to consider this series more carefully and because its innate nature, not a little abstruse, may become apparent, I am going to show its truth in a two-fold manner. Clearly in the first place, I will show, the sum assigned to be equal to the actual sum of the whole progression ; then I will show by a particular analysis, that not only will lead to the same series being produced, but also will indicate its sum.

Demonstration of this summation:

$$\frac{m}{m-\mu}=1+\frac{\mu}{m+n}+\frac{\mu(\mu+n)}{(m+n)(m+2n)}+\frac{\mu(\mu+n)(\mu+2n)}{(m+n)(m+2n)(m+3n)}+\text{etc.}$$

§.16. Here of course I will show, if all the terms of the series shall be subtracted successively from the sum of the series found $\frac{m}{m-\mu}$, finally truly nothing is going to remain. Indeed, with the first term 1 subtracted, $\frac{\mu}{m-\mu}$ will remain. Hence the second term taken away leaves $\frac{\mu(\mu+n)}{(m-\mu)(m+n)}$. Hence again the third term may be subtracted and $\frac{\mu(\mu+n)(\mu+2n)}{(m-\mu)(m+n)(m+2n)}$ will remain .

Hence now the fourth term taken away gives the following remainder

$$\frac{\mu(\mu+n)(\mu+2n)(\mu+3n)}{(m-\mu)(m+n)(m+2n)(m+3n)}$$

From which now it is clear enough, with all the terms taken away, only this product to be left after an infinite excursion

$$\frac{\mu(\mu+n)(\mu+2n)(\mu+3n)(\mu+4n)(\text{etc.}}{(m-\mu)(m+n)(m+2n)(m+3n)(m+4n)(\text{etc.}}.$$

§.17. But it is readily understood the value of this product can never actually become equal to zero. For with the first factor $\frac{m}{m-\mu}$ omitted, all the remaining factors are fractions less than one, because $\mu < m$, and because both the numerators as well as the denominators increase in arithmetical progressions, it is agreed well enough, the value of such a product actually vanish. But here the truth must be established, that a product of an infinite number of such fractions shall go to zero, it is not sufficient, that the individual terms shall be less than one, just as occurs in this form

$$\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{15}{16}\cdot\frac{24}{25}\cdot\frac{35}{36}\cdot\frac{48}{49}\cdot\text{etc.},$$

of which the product extended to infinity may be shown easily to have the value $=\frac{1}{2}$.

§.18. Because in our product the individual denominators exceed their numerators by the same quantity $m=\mu$, I will consider this more general form

$$\frac{a}{a+\Delta}\cdot\frac{b}{b+\Delta}\cdot\frac{c}{c+\Delta}\cdot\frac{d}{d+\Delta}\cdot\frac{e}{e+\Delta}\cdot\text{etc.}$$

and I will search, under which conditions of its value extended to infinity, which shall be $\Pi$, actually shall be going to zero. But it is evident, this happens, if with the same formula inverted

$$\frac{1}{\Pi}=\frac{a+\Delta}{a}\cdot\frac{b+\Delta}{b}\cdot\frac{c+\Delta}{c}\cdot\frac{d+\Delta}{d}\cdot\text{ etc.}$$

may be increased indefinitely. But if its value were infinite, also its logarithm by necessity must become infinite. Therefore since there shall be

$$l\frac{1}{\Pi}=l\frac{a+\Delta}{a}+l\frac{b+\Delta}{b}+l\frac{c+\Delta}{c}+l\frac{d+\Delta}{d}+\text{ etc.}$$

with the product expanded out there will be found

$$\begin{aligned}l\frac{1}{\Pi}=&\Delta\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\text{etc.}\right)\\ &-\frac{1}{2}\Delta^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2}+\text{etc.}\right)\\ &+\frac{1}{3}\Delta^3\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}+\frac{1}{e^3}+\text{etc.}\right)\\ &-\text{etc.}\end{aligned}$$

which expression will be infinite always, whenever the sum of the first series $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\text{etc}$ were infinite. But this sum will be infinite always, whenever the numbers *a, b, c, d,* etc. increase in an arithmetical progression, now it is to be observed and to be self evident, that it certainly pertains to our case ; the value of this infinite product to be vanishing.

§.19. But concerning our series this expression occurs which is worthy of note, because the letter *n* is not part of this sum $\frac{m}{m-\mu}$, thus so that its sum shall always remain the same, whatever values may be attributed to the letter $n$, which indeed for the case $n=0$ itself shall be evident at once, as whenever our series emerges

$$1 + \frac{\mu}{m} + \frac{\mu^2}{m^2} + \frac{\mu^3}{m^3} + \text{etc.},$$

which since it shall be a geometric progression, the sum of which shall be $\frac{m}{m-\mu}$. Truly since the sum shall remain the same always, whatever values may be attributed to $n$, may not be seen as easily, as the truth shall now be shown by us.

§. 20. Indeed this demonstration treated here may appear from a much broader point of view, since thus values of $n$ itself in the same form may be allowed to vary. Thus if by putting $\alpha$ in place of $n$, and for its multiples $2n$, $3n$, $4n$, $5n$, etc. $2n$, $3n$, $4n$, $5n$, etc. we may write the new letters $\beta,\ \gamma,\ \delta,\ \varepsilon,$ etc. in order that this series may be had

$$1+\frac{\mu}{m+\alpha}+\frac{\mu}{m+\alpha}\cdot\frac{\mu+\alpha}{m+\beta}+\frac{\mu}{m+\alpha}\cdot\frac{\mu+\alpha}{m+\beta}\cdot\frac{\mu+\beta}{m+\gamma}+\text{etc.},$$

the sum of this also now will be $\frac{m}{m-\mu}$. For with the first term subtracted there shall remain $\frac{\mu}{m-\mu}$. Hence the second term subtracted leaves

$$\frac{\mu(\mu+\alpha)}{(m-\mu)(m+\alpha)}$$

Hence the third term subtracted leaves

$$\frac{\mu(\mu+\alpha)(\mu+\beta)}{(m-\mu)(m+\alpha)(m+\beta)}$$

From which it is now apparent, finally produced indefinitely

$$\frac{\mu(\mu+\alpha)(\mu+\beta)(\mu+\gamma)(\mu+\delta)(\text{etc.}}{(m-\mu)(m+\alpha)(m+\beta)(m+\gamma)(m+\delta)(\text{etc.}},$$

the product of this value will vanish always, only if this series,

$$\frac{1}{\mu+\alpha} + \frac{1}{\mu+\beta} + \frac{1}{\mu+\gamma} + \frac{1}{\mu+\delta} + \text{etc.}$$

has become an infinitely great sum, as in the manner we have shown before.

The particular analysis leading directly to the series found above.

§.21. We may put

$$x^m\left(1-x^n\right)^\theta = \mathrm{A}\int x^{m-1}\partial x\left(1-x^n\right)^\theta + \mathrm{B}\int x^{m-1}\partial x\left(1-x^n\right)^{\theta-1},$$

and there shall be found $\mathrm{A} = m+\theta n$ and $\mathrm{B} = -\theta n$; hence therefore if we may put $\int x^{m-1}\partial x\left(1-x^n\right)^\theta = \mathrm{P}$ and $\int x^{m-1}\partial x\left(1-x^n\right)^{\theta-1} = \mathrm{Q}$, there will be

$x^m\left(1-x^n\right)^\theta=\left(m+\theta n\right)\mathrm{P}-\theta n\mathrm{Q}$, and thus

$$\mathrm{Q}=\tfrac{m+\theta n}{\theta n}\cdot\mathrm{P}-\tfrac{1}{\theta n}x^m\left(1-x^n\right)^\theta.$$

Because if now we may extend both the integrals P and Q from the limit $x=0$ as far as to $x=1$, there will be $\mathrm{Q}=\frac{m+\theta n}{\theta n}\times\mathrm{P};$ but only if there were both $m>0$ as well as $e>0$.

§. 22. Now since there shall be $\partial\mathrm{Q}=\frac{\partial\mathrm{P}}{1-x^n}$, with the denominator expanded out into a series, there will be

$$\partial Q=\partial\mathrm{P}\left(1+x^n+x^{2n}+x^{3n}+x^{4n}+\text{etc.}\right):$$

consequently there will be had

$$\mathrm{Q}=\mathrm{P}+\int x^n\partial\mathrm{P}+\int x^{2n}\partial\mathrm{P}+\int x^{3n}\partial\mathrm{P}+\text{etc.}$$

which individual integrals are prepared thus, so that any shall be able to be reduced to the previous, with the aid of this reduction:

$$x^\alpha\left(1-x^n\right)^{\theta+1}=\mathrm{A}\int x^{\alpha+n-1}\partial x\left(1-x^n\right)^\theta+\mathrm{B}\int x^{\alpha-1}\partial x\left(1-x^n\right)^\theta,$$

from which there is found $\mathrm{A}=-\alpha-n(\theta+1)$ and $\mathrm{B}=a$.

§. 23. But also these two integrals may be extended from the limit $x=0$ as far as to $x=1$, becoming

$$0=-\left[\alpha+n\left(\theta+1\right)\right]\int x^{\alpha+n-1}\partial x\left(1-x^n\right)^\theta+\alpha\int x^{\alpha-1}\partial x\left(1-x^n\right)^\theta,$$

but only if there were $\alpha>0$ and $\theta+1>0$. Now we may make $\alpha=m+\lambda n$, and because before we had put $x^{m-1}\left(1-x^n\right)^\theta=\partial\mathrm{P}$, this equation will be changed into this form

$$-\left[\alpha+n\left(\theta+1\right)\right]\int x^{(\lambda+1)n}\partial\mathrm{P}+\alpha\int x^{\lambda n}\partial\mathrm{P},$$

whereby we will have this reduction

$$\int x^{\lambda n+n}\partial\mathrm{P}=\frac{\alpha}{\alpha+n(\theta+1)}\int x^{\lambda n}\partial\mathrm{P}=\frac{m+\alpha n}{m+n(\lambda+\theta+1)}\int x^{\lambda n}\partial\mathrm{P}.$$

§.24. This general formula now supplies us with the following special integrals :

1°) If $\lambda=0$ | $\int x^n\partial\mathrm{P}=\frac{m}{m+n(\theta+1)}\mathrm{P}$,

2°) If $\lambda=1$ | $\int x^{2n}\partial\mathrm{P}=\frac{m+n}{n+n(\theta+2)}\int x^n\partial\mathrm{P}$, and thus

$$\int x^{2n}\partial\mathrm{P}=\frac{m}{m+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)}\mathrm{P},$$

3°) If $\lambda = 3$ | $\int x^{3n}\partial \mathrm{P} = \frac{m+2n}{n+n(\theta+3)}\int x^{2n}\partial \mathrm{P}$, and thus

$$\int x^{3n}\partial \mathrm{P} = \frac{m}{n+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)}\cdot\frac{m+2n}{m+n(\theta+3)}\mathrm{P},$$

4°) If $\lambda = 4$ | $\int x^{4n}\partial \mathrm{P} = \frac{m}{n+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)}\cdot\frac{m+2n}{m+n(\theta+3)}\cdot\frac{m+3n}{m+n(\theta+4)}\mathrm{P}.$

etc. | etc.

§. 25. Therefore since from the above there shall become

$$Q = \int \partial \mathrm{P}\left(1 + x^n + x^{2n} + x^{3n} + x^{4n} + \text{etc.}\right),$$

if, for the individual terms, we may only substitute the values found, and divide both sides by P, we will obtain this equation

$$\frac{\mathrm{Q}}{\mathrm{P}} = 1 + \frac{m}{m+n(\theta+1)} + \frac{m}{m+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)} + \frac{m}{m+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)}\cdot\frac{m+2n}{m+n(\theta+3)} + \text{etc.}$$

but above we have shown to be $\frac{\mathrm{Q}}{\mathrm{P}} = \frac{m+\theta n}{n\theta}$, which fraction therefore is the sum of this infinite series.

§. 26. Now so that we may show the agreement of this series with that found above, in the first place for $\theta$ there may write $\frac{\mu}{n}$, and our series found adopts this form :

$$\frac{m+\mu}{n} = \frac{m}{m+n+\mu} + \frac{m}{m+n+\mu}\cdot\frac{m+n}{m+2n+\mu} + \frac{m}{m+n+\mu}\cdot\frac{m+n}{m+2n+\mu}\cdot\frac{m+2n}{m+3n+\mu} + \text{etc.},$$

the truth of which can be demonstrated in the same manner which I have used before. For if the first term may be subtracted from the sum , there remains $\frac{\mu}{n}$. Hence with the second term subtracted $\frac{m(m+n)}{\mu(m+n+\mu)}$ remains. Hence again the third term subtracted leaves $\frac{m(m+n)(m+2n)}{\mu(m+n+\mu)(m+2n+\mu)}$ and thus so on, which operation if it may be continued indefinitely, the resultant value produced of this is $= 0$. Then truly it is evident, this series found to be the same as that we gave above to be transformed into this same one, if here in place of *m* there may be written $\mu$, and truly $m - \mu$ in place of P.

§. 27. In place of a colophon I may attach this much more general series of the same kind, the sum of which is equally allowed to be assigned, which I am going to include in the following problem.

## Problem I.

§. 28. *For this series proposed,* $A+B\frac{\alpha}{a}+C\frac{\alpha\beta}{ab}+D\frac{\alpha\beta\gamma}{abc}+etc.$*, to investigate the conditions under which its sum may be allowed to be assigned.*

## Solution.

This series therefore involves three series: the first of the letters $\alpha,\beta,\gamma,\delta,$etc. which constitute the numbers of the proposed series; the second of the letters *a, b, c, d,* etc., from which the denominators are formed; the third of the letters A, B, C, D, etc. which show the coefficients of the terms. Therefore just as these three series must be compared, so that the sum of the proposed series may be permitted to be assigned by a finite and thus rational expression, this I am going to investigate.

§. 29. We may establish the sum of this series to be $\frac{\mathrm{S}}{t}$ , and by the same method as we have used above, namely from this sum at first we subtract the first term A and since there shall remain $\frac{\mathrm{S}-\mathrm{A}t}{t}$, we may set $\mathrm{S}-At=\alpha$, so that we may have $\frac{\alpha}{t}$ ; hence we may subtract the second term $B\frac{\alpha}{a}$, and the remainder will be $\frac{\alpha(a-Bt)}{t\cdot a}$. Now here we may make $a-Bt=\beta$ , so that we shall have $\frac{\alpha\beta}{t\cdot a}$; so that if the third term $C\frac{\alpha\beta}{ab}$ may be subtracted, the remainder will be $\frac{\alpha\beta(b-Ct)}{t\cdot ab}$. Here there may be made $b-\mathrm{C}t=\gamma$, in order that we may have $\frac{\alpha\beta\gamma}{t\cdot ab}$, from which the fourth term removed leaves $\frac{\alpha\beta\gamma(c-Dt)}{t\cdot abc}$. Here there may again be put $c-Dt=\delta$ , so that we may have $\frac{\alpha\beta\gamma\delta}{t\cdot abc}$, so that by subtracting the fifth term there is deduced $\frac{\alpha\beta\gamma\delta(d-Et)}{t\cdot abcd}$. And these operations may be understood to be continued indefinitely.

§.30. Both the letter S as well as the letters *a, b, c, d,* etc. may be defined in the following manner from these determinations :

$$S=\alpha+\mathrm{A}t;\ a=\beta+\mathrm{B}t;\ b=\gamma+\mathrm{C}t;\ c=\delta+\mathrm{D}t;\ \text{etc.}$$

And the remainder from all these values introduced, after all the terms of the series shall have been removed from this formula $\frac{\mathrm{S}}{t}$, $\frac{\alpha\beta\gamma\delta\varepsilon\zeta\ etc.}{t\cdot abcdef\ etc.}$ will remain going off to infinity, which product therefore if it may go to zero, then the sum of the proposed series actually will be $=\frac{\mathrm{S}}{t}$ . Therefore we may consider under which conditions this product may vanish.

§. 31. We may designate this product by the letter Π, so that with the values substituted for *a, b, c,* etc. from the values found, there will be

$$\Pi = \tfrac{\mathrm{S}}{t}\left(\tfrac{\alpha}{\alpha+\mathrm{A}t}\cdot\tfrac{\beta}{\beta+\mathrm{B}t}\cdot\tfrac{\gamma}{\gamma+\mathrm{C}t}\cdot\tfrac{\delta}{\delta+\mathrm{D}t}\cdot\text{etc.}\right)$$

where clearly we have prefixed the factor $\frac{\mathrm{S}}{t}$. Now therefore it is sought under what conditions this same product shall be going to zero on being continued to infinity. But Moreover it is evident to eventuate here, if the product itself may be inverted, and its logarithm will become infinitely great.
This therefore arises, when the sum of these logarithms

$$l\left(1+\tfrac{\mathrm{A}t}{\alpha}\right)+l\left(1+\tfrac{\mathrm{B}t}{\beta}\right)+l\left(1+\tfrac{\mathrm{C}t}{\gamma}\right)+l\left(1+\tfrac{\mathrm{D}t}{\delta}\right)+\text{etc.}=\infty;$$

which happens always, if with so many terms taken first, which all have the common factor *t,* this series

$$\tfrac{\mathrm{A}}{\alpha}+\tfrac{\mathrm{B}}{\beta}+\tfrac{\mathrm{C}}{\gamma}+\tfrac{\mathrm{D}}{\delta}+\text{etc.}$$

will have an infinitely great sum, therefore moreover the sum of our proposed series will always be $\frac{\alpha+\mathrm{A}t}{t}$.

§.32. Nor indeed is it absolutely necessary, that the product Π vanishes completely, but may have had evidently some value Π, because this arises after the whole sum of the proposed series which we may put $=\mathrm{S}$, it may be taken from the formula $\frac{\mathrm{S}}{t}$, thus so that there shall be $\Pi=\frac{\mathrm{S}}{t}-\mathrm{S}$, from which it is evident there shall become $\mathrm{S}=\frac{\mathrm{S}}{t}-\Pi$.

§. 33. So that we may illustrate this by an example, to the letters $\alpha,\ \beta,\ \gamma,\ \delta,$ etc. we may attribute these values $\alpha=3$, $\beta=15$, $\gamma=35$, $\delta=63$, etc. in addition indeed there shall be $t=1$, and in addition $\mathrm{A}=\mathrm{B}=\mathrm{C}=\mathrm{D}=\text{etc.}=1$: hence therefore the determinations found provide :

$$\mathrm{S}=4,\ a=16,\ b=36,\ c=64,\ d=100,\ \text{etc.}$$

And thus our series will become now

$$1+\tfrac{3}{16}+\tfrac{3\cdot15}{16\cdot36}+\tfrac{3\cdot15\cdot35}{16\cdot36\cdot64}+\tfrac{3\cdot15\cdot35\cdot63}{16\cdot36\cdot64\cdot100}+\text{etc.}$$

for the sum of which it may be observed to be

$$\Pi=4\cdot\tfrac{3\cdot15\cdot35\cdot63\cdot99}{4\cdot16\cdot36\cdot64\cdot100}\,\text{etc.}$$

Moreover it is agreed from the quadrature of the circle by *Wallis* to be

$$\frac{3\cdot15\cdot35\cdot63\cdot99\ etc.}{4\cdot16\cdot36\cdot64\cdot100\ etc.}=\frac{2}{\pi},$$

with π being the periphery of the circle whose diameter is unity. Hence therefore $\Pi=\frac{8}{\pi}$, and thus the sum of our series $\mathrm{S}=4-\frac{8}{\pi}$, and thus approximately $\frac{16}{11}$.

§.34. But indeed the general series, which we have obtained in this manner, is most fertile in the formation of innumerable special series, since not only the series of letters $\alpha,\ \beta,\ \gamma,\ \delta,$ etc. but also of the letters A, B, C, D, etc. may be assumed as wished, since thence the letters *a, b, c, d,* etc. may be determined at once ; but as the sum of such series always can be assigned, only if the value of the product may be extended to infinity, which we have indicated by the letter Π, where it will be able to be defined likewise, whether that were a rational value or thus involving some transcending quadrature.

# SUPPLEMENTUM II. AD TOM. I. CAP. III.

## INTEGRATIONE FORMULARUM DIFFERENTIALIUM PER SERIES INFINITAS.

### De resolutione formulae integralis; $\int x^{m-1}\partial x\left(\Delta+x^{m-1}\right)^{\lambda}$ in seriem semper convergentem. Ubi simul plura insignia artificia circa serierum summationem explicantur. *M. S. Academiae exhib. die* 12 *Aug.* 1779.

§.1. Obtulit se mihi nuper haec formula integralis $\int\partial x\sqrt{\left(\Delta+x^4\right)}$, cujus valor, cum casu quo $\Delta=0$ sit $\frac{1}{3}x^3$, in mentem mihi venit, eos ejus valores investigare, quos induit, quando $\Delta$ est quantitas valde parva. Mox autem vidi , hoc vulgari evolutione praestari neutiquam posse. Cum enim sit

$$\sqrt{\left(\Delta+x^4\right)}=\sqrt{\Delta}\times\left(1+\frac{x^4}{\Delta}\right)^{\frac{1}{2}},$$

ideoque per seriem

$$\sqrt{\left(\Delta+x^4\right)}=\sqrt{\Delta}\left(1+\frac{1}{2}\cdot\frac{x^4}{\Delta}-\frac{1\cdot 1}{2\cdot 4}\cdot\frac{x^8}{\Delta^2}+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{x^{12}}{\Delta^3}-\text{etc.}\right).$$

erit valor formulae hujus integralis

$$\int\partial x\sqrt{\left(\Delta+x^4\right)}=x\sqrt{\Delta}\times\left(1+\frac{1}{2}\cdot\frac{x^4}{5\Delta}-\frac{1\cdot 1}{2\cdot 4}\cdot\frac{x^8}{9\Delta^2}+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{x^{12}}{13\Delta^3}-\text{etc.}\right).$$

quae series ergo manifesta maxime divergit, quoties $\Delta$ fuerit quantitas valde parva, atque adeo, quoties fractio $\frac{x^n}{\Delta}$ unitatem superaverit.

§. 2. Ut igitur ad scopum propositum pertingerem, ipsam hanc quaestionem sub hac forma sum contemplatus: *Valorem formulae integralis* $\int x^{m-1}\partial x\sqrt{\left(\Delta+x^4\right)}$ *a termina* $x=0$ *usque ad terminum* $x=a$ *extensum per seriem semper convergentem exprimere, quicunque valor litterae* $\Delta$ *tribuatur.* Hunc in finem formulam $\Delta+x^4$ sub hac specie repraesento

$$\Delta+a^4-\left(a^4-x^4\right),$$

sive hac

$$\left(\Delta+a^4\right)\left(1-\frac{a^4-x^4}{\Delta+a^4}\right).$$

Hinc igitur erit

$$\sqrt{\left(\Delta+x^4\right)}=\sqrt{\left(\Delta+a^4\right)}\times\left(1+\frac{1}{2}\cdot\frac{a^4-x^4}{\Delta+a^4}-\frac{1\cdot 1}{2\cdot 4}\cdot\left(\frac{a^4-x^4}{\Delta+a^4}\right)^2-\text{etc.}\right).$$

Sicque totum negotium huc redit ut harum formularum integralium

$$\int \partial x\left(a^4-x^4\right),\ \int \partial x\left(a^4-x^4\right)^2,\ \int \partial x\left(a^4-x^4\right)^3,\text{etc.}$$

valores ab $x=0$ usque ad $x=a$ extensi investigentur, unde primus terminus $\int \partial x$ dabit *a*.

§.3. Pro secundo termino habebitur integrando
$\int \partial x\left(a^4-x^4\right)=a^4x-\frac{1}{5}x^5$, cujus valor sumto $x=a$ erit $\frac{4}{5}a^5$. Pro tertio termino erit
$\int \partial x\left(a^4-x^4\right)^2=a^8x-\frac{2}{5}a^4x^5+\frac{1}{9}x^9$, quae expressio posito $x=a$ abit in $\frac{4\cdot 8}{5\cdot 9}a^9$. Simili modo pro quarto termina habebimus

$$\int \partial x\left(a^4-x^4\right)^3=a^{13}\left(1-\tfrac{3}{5}+\tfrac{3}{9}-\tfrac{1}{13}\right)^3=\tfrac{4\cdot 8\cdot 12}{5\cdot 9\cdot 13}a^{13}.$$

Eodemque modo reperitur fore

$$\int \partial x\left(a^4-x^4\right)^4=\tfrac{4\cdot 8\cdot 12\cdot 16}{5\cdot 9\cdot 13\cdot 17}a^{17},$$

et ita porro. Hanc autem elegantem progressionis legem infra sum demonstraturus.

§. 4. His igitur valoribus substitutis , totus valor integralis quaesitus reperietur fore

$$a\sqrt{\left(\Delta+a^4\right)}\times\left[1-\tfrac{1}{2}\cdot\tfrac{4}{5}\tfrac{a^4}{\Delta+a^4}-\tfrac{1\cdot 1}{2\cdot 4}\cdot\tfrac{4\cdot 8}{5\cdot 9}\cdot\left(\tfrac{a^4}{\Delta+a^4}\right)^2-\tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\tfrac{4\cdot 8\cdot 12}{5\cdot 9\cdot 13}\cdot\left(\tfrac{a^4}{\Delta+a^4}\right)^3-\text{etc.}\right].$$

Quoniam hic duplices coëfficientes occurrunt, si singulos factores priorum tam supra quam infra duplicemus, ista series contrahetur in sequentem

$$a\sqrt{\left(\Delta+a^4\right)}\times\left[1-\tfrac{2}{5}\tfrac{a^4}{\Delta+a^4}-\tfrac{2\cdot 2}{5\cdot 9}\cdot\left(\tfrac{a^4}{\Delta+a^4}\right)^2-\tfrac{2\cdot 2\cdot 6}{5\cdot 9\cdot 13}\cdot\left(\tfrac{a^4}{\Delta+a^4}\right)^3-\text{etc.}\right],$$

quae series manifesta semper convergit, propterea quod non solum coëfficientes haud mediocriter decrescunt, sed etiam formula $\frac{a^4}{\Delta+a^4}$ unitate est minor.

§. 5. Jam nihil obstat quo minus loco *a* restituamus ipsam quantitatem variabilem *x*, sicque valor hujus formulae integralis $\int \partial x\sqrt{\left(\Delta+x^4\right)}$ exprimetur per segmentem seriem semper convergentem

$$x\sqrt{\left(\Delta+x^4\right)}\times\left[1-\tfrac{2}{5}\tfrac{x^4}{\Delta+x^4}-\tfrac{2\cdot 2}{5\cdot 9}\cdot\left(\tfrac{x^4}{\Delta+x^4}\right)^2-\tfrac{2\cdot 2\cdot 6}{5\cdot 9\cdot 13}\cdot\left(\tfrac{x^4}{\Delta+x^4}\right)^3-\text{etc.}\right].$$

Hic casus quo ista series minime convergit, est ille ipse, quem initia commemoravimus, quo $\Delta = 0$, ipsumque integrale $= \frac{1}{3}x^3$. Posito igitur $\Delta = 0$ pervenimus ad sequentem seriem maxime notatu dignam

$$x^3\left(1 - \frac{2}{5} - \frac{2\cdot2}{5\cdot9} - \frac{2\cdot2\cdot6}{5\cdot9\cdot13} - \frac{2\cdot2\cdot6\cdot10}{5\cdot9\cdot13\cdot17} - \text{etc.}\right),$$

cujus adeo summam novimus esse $\frac{1}{3}x^3$, ita ut jam habeamus hanc summationem

$$\frac{1}{3} = 1 - \frac{2}{5} - \frac{2}{5}\cdot\frac{2}{9} - \frac{2}{5}\cdot\frac{2}{9}\cdot\frac{6}{13} - \frac{2}{5}\cdot\frac{2}{9}\cdot\frac{6}{13}\cdot\frac{10}{17} - \text{etc.}$$

cujus demonstratio altioris indaginis videtur. Interim tamen quoniam ejus summa est cognita, veritas sequenti modo ostendi potest.
Hinc enim erit

$$\frac{2}{5} + \frac{2\cdot2}{5\cdot9} + \frac{2\cdot2\cdot6}{5\cdot9\cdot13} + \text{etc.} = \frac{2}{3}.$$

quae aequatio in $\frac{5}{2}$ ducta dat

$$1 + \frac{2}{9} + \frac{2\cdot6}{9\cdot13} + \frac{2\cdot6\cdot10}{9\cdot13\cdot17} + \text{etc.} = \frac{5}{3}.$$

Transponatur hic primus terminus in alteram partem, et multiplicando per $\frac{9}{2}$ prodibit

$$1 + \frac{6}{13} + \frac{6\cdot10}{13\cdot17} + \frac{6\cdot10\cdot14}{13\cdot17\cdot21} + \text{etc.} = \frac{2}{3}.$$

Translata iterum primo termino ad alteram partem factaque multiplicatione per $\frac{13}{2}$ , colligitur

$$1 + \frac{10}{17} + \frac{10\cdot14}{17\cdot21} + \frac{10\cdot14\cdot18}{17\cdot21\cdot25} + \text{etc.} = \frac{13}{3}.$$

Simili modo progrediendo prodibit

$$1 + \frac{14}{21} + \frac{14\cdot18}{21\cdot25} + \frac{14\cdot18\cdot22}{21\cdot25\cdot29} + \text{etc.} = \frac{17}{3}.$$

$$1 + \frac{18}{25} + \frac{18\cdot22}{25\cdot29} + \frac{18\cdot22\cdot26}{25\cdot29\cdot33} + \text{etc.} = \frac{21}{3}.$$

Sicque innumerabiles nacti sumus series , quarum summa est cognita, et quoniam lege aequabili ulterius progrediuntur, signum hoc certum est summam primo datam esse justam. Hanc autem insignem veritatem infra, ubi rem in genere persequemur, accuratius demonstrabimus.

## Problema generale.

*Formulae integralis* $\int x^{m-1}\partial x(\Delta + x^n)^\lambda$ *valorem a termino* $x = 0$ *usque ad* $x = a$ *extensum per seriem semper convergentem exprimere.*

## Solution.

§.6. Formulam $\Delta + x^n$ sub hac forma repraesentemus $\Delta + x^n - (a^n - x^n)$ , quae reducitur ad hanc

$$\left(\Delta+x^n\right)\left(1-\tfrac{a^n-x^n}{\Delta+x^n}\right),$$

sicque formula integralis proposita erit

$$(\Delta+a^n)^\lambda\int x^{m-1}\partial x\left(1-\tfrac{a^n-x^n}{\Delta+x^n}\right)^\lambda.$$

At facta evolutione est

$$\left(1-\tfrac{a^n-x^n}{\Delta+x^n}\right)^\lambda=1-\tfrac{\lambda}{1}\left(\tfrac{a^n-x^n}{\Delta+x^n}\right)+\tfrac{\lambda(\lambda-1)}{2}\left(\tfrac{a^n-x^n}{\Delta+x^n}\right)^2-\text{etc.}$$

quae ergo series ducta in $x^{m-1}\partial x$ ita integrari debet, ut integrale ab $x=0$ usque ad $x=a$ extendatur. Hinc patet totum negotium reduci ad hanc integrationem $\int x^{m-1}\partial x(a^n-x^n)^\theta$, cujus valor casu quo $\theta=0$ manifesto est $\frac{x^m}{m}=\frac{a^m}{m}$. Casu vero quo $\theta=1$ erit

$$\int x^{m-1}\partial x(a^n-x^n)=\tfrac{a^nx^m}{m}-\tfrac{x^{m+n}}{m+n}$$

qui valor, posito $x=a$, evadit $\frac{n}{m(m+n)}a^{m+n}$. Ac casu quo $\theta=2$ erit

$$\int x^{m-1}\partial x(a^n-x^n)^2=a^{2n}\tfrac{x^m}{m}-2a^n\tfrac{x^{m+2n}}{m+2n}+\tfrac{x^{m+2n}}{m+2n},$$

quae expressio posito $x=a$ abit in hanc $\frac{n\cdot 2n}{m(m+n)(m+2n)}a^{m+2n}$.

Simili modo calcula subducto reperietur

$$\int x^{m-1}\partial x(a^n-x^n)^3=\tfrac{n\cdot 2n\cdot 3n}{m(m+n)(m+2n)(m+3n)}a^{m+3n}$$

Ne autem hic inductioni nimium tribuamus, hanc progressionem sequenti modo accuratius demonstrabimus.

§.7. Ponamus formulae $\int x^{m-1}\partial x(a^n-x^n)^\theta$ valorem jam esse inventum $=\mathrm{V}$, hincque quaeramus valorem formulae sequentis $\int x^{m-1}\partial x(a^n-x^n)^{\theta+1}$ Hunc in finem ponamus

$$\int x^{m-1}\partial x(a^n-x^n)^{\theta+1}=\mathrm{A}\int x^{m-1}\partial x(a^n-x^n)^\theta+\mathrm{B}x^m(a^n-x^n)^{\theta+1},$$

quae formula differentiata et per $x^{m-1}\partial x(a^n-x^n)^\theta$ divisa praebet

$$a^n-x^n=\mathrm{A}+m\mathrm{B}(a^n-x^n)-(\theta+1)n\mathrm{B}x^n;$$

unde nascuntur hae duae determinationes

$$\mathrm{A}+m\mathrm{B}a^n=a^n \text{ et } mB+(\theta+1)nB=1,$$

qui praebent

$$\mathrm{A}=\tfrac{(\theta+1)na^n}{m+(\theta+1)n}\quad\text{et } \mathrm{B}=\tfrac{1}{m+(\theta+1)n}$$

§.8. Quoniam igitur post integrationem fieri debet $x = a$, membrum littera B affectum evanescit, eritque

$$\int x^{m-1}\partial x(a^n - x^n)^{\theta+1} = \frac{(\theta+1)na^n}{m+(\theta+1)n}.\mathrm{V}.$$

Cum igitur casu $\theta = 0$ sit $\mathrm{V} = \frac{a^m}{m}$, erit

$$\int x^{m-1}\partial x(a^n - x^n) = \frac{n}{m(m+n)}a^{m+n},$$

$$\int x^{m-1}\partial x(a^n - x^n)^2 = \frac{n\cdot 2n}{m(m+n)(m+2n)}a^{m+2n},$$

$$\int x^{m-1}\partial x(a^n - x^n)^3 = \frac{n\cdot 2n\cdot 3n}{m(m+n)(m+2n)(m+3n)}a^{m+3n}.$$

Unde patet ordinem supra observatum in ipsa rei natura esse fundatum.

§. 9. Quia hic integralia ita capi debent, ut evanescant posito $x = 0$, in reductione generali, qua sumus usi ubi postremum membrum erat $\mathrm{B}x^m(a^n - x^n)^{\theta+1}$, evidens est, hoc membrum non evanescere, nisi fuerit $m > 0$ ; quamobrem, si forte ejusmodi formulae occurrant , ubi exponens *m* fuerit vel 0 vel adeo negativus, reductiones hic inventae locum habere nequeunt.

§.10. Singuli hi termini factorem involvunt comunem $\frac{a^m}{m}$, qui si cum multiplicatore generali conjungatur, series per integrationem orta erit

$$\frac{a^m}{m}\left(\Delta + a^n\right)^\lambda\left\{1 - \frac{\lambda}{1}\cdot\frac{n}{m+n}\left(\frac{a^n}{\Delta+x^n}\right) + \frac{\lambda(\lambda-1)}{1.2}\cdot\frac{m.n}{(m+n)(m+2n)}\left(\frac{a^n}{\Delta+x^n}\right)^2 - \text{etc}\right\}$$

ubi coëfficientes sequenti modo contrahi poterunt

$$\frac{a^m}{m}\left(\Delta + a^n\right)^\lambda\left\{1 - \frac{\lambda n}{m+n}\left(\frac{a^n}{\Delta+a^n}\right) + \frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\left(\frac{a^n}{\Delta+a^n}\right)^2 - \frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\frac{(\lambda-2)n}{m+3n}\cdot\left(\frac{a^n}{\Delta+a^n}\right)^3 + \text{etc.}\right\}.$$

Quod si jam hic loco *a* substituamus ipsam quantitatem variabilem *x* , haec series

$$\frac{x^m}{m}\left(\Delta + x^n\right)^\lambda\left\{1 - \frac{\lambda n}{m+n}\left(\frac{x^n}{\Delta+x^n}\right) + \frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\left(\frac{x^n}{\Delta+x^n}\right)^2 - \frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\frac{(\lambda-2)n}{m+3n}\cdot\left(\frac{x^n}{\Delta+x^n}\right)^3 + \text{etc.}\right\}.$$

exprimet valorem formulae. întegralis $\int x^{m-1}\partial x\left(\Delta + x^{m-1}\right)^\lambda$ a termina $x = 0$ sumtum.

§.11. Casibus quibus exponens $\lambda$ est numerus integer positivus, veritas seriei inventae sponte elucescit ; uti his casibus

1°) Si $\lambda = 1$, erit

$$\int x^{m-1}\partial x\left(\Delta+x^{n}\right)=\frac{x^{m}}{m}\left(\Delta+x^{n}\right)\left(1-\frac{n}{m+n}\frac{x^{n}}{\Delta+x^{n}}\right),$$

quae expressio reducitur ad hanc $\frac{x^{m}}{m}\left(\Delta+x^{n}-\frac{n}{m+n}x^{n}\right)$: integrale vero ordinario modo sumtum erit $\frac{\Delta x^{m}}{m}+\frac{x^{m+n}}{m+n}$, quod cum praecedente convenit.

2°) Si fuerit $\lambda=2$ , erit

$$\int x^{m-1}\partial x\left(\Delta+x^{n}\right)^{2}=\frac{x^{m}}{m}\left(\Delta+x^{n}\right)^{2}\left[1-\frac{2n}{m+n}\left(\frac{x^{n}}{\Delta+x^{n}}\right)+\frac{2n}{m+n}\cdot\frac{2}{m+n}\left(\frac{x^{n}}{\Delta+x^{n}}\right)^{2}\right]$$

quae expressio reducitur ad hanc

$$\frac{x^{m}}{m}\left\{\Delta\Delta+2\Delta x^{n}+x^{2n}-\frac{2n}{m+n}\Delta x^{n}-\frac{2n}{m+n}x^{2n}+\frac{n\cdot 2n}{(m+n)(m+2n)}x^{2n}\right\},$$

sive ad hanc concinniorem

$$\frac{x^{m}}{m}\left(\Delta\Delta+\frac{2n}{m+n}\Delta x^{n}+\frac{m}{m+2n}x^{2n}\right),$$

quod egregie convenit cum integrali more solito sumto. Caeterum hic meminisse juvabit, haec integralia locum habere non posse, nisi $m$ fuerit nihilo major, quia alioquin integrale non ita sumi posset, ut evanesceret casu $x=0$ .

§.12. Sin autem exponens $\lambda$ non fuerit numerus integer, series inventa in infinitum progreditur, ejusque veritas non amplius in oculos incurrit. His autem casibus forma nostri integralis simplicior et concinnior reddetur, si statuamus $\lambda=-\frac{\mu}{n}$ ; tum enim hujus formulae $\int x^{m-1}\partial x\left(\Delta+x^{n}\right)^{-\frac{\mu}{n}}$ integrale erit

$$\frac{x^{m}}{m\left(\Delta+x^{n}\right)^{\frac{\mu}{n}}}\left\{\begin{aligned}&1+\frac{\mu}{m+n}\left(\frac{x^{n}}{\Delta+x^{n}}\right)+\frac{\mu}{m+n}\cdot\frac{\mu}{m+2n}\left(\frac{x^{n}}{\Delta+x^{n}}\right)^{2}\\&+\frac{\mu}{m+n}\cdot\frac{\mu}{m+2n}\cdot\frac{\mu}{m+3n}\left(\frac{x^{n}}{\Delta+x^{n}}\right)^{3}+\text{etc.}\end{aligned}\right\}.$$

§.13. Hinc jam summam hujus seriei generalis assignare licebit

$$1+\frac{a}{b}\chi+\frac{a}{b}\cdot\frac{a+n}{b+n}\chi^{2}+\frac{a}{b}\cdot\frac{a+n}{b+n}\cdot\frac{a+2n}{b+2n}\chi^{3}+\text{etc.}$$

Si enim hanc seriem cum inventa comparemus, erit $\mu=a$, et $m+n=b$, ideoque $m=b-n$ ; tum vero erit $\chi=\frac{x^{n}}{\Delta+x^{n}}$ , unde relatio inter $\chi$ et $x$ innotescit. Tum igitur hujus seriei summa aequabitur huic formulae integrali

$$\int\frac{x^{b-n-1}\partial x}{\left(\Delta+x^{n}\right)^{\frac{a}{n}}}$$

divisae per hanc quantitatem $\frac{x^{b-n}}{(b-n)\left(\Delta+x^n\right)^{\frac{a}{n}}}$; ideoque ista summa erit

$$\frac{(b-n)\left(\Delta+x^n\right)^{\frac{a}{n}}}{x^{b-n}}\cdot\int\frac{x^{b-n-i}\partial x}{\left(\Delta+x^n\right)^{\frac{a}{n}}},$$

quae autem summa subsistere nequit, nisi fuerit $b>n$. Caeterum evidens est, istam seriem semper esse convergentem, cum non solum fractio $\frac{x^n}{\Delta+x^n}$ sit unitate minor, sed etiam coëfficientes omnes sint unitate minores.

§.14. Casus autem maxime memorabilîs, qui hic occurrit, est quando $\Delta=0$ ; tum enim nostra formula integralis erit

$$\int x^{m-\mu-1}\partial x=\frac{x^{m-\mu}}{m-\mu},$$

huic ergo quantitati semper aequabitur sequens series

$$\frac{x^{m-\mu}}{m}\left(1+\frac{\mu}{m+n}+\frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}+\frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\cdot\frac{\mu+2n}{m+3n}+\text{etc.}\right)$$

si modo fuerit *m* numerus positivus, uti jam aliquoties est animadversum. Consequenter hujus seriei

$$1+\frac{\mu}{m+n}+\frac{\mu(\mu+n)}{(m+n)(m+2n)}+\frac{\mu(\mu+n)(\mu+2n)}{(m+n)(m+2n)(m+3n)}+\text{etc.}$$

summa est $\frac{m}{m-\mu}$, quae summatio est eo magis notatu digna, quod vix ulla via patet, ejus veritatem investigandi.

§.15. Statim autem apparet, hanc summam subsistere non posse, nisi tam *n* quam $m-\mu$ fuerit numerus positivus. Cum enim formula nostra integralis casu $\Delta=0$ sit $\int x^{m-1-\mu}\partial x$ quam ab $x=0$ inchoari oportet, evidens est hoc fieri non posse, nisi $m-\mu$ fuerit numerus positivus ; praeterea etiam notandum est; exponentem *n* necessario positivum esse debere. Cum enim in Analysi supra exposita hoc integrale occurrat $\int x^{m-1}\partial x\left(a^n-x^n\right)^{\theta}$, manifestum est , si *n* esset numerus negativus , integrationem non ita institui posse , ut casu $x=0$ evanescat. His notatis istam seriem accuratius sum contemplaturus et quoniam ejus indoles non parum abscondita videtur, ejus veritatem duplici modo sum ostensurus. Primo scilicet ostendam, summam assignatam revera aequari summae totius progressionis ; deinde analysin prorsus singularem apperiam, quae non solum directe ad ipsam hanc seriem perducet, sed etiam ejus summam indicabit.

Demonstratio hujus summationis:

$$\frac{m}{m-\mu}=1+\frac{\mu}{m+n}+\frac{\mu(\mu+n)}{(m+n)(m+2n)}+\frac{\mu(\mu+n)(\mu+2n)}{(m+n)(m+2n)(m+3n)}+\text{etc.}$$

§.16. Hic scilicet ostendam, si omnes hujus seriei termini a summa inventa $\frac{m}{m-\mu}$ successive subtrahantur, tandem revera nihil relictum iri. Subtracto enim primo termino 1 remanet $\frac{\mu}{m-\mu}$. Hinc terminus secundus ablatus relinquit $\frac{\mu(\mu+n)}{(m-\mu)(m+n)}$. Hinc porro subtrahatur tertius terminus ac remanebit

$$\frac{\mu(\mu+n)(\mu+2n)}{(m-\mu)(m+n)(m+2n)}.$$

Hinc jam quartus terminus ablatus residuum praebet sequens

$$\frac{\mu(\mu+n)(\mu+2n)(\mu+3n)}{(m-\mu)(m+n)(m+2n)(m+3n)}$$

Unde jam satis liquet, omnibus terminis ablatis tandem remansurum esse hoc productum in infinitum excurrens

$$\frac{\mu(\mu+n)(\mu+2n)(\mu+3n)(\mu+4n)(\text{etc.}}{(m-\mu)(m+n)(m+2n)(m+3n)(m+4n)(\text{etc.}}.$$

§.17. Facile autem intelligitur valorem hujus producti revera nihilo esse aequalem. Omisso enim primo factore $\frac{m}{m-\mu}$, omnes reliqui factores sunt fractiones unitate minores , quia $\mu<m,$ et quoniam tam numeratores quam denominatores in arithmetica progressione increscunt, jam satis constat, valorem talis producti revera evanescere. Hic autem probe tenendum est, ut productum infinitarum talium fractionum in nihilum abeat, non sufficere, ut singulae fractiones sint unitate minores, veluti evenit in hac forma

$$\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{15}{16}\cdot\frac{24}{25}\cdot\frac{35}{36}\cdot\frac{48}{49}\cdot\text{etc.}$$

cujus producti in infinitum protensi valor facile ostenditur esse $=\frac{1}{2}$.

§.18. Quoniam in nostro producto singuli denominatores superant suos numeratores eadem quantitate $m=\mu$, istam formam generaliorem considerabo

$$\frac{a}{a+\Delta}\cdot\frac{b}{b+\Delta}\cdot\frac{c}{c+\Delta}\cdot\frac{d}{d+\Delta}\cdot\frac{e}{e+\Delta}\cdot\text{etc.}$$

et perscrutabor, sub quibusnam conditionibus ejus valor in infinitum extensus , qui sit $\Pi$ , revera in nihilum sit abiturus. Evidens autem est , hoc evenire , si eadem forma inversa

$$\frac{1}{\Pi}=\frac{a+\Delta}{a}\cdot\frac{b+\Delta}{b}\cdot\frac{c+\Delta}{c}\cdot\frac{d+\Delta}{d}\cdot\ \text{etc.}$$

in infinitum excrescit. Sin autem ejus valor fuerit infinitus , etiam ejus logarithmus infinitus evadat necesse est. Cum igitur sit

$$l\frac{1}{\Pi}=l\frac{a+\Delta}{a}+l\frac{b+\Delta}{b}+l\frac{c+\Delta}{c}+l\frac{d+\Delta}{d}+\text{ etc.}$$

facta evolutione reperietur

$$\begin{aligned} l\frac{1}{\Pi}=&\Delta\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\text{etc.}\right)\\ &-\frac{1}{2}\Delta^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2}+\text{etc.}\right)\\ &+\frac{1}{3}\Delta^3\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}+\frac{1}{e^3}+\text{etc.}\right)\\ &-\text{etc.}\end{aligned}$$

quae expressio semper erit infinita, quoties summa primae seriei $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\text{etc}$ fuerit infinita. Hanc autem summam semper esse infinitam, quoties numeri *a, b, c, d,* etc. in progressione arithmetica crescunt, jam notum est et per se perspicuum, quod cum in nostra serie contingat, certum est; illius producti infiniti valorem esse evanescentem.

§.19. Circa nostram autem seriem id imprimis notatu dignum occurrit, quod ejus summa $\frac{m}{m-\mu}$ litteram *n* non involvat, ita ut ejus summa semper maneat eadem, quicunque valores litterae *n* tribuantur, quod quidem pro casu $n=0$ per se statim fit manifestum , quandoquidem tum series nostra evadit

$$1+\frac{\mu}{m}+\frac{\mu^2}{m^2}+\frac{\mu^3}{m^3}+\text{ etc.}$$

quae cum sit progressio geometrica, ejus summa erit $\frac{m}{m-\mu}$. Quod vero summa perpetua maneat eadem, quicunque valores ipsi *n* tribuantur, non tam facile perspicitur, etsi veritas a nobis jam sit demonstrata.

§. 20. Quin etiam demonstratio hic tradita multo latius patet, cum adeo in eadem forma valores ipsius *n* variare liceat. Ita si posito $\alpha$ loco *n*, pro ejus multiplis 2*n,* 3*n,* 4*n,* 5*n*, etc. scribamus novas litteras $\beta,\ \gamma,\ \delta,\ \varepsilon,$ etc. ut habeatur, ista series

$$1+\frac{\mu}{m+\alpha}+\frac{\mu}{m+\alpha}\cdot\frac{\mu+\alpha}{m+\beta}+\frac{\mu}{m+\alpha}\cdot\frac{\mu+\alpha}{m+\beta}\cdot\frac{\mu+\beta}{m+\gamma}+\text{etc.}$$

ejus summa etiam nunc erit $\frac{m}{m-\mu}$. Subtracto enim termine primo remanet $\frac{\mu}{m-\mu}$. Hinc terminus secundus subtractus relinquit

$$\frac{\mu(\mu+\alpha)}{(m-\mu)(m+\alpha)}$$

Hinc tertius terminus subtractus

$$\frac{\mu(\mu+\alpha)(\mu+\beta)}{(m-\mu)(m+\alpha)(m+\beta)}$$

Unde jam patet, in infinitum tandem prodiri productum

$$\frac{\mu(\mu+\alpha)(\mu+\beta)(\mu+\gamma)(\mu+\delta)(\text{etc.}}{(m-\mu)(m+\alpha)(m+\beta)(m+\gamma)(m+\delta)(\text{etc.}},$$

cujus valor semper erit evanescens, si modo haec series

$$\frac{1}{\mu+\alpha} + \frac{1}{\mu+\beta} + \frac{1}{\mu+\gamma} + \frac{1}{\mu+\delta} + \text{etc.}$$

habuerit summam infinite magnam, uti modo ante ostendimus.

Analysis singularis directe ad seriem supra inventam perducens.

§.21. Ponamus

$$x^m\left(1-x^n\right)^\theta = \mathrm{A}\int x^{m-1}\partial x\left(1-x^n\right)^\theta + \mathrm{B}\int x^{m-1}\partial x\left(1-x^n\right)^{\theta-1},$$

et reperietur $\mathrm{A} = m+\theta n$ et $\mathrm{B} = -\theta n$ ; hinc ergo si ponamus
$\int x^{m-1}\partial x\left(1-x^n\right)^\theta = \mathrm{P}$ et $\int x^{m-1}\partial x\left(1-x^n\right)^{\theta-1} = \mathrm{Q}$, erit

$x^m\left(1-x^n\right)^\theta = \left(m+\theta n\right)\mathrm{P} - \theta n\mathrm{Q}$, ideoque

$$\mathrm{Q} = \frac{m+\theta n}{\theta n}\cdot\mathrm{P} - \frac{1}{\theta n}x^m\left(1-x^n\right)^\theta .$$

Quodsi jam ambo integralia P et Q a termino $x=0$ usque ad $x=1$ extendamus , erit
$\mathrm{Q} = \frac{m+\theta n}{\theta n}\times\mathrm{P};$ si modo fuerit tam $m > 0$ quam $e > 0$.

§. 22. Cum jam sit $\partial\mathrm{Q} = \frac{\partial\mathrm{P}}{1-x^n}$, denominatore in seriem evoluto erit

$$\partial Q = \partial\mathrm{P}\left(1+x^n+x^{2n}+x^{3n}+x^{4n}+\text{etc.}\right):$$

consequenter habebitur

$$\mathrm{Q} = \mathrm{P} + \int x^n\partial\mathrm{P} + \int x^{2n}\partial\mathrm{P} + \int x^{3n}\partial\mathrm{P} + \text{etc.}$$

quae singula integralia ita sunt comparata, ut quodlibet ad praecedens reduci possit, ope hujus reductionis

$$x^\alpha\left(1-x^n\right)^{\theta+1} = \mathrm{A}\int x^{\alpha+n-1}\partial x\left(1-x^n\right)^\theta + \mathrm{B}\int x^{\alpha-1}\partial x\left(1-x^n\right)^\theta,$$

pro qua reperitur $\mathrm{A} = -\alpha - n(\theta+1)$ et $\mathrm{B} = a$.

§. 23. Si etiam haec duo integralia a termino $x=0$ usque ad $x=1$ extendantur, fiet

$$0 = -\left[\alpha + n\left(\theta+1\right)\right]\int x^{\alpha+n-1}\partial x\left(1-x^n\right)^\theta + \alpha\int x^{\alpha-1}\partial x\left(1-x^n\right)^\theta,$$

si modo fuerit $\alpha > 0$ et $\theta + 1 > 0$. Faciamus nunc $\alpha = m + \lambda n$, et quia ante posueramus $x^{m-1}\left(1-x^{n}\right)^{\theta} = \partial \mathrm{P}$, haec aequatio abibit in hanc formam

$$-\left[\alpha + n\left(\theta+1\right)\right]\int x^{(\lambda+1)n}\partial \mathrm{P} + \alpha\int x^{\lambda n}\partial \mathrm{P},$$

quocirca habebimus hanc reductionem

$$\int x^{\lambda n+n}\partial \mathrm{P} = \frac{\alpha}{\alpha+n(\theta+1)}\int x^{\lambda n}\partial \mathrm{P} = \frac{m+\alpha n}{m+n(\lambda+\theta+1)}\int x^{\lambda n}\partial \mathrm{P}.$$

§.24. Haec formula generalis nobis jam suppeditat sequentes integrationes speciales :

1°) Si $\lambda = 0$ $\quad \int x^{n}\partial \mathrm{P} = \frac{m}{m+n(\theta+1)}\mathrm{P},$

2°) Si $\lambda = 1$ $\quad \int x^{2n}\partial \mathrm{P} = \frac{m+n}{n+n(\theta+2)}\int x^{n}\partial \mathrm{P}$, ideoque

$$\int x^{2n}\partial \mathrm{P} = \frac{m}{m+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)}\mathrm{P},$$

3°) Si $\lambda = 3$ $\quad \int x^{3n}\partial \mathrm{P} = \frac{m+2n}{n+n(\theta+3)}\int x^{2n}\partial \mathrm{P}$, ideoque

$$\int x^{3n}\partial \mathrm{P} = \frac{m}{n+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)}\cdot\frac{m+2n}{m+n(\theta+3)}\mathrm{P},$$

4°) Si $\lambda = 4$ $\quad \int x^{4n}\partial \mathrm{P} = \frac{m}{n+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)}\cdot\frac{m+2n}{m+n(\theta+3)}\cdot\frac{m+3n}{m+n(\theta+4)}\mathrm{P}.$

etc. etc.

§. 25. Cum igitur ex superioribus fuisset

$$Q = \int \partial \mathrm{P}\left(1 + x^{n} + x^{2n} + x^{3n} + x^{4n} + \text{etc.}\right),$$

si pro singulis terminis valores modo inventas substituamus, atque utrinque per P dividamus, nanciscemur hanc aequationem

$$\frac{\mathrm{Q}}{\mathrm{P}} = 1 + \frac{m}{m+n(\theta+1)} + \frac{m}{m+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)} + \frac{m}{m+n(\theta+1)}\cdot\frac{m+n}{m+n(\theta+2)}\cdot\frac{m+2n}{m+n(\theta+3)} + \text{etc.}$$

supra autem ostendimus esse $\frac{\mathrm{Q}}{\mathrm{P}} = \frac{m+\theta n}{n\theta}$, quae ergo fractio est summa istius seriei infinitae.

§. 26. Ut jam consensum hujus seriei cum supra inventa ostendamus, primo loco $\theta$ scribamus $\frac{\mu}{n}$, atque series nostra inventa hanc induet formam

$$\frac{m+\mu}{n} = \frac{m}{m+n+\mu} + \frac{m}{m+n+\mu}\cdot\frac{m+n}{m+2n+\mu} + \frac{m}{m+n+\mu}\cdot\frac{m+n}{m+2n+\mu}\cdot\frac{m+2n}{m+3n+\mu} + \text{etc.},$$

cujus veritas eodem modo quo ante fueram usus, demonstrari potest. Si enim a summa subtrahatur terminus primus relinquitur $\frac{\mu}{n}$. Subtracto hinc termino secundo remanet $\frac{m(m+n)}{\mu(m+n+\mu)}$. Hinc porro tertius terminus sutractus relinquit $\frac{m(m+n)(m+2n)}{\mu(m+n+\mu)(m+2n+\mu)}$ et ita porro, quae operatio si in infinitum continuetur, producti hujus resultantis valor est $=0$. Tum vero evidens est, seriem hanc inventam in eam ipsam quam supra dedimus transmutari, si hic loco *m* scribatur $\mu$, at vero $m-\mu$ loco P.

§. 27. Coronidis loco hic subjungam seriem multo generaliorem ejusdem generis, cujus summam pariter assignare licet, quam sequenti problemate sum complexurus.

Problema I.

§. 28. *Proposita hac serie* $A+B\frac{\alpha}{a}+C\frac{\alpha\beta}{ab}+D\frac{\alpha\beta\gamma}{abc}+etc.$ *investigare conditiones, sub quibus ejus summam assignare liceat.*

Solutio.

Haec ergo series involvit ternas series : primam litterarum $\alpha,\beta,\gamma,\delta,$etc. quae numeratores seriei propositae constituunt; secundam litterarum *a, b, c, d,* etc. ex quibus denominatores formantur ; tertiam litterarum A, B, C, D, etc. quae coëfficientes terminorum exhibent. Quemadmodum igitur ternae istae series comparatae esse debeant, ut seriei propositae summam per expressionem finitam atque adeo rationalem assignare liceat, hic investigabo.

§. 29. Statuamus hujus seriei summam esse $\frac{S}{t}$, atque eadem methodo utamur quam jam supra adhibuimus, scilicet ab hac summa primo subtrahamus primum terminum A et cum remaneat $\frac{S-At}{t}$, statuamus $S-At=\alpha$, ut habeamus $\frac{\alpha}{t}$ ; hinc subtrahamus secundum terminum $B\frac{\alpha}{a}$, et residuum erit $\frac{\alpha(a-Bt)}{t\cdot a}$. Hic jam faciamus $a-Bt=\beta$, ut habeamus $\frac{\alpha\beta}{t\cdot a}$; unde si subtrahatur tertius terminus $C\frac{\alpha\beta}{ab}$, residuum erit $\frac{\alpha\beta(b-Ct)}{t\cdot ab}$. Fiat hic $b-Ct=\gamma$, ut habeamus $\frac{\alpha\beta\gamma}{t\cdot ab}$, unde terminus quartus ablatus relinquit $\frac{\alpha\beta\gamma(c-Dt)}{t\cdot abc}$. Fiat hic iterum $c-Dt=\delta$, ut habeamus $\frac{\alpha\beta\gamma\delta}{t\cdot abc}$, unde quintum terminum subtrahendo colligitur $\frac{\alpha\beta\gamma\delta(d-Et)}{t\cdot abcd}$. Haecque operationes in infinitum continuari intelligantur.

§.30. Ex his igitur determinationibus tam littera S quam litterae *a, b, c, d,* etc. sequenti modo definientur

$$S=\alpha+At;\ a=\beta+Bt;\ b=\gamma+Ct;\ c=\delta+Dt;\ \text{etc.}$$

Atque his valoribus introductis residuum, postquam omnes seriei termini fuerint a formula $\frac{S}{t}$ ablati, remanebit hoc productum in infinitum excurrens $\frac{\alpha\beta\gamma\delta\varepsilon\zeta\ etc.}{t\cdot abcdef\ etc.}$ quod ergo productum si in nihilum abeat, tum summa seriei propositae revera erit $=\frac{S}{t}$. Videamus igitur sub quibusnam conditionibus hoc productum evanescat.

§. 31. Designemus hoc productum littera Π, ut substitutis pro *a, b, c,* etc. valoribus inventis erit

$$\Pi=\frac{S}{t}\left(\frac{\alpha}{\alpha+At}\cdot\frac{\beta}{\beta+Bt}\cdot\frac{\gamma}{\gamma+Ct}\cdot\frac{\delta}{\delta+Dt}\cdot\text{etc.}\right)$$

ubi scilicet factorem $\frac{S}{t}$ praefiximus. Nunc igitur quaeritur sub quibusnam conditionibus istud productum in infinitum continuatum in nihilum sit abiturum. Evidens autem est hoc evenire, si productum istud invertatur, ejusque logarithmus eveniat infinite magnus. Hoc ergo locum inveniet, quando summa horum logarithmorum

$$l\left(1+\frac{At}{\alpha}\right)+l\left(1+\frac{Bt}{\beta}\right)+l\left(1+\frac{Ct}{\gamma}\right)+l\left(1+\frac{Dt}{\delta}\right)+\text{etc.}=\infty;$$

id quod semper continget , si sumtis tantum primis terminis, qui omnes factorem comunem habent *i,* series haec

$$\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}+\frac{D}{\delta}+\text{etc.}$$

habuerit summam infinite magnam, tum igitur nostrae seriei propositae summa semper erit $\frac{\alpha+At}{t}$ .

§.32. Neque vero absolute necesse est, ut productum Π penitus evanescat, sed quemcunque habuerit valorem scilicet Π, quoniam is oritur postquam tota summa seriei propositae quam ponamus $=S$, ablata fuerit a formula $\frac{S}{t}$, ita ut sit $\Pi=\frac{S}{t}-S$, unde manifesta fit $S=\frac{S}{t}-\Pi$.

§. 33. Ut hoc exemple illustramus, litteris $\alpha,\ \beta,\ \gamma,\ \delta,$ etc. hos tribuemus valores $\alpha=3$ , $\beta=15,\ \gamma=35,\ \delta=63$ , etc. praeterea vero sit $t=1$, atque insuper $A=B=C=D=\text{etc.}=1$ : hinc ergo determinationes inventae praebebunt

$$S=4,\ a=16,\ b=36,\ c=64,\ d=100,\ \text{etc.}$$

Sicque series nostra jam erit

$$1+\frac{3}{16}+\frac{3\cdot15}{16\cdot36}+\frac{3\cdot15\cdot35}{16\cdot36\cdot64}+\frac{3\cdot15\cdot35\cdot63}{16\cdot36\cdot64\cdot100}+\text{ etc.}$$

pro cujus summa notetur esse

$$\Pi=4\cdot\frac{3\cdot15\cdot35\cdot63\cdot99}{4\cdot16\cdot36\cdot64\cdot100}\text{etc.}$$

Constat autem ex quadratura circuli *Wallisiana* esse $\frac{3\cdot15\cdot35\cdot63\cdot99\ etc.}{4\cdot16\cdot36\cdot64\cdot100\ etc.}=\frac{2}{\pi}$, existente π peripheria circuli cujus diameter est unitas. Hinc ergo $\Pi=\frac{8}{\pi}$, ideoque summa nostrae serei $S=4-\frac{8}{\pi}$, ideoque proxime $\frac{16}{11}$.

§.34. At vero series generalis, quam hoc modo sumus adepti, maxime est faecunda in formatione innumerabilium serierum specialium, cum non modo tam seriem litterarum $\alpha,\ \beta,\ \gamma,\ \delta,$ etc. sed etiam litterarum A, B, C, D, etc. prorsus pro lubitu assumere liceat , quandoquidem inde litterae *a, b, c, d,* etc. sponte determinantur ; tum autem talium serierum summam semper assignare licebit, si modo valor producti in infinitum excurrentis , quod littera Π indicavimus definiri poterit, ubi perinde est, utrum iste valor fuerit rationalis sive adeo transcendens quadraturam quamcunque involvens.