A NEW AND EASY METHOD OF TREATING THE CALCULUS OF VARIATIONS

New Commentaries of the Petersburg Academy of Sciences 16 (1771), 1772, p. 35-70

[E420]

§. 1. If some equation may be given between the two variables x and y, or, what amounts to the same, if y were some function of x, then all the expressions formed in some manner from these two quantities x and y may be put together so that they can be considered as if functions of the one variable x, thus so that for any value determined of x , several values or y may be shared.

§. 2. But it is agreed three kinds of expressions of this sort are to be set up, formed from the quantities x and y ; to the first of which we have referred all these expressions, in which only these quantities x and y themselves occur and are related by some operations either algebraic, such as $\alpha x^3+\beta xy+\gamma y^3$, or transcending, such as $e^{ax}\text{Arcsin}\, y$, in which the former are considered to be algebraic and the latter transcending operations. However, the second kind involves these expressions in which besides these quantities x and y also a differential ratio occurs, which ratio we have extended thus to differentials of any order, and so that we may examine the nature of expressions of this kind more clearly, there may be put in the customary manner :

$$\partial y=p\partial x,\ \partial p=q\partial x,\ \partial q=r\partial x \text{ etc.}$$

and such expressions will be functions of the quantities x, y, p, q, r etc. The third and final kind contains expressions of some such kind, in which in addition integral formulas are involved, according to which these expressions arise to be considered pertaining especially to the calculus of variations, which are represented by this form $\int Vdx$, where V is some function not only of x and y themselves, but also of the quantities p, q, r etc., since it can involve in addition these other integral formulas.

§. 3. From the general expression concerned with these three expressions established of this kind, we will be able to explain the nature of the calculus of variations. For the whole problem reverts to this, so that, if some relation were proposed between x and y and that may be varied a little, or in its place a certain other relation between x and y may be used differing by an infinitely small part in some manner, it shall be required to be found, how great a change all these expressions, both of the first, second and of the third kinds shall be entered into, according to what is found in the calculation of the variation, as indeed I have treated that at one time, besides the differential ∂y, by which the quantity y is increased, while x goes to $x+dx$, to this quantity y itself another increment δy is attributed depending completely on our choice and not determined by

x, to which increment I gave the name variation and the method of setting out the numerous variations thence in a single general expression requiring to be found.

§. 4. Therefore the calculus of variations was seen to constitute generally a singular kind of calculation, truly after I had examined its nature more carefully, I observed this whole calculation able to be deduced with a small change made to the calculation of the integral, the elements of which concerning this argument can be found in the third volume of my work set out by me. Moreover in this following part I have worked through these integrations, which are involved with functions of two variables, in which calculation generally until now it was scarcely permitted to progress beyond the first elements.

§. 5. Clearly in place of this increment, which I have called the variation, I no longer consider the quantity y as a function of the variable x only, but as a function of the two variables x and t, that I introduce into the calculation, thus indeed, while $\partial x\left(\frac{\partial y}{\partial x}\right)$ indicates the true differential of y, this formula $\partial t\left(\frac{\partial y}{\partial t}\right)$ will be able to indicate the same, which before we indicated by the sign δy. So that these may be rendered clearer, we may consider y as the applied line of some curve corresponding to the abscissa x, and in the calculation of the variation another relation is required, which may include perhaps all the other curves close to this, but all the curves of this kind, if X may denote that function, to which y may be equal, it is evident able to be contained by such an equation : $y = \mathrm{X} + t\mathrm{V}$, with V denoting some function of x. For with t assumed infinitely small this equation may be understood generally to be near all these curved lines proposed and thus this form may be allowed to be returned much more generally, thus so that for some function y of the two variables x and t it may be able to be seized upon, as long as that will have been prepared thus, so that on putting $t = 0$, that same function proposed $y = \mathrm{X}$ may be produced.

§. 6. Therefore for finding the variation, the quantity x may be regarded as constant, truly the differential of this y must be chosen from the variability of t itself only ; from which, if the expression proposed were a function of the first kind, evidently of x and y only, which we may signify by the letter Z, we may put by the customary differentiation to produce $\mathrm{M}\partial x + \mathrm{N}\partial y$, and now for the variation being found there may become $\partial x = 0$, and in place of ∂y there may be written $\partial t\left(\frac{\partial y}{\partial t}\right)$, which certainly is the increment arising from the variation of t alone. With which done the variation sought of this expression Z will be $= \mathrm{N}\partial t\left(\frac{\partial y}{\partial t}\right)$. Whereby, if the same variation may be indicated likewise by $\partial t\left(\frac{\partial \mathrm{Z}}{\partial t}\right)$, we will have $\left(\frac{\partial \mathrm{Z}}{\partial t}\right) = \mathrm{N}\left(\frac{\partial y}{\partial t}\right)$.

§. 7. Now we may progress to expressions of the second kind, in which since the quantities p, q, r etc. may occur besides x and y, the variations of which, in as much as

y also depends on the variable t , may be expressed by the general rule for these formulas

$$\partial t\left(\frac{\partial p}{\partial t}\right);\ \partial t\left(\frac{\partial q}{\partial t}\right);\ \partial t\left(\frac{\partial r}{\partial t}\right);\ \text{etc.}$$

But since, for the variable x alone, there shall become :

$$p=\left(\frac{\partial y}{\partial x}\right);\ q=\left(\frac{\partial p}{\partial x}\right);\ p=\left(\frac{\partial\partial y}{\partial x^2}\right);\ \text{ et}$$
$$r=\left(\frac{\partial q}{\partial x}\right)=\left(\frac{\partial\partial p}{\partial x^2}\right)=\left(\frac{\partial^3 y}{\partial x^3}\right);\ \text{etc.}$$

there will become by the general rules of differentiating functions of two variables

$$\left(\frac{\partial p}{\partial t}\right)=\left(\frac{\partial\partial y}{\partial x\partial t}\right);\ \left(\frac{\partial q}{\partial t}\right)=\left(\frac{\partial^3 y}{\partial x^2\partial t}\right);\ \ \left(\frac{\partial r}{\partial t}\right)=\left(\frac{\partial^4 y}{\partial x^3\partial t}\right);\ \text{ etc.}$$

where it will help to be remembering the formula, for example, to produce $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$, if the function y may be differentiated three times and by two in turn for x alone, together with only once with t taken variable, then truly any differentiation may be derived from the simple differentials ∂x or ∂t .

§. 8. Now with these in place, Z shall be some function of x, y, p, q, r etc., indeed here at present with nothing had with respect to the variable t, certainly which is introduced only in aid of the variation, and with the differentiation made in the usual manner there may be produced

$$\partial \mathrm{Z}=\mathrm{M}\partial x+\mathrm{N}\partial y+\mathrm{P}\partial p+\mathrm{Q}\partial q+\mathrm{R}\partial r+\ \text{etc.}\ ;$$

therefore now for the variation, or $\partial t\left(\frac{\partial \mathrm{Z}}{\partial t}\right)$, being found it will be necessary to write as follows :

$$\partial x=0,\ \partial y=\partial t\left(\frac{\partial y}{\partial t}\right),\ \partial p=\partial t\left(\frac{\partial p}{\partial t}\right)=\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right)$$
$$\partial q=\partial t\left(\frac{\partial^3 y}{\partial x^2\partial t}\right);\ \partial r=\partial t\left(\frac{\partial^4 y}{\partial x^3\partial t}\right);\ \text{etc.}$$

and the variation sought will be

$$\partial t\left(\frac{\partial \mathrm{Z}}{\partial t}\right)=N\partial t\left(\frac{\partial y}{\partial t}\right)+\mathrm{P}\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\mathrm{Q}\partial t\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\mathrm{R}\partial t\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)+\text{etc.}$$

from which with the division made by ∂t there follows to become :

$$\left(\frac{\partial Z}{\partial t}\right)=N\left(\frac{\partial y}{\partial t}\right)+P\left(\frac{\partial\partial y}{\partial x\partial t}\right)+Q\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+R\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)+\text{etc.}$$

§. 9. Now let some expression be proposed of the third order $\int Z\partial x$, where Z shall be some function of these *x, y, p, q, r* etc., thus so that by ordinary differentiation there may be had :

$$\partial Z=M\partial x+N\partial y+P\partial p+Q\partial q+R\partial r+\text{etc.}$$

where indeed at this point no new ratio of the variable *t* has been had, and the integration of the proposed formula $\int Z\partial x$ by the variable *x* alone is being set out, with which sought noted it corresponds here, in order that if now *y* may be considered as a function of the two variables *x* and *t* and everywhere the quantity *y* may be increased by the element $\partial t\left(\frac{\partial Z}{\partial t}\right)$, the increase may be defined, which the formula of the integral $\int Z\partial x$ thence may take, for this increase itself will be the variation of the proposed integral formula.

§. 10. Whereby for this variation being found in that function Z, in place of y its value may be increased everywhere by $y+\partial t\left(\frac{\partial y}{\partial t}\right)$ and thus, as we have seen before, the function Z will take the increase $\partial t\left(\frac{\partial Z}{\partial t}\right)$, from which this formula of the integral will take this increase $\int\partial t\left(\frac{\partial Z}{\partial t}\right)\partial x$, which will be that variation sought. Truly since in this integration x alone may be had to be variable, the element ∂t will be placed before the integral sign, thus so that now the variation shall become $=\partial t\int\partial x\left(\frac{\partial Z}{\partial t}\right)$.

§.11. Whereby therefore in paragraph 8 the value of $\left(\frac{\partial Z}{\partial t}\right)$ now will be obtained set out, if with that substituted here, the variation of the formula $\int Z\partial x$ will be produced expressed thus :

$$\partial t\int\partial x\left[N\left(\frac{\partial y}{\partial t}\right)+P\left(\frac{\partial\partial y}{\partial x\partial t}\right)+Q\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+R\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)+\text{etc.}\right]$$

as also it will help to be represented by the parts in the following manner

$$\partial t\int N\partial x\left(\frac{\partial y}{\partial t}\right)+\partial t\int P\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\partial t\int Q\partial x\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\partial t\int R\partial x\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)+\text{etc.},$$

we may be able to be satisfied with which expression, if a question concerning some determined case may be put in place, where *y* may be equated not only to some given function of *x*, but also the new variable *t* may be introduced in the determined manner ; for then all these formulas $\left(\frac{\partial y}{\partial t}\right)$; $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$; $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$;etc. actually may be allowed to be set out, thus so that then the element ∂x may be affected by a single function of x, if indeed we agree, with the expansion again put in place, initially there must be put $t = 0$.

§. 12. But truly such determinate questions at no time are accustomed to occur, but rather the relation between y and *x* always is accustomed to be unknown, thence requiring to be determined at last, because the variation must go to zero, certainly in which the method of maxima and minima is involved. Therefore it will be convenient thus to enunciate questions of this kind, such a relation must exist between the quantities *x* and *y*, so that the variation of the proposed integral formula $\int Z\partial x$ may become zero, also in whatever way may the new variable *t* be introduced into the calculation? But if nevertheless it may be set up with this ratio sought, it is evident from the formulas $\left(\frac{\partial y}{\partial t}\right)$; $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$; $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$;etc. that no certain values can be attributed.

§. 13. Truly help may be provided here by a singular artifice, with the aid of which formulas the latter integrals in paragraph 11 will be allowed to be reduced to the earlier form, thus so that in everything the same formula $\left(\frac{\partial y}{\partial t}\right)$ may occur. Whereas indeed $\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right)$ shall be the differential of the formula $\left(\frac{\partial y}{\partial t}\right)$ taken with *x* alone to be variable, there will be by the usual reduction of the integral :

$$\int P\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right) = P\left(\frac{\partial y}{\partial t}\right) - \int \partial x\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right),$$

in a similar manner, because $\partial x\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$ is the differential of the formula $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$, we will have this reduction at once :

$$\int Q\partial x\left(\frac{\partial^3 y}{\partial x^2\partial t}\right) = Q\left(\frac{\partial\partial y}{\partial x\partial t}\right) - \int \partial x\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right),$$

now truly by the preceding reduction there shall become :

$$\int \partial x\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right) = \left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right) - \int \partial x\left(\frac{\partial\partial Q}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right),$$

and thus generally we shall have

$$\int Q\partial x\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)=Q\left(\frac{\partial\partial y}{\partial x\partial t}\right)-\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)+\int\partial x\left(\frac{\partial\partial Q}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right),$$

and now it is seen well enough the following integral formula can the reduced thus to go to :

$$\int R\partial x\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)=R\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)-\left(\frac{\partial R}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\left(\frac{\partial\partial R}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right)-\int\partial x\left(\frac{\partial^3 R}{\partial x^3}\right)\left(\frac{\partial y}{\partial t}\right),$$

and in addition if such a formula may be present, there may become :

$$\int S\partial x\left(\frac{\partial^5 y}{\partial x^4\partial t}\right)=S\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)-\left(\frac{\partial S}{\partial x}\right)\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\left(\frac{\partial\partial S}{\partial x^2}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right)$$
$$-\left(\frac{\partial^3 S}{\partial x^3}\right)\left(\frac{\partial y}{\partial t}\right)+\int\partial x\left(\frac{\partial^4 S}{\partial x^4}\right)\left(\frac{\partial y}{\partial t}\right).$$

§. 14. But if now we may substitute these reduced formulas into the variation of the formula $\int Z\partial x$ sought, then this variation not only will agree with the formulas of the integral, but also will contain these absolute parts, of which the parts of other formula $\left(\frac{\partial y}{\partial t}\right)$, as well as of this $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$, and truly of this $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$ etc. will be contained, while on the other hand all the integrals involve the same formula $\left(\frac{\partial y}{\partial t}\right)$, on account of which the variation sought of the proposed formula $\int Z\partial x$ will be obtained expressed in the following manner :

$$\partial t\int\partial x\left(\frac{\partial y}{\partial t}\right)\left(N-\left(\frac{\partial P}{\partial x}\right)+\left(\frac{\partial\partial Q}{\partial x^2}\right)-\left(\frac{\partial^3 R}{\partial x^3}\right)+\left(\frac{\partial^4 S}{\partial x^4}\right)-\text{etc.}\right)$$
$$+\partial t\left(\frac{\partial y}{\partial t}\right)\left(P-\left(\frac{\partial Q}{\partial x}\right)+\left(\frac{\partial\partial R}{\partial x^2}\right)-\left(\frac{\partial^3 S}{\partial x^3}\right)-\text{etc.}\right)$$
$$+\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right)\left(Q-\left(\frac{\partial R}{\partial x}\right)+\left(\frac{\partial\partial S}{\partial x^2}\right)-\text{etc.}\right)$$
$$+\partial t\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)\left(R-\left(\frac{\partial S}{\partial x}\right)+\text{etc.}\right)$$
$$+\partial t\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)(S-\text{etc.})$$
$$+\text{etc.}$$

§. 15. Although here my own investigation is not the method of maxima and minima treated, since this has been done now abundantly well by the other method, yet I am not going to overlook this, since I may note, if the variation of the formula $\int Z\partial x$ must vanish, also in what manner the new variable t may be entered into the calculation, that cannot happen in any way, unless the whole part of the first integral itself may vanish, from which it is necessary this equation be constructed between x and y

$$0 = \mathrm{N} - \left(\frac{\partial \mathrm{P}}{\partial x}\right) + \left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right) - \left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right) + \left(\frac{\partial^4 \mathrm{S}}{\partial x^4}\right) - \text{etc.}$$

and because now the variable t may have no further ratio and thus will be left with the single variable x only at this point, with the brackets omitted we will have this equation :

$$0 = \mathrm{N} - \frac{\partial \mathrm{P}}{\partial x} + \frac{\partial\partial \mathrm{Q}}{\partial x^2} - \frac{\partial^3 \mathrm{R}}{\partial x^3} + \frac{\partial^4 \mathrm{S}}{\partial x^4} - \text{etc.}$$

by which the required relation may be expressed between x and y. But the absolute parts refer only to the extreme terms, concerning which these must be observed, which now have been presented further elsewhere.

§. 16. Here also I shall not dwell on these cases, in which the quantity Z itself in addition involves integral formulas, since also this argument has been handled satisfactorily elsewhere, truly here I struggle with a much harder task, while thus I may try to extend this same method to functions of two variables, which indeed in that dissertation, which I composed at one time concerning the calculus of variations, then at the time I was not able to perform adequately, deterred by the amount of so many different kinds of quantities.

Application of the preceding method to functions of two variables.

§. 17. If some equation may be had between the three variables x, y et z, we consider to express the nature of some surface, where indeed we may understand the two coordinates x et y to be set up in the horizontal plane, truly the third z vertical, and thus this third z can be regarded as a function of the two x and y ; from which in the customary manner twice as many increments occur requiring to be considered, clearly in as much as it is arising from the variability of x or of y . Truly that increment of z, which arises from the variation of x , is accustomed to be indicated by this formula: $\partial x\left(\frac{\partial z}{\partial x}\right)$, truly that which arises from the variability of y : $\partial y\left(\frac{\partial z}{\partial y}\right)$.

§ 18. But if now this surface with the equation expressed between x, y and z must be compared with some other surfaces nearby to this, that may be done most conveniently by introducing a new variable t , thus so that now z shall be regarded as a function of the three variables x, y and t, which indeed with $t = 0$ may go into the above function, but, while infinitely small values may be attributed to t , all the nearby surfaces may be included, with which in place it is clear, since the variables x and y by no means may depend on t, the differentials dx and dy in no way may be mixed with dt , truly it can be taken only with the increment from the third kind of coordinate z; indeed besides these two now mentioned before, which proceed from x or from y, and if in addition such a formula may be present, it shall be able to accept an increment arising from the variability of t itself, which is being represented by such a formula $\partial t\left(\frac{\partial z}{\partial t}\right)$.

§ 19. Now we may put V to be some expression composed from the coordinates x, y and z, or by pure operations either algebraic or also transcending formats, which in the customary manner may give the differentials : $\partial \mathrm{V} = \mathrm{L}\partial x + \mathrm{M}\partial y + \mathrm{N}\partial z$, and if the same increment may be desired arising from the new variable t alone, it is evident there must be put in place $\partial x = 0$ and $\partial y = 0$, but in place of ∂z there must be written $\partial t\left(\frac{\partial z}{\partial t}\right)$ and thus this being indicated in the usual manner we will have

$$\partial t\left(\frac{\partial \mathrm{V}}{\partial t}\right) = \mathrm{N}\partial t\left(\frac{\partial z}{\partial t}\right) \text{ and thus } \left(\frac{\partial \mathrm{V}}{\partial t}\right) = \mathrm{N}\left(\frac{\partial z}{\partial t}\right).$$

Moreover such expressions as before constitute the first kind.

§. 20. Therefore we may progress to the second kind, where the expression v besides there coordinates x, y, z also involve the ratios of these differentials; and here before everything it is necessary to consider carefully the form of more accurate expressions of this kind. But since here the quantity z can take at once two-fold increments (for here we do not yet consider the new variable t), for the sake of brevity we may put

$$\left(\frac{\partial z}{\partial x}\right) = p \text{ and } \left(\frac{\partial z}{\partial y}\right) = p'$$

which two letters deal with differentials of the first order, then for differentials of the second order we may put :

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = q,\ \left(\frac{\partial\partial z}{\partial x\partial y}\right) = q',\ \left(\frac{\partial\partial z}{\partial y^2}\right) = q'',$$

from which it will help to be noting the following relations between these letters and the preceding ones :

$$\left(\frac{\partial p}{\partial x}\right) = q,\ \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right) = q',\ \left(\frac{\partial p'}{\partial y}\right) = q'';$$

in a similar manner we obtain the differentials of the third order from these formulas :

$$\left(\frac{\partial^3 z}{\partial x^3}\right) = r,\ \left(\frac{\partial^3 z}{\partial x^3\partial y}\right) = r',\ \left(\frac{\partial^3 z}{\partial x\partial y^2}\right) = r'',\ \left(\frac{\partial^3 z}{\partial y^3}\right) = r''',$$

where these relations are to be noted :

$$r = \left(\frac{\partial q}{\partial x}\right),\ r' = \left(\frac{\partial q}{\partial y}\right) = \left(\frac{\partial q'}{\partial x}\right),\ r'' = \left(\frac{\partial q'}{\partial y}\right) = \left(\frac{\partial q''}{\partial x}\right),\ r''' = \left(\frac{\partial q''}{\partial y}\right);$$

moreover the differentials of the fourth order give these formulas :

$$s=\left(\frac{\partial^4 z}{\partial x^4}\right),\ s'=\left(\frac{\partial^4 z}{\partial x^3\partial y}\right),\ s''=\left(\frac{\partial^4 z}{\partial x^2\partial y^2}\right),\ \ s'''=\left(\frac{\partial^4 z}{\partial x\partial y^3}\right),\ \ s''''=\left(\frac{\partial^4 z}{\partial y^4}\right)$$

and thus beyond, as far as it pleases.

§. 21. With these expressions of the second kind set out besides these coordinates themselves *x, y* and *z* also the quantities *p, p′, q, q′, q″, r, r′, r″, r′″* etc. can be involved somehow, from which, if V may denote some expression of this kind, we may show its differential obtained in the customary manner with the following form:

$$\begin{aligned}\partial \mathrm{V}=\mathrm{L}\partial x+\mathrm{M}\partial y+\mathrm{N}\partial z+\mathrm{P}\partial p\quad &+\mathrm{Q}\partial q\quad &&+\mathrm{R}\partial r\\ +\mathrm{P}'\partial p'\quad &+\mathrm{Q}'\partial q'\quad &&+\mathrm{R}'\partial r'\\ &+\mathrm{Q}''\partial q''\quad &&+\mathrm{R}''\partial r''\\ &&&+\mathrm{R}'''\partial r'''\\ &&&\quad \text{etc.,}\end{aligned}$$

which form it may be convenient to remember, lest there may be a need to repeat that more often.

§ 22. But if now a variation of this expression or that increment must be found, which results from the variation of the new variable *t,* as in the value of the coordinate *z* we introduce, now we see there must be taken $\partial x=0$ and $\partial y=0$, then truly there becomes $\partial z=\partial t\left(\frac{\partial z}{\partial t}\right)$, on account of which truly by the same reasoning the following differentials will be expressed in the same manner, which by their own transformations thus will themselves become clear :

$$\partial p=\partial t\left(\frac{\partial p}{\partial t}\right)=\partial t\left(\frac{\partial\partial z}{\partial x\partial t}\right),\ \partial p'=\partial t\left(\frac{\partial p'}{\partial t}\right)=\left(\frac{\partial\partial z}{\partial y\partial t}\right),$$

$$\partial q=\partial t\left(\frac{\partial q}{\partial t}\right)=\partial t\left(\frac{\partial^3 z}{\partial x^2\partial t}\right),$$

$$\partial q'=\partial t\left(\frac{\partial q'}{\partial t}\right)=\partial t\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right),\ \partial q''=\partial t\left(\frac{\partial q''}{\partial t}\right)=\partial t\left(\frac{\partial^3 z}{\partial y^2\partial t}\right),$$

$$\partial r=\partial t\left(\frac{\partial r}{\partial t}\right)=\partial t\left(\frac{\partial^4 z}{\partial x^3\partial t}\right),\qquad \partial r'=\partial t\left(\frac{\partial r'}{\partial t}\right)=\partial t\left(\frac{\partial^4 z}{\partial x^2\partial y\partial t}\right),$$

$$\partial r''=\partial t\left(\frac{\partial r''}{\partial t}\right)=\partial t\left(\frac{\partial^4 z}{\partial x\partial y^3\partial t}\right),\ \partial r'''=\partial t\left(\frac{\partial r'''}{\partial t}\right)=\partial t\left(\frac{\partial^4 z}{\partial y^3\partial t}\right)\ \text{etc.}$$

§ 23. Therefore the whole transaction returns to this, so that in that formula given of the differential for $\partial \mathrm{V}$, these values may be substituted in place of the individual differentials , and in this manner the variation of the expression V arises from the variability of t alone, or the value of this formula $\partial t\left(\frac{\partial V}{\partial t}\right)$, but since the individual parts will be affected by the element ∂t, with those other parts omitted, we arrive at the following form :

$$\begin{aligned}\left(\frac{\partial V}{\partial t}\right)=\mathrm{N}\left(\frac{\partial z}{\partial t}\right)+\mathrm{P}\left(\frac{\partial\partial z}{\partial x\partial t}\right)+\ \mathrm{Q}\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)&+\mathrm{R}\left(\frac{\partial^4 z}{\partial x^3\partial t}\right)\\ +\mathrm{P}'\left(\frac{\partial\partial z}{\partial y\partial t}\right)+\mathrm{Q}'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)&+\mathrm{R}'\left(\frac{\partial^4 z}{\partial x^2\partial y\partial t}\right)\\ +\mathrm{Q}''\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)&+\mathrm{R}''\left(\frac{\partial^4 z}{\partial x\partial y^2\partial t}\right)\\ &+\mathrm{R}'''\left(\frac{\partial^4 z}{\partial y^3\partial t}\right),\\ &\text{etc.,}\end{aligned}$$

which suffice for the variations of any expressions of the second kind being required to be found.

§ 24. Now we will be able to approach expressions of the third kind involving integral formulas ; by which chiefly the strength of this method is decided. For when the question is concerned with maxima or minima, which can occur in surfaces, that formula may be turned around, which must return the maximum or minimum, by necessity it is an integral formula and thus the formula for a double integral, the nature of which it will be convenient to explain with a few examples. Just as in the preceding part simple integral formulas are considered, which are related to a given abscissa x, thus here on the surfaces sought always are not referred to the abscissa x alone, but also are being referred to a certain whole area in the horizontal plane as the base, to which a part of the surface may be considered, which must entertain the certain property of a maximum or minimum. Whereby, when since a twofold dimension of such a base may be had, the one depending on x, the other truly on y , the integral formulas of this kind are twofold expressed in the customary manner $\iint V\partial x\partial y$; clearly these demand a twofold integration and in at first either the coordinate x or the coordinate y is had as a variable and the integration is extended as far as the limits of the base, then truly finally also the other variable is assumed and the other integration is performed. And because likewise, either may be had first for the variable, without distinguishing between both we may indicate that integration with the twofold sign $\iint$; nor indeed is this the place to set out everything further, which are required to be observed about double integrations of this kind, certainly because the argument in Book XIV of these Commentaries had now bee treated carefully enough [See E391 in Supp. 6 above].

§ 25. But if therefore the variation of integral formulas of this kind $\iint V\partial x\partial y$ must be sought, where V may denote some expression either of the first or second kind, from the above it is clear enough this variation is going to be expressed thus :

$$\partial t\iint\left(\frac{\partial V}{\partial t}\right)\partial x\partial y\,,$$

which form again is a double integral and, just as x or y in the first integration may be regarded as constant, this formula can be shown either in this way

$$\partial t\int\partial x\int\left(\frac{\partial V}{\partial t}\right)\partial y\,,$$

or in this way

$$\partial t\int\partial y\int\left(\frac{\partial V}{\partial t}\right)\partial x\,.$$

§ 26. Now V shall be such an expression, of the kind we have described above in paragraph 19 and whose variation or value $\left(\frac{\partial V}{\partial t}\right)$ we have established in paragraph 23, only there will be a need to substitute the individual members set out here in place of $\left(\frac{\partial V}{\partial t}\right)$; from which the following collections of formulas will arise, with which taken together the variation sought $\partial t\iint\left(\frac{\partial V}{\partial t}\right)\partial x\partial y$ may be expressed :

$$\begin{aligned}
&\partial t\iint N\left(\frac{\partial z}{\partial t}\right)\partial x\partial y+\partial t\iint P\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x\partial y+\partial t\iint Q\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial x\partial y+\partial t\iint R\left(\frac{\partial^4 z}{\partial x^3\partial t}\right)\partial x\partial y\\
&\qquad+\partial t\iint P'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial x\partial y+\partial t\iint Q'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x\partial y+\partial t\iint R'\left(\frac{\partial^4 z}{\partial x^2\partial y\partial t}\right)\partial x\partial y\\
&\qquad\qquad+\partial t\iint Q''\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\partial x\partial y+\partial t\iint R''\left(\frac{\partial^4 z}{\partial x\partial y^2\partial t}\right)\partial x\partial y\\
&\qquad\qquad\qquad+\partial t\iint R'''\left(\frac{\partial^4 z}{\partial y^3\partial t}\right)\partial x\partial y,\\
&\qquad\qquad\qquad\qquad\text{etc.},
\end{aligned}$$

§ 27. Now these individual terms after the first admit to special reductions, which it will help to be noted properly. For the second term we may take initially x as variable, and there will be :

$$\int P\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x=P\left(\frac{\partial z}{\partial t}\right)-\int\left(\frac{\partial z}{\partial t}\right)\partial x\left(\frac{\partial P}{\partial x}\right),$$

from which also the other integration on being added will be :

$$\iint \mathrm{P}\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x\partial y = \int \mathrm{P}\left(\frac{\partial z}{\partial t}\right)\partial y - \iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial \mathrm{P}}{\partial x}\right)\partial x\partial y.$$

For the third term only the first *y* may be taken variable, and there will be :

$$\int \mathrm{P}'\left(\frac{\partial\partial z}{\partial y\cdot\partial t}\right)\partial y = \mathrm{P}'\left(\frac{\partial z}{\partial t}\right) - \int\left(\frac{\partial z}{\partial t}\right)\partial y\left(\frac{\partial \mathrm{P}'}{\partial y}\right),$$

from which the third term itself will be changed into

$$\iint \mathrm{P}'\left(\frac{\partial\partial z}{\partial y\cdot\partial t}\right)\partial x\partial y = \int \mathrm{P}'\left(\frac{\partial z}{\partial t}\right)\partial x - \iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial \mathrm{P}'}{\partial y}\right)\partial x\partial y.$$

§ 28. For the following terms these following reductions will give the transformations, clearly for the fourth we will have from the second :

$$\iint \mathrm{Q}\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial x\partial y = \int \mathrm{Q}\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial y - \iint\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\partial x\partial y,$$

but truly this latter term is reduced in this manner to a likeness of the second, where finally in place of P there must be $\left(\frac{\partial \mathrm{Q}}{\partial x}\right)$:

$$\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\partial y - \iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)\partial x\partial y,$$

thus so that now the fourth term will give this form:

$$\int \mathrm{Q}\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial y - \int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\partial y + \iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)\partial x\partial y.$$

In a similar manner the fifth term is reduced with the aid of the second, where in place of P there may be written Q' and in place of $\left(\frac{\partial\partial z}{\partial x\partial t}\right), \left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)$, or in place of $\left(\frac{\partial z}{\partial t}\right)$ by writing $\left(\frac{\partial\partial z}{\partial y\partial t}\right)$, and thus there will be had

$$\iint \mathrm{Q}'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x\partial y = \int \mathrm{Q}'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial y - \iint\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial \mathrm{Q}'}{\partial x}\right)\partial x\partial y,$$

which latter term may be brought together with the third, where in place of P′ only there must be written $\left(\frac{\partial \mathrm{Q}'}{\partial x}\right)$, with which agreed on the whole term will adopt this form:

$$\int \mathrm{Q}'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial y-\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial \mathrm{Q}'}{\partial x}\right)\partial x+\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{Q}'}{\partial x\partial y}\right)\partial x\partial y,$$

truly the sixth term taken twice with the second is reduced to this form :

$$\int \mathrm{Q}''\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial x-\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial \mathrm{Q}''}{\partial y}\right)\partial x+\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{Q}''}{\partial y^2}\right)\partial x\partial y,$$

§29. If we may progress further in this manner to the following terms, the seventh term is resolved into the following parts :

$$\int \mathrm{R}\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial y-\int\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial \mathrm{R}}{\partial x}\right)\partial y+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)\partial y-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right)\partial x\partial y,$$

then the eighth term :

$$\int \mathrm{R}'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial y-\int\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial \mathrm{R}'}{\partial x}\right)\partial x+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}'}{\partial x\partial y}\right)\partial y-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 \mathrm{R}'}{\partial x^2\partial y}\right)\partial x\partial y,$$

moreover the ninth term will become:

$$\int \mathrm{R}''\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x-\int\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial \mathrm{R}''}{\partial y}\right)\partial y+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}''}{\partial x\partial y}\right)\partial x-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 \mathrm{R}''}{\partial x\partial y^2}\right)\partial x\partial y,$$

and the tenth

$$\int \mathrm{R}'''\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\partial x-\int\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial \mathrm{R}'''}{\partial y}\right)\partial x+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}'''}{\partial y^2}\right)\partial x-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 \mathrm{R}'''}{\partial y^3}\right)\partial x\partial y.$$

§30. Now we may gather all these formulas into one sum and the variation sought will be agreed on from the various members, of which the first double integral formulas, with the rest truly simply enclosed in brackets, with this agreed on the variation sought will be expressed in the following manner :

$$\partial t\iint \partial x\partial y\left(\frac{\partial z}{\partial t}\right)\left\{\begin{array}{l} \mathrm{N}-\left(\frac{\partial \mathrm{P}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)-\left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right)\\ \quad -\left(\frac{\partial \mathrm{P}'}{\partial y}\right)+\left(\frac{\partial\partial \mathrm{Q}'}{\partial x\partial y}\right)-\left(\frac{\partial^3 \mathrm{R}'}{\partial x^2\partial y}\right)\\ \qquad\qquad +\left(\frac{\partial\partial \mathrm{Q}''}{\partial y^2}\right)-\left(\frac{\partial^3\mathrm{R}''}{\partial x\partial y^2}\right)\\ \qquad\qquad\qquad\qquad -\left(\frac{\partial^3 \mathrm{R}'''}{\partial y^3}\right)\\ \qquad\qquad\qquad\qquad\quad \text{etc.}\end{array}\right\}$$

$$+\partial t\left\{\begin{array}{l} \int\left(\frac{\partial z}{\partial t}\right)\mathrm{P}\partial y+\int \mathrm{Q}\partial y\left(\frac{\partial\partial z}{\partial x\partial t}\right)-\int\partial y\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right)+\int \mathrm{R}\partial y\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\\ \int\left(\frac{\partial z}{\partial t}\right)\mathrm{P}'\partial x+\int \mathrm{Q}'\partial y\left(\frac{\partial\partial z}{\partial y\partial t}\right)-\int\partial x\left(\frac{\partial \mathrm{Q}'}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right)+\int \mathrm{R}'\partial y\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\\ \qquad +\int \mathrm{Q}''\partial x\left(\frac{\partial\partial z}{\partial y\partial t}\right)-\int\partial x\left(\frac{\partial \mathrm{Q}''}{\partial y}\right)\left(\frac{\partial z}{\partial t}\right)+\int \mathrm{R}''\partial x\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\\ \qquad\qquad\qquad +\int \mathrm{R}'''\partial x\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\\ \qquad\qquad -\int\partial y\left(\frac{\partial \mathrm{R}}{\partial x}\right)\left(\frac{\partial\partial z}{\partial x\partial t}\right)+\int\partial y\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)\\ \qquad\qquad -\int\partial x\left(\frac{\partial \mathrm{R}'}{\partial x}\right)\left(\frac{\partial\partial z}{\partial x\partial t}\right)+\int\partial y\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}'}{\partial x\partial y}\right)\\ \qquad\qquad -\int\partial y\left(\frac{\partial \mathrm{R}''}{\partial y}\right)\left(\frac{\partial\partial z}{\partial y\partial t}\right)+\int\partial x\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}''}{\partial x\partial y}\right)\\ \qquad\qquad -\int\partial x\left(\frac{\partial \mathrm{R}'''}{\partial y}\right)\left(\frac{\partial\partial z}{\partial y\partial t}\right)+\int\partial x\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}'''}{\partial y^2}\right)\\ \qquad\qquad\qquad \text{etc.}\end{array}\right\}$$

§ 31. Truly at this point it is by no means clear what these individual terms may indicate properly, and to whatever use they may be able to be put ; it being agreed to demand all the attention of the geometers, and it would seem to demand a much more precise investigation of which the first foundations even now have scarcely been laid ; as even before the problem can be allowed to be undertaken, all the particular cases will need to have been established with care and diligence; even the first part, which may involve only functions of one variable, by no means has been enunciated clearly and distinctly enough at this stage; thus so that we may understand clearly the true disposition and nature of the individual parts we have found, to be contained by which variation, which finally the following elucidations has been considered to be added here.

§32. The questions, which occur here, may be applied to this general problem:

The exposition of this Problem

If y were some function of x and thence the value may be defined of a certain given integral formula $\int Z\partial x$, *with* Z *denoting an expression composed from these quantities x et y, and of their differentials composed in some manner from the ratios; the question is, if in place of this function y, some other may be used one nearby, or disagreeing with that only by an infinitely small amount, then by how much greater or smaller a value shall the same integral formula* $\int Z\partial x$ *be going to follow.*

§33. But because this same question enunciated in this way may be seen to be exceedingly abstract, we may recall that customary solution for the geometer. Therefore on the axis AP (Fig. 1) there shall be some proposed curve AM with the equation expressed between the abscissa $\mathrm{AP}=x$ and the applied line $\mathrm{PM}=y$, for which the value of the formula of some integral may be required to be defined $\int Z\partial x$, which shall be $=\mathrm{W}$, with which in place we may consider some other curve $\alpha\mu$ differing an infinitely small amount from the given curve, and if for this curve likewise the value of the formula $\int Z\partial x$ may be sought, how much this value shall be going to differ from the preceding ; indeed it is evident this discrepancy provides the same variation of the quantity W, as we have shown above with the aid of the calculus of variations.

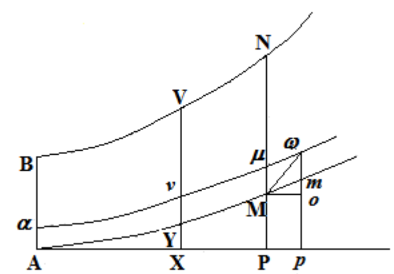

Fig. 1

§ 34. So that this may appear at this point, we may advance some example, where the curve proposed AM and with its axis AP being considered as vertical, the time is sought, in which the body descending from the point A on this curve AM reaches as far as to the point M. Now, because the speed of the body at M is as $\sqrt{\mathrm{AP}}=\sqrt{x}$ and the element of the curve itself $=dx\sqrt{(1+pp)}$, evidently on putting $dy=pdx$, as has been set out in the general solution, the time for the element $\mathrm{M}m=dx\frac{\sqrt{(1+pp)}}{\sqrt{x}}$, from which the formula of the integral $\int Z\partial x$ for this case will be changed into $\int \partial x\frac{\sqrt{(1+pp)}}{\sqrt{x}}$, thus so that there may be had $Z=\frac{\sqrt{(1+pp)}}{\sqrt{x}}$, whereby now the time will be required to be defined, where the body descending on some nearby curve $\alpha\mu$ from α will come to μ, where the difference will give the same variation of the formula $\int \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$ appropriate for this case.

§ 35. Because here the formula of the integral being considered arises, before everything it is required to be seen clearly, how that ought to be determined. Indeed in the example advanced it is evident the integral of the formula $\int \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$ thus must be taken, so that it may vanish on putting $x=0$, from which also in general it is understood always for the integration of the formula $\int \mathrm{Z}\partial x$ some certain limit or point A to be used as if the beginning of the integration to be put in place and the integral $\int \mathrm{Z}\partial x$ must vanish on putting $x=0$ or, if perhaps if circumstances were prepared otherwise, on some given value being attributed to x ; then truly the initial value of the formula $\int \mathrm{Z}\partial x = \mathrm{W}$ put in place, will correspond to the abscissa $\mathrm{AP} = x$.

§ 36. We may see from these observations about the integral formula $\int \mathrm{Z}\partial x$, how we ourselves may form an idea about these nearby curves $\alpha\mu$. And indeed in the first place it is apparent these curves must be drawn with a certain force, thus so that nowhere within these does some angle or another leap out to be seized upon ; only here likewise it has been observed, either these curves may be from some law of continuation or they may be contained by some equation, or thus they may be discontinuous, as if they may be drawn by the motion of a free hand.

§ 37. The formation of curved lines of this kind may be brought to mind, and represented most conveniently in the following way. Clearly, some curved line BN overhanging the same abscissa AP may be drawn as it pleases, and with the applied line XYV drawn to the singular point X of the axis, they may cut at v with the singular interval YV in some finite ratio to an infinitesimally small part Yv, thus so that Yv shall be as if an infinitesimal part of the interval YV. For in this manner the curve $\alpha v\mu$ will be obtained placed at all the infinitely small intervals from the proposed curve AM, such as we require according to our setting up. Yet besides it is required to be observed on that arbitrary curve BN nowhere the tangent must be normal to the axis AP, because in this way the division of these intervals would be disturbed. And now it is evident not only are the intervals Yv to be infinitely small, but also the tangents at the points Y and v depart from being parallel by in infinitely small amount.

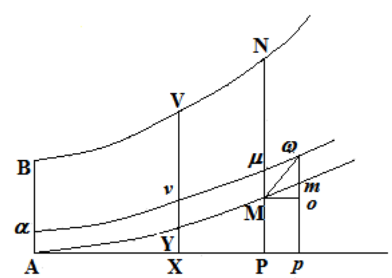

Fig. 1

Exposition of the first part in the variation.

§ 38. With these questions noted about the proposition, now also we may consider more carefully the solution found above and its individual parts, so that we may understand clearly, whatever any of these may indicate and for whatever use it may be adopted ; moreover we will be considering here the solution given in paragraph 15. Therefore we will examine at once the first part of the variation found there, which this integral formula may contain:

$$\partial t\int \partial x\left(\frac{\partial y}{\partial t}\right)\left(\mathrm{N}-\left(\frac{\partial \mathrm{P}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)-\left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right)+\left(\frac{\partial^4 \mathrm{S}}{\partial x^4}\right)-\text{etc.}\right),$$

the integration of which must be taken thus, so that A may vanish in the initial limit, by which condition the arbitrary constant may be determined; but therefore if this formula may be understood to be applied at the particular points XY, the sum of all of these elements from the beginning A extended as far as to the limit M will give the first part of the variation sought, and this indeed is apparent in the Figure to express the small interval Yv, the increment of the applied line y arising from the variation t alone, thus so that there shall be $\mathrm{Y}v = dt\left(\frac{\partial y}{\partial t}\right)$.

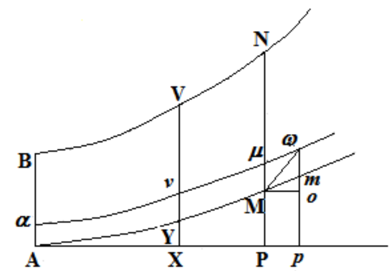

Fig. 1

§ 39. This first part of the variation therefore has involved all the small interval v contained between the limits A and M, which since they may be varied indefinitely and thus may be able to change from positive to negative, here the maximum variations can happen. Yet truly a single case hence must be excepted, where the curve AM has been prepared thus, so that there shall be

$$0=\mathrm{N}-\left(\frac{\partial \mathrm{P}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)-\left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right)+\left(\frac{\partial^4 \mathrm{S}}{\partial x^4}\right)-\text{etc.},$$

since then, whatever nearest curves were prepared, that first part of the variation will become zero always. Nor does a deviation of the nearest curves $\alpha\mu$ bring anything to the principal AM between the limits A and M ; from which this curve is especially memorable with regard to the integral formula $\int \mathrm{Z}\partial x$, since in that this integral formula will obtain either a maximum or minimum value.

Exposition of the second part in the variation.

§ 40. Now we may progress to the second part of the variation found above, which is

$$\partial t\left(\frac{\partial y}{\partial t}\right)\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\left(\frac{\partial^3 \mathrm{S}}{\partial x^3}\right)+\text{etc.}\right),$$

about which I observe first, since that is referred back to the limit M by the integration duly put in place, in addition a similar expression must be added to the first limit A referred to, but truly associated with an opposite sign, that which thus is necessary, so that on making $x=0$ this expression is completely removed. Moreover this part is referred to

$$\partial t\left(\frac{\partial y}{\partial t}\right)\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\text{etc.}\right)$$

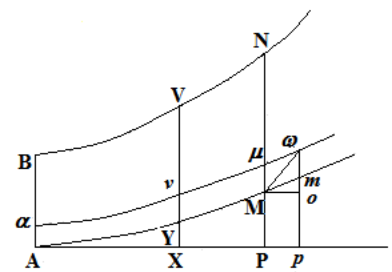

Fig. 1

especially to the last term M, where $\partial t\left(\frac{\partial y}{\partial t}\right)$ itself expresses the small interval Mμ, and in a similar manner in the other part for the initial A, the small interval Aα may be introduced. Hence it is apparent, if all the nearest curves $\alpha\mu$ may be drawn through both these terms A and M, then the variation of the second parts go to zero.

§ 41. But we must consider the case, where the nearest curve $\alpha\mu$ indeed passes through the limit A, but truly not also through the other M, but the point μ shall be its limit, and the variation produced from the second part will be

$$=\mathrm{M}\mu\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\text{etc.}\right).$$

And hence also we will be able to define the variation arising from the same source, if the nearest curve Aμ may not end in the point μ itself, but at some other point ω, with the interval present $\mu\omega$ always infinitely small. For with the applied line ωmp drawn, the variation in the manner found above must be increased by a small part of the formula $\int \mathrm{Z}\partial x$, which corresponds to the element $\mathrm{P}p = dx$, which since the small part shall be $=\mathrm{Z}\cdot\mathrm{P}p$, for the arc of the nearest curve Aω the variation arising from the second part will be

$$=\mathrm{M}\mu\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\text{etc.}\right)+\mathrm{Z}\cdot\mathrm{P}p.$$

§ 42. The right line Mω may be drawn and we may seek the angle *wMm,* which this right line Mω makes with the main curve, this angle ωMm may be put $=\omega$ and with MO drawn parallel to Pp itself, because there is approximately $m\omega = \mathrm{M}\mu$ and the tangent of the angle mMo p and thus $om = p\cdot Pp$, there will be had $o\omega = M\mu + p\cdot Pp$, from which there becomes :

$$\text{tang } \omega Mo = \frac{M\mu}{Pp} + p,$$

and hence there is deduced

$$\omega Mm = \text{tang } \omega = \frac{M\mu}{Pp(1+pp)+M\mu\cdot p}.$$

Now we make use of this in the calculation of the angle ω and hence we will have the small interval :

$$M\mu = \frac{Pp(1+pp)\text{tang } \omega}{1-p\text{tang } \omega},$$

with the value substituted the variation for the arc $A\omega$ will become :

$$Pp\left(Z + \frac{(1+pp)\text{tang } \omega}{1-p\text{tang } \omega}\cdot\left(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \text{etc.}\right)\right).$$

§ 43. Now it will be worth the effort to define that angle ω itself, so htat this variation may disappear into nothing, that which will happen, if there may be taken

$$\text{tang } \omega = \frac{Z}{pZ-(1+pp)\left(P-\frac{\partial Q}{\partial x}+\frac{\partial\partial R}{\partial x^2}-\text{etc.}\right)},$$

whereby with this angle thus put in place for all the closest lines, and everywhere on the right line $M\omega$ the variation of the limit will vanish arising from the second part. This case before all the rest is generally merited to be considered noteworthy, where the right line $M\omega$ becomes normal to the principal curve at the point M, which happens, if there were

$$pZ-(1+pp)\left(P-\frac{\partial Q}{\partial x}+\frac{\partial\partial R}{\partial x^2}-\text{etc.}\right)=0,$$

from which equation certainly the condition of its integral formula $\int Z\partial x$, or the nature of the expression Z is defined.

§ 44. Therefore there will be no annoyance to have examined such an expression Z, and indeed in the first place it appears that besides the coordinates x and y the quantity p must be involved. But we may assume besides that the letters q, r etc. are not to be present in Z, thus so that there shall be $Q = 0$, $R = 0$, and our equation required to be resolved will be :

$$pZ = (1+pp)P,$$

where it is to be observed :

$$\partial Z = M\partial x + N\partial y + P\partial p,$$

whereby, if both the coordinates x and y may be treated as constants, there will become

$$\partial Z = P\partial p \text{ and thus } P = \tfrac{\partial Z}{\partial p},$$

with which value introduced everywhere this equation will be produced :

$$\tfrac{dZ}{Z} = \tfrac{pdp}{1+pp},$$

which integrated will give :

$$LZ = L\sqrt{(1+pp)} + LC,$$

which constant can be some function of x et y , V shall be such a function and we will have

$$Z = V\sqrt{(1+pp)}$$

and thus the formula of the integral

$$= \int V\partial x\sqrt{(1+pp)} .$$

The significance of this formula can be expressed elegantly enough by the time, in which some body may be moved by the curve AM. If indeed the speed at some point M were $= \frac{1}{V}$, that is if the speed at the individual points were proportional to some function of the two variables x and y, then $V\partial x\sqrt{(1+pp)}$ expresses and element of time and thus the formula

$$= \int V\partial x\sqrt{(1+pp)}$$

the whole time, in which the body arrives at M from A.

Exposition of the third part in the variation.

§ 45. Which pertains to the third part of the variation, namely

$$\partial t\left(\tfrac{\partial\partial y}{\partial x\partial t}\right)\left(Q - \tfrac{\partial R}{\partial x} + \tfrac{\partial\partial S}{\partial x^2} - \text{etc.}\right),$$

this may not have a place, unless the expression Z may also involve differentials of the second order, which indeed is accustomed to be the rarest in use to arise. But here it is required to be observed, since $M\mu = \partial t\left(\frac{\partial y}{\partial t}\right)$, for the following element to become :

$$m\omega = \partial t\left(\frac{\partial y}{\partial t}\right) + \partial t\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right),$$

from which there is deduced :

$$\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right) = \frac{m\omega - \mathrm{M}\mu}{\partial x} = \frac{\mathrm{M}\omega - \mathrm{M}\mu}{\mathrm{P}p},$$

but this formula is expressing the deviation of the direction $\mu\omega$ from the direction M*m*, which indeed, as now we have observed before, always is very small.

§ 46. But if therefore the tangent at μ were perfectly parallel to the tangent at M, which happens, if also the tangent BN being generated on the curve at N were parallel to this, then the variation arising from the third part evidently vanishes, which also is to be understood from the initial limit A, if the tangents at A and B were parallel to each other, and hence now it is seen, to be necessary that the variations arising from the fourth part vanish, so that besides also the radii of osculation at the points M and μ become equal.

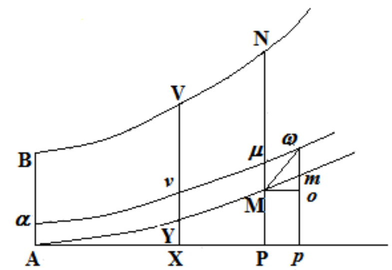

Fig. 1

§ 47. It is clear enough now from these that the variations arising from the second part vanish, if all the closest curves $\alpha\mu$ may be drawn through the limits M and A, then truly in addition the variations of the third part, if all the closest curves likewise at each limit A and M may have common tangents with the principal curve AM. Besides truly the variations of the fourth part too go to zero, if in addition the account of curvature of all the closest curves at the limits A and M may agree with the main curve. But here it will help to remember properly the variations of the third part themselves vanish, only if the quantity Z may not involve differentials of the second order ; truly of the fourth part vanish always, unless the differentials of the third order may enter into the quantity Z, and so thus. From which we have shown from the beginning, how the variation of the first parts shall be reduced to zero, now we understand most clearly, under what conditions all the variations of the parts likewise may vanish.

Explanations concerning curves endowed with the property of maxima or minima

§ 48. If the integral formula $\int \mathrm{Z}\partial x$ sought in a curve must be either a maximum or minimum, now above we have shown on putting

$$\partial\mathrm{Z} = \mathrm{M}\partial x + \mathrm{N}\partial y + \mathrm{P}\partial p + \mathrm{Q}\partial q + \mathrm{R}\partial r + \text{etc.}$$

the nature of this curve to be expressed by this equation :

$$0 = \mathrm{N} - \frac{\partial\mathrm{P}}{\partial x} + \frac{\partial\partial\mathrm{Q}}{\partial x^2} - \frac{\partial^3\mathrm{R}}{\partial x^3} + \text{etc.},$$

which equation, unless the quantities P, Q, R may vanish or shall be constants, always is a differential, either of the second, fourth, sixth, or of some higher even degree. Therefore here at once it happens worth mentioning, because this equation at no time may emerge either with a simpler differential, or of the third, fifth or of any higher odd order, that which soon we may set out more clearly.

§ 49. Therefore here related questions may be divided at once into various classes for the grade of the differentials, for which equations arise, since the nature of the solution depends mainly on this grade, therefore that always involves just as many arbitrary constants. Therefore to the first class we will refer these cases, in which the equation for the maximum or minimum found is absolutely finite. But for the second class these, in which this equation shall be a differential of the second order, those to the third in which the equation shall rise to the fourth order, and so on thus, which individual classes we may describe in order.

Class I.

§ 50. The formula $\int \mathrm{Z}\partial x$ leads at once to a solution of the first class, when the expression Z may be determined by the coordinates x and y with all the differential ratios excluded ; because indeed in this case there shall become simply $\partial \mathrm{Z} = \mathrm{M}\partial x + \mathrm{N}\partial y$, the equation for the maxima or minima for curves will be $\mathrm{N} = 0$, therefore which equation generally is required to be determined, and thus satisfying the individual curves of their own kind. Just as, if the line may be sought, in which the value of the formula $\int \partial x(2xy - yy)$ may become a maximum or a minimum, on account of $\mathrm{Z} = 2xy - yy$ and thus $\mathrm{N} = 2(x - y)$ the equation sought will be $x - y = 0$ or the line sought will be a right line inclined to the axis by a semi-right angle, therefore for which the value of the proposed integral is x , which certainly is smaller, than if any other curved line may be assumed clearly for the same abscissa.

§ 51. But the first class is not yet exhausted by these cases, moreover others likewise are given at this stage leading to finite equations, for which there shall be required to be shown some function $\mathfrak{Z}$ of x and y and $\partial \mathfrak{Z} = \mathfrak{M}\partial x + \mathfrak{N}\partial y$, and now there may be put $\mathrm{Z} = \mathfrak{Z}p$, and there will be $\mathrm{M} = \mathfrak{M}p,\ \mathrm{N} = \mathfrak{N}p,\ \mathrm{P} = \mathfrak{Z}$, whereby, so that the formula $\int \mathrm{Z}\partial x$ may become a maximum or minimum, the equation is found:

$$0 = \mathfrak{N}p - \frac{\partial \mathfrak{Z}}{\partial x} = \mathfrak{N}p - \mathfrak{M} - \frac{\mathfrak{N}\cdot\partial y}{\partial x} = -\mathfrak{M},$$

which likewise is a finite equation. Which indeed also may be allowed to be foreseen, since indeed there shall be $p\partial x = \partial y$, this integral formula $\int \mathfrak{Z}\partial y$ does not differ much from the preceding $\int Z\partial x$, except that the coordinates x and y shall be interchanged, from which, since this was confirmed with the former, also it will prevail with the latter.

Hence the nature of the first class at this stage can thus be described more generally, as that may include all the integral formulas of this kind $\int (Z+\mathfrak{Z}p)\partial x$, where the letters Z et $\mathfrak{Z}$ may denote any functions of x and y, then indeed the equation for the curve of the maximum or minimum will be :

$$0 = N - \mathfrak{M},$$

which is the equation generally determined.

Class II.

§ 52. We will refer these integral formulas $\int Z\partial x$ to the second class, which are deduced for a differential equation of the second order; therefore here initially the cases pertain, in which Z is composed only from the letters x, y and p , thus so that there shall become :

$$\partial Z = M\partial x + N\partial y + P\partial p,$$

from which indeed it will be required to remove the latter of the first case, which certainly comes about, if P were a function of x and y only, thus so that for the present case the quantity P besides x and y also must include the letter p . Moreover then the equation for the curve sought will be $0 = N - \frac{\partial P}{\partial x}$, where, since P may include p and thus $\frac{\partial y}{\partial x}$, the formula $\frac{\partial P}{\partial x}$ will contain differentials of the second order ; therefore this equation can never be determined, since indeed it may receive two arbitrary constants, from which it can be put into effect, so that the curve may pass through two given points, and thus indeed the questions of this class are required to be defined more precisely, so that the curves to be investigated, which may entertain the prescribed property of maxima or minima, may not be between all plane curves, but only between these, which may be drawn through the same two points ; but questions of this class always are to be prepared thus, as by its nature demand this restriction.

§ 53. Truly besides it will be necessary also to refer to the second class the case, in which $Z = \mathfrak{Z}q$ with $\mathfrak{Z}$ being some function of x, y and p, if indeed there were

$$\partial\mathfrak{Z} = \mathfrak{M}\partial x + \mathfrak{N}\partial y + \mathfrak{P}\partial p,$$

we will have

$$M = \mathfrak{M}q,\ N = \mathfrak{N}q,\ P = \mathfrak{P}q \text{ and } Q = \mathfrak{Z},$$

whereby, since the equation for the curve shall be

$$0 = \mathrm{N} - \frac{\partial \mathrm{P}}{\partial x} + \frac{\partial\partial \mathrm{Q}}{\partial x^2} \text{ or } 0 = \mathrm{N} - \frac{\partial}{\partial x}\partial \cdot \left(\mathrm{P} - \frac{\partial \mathrm{Q}}{\partial x}\right),$$

this formula $\mathrm{P} - \frac{\partial \mathrm{Q}}{\partial x}$ will be changed into

$$\mathfrak{P}q - \frac{\partial \mathfrak{Z}}{\partial x} = \mathfrak{P}q - \mathfrak{M} - \mathfrak{N}p - \mathfrak{P}q = -\mathfrak{M} - \mathfrak{N}p,$$

from which our equation becomes :

$$0 = \mathrm{N} + \frac{\partial}{\partial x}\partial\left(\mathfrak{M} + \mathfrak{N}p\right) = 2\mathfrak{N}q + \frac{\partial \mathfrak{M}}{\partial x} + p\frac{\partial \mathfrak{N}}{\partial x},$$

which evidently contains a differential of the second order only. Therefore more generally according to this, if the proposed integral formula were $\int\left(\mathrm{Z} + \mathfrak{Z}p\right)\partial x$, where Z and $\mathfrak{Z}$ may be composed in some manner from the quantities *x, y* and p, the equation for the curve sought will be :

$$0 = \mathrm{N} - \frac{\partial \mathrm{P}}{\partial x} + 2\mathfrak{N}q + \frac{\partial \mathfrak{M}}{\partial x} + p\frac{\partial \mathfrak{N}}{\partial x},$$

or also

$$0 = \mathrm{N}\partial x - \partial \mathrm{P} + 2\mathfrak{N}\partial p + \partial \mathfrak{M} + p\partial \mathfrak{N},$$

which clearly is a differential of the second order only.

Class III.

§ 54. But if the quantity Z thus were composed from the letters *x, y, p* and q, so that on putting

$$\partial \mathrm{Z} = \mathrm{M}\partial x + \mathrm{N}\partial y + \mathrm{P}\partial p + \mathrm{Q}\partial q$$

also the quantity Q may involve the letter *q,* then cases of this kind being referred to the third class, and since the equation for the curve sought may be found :

$$0 = \mathrm{N} - \frac{\partial \mathrm{P}}{\partial x} + \frac{\partial\partial \mathrm{Q}}{\partial x^2},$$

it is evident the limit $\frac{\partial\partial \mathrm{Q}}{\partial x^2}$ involves differentials of the fourth order, from which a finite equation for the curve will contain four arbitrary constants, from which therefore it can be put into effect, so that the desired curve not only may pass through the two given limits [of integration], but also its tangents at each limit may obtain a given position, in which by a fourfold determination the nature sought pertaining to this class may be put in place, and is to be considered most carefully.

§ 55. I do not tarry with the remaining cases pertaining to this class, truly rather for the sake of illustration I may advance a significant example, by which elastic curves are accustomed to be investigated. Clearly, (Fig. 1), if the letter ρ may denote the radius of osculation of the curve sought at the point M, all these curves entertain this property, so that in these the formula $\int \frac{\partial x\sqrt{(1+pp)}}{\rho\rho}$ shall be a minimum and thus there may be had

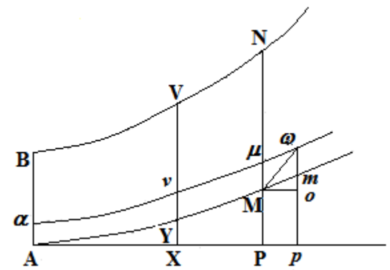

Fig. 1

$\mathrm{Z}=\frac{\sqrt{(1+pp)}}{\rho\rho}$, since truly there shall be $\rho=\frac{(1+pp)^{3:2}}{q}$,

we will have $\mathrm{Z}=\frac{qq}{(1+pp)^{3:2}}$, from which there shall become

$$\mathrm{M}=0,\ \mathrm{N}=0,\ \mathrm{P}=-\frac{5pqq}{(1+pp)^{7:2}} \text{ and } \mathrm{Q}=+\frac{2q}{(1+pp)^{5:2}},$$

whereby, as on account of $\mathrm{N}=0$, the equation of the curve sought shall be :

$$0=-\frac{\partial \mathrm{P}}{\partial x}+\frac{\partial\partial \mathrm{Q}}{\partial x^2},$$

of which the integral at once becomes:

$$\mathrm{P}-\frac{\partial \mathrm{Q}}{\partial x}=\mathrm{A},$$

which according to this is a differential of the third order.

§ 56. Truly at this point the equation can be integrated in general, for it may be multiplied by $qdx=dp$, so that this equation may be had $\mathrm{P}dp-qd\mathrm{Q}=\mathrm{A}dp$, since truly there shall be $d\mathrm{Z}=\mathrm{P}dp+\mathrm{Q}dq$, there will be $\mathrm{P}dp=dZ-\mathrm{Q}dq$, with which value substituted this equation will result :

$$\partial \mathrm{Z}-\mathrm{Q}\partial q-q\partial \mathrm{Q}=\mathrm{A}\partial p\ ,$$

the integral of which clearly is :

$$\mathrm{Z}-\mathrm{Q}q=\mathrm{A}p+\mathrm{B},$$

now therefore the values given above for Z and Q may be substituted, and we will obtain the following equation :

$$-\frac{qq}{(1+pp)^{5:2}}=\mathrm{A}p+\mathrm{B};$$

therefore with the signs of the constants changed we will deduce

$$qq=(\mathrm{A}p+\mathrm{B})(1+pp)^{5:2}$$

and thus

$$q=(1+pp)^{5:4}\sqrt{(\mathrm{A}p+\mathrm{B})}=\frac{\partial p}{\partial x},$$

and thus we conclude

$$\partial x=\frac{\partial p}{(1+pp)^{5:4}\sqrt{(\mathrm{A}p+\mathrm{B})}}$$

and hence again

$$\partial y=\frac{p\partial p}{(1+pp)^{5:4}\sqrt{(\mathrm{A}p+\mathrm{B})}}$$

from which two equations the construction of the curve is resolved.

§ 57. Formerly when this method of treating maxima and minima was started, not only curves of this kind were investigated, in which some integral formula $\int \mathrm{Z}\partial x$ were to be either a maximum or a minimum, but also questions of this kind were being proposed, as that may be sought, not between all plane curves, but only between these, which may have the same length, in which that formula may become a maximum or minimum, from which case itself, the name of the Isoperimetric Problem has arisen ; but this name has not impeded, nevertheless, how more general questions of this kind may be proposed, so that between all these curves, in which some value of a certain integral formula $\int \mathrm{V}\partial x$ equally may be agreed on, that may be defined, in which the formula $\int \mathrm{Z}\partial x$ may draw out a maximum or minimum value, when also conditions according to this were to be multiplied in this way, so that only between all these curves, in which not only the formula $\int \mathrm{V}\partial x$, but also however many of these $\int \mathrm{V}'\partial x$, $\int \mathrm{V}''\partial x$ etc. equally may come together, that may be defined, in which $\int \mathrm{Z}\partial x$ shall be a maximum or a minimum, problems of this kind at the time were seen to demand much hard work. Truly afterwards in my treatment concerning this problem, I may have shown that problems of this kind can be reduced always to this simple problem, where between all plane curves these may be investigated, in which this integral formula

$$\int \partial x\left(\mathrm{Z}+\alpha\mathrm{V}+\beta\mathrm{V}'+\gamma\mathrm{V}''+\text{etc.}\right)$$

may become a maximum or a minimum, problems of this kind have no further difficulty.

METHODUS NOVA ET FACILIS CALCULUM VARIATIONUM TRACTANDI

Commentatio indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 16 (1771), 1772, p. 35-70
[E420]

§ 1. Si detur aequatio quaecunque inter binas variabiles x et y seu, quod eodem redit, si y fuerit functio quaecunque ipsius x, tum omnes expressiones quomodocunque ex his duabus quantitatibus x et y formatae et compositae tamquam functiones unius variabilis x spectari poterunt, ita ut pro quovis valore determinato ipsius x determinatos quoque valores sortiantur.

§ 2. Huiusmodi autem expressionum ex quantitatibus x et y formatarum tria genera constitui convenit; ad quorum primum referimus omnes illas expressiones, in quibus tantum ipsae quantitates x et y occurrunt et per operationes quascunque sive algebraicas sive etiain transcendentes inter se sunt complicatas, cuiusmodi sunt $\alpha x^3+\beta xy+\gamma y^3$, item $e^{ax}\text{Arcsin}\, y$, in qua posteriore operationes transcendentes cernuntur. Secundum autem genus eas complectitur expressiones in quibus praeter ipsas quantitates x et y etiam ratio differentialium occurrit, quam rationem adeo ad differentialia cuiusque gradus extendimus, cuiusmodi expressionum indolem quo clarius perspiciamus, ponatur more solito

$$\partial y=p\partial x,\ \partial p=q\partial x,\ \partial q=r\partial x \text{ etc.}$$

ac tales expressiones erunt functiones quantitatum x, y, p, q, r etc. Tertium denique genus eiusmodi expressiones continet, in quibus praeterea formulae integrales involvuntur, quorsum pertinent expressiones illae in calculo variationum imprimis consideratae, quae hac forma sunt repraesentatae $\int V dx$, ubi V est functio quaecunque non solum ipsarum x et y, sed etiam quantitatum p, q, r etc., quin etiam ea alias insuper formulas integrales involvere potest.

§ 3. His circa terna huiusmodi expressionum genera constitutis facilius indolem calculi variationum explicare poterimus. Totum enim negotium huc redit, ut, si proposita fuerit relatio quaecunque inter x et y eaque aliquantillum varietur seu eius loco alia quaepiam relatio inter x et y ab illa infinite parum quomodocunque discrepans adhibeatur, investigari oporteat, quantam mutationem omnes illae expressiones, tam primi, quam secundi et tertii generis sint subiturae, ad quod inveniendum in calculo variationum, prouti equidem eum olim tractavi, praeter differentiale ∂y, quo quantitas y augetur, dum x in $x+dx$ abit, ipsi quantitati y aliud incrementum δy tribuitur penitus ab arbitrio nostro pendens neque per x determinatum, cui incremento variationis nomen indideram atque methodum exposueram variationes inde in singula expressionum genera redundantes inveniendi.

§ 4. Videbatur igitur calculus variationum omnino singulare calculi genus constituere, verum postquam eius indolem accuratius essem perscrutatus, universum hunc calculum perspexi levi facta immutatione ad secundam partem calculi integralis, cuius elementa in tertio volumine operis mei de hoc argumento exposui, reduci posse. Pertractavi autem in ista secunda parte eas integrationes, quae circa functiones duarum variabilium versantur, in quo calculi genere etiam nunc vix ultra prima elementa progredi licuit.

§ 5. Illius scilicet incrementi loco, quod variationem appellavi, ipsam quantitatem y non amplius tamquam functionem solius variabilis x considero, sed eam tamquam functionem binarum variabilium x et t in calculum introduco, sic enim, dum $\partial x\left(\frac{\partial y}{\partial x}\right)$ significat verum differentiale ipsius y, haec formula $\partial t\left(\frac{\partial y}{\partial t}\right)$ idem significare poterit, quod antea signo δy indicavimus. Quo haec reddantur clariora, concipiamus y ut applicatam cuiuspiam curvae abscissae x respondentem, atque in calculo variationum alia relatio requiritur, quae omnes alias curvas huic saltem proximas complectatur, omnes autem huiusmodi curvas, si X denotet illam functionem, cui y aequatur, tali aequatione contineri posse : $y = \mathrm{X} + t\mathrm{V}$ manifestum est, denotante V functionem quamcunque ipsius x. Sumta enim t infinite parva haec aequatio omnes omnino lineas curvas propositae proximas in se comprehendet atque adeo hanc formam multo generaliorem reddere licet, ita ut pro y functio quaecunque binarum variabilium x et t usurpari possit, dummodo ea ita fuerit comparata, ut posito $t = 0$ prodeat ipsa functio proposita $y = \mathrm{X}$.

§ 6. Pro variatione igitur invenienda, quantitas x ut constans spectari, ipsius vero y differentiale tantum ex variabilitate ipsius t desumi debet; unde, si expressio proposita fuerit primi generis functio scilicet ipsarum x et y tantum, quam littera Z designemus, ponamus differentiatione consueta prodire $\mathrm{M}\partial x + \mathrm{N}\partial y$, atque nunc pro variatione invenienda fiat $\partial x = 0$, ad loco ∂y scribatur $\partial t\left(\frac{\partial y}{\partial t}\right)$, quippe quod est incrementum ex sola variabilitate t oriundum. Quo facto variatio quaesita huius expressionis Z erit $= \mathrm{N}\partial t\left(\frac{\partial y}{\partial t}\right)$. Quare, si ipsa variatio simili modo per $\partial t\left(\frac{\partial \mathrm{Z}}{\partial t}\right)$ indicetur, habebimus $\left(\frac{\partial \mathrm{Z}}{\partial t}\right) = \mathrm{N}\left(\frac{\partial y}{\partial t}\right)$.

§ 7. Nunc ad expressiones secundi generis progrediamur, in quibus quum praeter x et y occurrant quantitates p, q, r etc., harum variationes, quatenus y etiam a variabili t pendet, per legem generalem his formulis exprimentur

$$\partial t\left(\frac{\partial p}{\partial t}\right);\ \partial t\left(\frac{\partial q}{\partial t}\right);\ \partial t\left(\frac{\partial r}{\partial t}\right);\ \text{etc.}$$

Quum autem pro sola variabili x, sit

$$p=\left(\frac{\partial y}{\partial x}\right);\ q=\left(\frac{\partial p}{\partial x}\right);\ p=\left(\frac{\partial\partial y}{\partial x^2}\right);\ \text{ et}$$
$$r=\left(\frac{\partial q}{\partial x}\right)=\left(\frac{\partial\partial p}{\partial x^2}\right)=\left(\frac{\partial^3 y}{\partial x^3}\right);\ \text{etc.}$$

erit per regulas generales differentiandi functiones duarum variabilium

$$\left(\frac{\partial p}{\partial t}\right)=\left(\frac{\partial\partial y}{\partial x\partial t}\right);\ \left(\frac{\partial q}{\partial t}\right)=\left(\frac{\partial^3 y}{\partial x^2\partial t}\right);\ \ \left(\frac{\partial r}{\partial t}\right)=\left(\frac{\partial^4 y}{\partial x^3\partial t}\right);\ \text{ etc.}$$

ubi meminisse iuvabit formulam verbi gratia $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$ prodire, si functio *y* ter differentietur et duabus vicibus sola *x,* una vice autem sola *t* variabilis sumatur, tum vero qualibet differentiatione differentialia simplicia ∂x vel ∂t abiiciantur.

§ 8. His expeditis sit iam Z functio quaecunque ipsarum *x, y, p, q, r* etc., hic quidem nullo adhuc respectu habito ad variabilem *t,* quippe quae tantum in subsidium variationis introducitur, atque differentiatione more solito facta prodeat

$$\partial \mathrm{Z}=\mathrm{M}\partial x+\mathrm{N}\partial y+\mathrm{P}\partial p+\mathrm{Q}\partial q+\mathrm{R}\partial r+\text{ etc. };$$

nunc igitur pro variatione seu $\partial t\left(\frac{\partial \mathrm{Z}}{\partial t}\right)$ invenienda scribi debebit, ut sequitur

$$\partial x=0,\ \partial y=\partial t\left(\frac{\partial y}{\partial t}\right),\ \partial p=\partial t\left(\frac{\partial p}{\partial t}\right)=\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right)$$
$$\partial q=\partial t\left(\frac{\partial^3 y}{\partial x^2\partial t}\right);\ \partial r=\partial t\left(\frac{\partial^4 y}{\partial x^3\partial t}\right);\text{ etc.}$$

atque variatio quaesita erit

$$\partial t\left(\frac{\partial \mathrm{Z}}{\partial t}\right)=N\partial t\left(\frac{\partial y}{\partial t}\right)+\mathrm{P}\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\mathrm{Q}\partial t\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\mathrm{R}\partial t\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)+\text{etc.}$$

unde sequitur divisione per ∂t facta fore:

$$\left(\frac{\partial \mathrm{Z}}{\partial t}\right)=N\left(\frac{\partial y}{\partial t}\right)+\mathrm{P}\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\mathrm{Q}\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\mathrm{R}\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)+\text{etc.}$$

§ 9. Sit nunc etiam expressio quaecunque tertii generis proposita $\int \mathrm{Z}\partial x$, ubi Z sit functio quaecunque ipsarum *x, y, p, q, r* etc., ita ut per differentiationem ordinariam habeatur:

$$\partial \mathrm{Z}=\mathrm{M}\partial x+\mathrm{N}\partial y\mathrm{P}\partial p+\mathrm{Q}\partial q+\mathrm{R}\partial r+\text{etc.}$$

ubi quidem hactenus nulla ratio novae variabilis t est habita atque integratio formulae propositae $\int Z\partial x$ per solam variabilem x est expedienda, quo observato quaestio huc redit, ut si iam y ut functio binarum variabilium x et t consideretur et ubique quantitas y elemento $\partial t\left(\frac{\partial Z}{\partial t}\right)$ augeatur, augmentum, quod ipsa formula integralis $\int Z\partial x$ inde capiet, definiatur, hoc enim augmentum ipsa erit variatio formulae integralis propositae.

§ 10. Quare ad hanc variationem inveniendam in functione illa Z ubique loco y scribatur eius valor auctus $y+\partial t\left(\frac{\partial y}{\partial t}\right)$ sicque, ut ante vidimus, ipsa functio Z augmentum capiet $\partial t\left(\frac{\partial Z}{\partial t}\right)$, ex quo ipsa formula integralis augmentum capiet hoc $\int \partial t\left(\frac{\partial Z}{\partial t}\right)\partial x$, quod erit ipsa variatio quaesita. Quoniam vero in hac integratione sola x pro variabili habetur, elementum ∂t ante signum poni poterit, ita ut iam variatio futura sit $=\partial t\int \partial x\left(\frac{\partial Z}{\partial t}\right)$.

§11. Quoniam igitur in paragrapho 8 valor ipsius $\left(\frac{\partial Z}{\partial t}\right)$ iam evolutus habetur, si ille hic substituatur, formulae $\int Z\partial x$ variatio prodibit ita expressa:

$$\partial t\int \partial x\left[\mathrm{N}\left(\frac{\partial y}{\partial t}\right)+\mathrm{P}\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\mathrm{Q}\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\mathrm{R}\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)+\text{etc.}\right]$$

quam etiam sequenti modo per partes repraesentasse iuvabit

$$\partial t\int \mathrm{N}\partial x\left(\frac{\partial y}{\partial t}\right)+\partial t\int \mathrm{P}\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\partial t\int \mathrm{Q}\partial x\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\partial t\int \mathrm{R}\partial x\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)+\text{etc}$$

qua expressione contenti esse possemus, si quaestio circa casum aliquem determinatum institueretur, ubi y non solum functioni cuipiam datae ipsius x aequaretur, sed etiam nova variabilis t modo determinato introduceretur; tum enim omnes istas formulas $\left(\frac{\partial y}{\partial t}\right)$; $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$; $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$; etc. actu evolvere liceret, ita ut tum elementum ∂x per solam functionem ipsius x afficeretur, siquidem uti initio innuimus, evolutione facta iterum poni debet $t=0$.

§ 12. At vero tales quaestiones determinatae nunquam occurrere solent, sed potius relatio inter y et x semper incognita esse solet inde demum determinanda, quod variatio in nihilum abire debeat, quippe in quo Methodus maximorum et minimorum versatur. Huiusmodi quaestiones ergo ita enunciari convenit, qualis relatio inter quantitates x et y intercedere debeat, ut formulae integralis propositae $\int Z\partial x$ variatio in nihilum abeat, quomodocunque etiam nova variabilis t in calculum introducatur? Quodsi autem quaestio hac ratione instituatur, perspicuum est formulis $\left(\frac{\partial y}{\partial t}\right)$; $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$; $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$; etc. nullos certos valores tribui posse.

§ 13. Verum hic prorsus singulare artificium in subsidium vocari potest, cuius ope formulas integrales posteriores in paragrapho 11 ad formam prioris reducere licet, ita ut in omnibus eadem formula $\left(\frac{\partial y}{\partial t}\right)$ occurrat. Quum enim $\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right)$ sit differentiale formulae $\left(\frac{\partial y}{\partial t}\right)$ sumta sola x variabili, erit per consuetam integralium reductionem:

$$\int \mathrm{P}\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right)=\mathrm{P}\left(\frac{\partial y}{\partial t}\right)-\int\partial x\left(\frac{\partial \mathrm{P}}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right),$$

simili modo, quia $\partial x\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$ est differentiale formulae $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$, habebimus statim hanc reductionem

$$\int \mathrm{Q}\partial x\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)=\mathrm{Q}\left(\frac{\partial\partial y}{\partial x\partial t}\right)-\int\partial x\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right),$$

nunc vero per praecedentem reductionem fit

$$\int\partial x\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right)=\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)-\int\partial x\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right),$$

sicque omnino habebimus

$$\int \mathrm{Q}\partial x\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)=\mathrm{Q}\left(\frac{\partial\partial y}{\partial x\partial t}\right)-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)+\int\partial x\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right),$$

atque nunc satis perspicuum est sequentem formulam integralem ita reductum iri:

$$\int \mathrm{R}\partial x\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)=\mathrm{R}\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)-\left(\frac{\partial \mathrm{R}}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right)-\int\partial x\left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right)\left(\frac{\partial y}{\partial t}\right),$$

ac si insuper talis formula adesset, foret

$$\int \mathrm{S}\partial x\left(\frac{\partial^5 y}{\partial x^4\partial t}\right)=\mathrm{S}\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)-\left(\frac{\partial \mathrm{S}}{\partial x}\right)\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\left(\frac{\partial\partial \mathrm{S}}{\partial x^2}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right)$$
$$-\left(\frac{\partial^3 \mathrm{S}}{\partial x^3}\right)\left(\frac{\partial y}{\partial t}\right)+\int\partial x\left(\frac{\partial^4 \mathrm{S}}{\partial x^4}\right)\left(\frac{\partial y}{\partial t}\right).$$

§ 14. Quodsi nunc has formulas reductas substituamus in expressione variationis quaesitae formulae $\int Z\partial x$, tum haec variatio non solum formulis constabit integralibus, sed etiam continebit partes absolutas, quarum aliae formula $\left(\frac{\partial y}{\partial t}\right)$, aliae hanc $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$, aliae vero hanc $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$ etc. continebunt, dum contra omnes integrales eandem formulam $\left(\frac{\partial y}{\partial t}\right)$ involvunt, quocirca variatio quaesita formulae propositae $\int Z\partial x$ sequenti modo habebitur expressa

$$\begin{aligned}
&\partial t\int \partial x\left(\frac{\partial y}{\partial t}\right)\left(\mathrm{N}-\left(\frac{\partial \mathrm{P}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)-\left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right)+\left(\frac{\partial^4 \mathrm{S}}{\partial x^4}\right)-\text{etc.}\right)\\
&+\partial t\left(\frac{\partial y}{\partial t}\right)\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\left(\frac{\partial^3 \mathrm{S}}{\partial x^3}\right)+\left(\frac{\partial^4 \mathrm{T}}{\partial x^4}\right)-\text{etc.}\right)\\
&+\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right)\left(\mathrm{Q}-\left(\frac{\partial \mathrm{R}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{S}}{\partial x^2}\right)-\text{etc.}\right)\\
&+\partial t\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)\left(\mathrm{R}-\left(\frac{\partial \mathrm{S}}{\partial x}\right)+\text{etc.}\right)\\
&+\partial t\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)\left(\mathrm{S}-\text{etc.}\right)\\
&+\text{etc.}
\end{aligned}$$

§ 15. Quamquam hic meum institutum non est methodum maximorum et minimorum pertractare, quoniam hoc alibi iam satis copiose est factum, tamen hic praetermittere non possum, quin observem, si variatio formulae $\int Z\partial x$ evanescere debeat, quomodocunque etiam nova variabilis tin calculum ingrediatur, id nullo modo fieri posse, nisi tota pars prima integralis seorsim evanescat, ex quo necesse est inter x et y hanc aequationem constitui

$$0=\mathrm{N}-\left(\frac{\partial \mathrm{P}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)-\left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right)+\left(\frac{\partial^4 \mathrm{S}}{\partial x^4}\right)-\text{etc.}$$

et quia nunc variabilis t nulla amplius ratio habetur sicque tantum unica adhuc variabilis x superest, clausulis omissis hanc habebimus aequationem :

$$0=\mathrm{N}-\frac{\partial \mathrm{P}}{\partial x}+\frac{\partial\partial \mathrm{Q}}{\partial x^2}-\frac{\partial^3 \mathrm{R}}{\partial x^3}+\frac{\partial^4 \mathrm{S}}{\partial x^4}-\text{etc.}$$

qua desiderata relatio inter x et y exprimitur. Partes autem absolutae tantum ad terminos extremos referuntur, circa quas ea observari debent, quae iam alibi fusius sunt praecepta.

§ 16. Hic etiam non immoror iis casibus, quibus quantitas Z ipsa insuper formulas integrales involvit, quoniam etiam hoc argumentum alibi satis est pertractatum, verum hic opus multo magis arduum molior, dum eandem hanc methodum ad functiones adeo duarum variabilium extendere conabor, quod equidem in dissertatione illa, quam olim de calculo variationum conscripseram, tunc temporis praestare non potui, multitudine tot quantitatum diversi generis deterritus.

Applicatio methodi pracedentis ad functiones duarum variabilium

§ 17. Si habeatur aequatio quaecunque inter ternas variabiles x, y et z, ea naturam cuiuspiam superficiei exprimi censemus, ubi quidem binas coordinatas x et y in plano horizontali constitui intelligamus, tertiam vero z verticalem, sicque haec tertia z ut functio spectari potest binarum x et y ; unde more solito duplicia incrementa consideranda occurrunt, quatenus scilicet a variabilitate ipsius x vel ipsius y nascuntur. Illud nempe incrementum ipsius z, quod ex variatione ipsius x oritur, hac formula: $\partial x\left(\frac{\partial z}{\partial x}\right)$, hoc vero ex variatione ipsius y oriundum ista : $\partial y\left(\frac{\partial z}{\partial y}\right)$ indicari solet.

§ 18. Quodsi iam haec superficies aequatione inter x, y et z expressa cum aliis quibuscunque superficiebus ipsi proximis comparari debeat, id commodissime fiet novam variabilem t introducendo, ita ut iam z spectanda sit ut functio trium variabilium x, y et t, quae quidem sumto $t=0$ in functionem superiorem abeant, at, dum ipsi t valores infinite parvi tribuuntur, omnes superficies proximas complectatur, quo posito perspicuum est, quoniam variabiles x et y a nova t neutiquam pendent, earum differentialia dx et dy nullo modo cum dt permisceri, sola vero coordinata z triplicis generis incrementa capere potest; praeter bina enim iam ante commemorata, quae vel ab x vel ab y proficiscuntur, accipere poterit incrementum a variabilitate ipsius t oriundum, quod tali formula $\partial t\left(\frac{\partial z}{\partial t}\right)$ est repraesentandum.

§ 19. Ponamus nunc V esse expressionem utcunque ex ipsis coordinatis x, y et z compositam, sive per meras operationes algebraicas sive etiam transcendentes formatas, quae more solito differentiata praebeat : $\partial \mathrm{V}=\mathrm{L}\partial x+\mathrm{M}\partial y+\mathrm{N}\partial z$, atque si eiusdem incrementum desideretur a nova variabili t sola oriundum, manifestum est statui debere $\partial x=0$ et $\partial y=0$, at loco ∂z scribi debere $\partial t\left(\frac{\partial z}{\partial t}\right)$ sicque hoc signandi modo usurpato habebimus

$$\partial t\left(\frac{\partial \mathrm{V}}{\partial t}\right)=\mathrm{N}\partial t\left(\frac{\partial z}{\partial t}\right) \text{ ideoque } \left(\frac{\partial \mathrm{V}}{\partial t}\right)=\mathrm{N}\left(\frac{\partial z}{\partial t}\right).$$

Tales autem expressiones ut ante primum genus constituunt.

§ 20. Progrediamur ergo ad secundum genus, quo expressio v praeter ipsas coordinatas x, y, z etiam rationes differentialium earum involvat; atque hic quidem ante omnia formam huiusmodi expressionum accuratius perpendi oportet. Quoniam autem hic statim quantitas z duplicia incrementa capere potest (hic enim nondum ad novam variabilem t respicimus), ponamus brevitatis gratia

$$\left(\frac{\partial z}{\partial x}\right)=p \text{ et } \left(\frac{\partial z}{\partial y}\right)=p'$$

quae duae litterae differentialia primi gradus comprehendunt, deinde pro differentialibus secundi gradus ponamus:

$$\left(\frac{\partial\partial z}{\partial x^2}\right)=q,\ \left(\frac{\partial\partial z}{\partial x\partial y}\right)=q',\ \left(\frac{\partial\partial z}{\partial y^2}\right)=q'',$$

unde sequentes relationes inter has litteras et praecedentes notasse iuvabit :

$$\left(\frac{\partial p}{\partial x}\right)=q,\ \left(\frac{\partial p}{\partial y}\right)=\left(\frac{\partial p'}{\partial x}\right)=q',\ \left(\frac{\partial p'}{\partial y}\right)=q'';$$

simili modo differentialia tertii gradus his formulis complectamur :

$$\left(\frac{\partial^3 z}{\partial x^3}\right)=r,\ \left(\frac{\partial^3 z}{\partial x^3\partial y}\right)=r',\ \left(\frac{\partial^3 z}{\partial x\partial y^2}\right)=r'',\ \left(\frac{\partial^3 z}{\partial y^3}\right)=r''',$$

ubi hae relationes sunt notandae :

$$r=\left(\frac{\partial q}{\partial x}\right),\ r'=\left(\frac{\partial q}{\partial y}\right)=\left(\frac{\partial q'}{\partial x}\right),\ r''=\left(\frac{\partial q'}{\partial y}\right)=\left(\frac{\partial q''}{\partial x}\right),\ r'''=\left(\frac{\partial q''}{\partial y}\right);$$

quarta autem differentialia has formulas praebent:

$$s=\left(\frac{\partial^4 z}{\partial x^4}\right),\ s'=\left(\frac{\partial^4 z}{\partial x^3\partial y}\right),\ s''=\left(\frac{\partial^4 z}{\partial x^2\partial y^2}\right),\ s'''=\left(\frac{\partial^4 z}{\partial x\partial y^3}\right),\ s''''=\left(\frac{\partial^4 z}{\partial y^4}\right)$$

et sic ultra, quousque libuerit.

§ 21. His explicatis expressiones secundi generis praeter ipsas coordinatas *x, y* et *z* etiam quantitates *p, p', q, q', q'', r, r', r'', r'''* etc. utcunque involvere possunt, ex quo, si V denotet quamcunque huiusmodi expressionem, eius differentiale more solito sumtum sequenti forma exhibeamus:

$$\begin{aligned}\partial V = L\partial x + M\partial y + N\partial z + P\partial p &+ Q\partial q &+ R\partial r\\ + P'\partial p' &+ Q'\partial q' &+ R'\partial r'\\ &+ Q''\partial q'' &+ R''\partial r''\\ &&+ R'''\partial r'''\\ &&\text{etc.,}\end{aligned}$$

quam formam animo imprimi conveniet, ne opus sit eam saepius repetere.

§ 22. Quodsi iam huiusmodi expressionum variatio seu id incrementum inveniri debeat, quod resultat ex variatione novae variabilis *t,* quam in valorem coordinatae *z* introducimus, iam vidimus sumi debere $\partial x = 0$ et $\partial y = 0$, tum vero fieri $\partial z = \partial t\left(\frac{\partial z}{\partial t}\right)$, ob eandem vero rationem sequentia differentialia simili modo erunt exprimenda, quae cum suis transformationibus per se claris ita se habebunt :

$$\partial p = \partial t\left(\frac{\partial p}{\partial t}\right) = \partial t\left(\frac{\partial\partial z}{\partial x\partial t}\right),\ \partial p' = \partial t\left(\frac{\partial p'}{\partial t}\right) = \left(\frac{\partial\partial z}{\partial y\partial t}\right),$$
$$\partial q = \partial t\left(\frac{\partial q}{\partial t}\right) = \partial t\left(\frac{\partial^3 z}{\partial x^2\partial t}\right),$$
$$\partial q' = \partial t\left(\frac{\partial q'}{\partial t}\right) = \partial t\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right),\ \partial q'' = \partial t\left(\frac{\partial q''}{\partial t}\right) = \partial t\left(\frac{\partial^3 z}{\partial y^2\partial t}\right),$$
$$\partial r = \partial t\left(\frac{\partial r}{\partial t}\right) = \partial t\left(\frac{\partial^4 z}{\partial x^3\partial t}\right),\quad \partial r' = \partial t\left(\frac{\partial r'}{\partial t}\right) = \partial t\left(\frac{\partial^4 z}{\partial x^2\partial y\partial t}\right),$$
$$\partial r'' = \partial t\left(\frac{\partial r''}{\partial t}\right) = \partial t\left(\frac{\partial^4 z}{\partial x\partial y^3\partial t}\right),\ \partial r''' = \partial t\left(\frac{\partial r'''}{\partial t}\right) = \partial t\left(\frac{\partial^4 z}{\partial y^3\partial t}\right) \text{ etc.}$$

§ 23. Totum ergo negotium huc redit, ut in formula illa differentiali pro $\partial \mathrm{V}$ data loco singulorum differentialium isti valores substituantur, hocque modo prodibit variatio expressionis V ex sola variabilitate ipsius *t* oriunda seu valor huius formulae $\partial t\left(\frac{\partial \mathrm{V}}{\partial t}\right)$, quoniam autem singula membra elemento ∂t erunt affecta, eo omisso adipiscimur sequentem formam :

$$\begin{aligned}\left(\frac{\partial \mathrm{V}}{\partial t}\right) = \mathrm{N}\left(\frac{\partial z}{\partial t}\right) + \mathrm{P}\left(\frac{\partial\partial z}{\partial x\partial t}\right) + \mathrm{Q}\left(\frac{\partial^3 z}{\partial x^2\partial t}\right) + \mathrm{R}\left(\frac{\partial^4 z}{\partial x^3\partial t}\right)&\\ + \mathrm{P}'\left(\frac{\partial\partial z}{\partial y\partial t}\right) + \mathrm{Q}'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right) + \mathrm{R}'\left(\frac{\partial^4 z}{\partial x^2\partial y\partial t}\right)&\\ + \mathrm{Q}''\left(\frac{\partial^3 z}{\partial y^2\partial t}\right) + \mathrm{R}''\left(\frac{\partial^4 z}{\partial x\partial y^2\partial t}\right)&\\ + \mathrm{R}'''\left(\frac{\partial^4 z}{\partial y^3\partial t}\right),&\\ \text{etc.,}&\end{aligned}$$

quae ad variationes quarumcunque expressionum secundi generis inveniendas sufficit.

§ 24. Nunc expressiones tertii generis adgredi poterimus formulas integrales involventes; in quibus potissimum vis huius methodi cernitur. Quando enim quaestio circa maxima vel minima, quae in superficiebus occurrere possunt, versatur, formula illa, quae maximum vel minimum reddi debet, necessario est formula integralis atque adeo formula integralis duplicata, cuius indolem hic paucis explicari convenit. Quemadmodum enim in praecedente parte formulae integrales simplices sunt consideratae, quae ad datam abscissam *x* sunt relatae, ita hic in superficiebus quaestiones semper non ad solam abscissam *x,* sed ad totum quoddam spatium in plano horizontali tanquam basem sunt referendae, cui portio superficiei, quae maximi minimive quadam proprietate gaudere debet, immineat. Quare, quum talis basis duplicem habeat dimensionem, alteram ab *x,* alteram vero ab *y* pendentem, huiusmodi formulae integrales erunt duplicatae hoc modo exprimi solitae $\iint V\partial x\partial y$; eae scilicet duplicem integrationem postulant atque in priore sola coordinata *x* vel sola *y* pro variabili habetur et integratio usque ad terminos basis propositae extenditur, tum vero demum etiam altera variabilis assumitur atque altera integratio absolvitur. Et quoniam perinde est, utra prius pro variabili habeatur, sine discrimine geminam illam integrationem signo duplicato $\iint$ indicamus; neque vero hic loci est omnia, quae circa huiusmodi integrationes duplicatas sunt observanda, fusius exponere, quippe quod argumentum in Tomo XIV. horum Commentariorum iam satis accurate est pertractatum.

§ 25. Quodsi ergo huiusmodi formulae integralis $\iint V\partial x\partial y$ variatio quaeri debeat, ubi V denotat expressionem quamcunque vel primi vel secundi generis, ex superioribus satis liquet hanc variationem ita expressum iri:

$$\partial t\iint\left(\frac{\partial V}{\partial t}\right)\partial x\partial y\,,$$

quae forma iterum est integralis duplicata et, prouti vel *x* vel *y* priore integratione ut constans spectatur, ea formula vel hoc modo

$$\partial t\int\partial x\int\left(\frac{\partial V}{\partial t}\right)\partial y$$

vel hoc modo

$$\partial t\int\partial y\int\left(\frac{\partial V}{\partial t}\right)\partial x$$

exhiberi potest.

§ 26. Sit nunc V talis expressio, qualem supra paragrapho 19 descripsimus et cuius variationem seu valorem $\left(\frac{\partial V}{\partial t}\right)$ in paragrapho 23 evolvimus, tantum opus erit singula membra ibi exposita hoc loco $\left(\frac{\partial V}{\partial t}\right)$ substituere; unde sequens congeries formularum integralium nascetur, quibus iunctim sumtis variatio quaesita $\partial t\iint\left(\frac{\partial V}{\partial t}\right)\partial x\partial y$ exprimetur:

$$\partial t\iint \mathrm{N}\left(\tfrac{\partial z}{\partial t}\right)\partial x\partial y+\partial t\iint \mathrm{P}\left(\tfrac{\partial\partial z}{\partial x\partial t}\right)\partial x\partial y+\partial t\iint \mathrm{Q}\left(\tfrac{\partial^3 z}{\partial x^2\partial t}\right)\partial x\partial y+\partial t\iint \mathrm{R}\left(\tfrac{\partial^4 z}{\partial x^3\partial t}\right)\partial x\partial y$$
$$+\partial t\iint \mathrm{P}'\left(\tfrac{\partial\partial z}{\partial y\partial t}\right)\partial x\partial y+\partial t\iint \mathrm{Q}'\left(\tfrac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x\partial y+\partial t\iint \mathrm{R}'\left(\tfrac{\partial^4 z}{\partial x^2\partial y\partial t}\right)\partial x\partial y$$
$$+\partial t\iint \mathrm{Q}''\left(\tfrac{\partial^3 z}{\partial y^2\partial t}\right)\partial x\partial y+\partial t\iint \mathrm{R}''\left(\tfrac{\partial^4 z}{\partial x\partial y^2\partial t}\right)\partial x\partial y$$
$$+\partial t\iint \mathrm{R}'''\left(\tfrac{\partial^4 z}{\partial y^3\partial t}\right)\partial x\partial y,$$
etc.,

§ 27. Nunc singula haec membra post primum peculiares reductiones admittunt, quas probe notasse iuvabit. Pro secunda membro sumamus primo x tantum variabile, eritque:

$$\int \mathrm{P}\left(\tfrac{\partial\partial z}{\partial x\partial t}\right)\partial x=\mathrm{P}\left(\tfrac{\partial z}{\partial t}\right)-\int\left(\tfrac{\partial z}{\partial t}\right)\partial x\left(\tfrac{\partial \mathrm{P}}{\partial x}\right),$$

unde etiam alteram integrationem adiiciendo erit :

$$\iint \mathrm{P}\left(\tfrac{\partial\partial z}{\partial x\partial t}\right)\partial x\partial y=\int \mathrm{P}\left(\tfrac{\partial z}{\partial t}\right)\partial y-\iint\left(\tfrac{\partial z}{\partial t}\right)\left(\tfrac{\partial \mathrm{P}}{\partial x}\right)\partial x\partial y.$$

Pro tertio membra sumatur prima sola y variabilis, eritque:

$$\int \mathrm{P}'\left(\tfrac{\partial\partial z}{\partial y\cdot\partial t}\right)\partial y=\mathrm{P}'\left(\tfrac{\partial z}{\partial t}\right)-\int\left(\tfrac{\partial z}{\partial t}\right)\partial y\left(\tfrac{\partial \mathrm{P}'}{\partial x}\right),$$

unde ipsum tertium membrum transibit in

$$\iint \mathrm{P}'\left(\tfrac{\partial\partial z}{\partial y\cdot\partial t}\right)\partial x\partial y=\int \mathrm{P}'\left(\tfrac{\partial z}{\partial t}\right)\partial x-\iint\left(\tfrac{\partial z}{\partial t}\right)\left(\tfrac{\partial \mathrm{P}'}{\partial y}\right)\partial x\partial y.$$

§ 28. Pro sequentibus membris hae ipsae reductiones sequentes dabunt transformationes, pro quarto scilicet habebimus ex secunda

$$\iint \mathrm{Q}\left(\tfrac{\partial^3 z}{\partial x^2\partial t}\right)\partial x\partial y=\int \mathrm{Q}\left(\tfrac{\partial\partial z}{\partial x\partial t}\right)\partial y-\iint\left(\tfrac{\partial\partial z}{\partial x\partial t}\right)\left(\tfrac{\partial \mathrm{Q}}{\partial x}\right)\partial x\partial y,$$

at vero hoc membrum posterius ad similitudinem secundi reducitur hoc modo, ubi tantum loco P scribi debet $\left(\tfrac{\partial \mathrm{Q}}{\partial x}\right)$:

$$\int\left(\tfrac{\partial z}{\partial t}\right)\left(\tfrac{\partial \mathrm{Q}}{\partial x}\right)\partial y-\iint\left(\tfrac{\partial z}{\partial t}\right)\left(\tfrac{\partial\partial \mathrm{Q}}{\partial x^2}\right)\partial x\partial y,$$

ita ut nunc quartum membrum praebeat hanc formam :

$$\int Q\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial y-\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q}{\partial x}\right)\partial y+\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q}{\partial x^2}\right)\partial x\partial y.$$

Simili modo quintum membrum ope secundi reducitur, ubi loco P scribitur Q' et loco $\left(\frac{\partial\partial z}{\partial x\partial t}\right)$, $\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)$, sive loco $\left(\frac{\partial z}{\partial t}\right)$ scribendo $\left(\frac{\partial\partial z}{\partial y\partial t}\right)$, sicque habebitur

$$\iint Q'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x\partial y=\int Q'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial y-\iint\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial Q'}{\partial x}\right)\partial x\partial y,$$

quod posterius membrum cum tertio conferatur, ubi loco tantum P′ scribi debet $\left(\frac{\partial Q'}{\partial x}\right)$, quo pacto totum membrum induet hanc formam:

$$\int Q'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial y-\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q'}{\partial x}\right)\partial x+\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q'}{\partial x\partial y}\right)\partial x\partial y,$$

sextum vero membrum bis cum secundo collatum reducitur ad hanc formam:

$$\int Q''\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial x-\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q''}{\partial y}\right)\partial x+\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q''}{\partial y^2}\right)\partial x\partial y,$$

§29. Si hoc modo ulterius progrediamur ad sequentia membra, septimum membrum in sequentes partes resolvitur :

$$\int R\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial y-\int\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial R}{\partial x}\right)\partial y+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R}{\partial x^2}\right)\partial y-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R}{\partial x^3}\right)\partial x\partial y,$$

deinde octavum membrum

$$\int R'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial y-\int\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial R'}{\partial x}\right)\partial x+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R'}{\partial x\partial y}\right)\partial y-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R'}{\partial x^2\partial y}\right)\partial x\partial y,$$

tum nonum membrum fiet

$$\int R''\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x-\int\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial R''}{\partial y}\right)\partial y+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R''}{\partial x\partial y}\right)\partial x-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R''}{\partial x\partial y^2}\right)\partial x\partial y,$$

et decimum

$$\int R'''\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\partial x-\int\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial R'''}{\partial y}\right)\partial x+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R'''}{\partial y^2}\right)\partial x-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R'''}{\partial y^3}\right)\partial x\partial y.$$

§30. Colligamus nunc omnes istas formulas in unam summam atque variatio quaesita pluribus constabit membris, quarum primum formulas integrales duplicatas, reliqua vero simplices complectentur, hoc pacto variatio quaesita sequenti modo erit expressa :

$$\partial t\iint \partial x\partial y\left(\frac{\partial z}{\partial t}\right)\left\{\begin{array}{r} \mathrm{N}-\left(\frac{\partial \mathrm{P}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right)-\left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right)\\ -\left(\frac{\partial \mathrm{P'}}{\partial y}\right)+\left(\frac{\partial\partial \mathrm{Q'}}{\partial x\partial y}\right)-\left(\frac{\partial^3 \mathrm{R'}}{\partial x^2\partial y}\right)\\ +\left(\frac{\partial\partial \mathrm{Q''}}{\partial y^2}\right)-\left(\frac{\partial^3 \mathrm{R''}}{\partial x\partial y^2}\right)\\ -\left(\frac{\partial^3 \mathrm{R'''}}{\partial y^3}\right)\\ \text{etc.}\end{array}\right\}$$

$$+\partial t\left\{\begin{array}{r} \int\left(\frac{\partial z}{\partial t}\right)\mathrm{P}\partial y+\int \mathrm{Q}\partial y\left(\frac{\partial\partial z}{\partial x\partial t}\right)-\int\partial y\left(\frac{\partial \mathrm{Q}}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right)+\int \mathrm{R}\partial y\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\\ \int\left(\frac{\partial z}{\partial t}\right)\mathrm{P'}\partial x+\int \mathrm{Q'}\partial y\left(\frac{\partial\partial z}{\partial y\partial t}\right)-\int\partial x\left(\frac{\partial \mathrm{Q'}}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right)+\int \mathrm{R'}\partial y\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\\ +\int \mathrm{Q''}\partial x\left(\frac{\partial\partial z}{\partial y\partial t}\right)-\int\partial x\left(\frac{\partial \mathrm{Q''}}{\partial y}\right)\left(\frac{\partial z}{\partial t}\right)+\int \mathrm{R''}\partial x\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\\ +\int \mathrm{R'''}\partial x\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\\ -\int\partial y\left(\frac{\partial \mathrm{R}}{\partial x}\right)\left(\frac{\partial\partial z}{\partial x\partial t}\right)+\int\partial y\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)\\ -\int\partial x\left(\frac{\partial \mathrm{R'}}{\partial x}\right)\left(\frac{\partial\partial z}{\partial x\partial t}\right)+\int\partial y\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R'}}{\partial x\partial y}\right)\\ -\int\partial y\left(\frac{\partial \mathrm{R''}}{\partial y}\right)\left(\frac{\partial\partial z}{\partial y\partial t}\right)+\int\partial x\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R''}}{\partial x\partial y}\right)\\ -\int\partial x\left(\frac{\partial \mathrm{R'''}}{\partial y}\right)\left(\frac{\partial\partial z}{\partial y\partial t}\right)+\int\partial x\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial \mathrm{R'''}}{\partial y^2}\right)\\ \text{etc.}\end{array}\right\}$$

§ 31. Verum quid haec singula membra proprie significent et ad quemnam usum adhiberi queant, neutiquam adhuc perspicere licet, unde hoc argumentum, cuius prima fundamenta etiamnunc vix iacta sunt, censenda omnem Geometrarum attentionem atque multo accuratiorem investigationem postulare videtur, quod negotium vix ante suscipere licet, quam casus nonnulli particulares omni studio et diligentia fuerint evoluti, quin etiam ipsa pars prior, quae tantum circa functiones unius variabilis versatur, neutiquam adhuc satis clare et distincte est enucleata, ita ut perspicue intelligeremus veram indolem atque naturam singularum partium, quibus variationem contineri invenimus, quem in finem dilucidationes sequentes hic adiungere visum est.

Dilicudationes super theoria variationum ad
functiones saltem unius variabilis
accommodata.

§32. Quaestiones, quae hic occurrunt, ad hoc problema generale revocare licet:

Explicatio ipsius Problematis

Si y fuerit functio quaecunque ipsius x indeque definiatur valor cuiuspiam formulae integralis datae $\int Z\partial x$, *denotante* Z *expressionem ex ipsis quantitatibus x et y earumque differentialium rationibus utcunque compositam, quaestio est, si loco illius functionis y alia quaecunque illi proxima seu infinite parum tantum ab ea discrepans adhibeatur, quanto maiorem minoremve valorem tum eadem formula integralis* $\int Z\partial x$ *sit consecutura.*

§33. At quia hoc modo ista quaestio enunciata nimis videri posset abstracta, eam more soluto ad Geometriam revocemus. Sit igitur super axe AP (Fig. 1) proposita curva quaecunque AM aequatione inter abscissam $\mathrm{AP} = x$ et applicatam $\mathrm{PM} = y$ expressa, pro qua definiri oporteat valorem formulae cuiuspiam integralis $\int Z\partial x$, qui sit $= \mathrm{W}$, quo posito consideretur alia curva quaecunque $\alpha\mu$ infinite parum a data discrepans, ac si pro hac curva itidem definiatur valor formulae $\int Z\partial x$, quaeritur, quantum iste valor a praecedente sit discrepaturus ; evidens enim est hoc discrimen praebere ipsam variationem quantitatis W, quam supra ope calculi variationum exhibuimus.

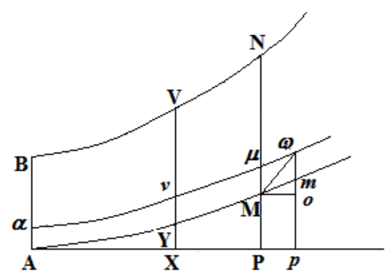

Fig. 1

§ 34. Quo haec adhuc clariora evadant, exemplum quodpiam proferamus, quo proposita curva AM eiusque axe AP tamquam verticali considerato quaeritur tempus, quo corpus ex puncto A super hac curva AM descendens usque ad punctum M pertingit. Iam, quia celeritas corporis in M est ut $\sqrt{\mathrm{AP}} = \sqrt{x}$ et ipsum curvae elementum $= dx\sqrt{(1+pp)}$, posito scilicet $dy = pdx$, uti in solutione generali est praeceptum, erit tempus per elementum $\mathrm{M}m = dx\frac{\sqrt{(1+pp)}}{\sqrt{x}}$, unde formula integralis $\int Z\partial x$ pro hoc casu abit in $\int \partial x\frac{\sqrt{(1+pp)}}{\sqrt{x}}$ ita ut habeatur $Z = \frac{\sqrt{(1+pp)}}{\sqrt{x}}$ quare nunc tempus erit definiendum, quo corpus super curva quacunque proxima $\alpha\mu$ descendens ab α usque ad μ perveniet, ubi discrimen dabit ipsam variationem formulae $\int \partial x\frac{\sqrt{(1+pp)}}{\sqrt{x}}$ huic casui convenientem.

§ 35. Quoniam hic formula integralis consideranda venit, ante omnia dispiciendum est, quomodo eam determinari oporteat. In exemplo quidem allato manifestum est formulae $\int \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$ integrale ita capi debere, ut evanescat posito $x=0$, unde etiam in genere intelligitur semper pro integratione formulae $\int \mathrm{Z}\partial x$ certum aliquem terminum vel uti punctum A tamquam principium integrationis statui atque integrale $\int \mathrm{Z}\partial x$ evanescere debere posito $x=0$ vel, si forte circumstantiae aliter fuerint comparatae, tribuendo ipsi x valorem quempiam datum ; deinde vero initio constituto valor formulae $\int \mathrm{Z}\partial x = \mathrm{W}$ abscissae $\mathrm{AP}=x$ respondebit.

§ 36. His circa formulam integralem $\int \mathrm{Z}\partial x$ observatis videamus, quamnam ideam nobis de curvis illis proximis $\alpha\mu$, formare debeamus. Ac primo quidem patet has curvas continuo quodam tractu ductas esse debere, ita ut in iis nusquam anguli aliive saltus deprehendantur; hoc solo notato perinde est, sive istae curvae lege quapiam continuitatis vel aequatione quapiam contineantur, sive sint adeo discontinuae, quasi libero manus motu ductae.

§ 37. Huiusmodi lineae curvae commodissime sequenti modo formatae menti repraesentari possunt. Ducatur scilicet pro lubitu linea curva quaecunque BN eidem abscissae AP imminens, ac ductis ad singula axis puncta X applicatis XYV singula intervalla YV in ratione finiti ad infinite parvum secentur in v, ita ut Yv sit quasi pars infinitesima intervalli YV. Hoc enim modo curva $\alpha v\mu$ obtinebitur a curva proposita AM in omnibus punctis infinite parum dissita, qualem ad institutum nostrum requirimus. Praeterea tamen notandum est in curva illa arbitraria BN nusquam tangentem ad axem AP normalem esse debere, quia hoc modo divisio illorum intervallorum turbaretur. Atque nunc evidens est non solum intervalla Yv esse infinite parva, sed etiam tangentes in punctis Y et v infinite parum a parallelismo deficere.

Explicatio partis primae in variatione

§ 38. His circa ipsam quaestionis propositionem annotatis contemplemur nunc accuratius quoque solutionem supra inventam eiusque singulas partes, ut, quid quaelibet earum innuat et ad quemnam usum sit transferenda, perspicue intelligamus; solutionem autem in paragrapho 14 datam hic contemplabimur. Statim igitur consideremus primam variationis ibi inventae partem, quae hac formula integrali continetur

$$\partial t \int \partial x \left(\frac{\partial y}{\partial t}\right)\left(\mathrm{N} - \left(\frac{\partial \mathrm{P}}{\partial x}\right) + \left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right) - \left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right) + \left(\frac{\partial^4 \mathrm{S}}{\partial x^4}\right) - \text{etc.}\right),$$

cuius integratio ita capi debet, ut in ipso termino initiali A evanescat, qua conditione constans arbitraria determinatur ; quodsi ergo in singulis punctis XY haec formula applicata intelligatur, aggregatum omnium istarum formularum elementarium ab initio

A usque ad terminum M extensum praebebit primam partem variationis quaesitae, atque hic quidem in Figura perspicuum est spatiolum Yv exprimere incrementum applicatae y a sola variabili t oriundum, ita ut sit $\mathrm{Y}v = dt\left(\frac{\partial y}{\partial t}\right)$.

§ 39. Haec igitur prima pars variationis involvit omnia spatiola v intra terminos A et M contenta, quae quum in infinitum variari possint atque adeo a positivis ad negativa transire queant, maximae variationes hic locum habere possunt. Verum tamen unicus casus hinc debet excipi, quo curva AM ita est comparata, ut sit

$$0 = \mathrm{N} - \left(\frac{\partial \mathrm{P}}{\partial x}\right) + \left(\frac{\partial\partial \mathrm{Q}}{\partial x^2}\right) - \left(\frac{\partial^3 \mathrm{R}}{\partial x^3}\right) + \left(\frac{\partial^4 \mathrm{S}}{\partial x^4}\right) - \text{etc.},$$

tum enim, utcunque curvae proximae fuerint comparatae, ista pars prima variationis semper in nihilum abit. Neque deviatio curvarum proximarum $\alpha\mu$ a principali AM intra terminos A et M quicquam ad variationem confert; ex quo haec curva respectu formulae integralis $\int \mathrm{Z}\partial x$ inprimis est memorabilis, quandoquidem in ea haec formula integralis vel maximum vel minimum obtinet valorem.

Explicatio partis secundae in variatione

§ 40. Progrediamur nunc ad secundam partem variationis supra inventae, quae est

$$\partial t\left(\frac{\partial y}{\partial t}\right)\left(\mathrm{P} - \left(\frac{\partial \mathrm{Q}}{\partial x}\right) + \left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right) - \left(\frac{\partial^3 \mathrm{S}}{\partial x^3}\right) + \text{etc.}\right),$$

circa quam primum observo, quoniam ea ad terminum M refertur per integrationem rite institutam, insuper adiici debere similem expressionem ad terminum priorem A relatam, at vero signo contrario affectam, id quod ideo est necessarium, ut facto $x = 0$ etiam haec expressio penitus tollatur. Refertur autem ista pars

$$\partial t\left(\frac{\partial y}{\partial t}\right)\left(\mathrm{P} - \left(\frac{\partial \mathrm{Q}}{\partial x}\right) + \left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right) - \text{etc.}\right)$$

unice ad ultimum terminum M, ubi $\partial t\left(\frac{\partial y}{\partial t}\right)$ ipsum spatiolum Mμ exprimit, similique modo in alteram partem pro initio A spatiolum Aα ingredietur. Hinc patet, si omnes curvae proximae $\alpha\mu$ per ipsos ambos terminos A et M ducantur, tum variationem secundae partis in nihilum abire.

§ 41. Consideremus autem casum, quo curva proxima $\alpha\mu$ per primum quidem terminum A transit, non vero quoque per alterum M, sed sit punctum μ eius terminus, atque variatio ex secunda parte nata erit

$$= \mathrm{M}\mu\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\text{etc.}\right).$$

Atque hinc etiam definire poterimus variationem ex eodem fonte oriundam, si curva proxima Aμ non in ipso puncto μ, sed alio quocunque ω terminetur, existente semper intervallo $\mu\omega$ infinite parvo. Ducta enim applicata ωmp variatio modo inventa insuper augeri debet particula formulae $\int \mathrm{Z}\partial x$, quae elemento $\mathrm{P}p = dx$ respondet, quae particula quum sit $= \mathrm{Z}\cdot \mathrm{P}p$, pro arcu curvae proximae Aω erit variatio ex secunda parte oriunda

$$= \mathrm{M}\mu\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\text{etc.}\right)+\mathrm{Z}\cdot\mathrm{P}p.$$

§ 42. Ducatur recta Mω et quaeramus angulum *wMm,* quem haec recta Mω cum curva principali constituit, ponatur iste angulus $\omega \mathrm{M}m = \omega$ et ducta MO ipsi Pp parallela, quia est proxime $m\omega = \mathrm{M}\mu$ et anguli $m\mathrm{M}o$ tangens p ideoque $om = p\cdot \mathrm{P}p$, habebitur $o\omega = \mathrm{M}\mu + p\cdot \mathrm{P}p$, unde fit:

$$\text{tang } \omega \mathrm{M}o = \frac{\mathrm{M}\mu}{\mathrm{P}p} + p,$$

atque hinc colligitur

$$\omega \mathrm{M}m = \text{tang } \omega = \frac{\mathrm{M}\mu}{\mathrm{P}p(1+pp)+\mathrm{M}\mu\cdot p}.$$

Servemus nunc in calculo hunc ipsum angulum ω atque hinc habebimus spatiolum:

$$\mathrm{M}\mu = \frac{\mathrm{P}p(1+pp)\text{tang } \omega}{1-p\text{tang } \omega},$$

quo valore substituto variatio pro arcu Aω erit :

$$\mathrm{P}p\left(\mathrm{Z}+\frac{(1+pp)\text{tang } \omega}{1-p\text{tang } \omega}\cdot\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\text{etc.}\right)\right).$$

§ 43. Nunc operae pretium erit eum angulum ω definire, ut ista variatio in nihilum abeat, id quod eveniet, si capiatur

$$\text{tang } \omega = \frac{\mathrm{Z}}{p\mathrm{Z}-(1+pp)\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\text{etc.}\right)},$$

quare hoc angulo ita constituto pro omnibus lineis proximis ubicunque in recta Mω terminatis variatio ex secunda parte oriunda evanescet. Hic casus prae caeteris omnino notatu dignus considerari meretur, quo recta Mω fit ad curvam principalem in puncto M normalis, quod evenit, si fuerit

$$p\mathrm{Z}-(1+pp)\left(\mathrm{P}-\left(\frac{\partial \mathrm{Q}}{\partial x}\right)+\left(\frac{\partial\partial \mathrm{R}}{\partial x^2}\right)-\text{etc.}\right)=0,$$

qua aequatione certa conditio ipsius formulae integralis $\int Z\partial x$ sive indoles expressionis *Z* definitur.

§ 44. Non igitur pigebit in talem expressionem *Z* inquisivisse, ac primo quidem patet eam praeter coordinatas *x* et *y* etiam quantitatem *p* involvere debere. Sumamus autem praeterea in *Z* non ingredi litteras *q, r* etc., ita ut sit $\mathrm{Q}=0,\ \mathrm{R}=0$, ac nostra aequatio resolvenda erit:

$$p\mathrm{Z}=(1+pp)\mathrm{P}\ ,$$

ubi notandum est esse

$$\partial \mathrm{Z}=\mathrm{M}\partial x+\mathrm{N}\partial y+\mathrm{P}\partial p,$$

quare, si ambae coordinatae *x* et *y* tanquam constantes tractentur, erit

$$\partial \mathrm{Z}=\mathrm{P}\partial p \text{ ideoque } \mathrm{P}=\frac{\partial \mathrm{Z}}{\partial p},$$

quo valore ibi introducto haec prodibit aequatio

$$\frac{d\mathrm{Z}}{\mathrm{Z}}=\frac{pdp}{1+pp},$$

quae integrata dat

$$\mathrm{LZ}=\mathrm{L}\sqrt{(1+pp)}+\mathrm{LC},$$

quae constans functio quaecunque ipsarum *x* et *y* esse potest, talis functio sit V atque habebimus

$$\mathrm{Z}=\mathrm{V}\sqrt{(1+pp)}$$

ideoque formula integralis

$$=\int \mathrm{V}\partial x\sqrt{(1+pp)}\ .$$

Huius formulae significatum satis eleganter per tempus, quo corpus quodpiam per curvam AM promovetur, exprimi potest. Si enim celeritas in puncto M fuerit $=\frac{1}{\mathrm{V}}$, hoc est si celeritas in singulis punctis proportionalis fuerit functioni cuicunque binarum variabilium *x* et *y,* tum $\mathrm{V}\partial x\sqrt{(1+pp)}$ exprimit elementum temporis ideoque formula

$$=\int \mathrm{V}\partial x\sqrt{(1+pp)}$$

totum tempus, quo corpus ab A ad M pervenit.

Explicatio partis tertiae in variatione

§ 45. Quod ad tertiam partem variationis attinet, scilicet

$$\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right)\left(\mathrm{Q}-\left(\frac{\partial\mathrm{R}}{\partial x}\right)+\left(\frac{\partial\partial\mathrm{S}}{\partial x^2}\right)-\text{etc.}\right),$$

ea locum non habet, nisi expressio Z etiam differentialia secundi gradus involvat, quod quidem rarissime usu venire solet. Hic autem observandum est, quoniam $\mathrm{M}\mu=\partial t\left(\frac{\partial y}{\partial t}\right)$, fore pro sequenti elemento

$$m\omega=\partial t\left(\frac{\partial y}{\partial t}\right)+\partial t\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right),$$

unde colligitur

$$\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right)=\frac{m\omega-\mathrm{M}\mu}{\partial x}=\frac{\mathrm{M}\omega-\mathrm{M}\mu}{\mathrm{P}p},$$

hac autem formula exprimitur declinatio directionis $\mu\omega$ a directione M*m,* quae quidem, ut iam ante observavimus, semper est quam minima.

§ 46. Quodsi ergo tangens in μ perfecte fuerit parallela tangenti in M*,* quod evenit, si etiam in curva generatrice BN tangens ad N huic fuerit parallela, tum variatio ex tertia parte oriunda prorsus evanescit, quod etiam de termino initiali A est intelligendum, si tangentes in A et B inter se fuerint parallelae, atque hinc iam perspicitur, ut variationes ex quarta parte oriundae evanescant, necesse esse, ut praeterea etiam radii osculi in punctis M et μ fiant aequales.

§ 47. Atque ex his iam satis perspicuum est variationes ex secunda parte oriundas evanescere, si omnes curvae proximae $\alpha\mu$ per utrumque terminum M et A ducantur, deinde vero insuper etiam variationes tertiae partis, si omnes curvae proximae simul in utroque termino A et M cum curva principali AM communes habeant tangentes. Praeterea vero quoque variationes quartae partis in nihilum abire, si omnes curvae proximae in terminis A et M insuper ratione curvaturae cum curva principali conveniant. Hic autem probe meminisse iuvabit variationes tertiae partis per se evanescere, si modo quantitas Z non differentialia secundi gradus involvat ; quartae vero partis semper evanescere, nisi differentialia tertii gradus in quantitatem Z ingrediantur, et ita porro. Unde quum initio ostenderimus, quomodo variatio primae partis ad nihilum sit redigenda, nunc evidentissime intelligimus, sub quibusnam conditionibus omnes variationis partes simul evanescant.

Dilucidationes circa curvas maximi minimive proprietate praeditas

§ 48. Si formula integralis $\int \mathrm{Z}\partial x$ in curva quaesita debeat esse vel maximum vel minimum, iam supra ostendimus posito

$$\partial Z = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

naturam huius curvae hac exprimi aequatione :

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \text{etc.},$$

quae aequatio, nisi quantitates P, Q, R evanescant vel sint constantes, semper est differentialis vel secundi vel quarti vel sexti aliusve gradus paris. Hic ergo statim memoratu dignum occurrit, quod ista aequatio nunquam vel simpliciter differentialis vel tertii vel quinti aliusve gradus imparis evadat, id quod mox clarius exponemus.

§ 49. Quaestiones ergo huc pertinentes sponte in varias dividuntur pro gradu differentialium, ad quem aequationes exsurgunt, quandoquidem ab hoc gradu natura solutionis maxime pendet, propterea quod ea semper totidem constantes arbitrarias involvit. Ad primam ergo classem referimus eos casus, quibus aequatio pro maximo vel minimo inventa prorsus est finita. Ad secundam autem classem eos, quibus haec aequatio fit differentialis secundi gradus, ad tertiam eos, quibus aequatio ad quartum gradum ascendit, et ita porro, quas singulas classes ordine describamus.

Classis I.

§ 50. Ad solutionem ergo primae classis formula $\int Z\partial x$ statim perducit, quando expressio Z tantum per coordinatas x et y exclusis omnium differentialium rationibus determinatur ; quia enim hoc casu simpliciter fit $\partial Z = M\partial x + N\partial y$, aequatio pro curva maximi vel minimi erit $N = 0$, quae ergo aequatio omnino est determinata atque adeo curva satisfaciens unica in suo genere. Veluti, si quaeratur linea, in qua valor formulae $\int \partial x(2xy - yy)$ fiat maximus vel minimus, ob $Z = 2xy - yy$ ideoque $N = 2(x - y)$ aequatio quaesita erit $x - y = 0$ seu linea quaesita erit recta ad axem angulo semirecto inclinata, pro qua ergo valor formulae propositae integralis est x, qui utique minor est, quam si ulla alia linea curva sumeretur pro eadem scilicet abscissa.

§ 51. His autem casibus prima classis nondum exhauritur, sed dantur adhuc alii perinde ad aequationes finitas ducentes, ad quod ostendendum sit $\mathfrak{Z}$ functio quaecunque ipsarum x et y atque $\partial\mathfrak{Z} = \mathfrak{M}\partial x + \mathfrak{N}\partial y$, iamque ponatur $Z = \mathfrak{Z}p$, eritque $M = \mathfrak{M}p,\ N = \mathfrak{N}p,\ P = \mathfrak{Z}$, quare, ut formula $\int Z\partial x$ fiat maximum vel minimum, aequatio reperitur:

$$0 = \mathfrak{N}p - \frac{\partial\mathfrak{Z}}{\partial x} = \mathfrak{N}p - \mathfrak{M} - \frac{\mathfrak{N}\cdot\partial y}{\partial x} = -\mathfrak{M},$$

quae itidem est aequatio finita. Quod quidem etiam statim praevidere licuisset, quum enim sit $p\partial x = \partial y$, haec formula integralis $\int \mathfrak{Z}\partial y$ a praecedente

$\int Z\partial x$ aliter non differt, nisi quod coordinatae x et y sint permutatae, unde, quod de priore erat affirmatum, etiam de posteriore valet.

Hinc natura primae classis adhuc generalius ita describi potest, ut ea complectatur omnes formulas integrales huiusmodi $\int(Z+\mathfrak{Z}p)\partial x$, ubi litterae Z et $\mathfrak{Z}$ denotant functiones quascunque ipsarum x et y, tum enim aequatio pro curva maximi vel minimi erit :

$$0=N-\mathfrak{M},$$

quae est aequatio omnino determinata.

Classis II.

§ 52. Ad classem secundam referimus eas formulas integrales $\int Z\partial x$, quae deducunt ad aequationem differentialem secundi gradus; huc ergo primo pertinent casus, quibus Z tantum ex litteris x, y et p componitur, ita ut sit

$$\partial Z=M\partial x+N\partial y+P\partial p,$$

unde quidem casum posteriorem primae classis excipere oportet, quippe quod evenit, si P fuerit functio tantum ipsarum x et y, ita ut pro praesenti casu quantitas P praeter x et y etiam litteram p complecti debeat. Tum autema equatio pro curva quaesita erit $0=N-\frac{\partial P}{\partial x}$, ubi, quum P involvat p ideoque $\partial\cdot\left(\frac{\partial x}{\partial y}\right)$, formula $\frac{\partial P}{\partial x}$ continebit differentialia secundi gradus ; haec ergo aequatio neutiquam est determinata, quum duas adeo constantes arbitrarias recipiat, quibus effici potest, ut curva per data duo puncta transeat, atque adeo quaestiones huius classis ita accuratius sunt definiendae, ut curvae investigentur, quae non inter omnes plane curvas, sed inter eas tantum, quae per eadem duo puncta ducuntur, praescripta maximi minimive proprietate gaudeant ; semper autem quaestiones huius classis ita sunt comparatae, ut per naturam suam hanc restrictionem postulent.

§ 53. Praeterea vero etiam ad secundam classem referri oportet casus, quibus $Z=\mathfrak{Z}q$ existente $\mathfrak{Z}$ functione quacunque ipsarum x, y et p, si enim fuerit

$$\partial\mathfrak{Z}=\mathfrak{M}\partial x+\mathfrak{N}\partial y+\mathfrak{P}\partial p,$$

habebimus

$$M=\mathfrak{M}q,\ N=\mathfrak{N}q,\ P=\mathfrak{P}q \text{ et } Q=\mathfrak{Z},$$

quare, quum aequatio pro curva sit

$$0=N-\tfrac{\partial P}{\partial x}+\tfrac{\partial\partial Q}{\partial x^2}\ \text{sive}\ 0=N-\tfrac{\partial}{\partial x}\partial\cdot\left(P-\tfrac{\partial Q}{\partial x}\right),$$

formula haec $\mathrm{P}-\frac{\partial \mathrm{Q}}{\partial x}$ abit in

$$\mathfrak{P}q-\frac{\partial \mathfrak{Z}}{\partial x}=\mathfrak{P}q-\mathfrak{M}-\mathfrak{N}p-\mathfrak{P}q=-\mathfrak{M}-\mathfrak{N}p,$$

unde aequatio nostra evadet

$$0=\mathrm{N}+\frac{\partial}{\partial x}\partial\left(\mathfrak{M}+\mathfrak{N}p\right)=2\mathfrak{N}q+\frac{\partial \mathfrak{M}}{\partial x}+p\frac{\partial \mathfrak{N}}{\partial x},$$

quae manifesto tantum differentialia secundi gradus continet. Generalius ergo adhuc, si formula integralis proposita fuerit $\int\left(\mathrm{Z}+\mathfrak{Z}p\right)\partial x$, ubi Z et $\mathfrak{Z}$ quomodocunque ex quantitatibus *x, y* et *p* sint compositae, aequatio pro curva quaesita erit

$$0=\mathrm{N}-\frac{\partial \mathrm{P}}{\partial x}+2\mathfrak{N}q+\frac{\partial \mathfrak{M}}{\partial x}+p\frac{\partial \mathfrak{N}}{\partial x}$$

sive etiam

$$0=\mathrm{N}\partial x-\partial \mathrm{P}+2\mathfrak{N}\partial p+\partial \mathfrak{M}+p\partial \mathfrak{N},$$

quae manifesto tantum est differentialis secundi gradus.

Classis III.

§ 54. At si quantitas Z ita ex litteris *x, y, p* et *q* fuerit composita, ut posito

$$\partial \mathrm{Z}=\mathrm{M}\partial x+\mathrm{N}\partial y+\mathrm{P}\partial p+\mathrm{Q}\partial q$$

etiam quantitas Q involvat litteram *q,* tum huiusmodi casus ad tertiam classem erunt referendi, et quum aequatio pro curva quaesita reperiatur

$$0=\mathrm{N}-\frac{\partial \mathrm{P}}{\partial x}+\frac{\partial\partial \mathrm{Q}}{\partial x^{2}},$$

evidens est terminum $\frac{\partial\partial \mathrm{Q}}{\partial x^{2}}$ involvere differentialia quarti gradus, unde aequatio finita pro curva implicabit quatuor constantes arbitrarias, quibus ergo effici potest, ut curva desiderata non solum per datos duos terminos transeat, sed etiam eius tangentes in utroque termino datam obtineant positionem, in qua quadruplici determinatione natura quaestionum ad hanc classem pertinentium continetur et accuratissime perspicitur.

§ 55. Reliquis casibus ad hanc classem pertinentibus non immoror, verum potius illustrationis caussa insigne adferam exemplum, quo curvae elasticae investigari solent. Scilicet (Fig. 1) si littera ρ denotet radium osculi curvae quaesitae in puncto M, omnes hae curvae hac gaudent proprietate, ut in iis haec formula $\int\frac{\partial x\sqrt{(1+pp)}}{\rho\rho}$ sit minimum ideoque habeatur

$$Z=\frac{\sqrt{(1+pp)}}{\rho\rho}, \text{ quum vero sit } \rho=\frac{(1+pp)^{3:2}}{q},$$

habebimus $Z=\frac{qq}{(1+pp)^{3:2}}$, unde fit

$$M=0,\ N=0,\ P=-\frac{5pqq}{(1+pp)^{7:2}} \text{ et } Q=+\frac{2q}{(1+pp)^{5:2}},$$

quare, quum ob $N=0$ aequatio pro curvis quaesitis sit:

$$0=-\frac{\partial P}{\partial x}+\frac{\partial\partial Q}{\partial x^2},$$

eius integrale statim praebet

$$P-\frac{\partial Q}{\partial x}=A,$$

quae adhuc est differentialis tertii gradus.

§ 56. Verum haec aequatio adhuc in genere integrari potest, multiplicetur enim per $qdx=dp$, ut habeatur haec aequatio $Pdp-qdQ=Adp$, quum vero sit $dZ=Pdp+Qdq$, erit $Pdp=dZ-Qdq$, quo valore substituto aequatio resultat haec:

$$\partial Z-Q\partial q-q\partial Q=A\partial p\ ,$$

cuius integrale manifesto est

$$Z-Qq=Ap+B,$$

nunc igitur pro Z et Q valores supra dati substituantur atque nanciscemur sequentem aequationem:

$$-\frac{qq}{(1+pp)^{5:2}}=Ap+B;$$

mutatis igitur signis constantium colligemus

$$qq=(Ap+B)(1+pp)^{5:2}$$

ideoque

$$q=(1+pp)^{5:4}\sqrt{(Ap+B)}=\frac{\partial p}{\partial x},$$

sicque concludimus

$$\partial x=\frac{\partial p}{(1+pp)^{5:4}\sqrt{(Ap+B)}}$$

hincque porro

$$\partial y=\frac{p\partial p}{(1+pp)^{5:4}\sqrt{(Ap+B)}}$$

quibus duabus aequationibus constructio curvae absolvitur.

§ 57. Cum olim haec Methodus maximorum et minimorum tractari est coepta, non solum eiusmodi curvae sunt investigatae, in quibus formula quaepiam integralis $\int Z\partial x$ esset vel maximum vel minimum, sed etiam eiusmodi quaestiones proponebantur, ut non inter omnes plane curvas, sed inter eas tantum, quae habeant eandem longitudinem, ea quaeratur, in qua illa formula fiat maxima vel minima, ex quo ipso casu nomen Problematis Isoperimetrici est natum; hoc autem nomen non impedivit, quominus eiusmodi quaestiones generaliores proponerentur, ut inter omnes eas curvas, quibus valor certae cuiuspiam formulae integralis $\int V\partial x$ aeque conveniat, ea definiatur, in qua formula $\int Z\partial x$ maximum minimumve sortiatur valorem, quin etiam conditiones adhuc fuerunt multiplicatae in hunc modum, ut tantum inter omnes eas curvas, quibus non solum formula $\int V\partial x$, sed etiam hae quotcunque $\int V'\partial x$, $\int V''\partial x$ etc. aequaliter competant, ea definiatur, in qua $\int Z\partial x$ sit maximum vel minimum, cuiusmodi problemata tum temporis summopere ardua sunt visa. Postquam vero in tractatu meo de hoc argumento ostendissem huismodi problemata semper reduci posse ad hoc problema simplex, quo inter omnes plane lineas ea investigetur, in qua haec formula integralis

$$\int \partial x\left(Z+\alpha V+\beta V'+\gamma V''+\text{etc.}\right)$$

fiat maximum vel minimum, huius generis problemata nullam amplius habent difficultatem.